HYPERPLANE SECTIONS OF POLYHEDRA, TOROIDAL MANIFOLDS, AND DISCRETE GROUPS IN LOBACHEVSKII SPACE

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A bounded polyhedron is called *simple*, if it is the intersection of half-spaces in general position. In this paper we estimate the number and the proportion of k-dimensional faces of a simple n-dimensional polyhedron, which can intersect a hyperplane not passing through a vertex of it. Here is an example of a result: a generic hyperplane cannot intersect more than $P/3 + 2$ edges of a simple three-dimensional polyhedron with P edges. This estimate is sharp, i.e., there exist polyhedra (with arbitrarily large number of edges) and their sections, for which the bound is achieved.

One has

THEOREM 1. In Lobachevskii space of dimension >995 there do not exist discrete groups generated by reflections with fundamental polyhedron of finite volume.

For groups with compact fundamental polyhedron the analogous result was found by Nikulin (arithmetic case [1, 2]) and Vinberg (general case [3, 4]). It is based on Nikulin's estimate of the average number of l-dimensional faces on a k-dimensional face of a multidimensional simple polyhedron. After the work of Prokhorov done in 1984 (cf. [5] and Sec. 6) to prove Theorem 1, only an analog of Nikulin's estimate for polyhedra whose simplicity fails only at vertices was lacking. Such an analog is found in Sec. 5 from the theorems on hyperplane sections of polyhedra (all the results of the paper arose from attempts to get it). Thus, Theorem 1 is proved (cf. Sec. 6) as a result of the combined conditions of Prokhorov and the author.

Nikulin's estimate is based on the properties of simple polyhedra, proved with the help of toroidal manifolds (cf. Sec. 1). The connection of toroidal geometry and the geometry of polyhedra, modeled on the polyhedral calculations of the cohomology of toroidal manifolds, lies at the foundation of the present paper. Each theorem on the combinatorics of hyperplane section of polyhedra given has a precise analog in the geometry of toroidal manifolds.

On the other hand, for all assertions about the geometry of polyhedra which occur (except for the assertion about the growth up to the middle coefficient of the H-polynomial) one can find elementary proofs in the paper. It is not necessary to be familiar with toroidal geometry to understand them.

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1. Polyhedra and Toroidal Manifolds

Here we give familiar facts about the geometry of simple manifolds and smooth toroidal manifolds in the form we need.

1.1. By slightly perturbing the highest dimensional faces of any convex polyhedron, it can be turned into a simple one. Through each vertex of a simple n-dimensional polyhedron in R^n pass exactly n highest dimensional faces and exactly n edges.

By the F-polynomial of a simple n-dimensional polyhedron is meant the generating polynomial of the sequence of numbers of its faces of different dimensions, i.e., $F(t) = \Sigma F_k t^k$, where F_k is the number of k-dimensional faces of the polyhedron. By the H-polynomial of a simple polyhedron is meant the polynomial $H(t) = F(t - 1)$. The coefficients of the F- and H-polynomials can be expressed in terms of one another, namely $F_k = \Sigma C_m^k H_m$ and $H_k = \Sigma(-1)^{m-k} \times C_m^k F_m$.

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By the P-polynomial of a simple n-dimensional polyhedron is meant the polynomial of degree $[n/2]$, which is generating for the sequence of numbers $P_0 = H_0$, $P_i = H_i - H_{i-1}$, $i = 1,\dots,[n/2]$.

Assertion 1. For an n-dimensional simple polyhedron:

1) the H-polynomial is recurrent, i.e., $H_i = H_{n-i}$;

2) the coefficients of the H-polynomial increase up to the middle (i.e., the coefficients of the P-polynomial are nonnegative), while $H_0 = H_n = P_0 = 1$.

COROLLARY 1. 1) From the P-polynomial one can reconstruct the F- and H-polynomials; 2) the coefficients of the H-polynomial are positive.

Properties 1) and 2) together with the inequality $P_{i+1} \leqslant Q^i(P_i)$ for $1 \leqslant i < [n/2]$, in which Q^i is a special function of a natural number argument, are necessary and sufficient conditions on H-polynomials of simple polyhedra (i.e., are a criterion for the existence of a simple polyhedron with given numbers of faces of all dimensions). This criterion was formulated by McMullen [6] and proved by Billera and Lee [7] and Stanley [8]. Property 1) has an elementary proof [9] and is called the Dehn–Sommerville theorem. The proof of property 2) noted in point 1.3 was discussed (in different terms) in 1977 at V. I. Arnol'd's seminar.

1.2. An n-dimensional algebra manifold (over C) on which the group $(C \setminus 0)^n$ acts, is called *toroidal*, if it has an open dense orbit isomorphic with the group.

By the F-polynomial of a projective toroidal manifold is meant the generating polynomial of the sequence of numbers of its orbits of different dimensions, i.e., $F(t) = F_k t^k$, where F_k is the number of k-dimensional orbits (over C). By the H-polynomial is meant the polynomial $H(t) = F(t - 1)$. By the P-polynomial of an n-dimensional manifold is meant the generating polynomial of the sequence of numbers $P_0 = H_0$, $P_i = H_i - H_{i-1}$, $i = 1,\dots,[n/2]$.

Assertion 2 (cf. [10, 11]). 1) The closure of a k-dimensional orbit of a smooth n-dimensional toroidal manifold is a smooth manifold, where each 0-dimensional orbit lies in the closure of exactly n orbits of dimension $(n - 1)$. 2) A smooth toroidal manifold has only even Betti numbers b_{2i}, and the polynomial $\Sigma b_{2i} t^i$ coincides with the H-polynomial of the manifold. 3) On a toroidal manifold there do not exist holomorphic forms of nonzero degree. 4) The coefficient P_i coincides with the dimension of the primitive cohomology of the manifold of dimension 2i.

Property 4) follows from property 2). Property 3) is easily proved directly. It follows from it that the numbers $h^{p,0}$ are equal to zero for $p > 0$ for smooth toroidal manifolds and also that the first Betti number $b_1 = h^{1,0} + h^{0,1} = 0$. The calculations in point 2.2 give, in particular, a derivation of property 2) from properties 1) and 3). The results of the present paper were guessed thanks to these calculations.

1.3. Let us assume in addition that there is fixed a Kähler metric on a toroidal manifold, which is invariant with respect to the action of the real torus $T^n \subset (C \setminus 0)^n$. To each vector ξ of the Lie algebra $\mathscr{L}T^n$ corresponds a Hamiltonian vector field in the symplectic structure connected with the Kähler metric, with, since $b_1 = 0$, globally defined Hamiltonian h_ξ. The Hamiltonians h_ξ depend linearly on the vector ξ, so to each point x of the manifold corresponds a linear form M_x on $\mathscr{L}T^n$ defined by $M_x(\xi) = h_\xi(x)$. This correspondence defines the *moment map* (defined up to a constant summand) of the toroidal manifold into the dual space of the algebra $\mathscr{L}T^n$.

Assertion 3 [12, 13]. 1) The critical points of the Hamiltonian h_ξ form a (disconnected) submanifold, while the restriction of the Hessian of the function h_ξ to each transversal to the tangent subspace to the submanifold is a nondegenerate quadratic form (i.e., h_ξ is a Bott function) with even index. 2) The image of the manifold under the moment map is a convex polyhedron, the moment map establishes a one-to-one correspondence between the set of orbits of the manifold and the set of faces of the polyhedron, while a k-dimensional orbit (over C) is mapped with constant rank onto an open k-dimensional face (over R) of the polyhedron.

COROLLARY 2. The image of a nonsingular toroidal manifold under the moment map is a simple polyhedron.

For the proof one must compare point 1) of Assertion 2 and point 2) of Assertion 3.

We prove Assertion 1) for a polyhedron which is the image under the moment map of a smooth toroidal manifold. The F-, H-, and P-polynomials of such a polyhedron coincide with the F-, H-, and P-polynomials of the toroidal manifold. The symmetry of the H-polynomial of a toroidal manifold follows from Poincaré duality, the increase of the coefficients to the middle follows from the Levshits theorem, and the equality $H_0 = H_n = 1$ from the connectedness of the manifold. This proof extends to polyhedra which are images under the moment maps of *quasismooth* (cf. [14]) toroidal manifolds (for such manifolds Assertions 1 and 2) remain valid. It is straightforward to show that any simple polyhedron can be turned by an arbitrarily small deformation into the image of a quasismooth manifold. Hence the arguments given are sufficient to prove Assertion 1.

2. Linear Function on a Simple Polyhedron

We call a linear function on a convex polyhedron generic, if it is not constant on any edge of the polyhedron. Any linear function can be turned by a slight change into a generic one.

2.1. We say that the vertex b of a simple n-dimensional polyhedron has *index* i(b) with respect to some generic linear function L, if the function L is decreasing on exactly i edges issuing from this vertex [and consequently, on the remaining (n − i) edges issuing from this vertex it is increasing]. We denote by h(i) the number of vertices of the polyhedron having index i. One has

THEOREM 2. The number h(i) coincides with the i-th coefficient H_i of the H-polynomial of the polyhedron. In other words, the number h(i) are connected with the numbers F_k of k-dimensional faces of the polyhedron by the relations $F_k = \Sigma C_i^k h(i)$ for $k = 0,\ldots,n$.

Proof. We consider the map which associates with each k-dimensional face of the polyhedron the vertex at which the linear function L achieves the maximum on this face. Under this map each vertex of index i is hit by exactly C_i^k k-dimensional faces. In fact, at the maximum point on a k-dimensional face, the function decreases on k edges issuing from this vertex. Conversely, to a collection of k edges issuing from one vertex, on each of which the function decreases, corresponds a k-dimensional face for which this vertex is maximum — in a simple polyhedron, each collection of k edges issuing from one vertex spans a k-dimensional face. Summing over all vertices the number of preimages of the mapping produced, we get the formula $F_k = \Sigma C_i^k h(i)$.

COROLLARY 1. The numbers h(i) are independent of the choice of generic linear function.

COROLLARY 2. The numbers H_i and H_{n-i} are equal.

Proof. A vertex of index i for the function L has index n − i for the function −L. Hence the number h(i), calculated for the function L, coincides with the number h(n − i), calculated for the function −L.

We have found a new proof of the theorem of Dehn–Sommerville on the equality of H_i and H_{n-i}. It models the proof of Poincaré duality on a smooth manifold by considering Morse functions f and −f.

COROLLARY 3. For a simple n-dimensional polyhedron the numbers H_i for $i = 0,\ldots,n$ are strictly positive.

In fact, for an arbitrary vertex of the polyhedron there exists a generic linear function having any index from 0 to n at this vertex.

COROLLARY 4. The number H_i for any face does not exceed the number H_i for the polyhedron.

In fact, let L be a function which achieves a minimum on a given face, and $\tilde{L}$ be a generic function close to L. Then for any vertex of this face the index of the function $\tilde{L}$ on the polyhedron coincides with its index for the restriction of $\tilde{L}$ to this face.

We note that the upper bound conjecture [8] is easily derived from Corollary 4.

2.2. We consider the moment map M of a Kählerian toroidal manifold into $\mathcal{L}^*T^n$ and a linear function L on $\mathcal{L}^*T^n$ which is generic relative to polyhedron-image.

LEMMA 1. The function $L \circ M$ is a Morse function on the toroidal manifold. The critical points of the function $L \cdot M$ are 0-dimensional orbits of the manifold. The index of a critical point of the function $L \circ M$ is equal to twice the index of the corresponding vertex of the polyhedron-image with respect to the function L.

Proof. The linear function L on $\mathcal{L}^*T^n$ corresponds to some vector ξ of the algebra $\mathcal{L}T^n$, and the function $L \circ M$ is the Hamiltonian h_ξ. The Bott function h_ξ has no critical points on orbits of positive dimension (since the function L is not constant on faces of positive dimension, and the map of an orbit of a face has constant rank). The critical points of the function h_ξ can only be zero-dimensional orbits, and hence it is a Morse function. Let the index of the function L on the polyhedron at the vertex b, corresponding to a critical point, be equal to i. In the polyhedron there exist i-dimensional and (n − i)-dimensional faces on which the function L achieves a maximum and minimum, respectively, at the vertex b. The preimages of these faces are nonsingular 2i-dimensional and 2(n − i)-dimensional submanifolds (over R), on which the function $L \circ M$ achieves a maximum and minimum, respectively, at the critical point. The existence of such submanifolds proves the last assertion of the lemma.

COROLLARY 6. The H-polynomial of a toroidal manifold is equal to $\Sigma b_{2i} t^i$, where b_{2i} is its 2i-dimensional Betti number.

In fact, the Hamiltonian $L \circ M$ has critical points of even indices. According to Morse theory the Betti number b_{2i} is equal to the number of critical points of index 2i, which, according to the lemma, is equal to h(i). By Theorem 2, $h(i) = H_i$.

Thus we have found a new proof of property 3 of Assertion 1 in point 1.2. (In fact we have proved more: we have actually produced a basis for the homology groups consisting of algebraic cycles. The equation $b_{2i} = h^{i,i}$ and the explicit description of the intersection ring follow from this (cf. [11])).

3. Linear-Fractional Programming

Here we formulate and solve the special problems of linear-fractional programming needed in Sec. 4. Linear-fractional programming is the maximization on a convex polyhedron of the ratio of two linear functions L_1/L_2 (it is assumed that the denominator L_2 does not vanish on the polyhedron). Linear-fractional programming differs slightly from linear programming: by a projective transformation carrying the hyperplane $L_2 = 0$ to infinity one can turn a linear-fractional function into a linear one. Under such a transformation a convex polyhedron goes into another convex polyhedron, and the problem of linear-fractional programmong goes into a problem of linear programming. *The maximum of a linear-fractional function is achieved on some face of the polyhedron.* (In the case of general position this face is zero-dimensional and is one of the vertices of the polyhedron.)

The following is obvious.

Assertion 1. Let B_1 and B_2 be positive and $A_1/B_1 > A_2/B_2$. Then for nonnegative numbers α_1, α_2, not both equal to zero, one has $A_1/B_1 \geqslant (\alpha_1 A_1 + \alpha_2 A_2)/(\alpha_1 B_1 + \alpha_2 B_2) \geqslant A_2/B_2$. Here the first (second) inequality becomes an equality if and only if $\alpha_2 = 0$ ($\alpha_1 = 0$).

Let B be a finite set of points with a fixed integer-valued function $i: B \to \mathbf{Z}$, assuming values from 0 to n. We consider the space $R^{|B|}$, whose coordinate functions are in one-to-one correspondence with the points of the set B. We denote the coordinate function corresponding to the point b by x_b. For nonnegative integers l and k, we define a function $\Phi_{l,k}$ on $R^{|B|}$ by

$$\Phi_{l,k} = \Big[\sum_{b\in B} x_b C^l_{i(b)} + (1 - x_b) C^l_{n-i(b)}\Big] \Big/ \Big[\sum_{b\in B} x_b C^k_{i(b)} + (1 - x_b) C^k_{n-i(b)}\Big].$$

The function $\Phi_{l,k}$ is defined if there exists at least one point $b \in B$, for which $k \leqslant \min \times (i(b), n - i(b))$. We shall assume this condition holds.

Problem 1. Find the maximum of the function $\Phi_{l,k}$ on the standard cube $0 \leqslant x_b \leqslant 1$ of the space $R^{|B|}$.

We denote by q the number [n/2] and by A the face of the standard cube defined by the conditions: $x_b(A) = 1$, if $i(b) < q$, and $x_b(A) = 0$, if $i(b) > q$.

THEOREM 3. For $0 \leqslant l < k < q$ the maximum of the function $\Phi_{l,k}$ on the standard cube of the space $R^{|B|}$ is achieved on the face A.

The proof of the theorem uses the following two properties of the binomial coefficients.

Assertion 2. 1) Let $l < k$. Then the ratio $\psi(m) = C^l_m / C^k_m$, defined for $m \geqslant k$, is strictly monotone decreasing for increasing m. 2) Let $A > Q \geqslant B$. Then the ratio $\varphi(h) = (C^h_A - C^h_B)/C^h_Q$, defined for $0 \leqslant h \leqslant Q$, is strictly monotone increasing for increasing h.

Proof of the Assertion. 1) $\psi(m) = k!(m - l)(m - l - 1) \ldots (m - k + 1) l!$. As m increases the denominator increases. 2) $\varphi(h) = A!/Q!\,(A - h) \ldots (Q - h + 1) - B!\,(Q - h) \ldots (B - h + 1)/Q!$, the products (A − h) ... (Q − h + 1) and (Q − h) ... (B − h + 1) contain numbers of factors which are independent of h, which decrease as h increases.

We proceed to the proof of the theorem which we give in three steps. 1. The value of the function $\Phi_{l,k}$ on the face A is greater than or equal to C_q^l/C_q^k. In fact, according to point 1 of Assertion 2, the ratios C_k^l/C_k^k, $C_{k+1}^l/C_{k+1}^k, \ldots, C_q^l/C_q^k$ decrease. It remains to use Assertion 1.2 several times. The value of the function $\Phi_{l,k}$ on the face A is greater than at any neighboring vertex (i.e., than at any vertex joined by an edge to the face A). In fact, with the exception of one coordinate, all coordinates of neighboring vertices satisfy the same restrictions as the coordinates of the points of the face A. Let this exceptional coordinate correspond to the point $b \in B$. The value i(b) of the functions at this point b we denote by i. Two cases are possible: i < q and i > q. In the first of these cases, upon passage to a neighboring vertex one adds to the numerator of the function $\Phi_{l,k}$ the number $C_{n-i}^l - C_i^l$, and to the denominator, the number $C_{n-i}^k - C_i^k$. We show that the ratio of these numbers is less than $\Phi_{l,k}(A)$. According to Assertion 1 this is sufficient to prove point 2 in the case considered. By point 1 we have $\Phi_{l,k}(A) \geqslant C_q^l/C_q^k$. Further, $(C_{n-i}^l - C_i^l)/(C_{n-i}^k - C_i^k) < C_q^l/C_q^k$. In fact, the inequality considered is equivalent with the relation $(C_{n-i}^l - C_i^l)/C_q^l < (C_{n-i}^k - C_i^k)/C_q^k$. The validity of this relation follows from point 2) of Assertion 2 for A = n − i, Q = q, B = i (i < [n/2], n − 1 > [n/2], q = [n/2]) and values of h equal to l and k. The second case (i > q) can be analyzed analogously to the first. 3) We verified that the values of the function $\Phi_{l,k}$ at vertices which neighbor the face A are smaller than on the face A. According to the theory of linear (and hence also linear-fractional) programming, this is possible only if the function achieves its maximum on the face A. The theorem is proved.

For what follows we are interested in the case of Problem 1 in which B is the set of vertices of a simple n-dimensional polyhedron Δ, and the function i is the index of some generic linear function on the polyhedron. We express the maximum in Problem 1 in terms of the H- and P-polynomials of the polyhedron. For i = 0,...,n, we set i* = min (i, n − i) and we denote by $M_j(k, n)$ the sum

$$\sum_{j \leqslant i \leqslant n-j} C_{i*}^k = C_{[(n+1)/2]}^{k+1} + C_{[(n+2)/2]}^{k+1} - 2C_j^{k+1},$$

in particular, $M_0(k, n) = C_{[(n+1)/2]}^{k+1} + C_{[(n+2)/2]}^{k+1}$. For nonnegative k ⩽ [n/2] we define the number $G_k(\Delta)$ by the formula

$$G_k(\Delta) = \sum_{i \leqslant [n/2]} C_i^k H_i + \sum_{i > [n/2]} C_{n-i}^k H_i.$$

In terms of the P-polynomial this formula assumes the form

$$G_k(\Delta) = \sum P_j M_j(k, n).$$

<u>Assertion 3.</u> In the case of Problem 1 described, the maximum is equal to $G_l(\Delta)/G_k(\Delta)$.

The proof follows quickly from Theorem 3.

<u>Problem 2.</u> Find the maximum of the function

$$\sum_{0 \leqslant j \leqslant q} P_j M_j(l, n) / \sum_{0 \leqslant j \leqslant q} P_j M_j(k, n)$$

under the conditions $P_0 > 0$, $P_1 \geqslant 0, \ldots, P_q \geqslant 0$.

<u>THEOREM 4.</u> For $0 \leqslant l < k \leqslant q$ the maximum of the function of Problem 2 is equal to $M_0(l, n)/M_0(k, n)$ and is achieved for $P_{l+1} = \ldots = P_q = 0$.

<u>Proof.</u> 1) The ratios $M_j(l, n)/M_j(k, n)$ decrease monotonically as j increases from k to q. In fact, according to point 1) of Assertion 2, the ratios C_m^l/C_m^k decreases monotonically as j increases from k to q. Now for the proof it suffices to use Assertion 1 several times. 2) The ratios $M_j(l, n)/M_j(k, n)$ decrease monotonically as j increases from 0 to k. In fact, for increasing j the numerators decrease and the denominators remain unchanged. 3) To complete the proof of the theorem it suffices to use Assertion 1 several times.

For integers k and j such that 0 < k ⩽ q, 0 ⩽ j ⩽ q, q = [n/2], we set $N_j(k, n) = \sum_{j \leqslant i \leqslant n-j} C_i^k = C_{n+1-j}^{k+1} - C_j^{k+1}$, and in particular, $N_0(k, n) = C_{n+1}^{k+1}$.

Problem 3. Find the maximum of the function

$$\sum_{0\leqslant j\leqslant q} P_j N_j(k,\ n) / \sum_{0\leqslant j\leqslant q} P_j M_j(k,\ n)$$

under the conditions $P_0 > 0$, $P_1 \geqslant 0, \ldots, P_q \geqslant 0$.

THEOREM 5. For $0 < k \leqslant q$ the maximum of the function of problem 3 is equal to $N_0(k,\ n)/M_0(k,\ n)$ and is achieved for $P_1 = \ldots = P_q = 0$.

Proof. 1) The ratio $N_j(k,\ n)/M_j(k,\ n)$ increases monotonically as j decreases from q to k. In fact, upon passing from j + 1 to j one adds $C^k_{n-j} + C^k_j$ to the numerator, and $2C^k$ to the denominator. The ratio of these is equal to $(1 + C^k_{n-j}/C^k_j)/2$, which grows as j decreases. Each of them is bigger than $N_q(k,\ n)/M_q(k,\ n)$. Point 1) follows from this. 2) The ratio $N_j(k,\ n)/M_j(k,\ n)$ increases monotonically as j decreases from k to 0 — only the numerator increases. 3) To complete the proof of the theorem it suffices to use Assertion 1 several times.

4. Hyperplane Sections of Simple Polyhedra

The goal of Sec. 4 is to get a lower bound for the number and proportion of k-dimensional faces of a simple polyhedron which are not intersected by a hyperplane section of the polyhedron which does not pass through its vertices.

4.1. By slightly moving a hyperplane which does not pass through a vertex one can get it to be a level surface L = c of a generic linear function L on the polyhedron. We denote by $O(c)$ and $\Pi(c)$ the set of vertices of the polyhedron at which the value of the function is, respectively, less than and greater than c. The union of these sets coincides with the set of vertices and the polyhedron since by hypothesis the hyperplane does not pass through vertices.

THEOREM 6. The number $F_k(c)$ of k-dimensional faces in a simple n-dimensional polyhedron which do not intersect the hyperplane L = c, is determined by

$$F_k(c) = \sum_{b\in O(c)} C^k_{i(b)} + \sum_{b\in \Pi(c)} C^k_{n-i(b)}. \tag{*}$$

Proof. The set of k-dimensional faces which do not intersect the hyperplane L = c, splits into two subsets: the subset of faces on which the function L is strictly less than c and the subset of faces on which it is strictly greater than c. The number of faces in the first set is equal to the first summand. For the proof it is necessary to associate with each face from the first set a vertex at which the restriction of the function L to this face achieves a maximum, and to calculate how many faces correspond to one vertex. (An analogous calculation is made in the proof of Theorem 2.) Associating to a face from the second set a minimum point, we see that the number of faces in the second set is equal to the second summand. The theorem is proved. We say that a hyperplane which does not pass through vertices of a simple n-dimensional polyhedron dissects it successfully if it intersects all faces of dimension greater than [n/2].

COROLLARY 1. The generic hyperplane L = c successfully dissects a simple n-dimensional polyhedron, if and only if all vertices of index less than n/2 lie in the set $O(c)$ and all vertices of index greater than n/2 lie in the set $\pi(c)$.

The proof is found by application of Theorem 6 for k = [n/2] + 1.

COROLLARY 2. A generic hyperplane does not intersect certain [n/2]-dimensional faces of a simple n-dimensional polyhedron.

Proof. A vertex of index [n/2] makes a nonzero contribution to the number $F_{[n/2]}(c)$ independently of which of the two sets $O(c)$ and $\Pi(c)$ it lies in.

Remark. The assertion of Corollary 2 is also valid for nonsimple polyhedra. Moreover, for any hyperplane intersecting a convex n-dimensional polyhedron, there exist two faces lying on different sides of the hyperplane, the total dimension of which is $\geqslant(n - 1)$.

THEOREM 7. Let the hyperplane L = c not pass through vertices of the simple n-dimensional polyhedron Δ. Then for $0 \leqslant l < k \leqslant [n/2]$ one has $F_k(c) \geqslant G_k(\Delta)$ and $F_l(c)/F_k(c) \leqslant G_l \times (\Delta)/G_k(\Delta)$ [cf. Sec. 3 for the definition of the number $G(\Delta)$]. For a successful section of the polyhedron all these estimates are sharp. Conversely, if one of these estimates is sharp (for some k or for some pair l, k from the range indicated), then the section is successful.

We say that simple polyhedra are F-equivalent, if they have the same number of faces of each dimension.

COROLLARY 3. For a successful section of a simple polyhedron the numbers $F_k(c)$ and $F_l(c)/F_k(c)$ for $0 \leqslant l < k \leqslant n/2$ in the class of generic sections of F-equivalent simple polyhedra achieve, respectively, the minimum and maximum.

Proof of Theorem 7. Theorem 6 calculates the numbers $F_k(c)$ in terms of the sets $O(c)$ and $\Pi(c)$. We split the set of vertices of the polyhedron into the union of two disjoint sets O and Π and for each partition and nonnegative number $k \leqslant [n/2]$ by the formula analogous to (*), in which instead of the sets $O(c)$ and $\Pi(c)$ one has the sets O and Π, respectively. It is clear that the number $F_k(O)$ is minimal for precisely those partitions for which all vertices of index less than n/2 lie in the set O, and all vertices of index greater than n/2 lie in the set Π. Precisely for these partitions the number $F_l(O)/F_k(O)$ is maximal — this follows from Theorem 3. To complete the proof it remains to use Corollary 1.

4.2. Examples. 1) For a convex n-gon in the plane $H_0 = H_2 = 1$, $H_1 = n - 2$. Lines not passing through vertices of an n-gon do not intersect at least $n - 2$ of its sides. The estimate is reached only for a successful section, i.e., only for a section which intersects the polyhedron. 2) For a simple three-dimensional polyhedron, having B vertices and P edges, $H_0 = H_3 = 1$, $H_1 = H_2 = n - 1$, where $n = P/3 = B/2$ (in particular, one has the identity $2P = 3B$). For $n = 1$ the polyhedron is a simplex, for $n \geqslant 2$ it is F-equivalent with a prism with an n-gon as its base. According to Theorem 7, a plane not passing through the vertices of a simple three-dimensional polyhedron intersects no more than $P/3 + 2$ of its edges. By tilting the middle section of a prism one can arrange that it intersect the upper and lower bases and all lateral faces would be intersected as before. Here we get a successful section for which the estimates given are reached. 3) For an n-dimensional simplex $H_0 = \ldots = H_n = 1$, $P_0 = 1$, $P_1 = \ldots = P_{[n/2]} = 0$. A section of a simplex by a hyperplane on one side of which there are exactly $[(n + 1)/2]$ vertices is successful. For this section $F_k(c) = M_l(k, n)$. (For fixed k and $n \to \infty$ this number is asymptotically equal to $2^k n^{k+1}/(k + 1)$.) For any other section and $0 \leqslant l < k \leqslant [n/2]$ the number $F_k(c)$ will be larger, and the number $F_l(c)/F_k$ will be smaller, than for the successful section. 4) We consider in R^n the standard cube defined by the inequalities $0 \leqslant x_1 \leqslant 1, \ldots, 0 \leqslant x_n \leqslant 1$. The hyperplane $x_1 + \ldots + x_n = c$, where c for odd n is equal to n/2, and for even n is equal to $(n + 1)/2$, dissects the cube successfully. We calculate how many k-dimensional faces of the cube intersect the given section. A face of the cube of dimension k is given by fixing the value 0 or 1 for $n - k$ of the coordinates (the other coordinates on this face run through all values from 0 to 1). In order that the section intersect the face it is necessary that there be m ones among the fixed coordinates, where $[n/2] \geqslant m > [n/2] - k$. From this we have that the number $Q_{k,n}$ of intersected k-dimensional faces is equal to $C_n^k \Sigma C_{n-k}^m$, where the summation is over numbers m, satisfying the inequalities written.

According to Theorem 7 a hyperplane section cannot intersect more than $Q_{k,n}$ k-dimensional faces of a simple polyhedron, F-equivalent with the n-dimensional cube. What proportion of the k-dimensional faces for $k \sim c\sqrt{n}$ in a polyhedron F-equivalent with an n-dimensional cube can be intersected by a hyperplane? Answer: as $n \to \infty$ the greatest amount of this proportion tends to $(2\pi)^{-1/2} \int_{-c}^{c} \exp(-x^2/2)\,dx$ (asymptotically the numbers $Q_{k,n}/C_n^k 2^{n-k}$ for large n are determined by a normal distribution).

4.3. THEOREM 8. Let the hyperplane L = c not pass through the vertices of a simple n-dimensional polyhedron and the numbers l, k satisfy the inequalities $0 \leqslant l < k \leqslant [n/2]$. Then the proportion of the k-dimensional faces which do not intersect the hyperplane [i.e., the number $F_k(c)/F_k$], is not less than $M_0(k, n)/N_0(k, n)$, and the ratio of the number of nonintersecting l-dimensional faces to the number of nonintersecting k-dimensional faces is not greater than $M_0(l, n)/N\ (k, n)$. Both estimates are sharp and are achieved for a successful section of a simplex, while the first of the estimates is achieved only for this case.

COROLLARY 4. A generic section of a simple n-dimensional polyhedron does not intersect at least the $[2^{-k} - \varepsilon_k(n)]$-th part of its k-dimensional faces (for fixed k and large n), where $\varepsilon_k(n)$ is independent of the choice of the polyhedron and its section and tends to zero as n tends to infinity.

To prove Corollary 4 it suffices to calculate the asymptotic behavior of the estimate of the number $F_k(c)/F_k$ as $n \to \infty$.

Remark. For nonsimple multidimensional polyhedra the proportion of nonintersecting k-dimensional faces can be arbitrarily small.

Proof of Theorem 8. From Theorem 7 it follows that $F_k/F_k(c) \leqslant \Sigma P_j N_j(k, n)/\Sigma P_j M_j(k, n)$ and $F_l(c)/F_k(c) \leqslant \Sigma P_j M_j(l, n)/\Sigma P_j M_j(k, n)$, in which the summation is over j from 0 to [n/2]. According to Theorems 4 and 5 the ratios written achieve a maximum when $P_1 = \ldots = P_{[n/2]} = 0$, while the first of them achieves the maximum only in this case. Theorem 8 is proved, since $P_1 = \ldots = P_{[n/2]} = 0$ only for a simplex.

Remark. From the F-polynomial of a simple n-dimensional polyhedron one can estimate the number of k-dimensional faces for its generic section by a plane of codimension l. We consider a simple $(n - l)$-dimensional polyhedron, for which the number of k-dimensional faces for $k \geqslant (n - l)/2$ is equal to F_{k+l}. (The numbers of faces of lower dimensions of such a polyhedron can be found by the Dehn–Sommerville theorem.) *The number of faces of any dimension for a section does not exceed the number of faces of the same dimension for the polyhedron considered.* The following conjecture on the upper bound [8] give a special case of this assertion: among all simple m-dimensional polyhedra having N faces of highest dimension, the polyhedron dual to the cyclic one has the largest number of faces of any dimension. In fact, a polyhedron having N faces of highest dimension is a section of an $(N - 1)$-dimensional simplex. But the number of k-dimensional faces of the polyhedron dual to a cyclic one, for $k \geqslant m/2$, as is known, is exactly equal to the number of $(k + N - 1 - m)$-dimensional faces of the simplex. To prove the assertion given it is necessary to reformulate Theorem 6. Let $H^{<c}$ be the polynomial $\Sigma f_k^{<c}(t - 1)^k$, where $f_k^{<c}$ is the number of k-dimensional faces lying in the half-space $L < c$. Then the polynomial $H^{<c}$ is equal to $\Sigma h_i(c)t^i$, where $h_i(c)$ is the number of vertices of index i lying in the half-space $L < c$. One defines and calculates the polynomial $H^{>c}$ analogously. Theorem 6 is included in these calculations as well as the equation $H = H^{<c} + H^{>c} + H^c(t - 1)$, where H^c is the H-polynomial of the section $L = c$. The polynomials $H^{<c}$ and $H^{>c}$ have nonnegative coefficients, from which we have the inequality: all the coefficients of the Laurent series at the point ∞ of the function $H(t - 1)^{-1} - H^c$ are nonnegative. We denote by $H[l]H$ the H-polynomial of a section of codimension l. Continuing by induction we get that all the coefficients of the Laurent series at the point ∞ of the function $H(t - 1)^{-l} - H[l]$ are nonnegative. This relation is equivalent with the assertion formulated.

4.4. Each result of Sec. 4 has an analog in the category of toroidal manifolds. Here is a "dictionary" for translating to the toroidal category: a simple polyhedron is a smooth (or quasismooth) Kähler toroidal manifold; an open k-dimensional face (over R) of a polyhedron is a k-dimensional orbit (over C) of the manifold; a face of a polyhedron which belongs to another face of it is an orbit on the manifold which belongs to the closure of another orbit; the F-, H-, and P-polynomials of a polyhedron are the F-, H-, and P-polynomials of the manifold; a linear function on a polyhedron is the Hamiltonian of a vector field on the toroidal manifold corresponding to the action of a one-parameter subgroup of the real torous $T^n \subset (C \setminus 0)^n$; a generic linear function is a Hamiltonian which is a Morse function; a section not passing through vertices is a nonsingular level surface of the Hamiltonian. As an example, we translate Corollary 4 into the toroidal category.

COROLLARY 4'. On a smooth n-dimensional Kähler toroidal manifold a nonsingular level surface of a Hamiltonian corresponding to the action of any one-parameter subgroup of T^n, does not intersect at least the $(2^{-k} - \varepsilon_k(n))$-th part of the k-dimensional orbits, where $\varepsilon_k(n)$ is independent of the choice of the manifold and one-parameter subgroup and tends to zero as $n \to \infty$.

To prove Corollary 4' it suffices to use Corollary 4 and the moment map. The other assertions translate into the toroidal category analogously.

5. Mean Complexity of Faces of a Polyhedron

In this section we generalize an estimate of V. V. Nikulin.

5.1. The number of pairs consisting of an l-dimensional face and a k-dimensional face containing it, of the polyhedron Δ, will be denoted by $F_{l,k}(\Delta)$. This number can be interpreted as the total number of l-dimensional faces of k-dimensional faces of the polyhedron. By the average number of l-dimensional faces on a k-dimensional face of the polyhedron Δ we mean the ratio of this number to the number $F_k(\Delta)$ of k-dimensional faces of the polyhedron.

How large can this ratio be for n-dimensional polyhedra? We show how to reduce the question to a problem of a smaller number of dimensions. Let $L = c$ be a hyperplane, not parallel to any edge of a convex n-dimensional polyhedron Δ. With each vertex b of the polyhedron Δ we associate a pair consisting of an $(n-1)$-dimensional polyhedron $\Delta(b)$ and its hyperplane section $L(b)$. This pair is defined up to a projective transformation. Here is its definition. Near the vertex b the polyhedron Δ is a convex n-dimensional cone. The pair $\Delta(b)$, $L(b)$ is the projectivization of the pair consisting of this cone and its hyperplane section passing through the vertex of the cone, parallel to the hyperplane $L = c$.

We denote by $F_{l,k}(\Delta(b), L(b))$ the number of pairs consisting of an l-dimensional face, not intersecting the hyperplane $L(b)$, and any k-dimensional face of the polyhedron $\Delta(b)$ containing it, by $F_k(\Delta(b), L(b))$ the number of k-dimensional faces of the polyhedron $\Delta(b)$, not intersecting the hyperplane $L(b)$.

THEOREM 9 (on reduction). 1) For $k \leqslant (n+1)/2$ the average number of vertices on a k-dimensional face of the polyhedron Δ does not exceed the number $\max 2F_{k-1}(\Delta(b))/F_{k-1}(\Delta(b), L(b))$. 2) For $0 < l < k \leqslant (n+1)/2$ the average number of l-dimensional faces on a k-dimensional face of the polyhedron Δ does not exceed the number $\max F_{l-1,k-1}(\Delta(b), L(b))/F_{k-1}(\Delta(b), L(b))$. In 1) and 2) the maximum is taken over the set of all vertices b of the polyhedron Δ.

Proof. We represent both the numerator and denominator of the fraction $F_{l,k}(\Delta)/F_k(\Delta)$ as a sum of nonnegative numbers over the set of vertices B of the polyhedron Δ. For this, to each pair $\Gamma_l \subset \Gamma_k$ (to each face Γ_k) we assign two vertices, at which the function L, restricted to the face Γ_l (restricted to the face Γ_k), achieves its maximum and minimum. For $k > 0$ these vertices coincide only for pairs consisting of a vertex ($l = 0$) and a k-dimensional face containing it. Summing over vertices the number of objects assigned to it, we get: $2F_{l,k}(\Delta) = \Sigma F_{l-1,k-1}(\Delta(b), L(b))$ for $l > 0$, $F_{0,k}(\Delta) = \Sigma F_{k-1}(\Delta(b))$, $2F_k(\Delta) = \Sigma F_{k-1}(\Delta(b), L(b))$. For $k \leqslant (n+1)/2$ the numbers $F_{k-1}(\Delta(b), L(b))$ are positive [cf. Corollary 2 and the Remark in Sec. 4; we shall only use Theorem 9 in the case of simple polyhedra $\Delta(b)$]. To complete the proof it remains to use Assertion 1 of Sec. 3.

Nikulin's Lemma [2]. The average number of l-dimensional faces on a k-dimensional face of a simple n-dimensional polyhedron for $0 \leqslant l < k \leqslant (n+1)/2$ does not exceed the number $C_{n-l}^{n-k}(C_{[n/2]}^{l} + C_{[(n+1)/2]}^{l})/(C_{[n/2]}^{k} + C_{[(n+1)/2]}^{k})$.

In face, for a simple polyhedron all the polyhedra $\Delta(b)$ are $(n-1)$-dimensional simplices. Hence Theorem 9 reduces Nikulin's lemma to an easy problem about a simplex (cf. the example from point 4.2).

A convex n-dimensional polyhedron is called *simple of the edges*, if each of its edges is contained in exactly $n-1$ faces of highest dimension.

THEOREM 10. The estimate of Nikulin's lemma is valid for polyhedra which are simple at the edges.

Proof. For each vertex b the polyhedron $\Delta(b)$ is simple, as follows from the simplicity at the edges of the original polyhedron Δ. According to the reduction theorem it suffices for the simple polyhedra $\Delta(b)$ to estimate the ratio of the number of $(l-1)$-dimensional faces, disjoint from the hyperplane $L(b)$, to the number of $(k-1)$-dimensional faces, disjoint from this hyperplane [each $(l-1)$-dimensional face lies in exactly C_{n-l}^{n-k} $(k-1)$-dimensional faces] and the ratio of the number of all $(k-1)$-dimensional faces to the number of $(k-1)$-dimensional faces, disjoint from the hyperplane $L(b)$. These estimates are given by Theorem k. Theorem 10 is proved.

5.2. The following assertion follows from Nikulin's lemma and Theorem 10.

Assertion. 1) For $0 \leqslant l < k \leqslant (n+1)/2$ the average number of l-dimensional orbits lying in the closure of a k-dimensional orbit of a quasismooth projective n-dimensional toroidal manifold does not exceed the number $C_{n-l}^{n-k}(C_{[n/2]}^{l}\ C_{[(n+1)/2]}^{l})/C_{[n/2]}^{k} + C_{[(n+1)/2]}^{k})$. 2) The estimate of 1) remains valid if the quasismoothness of the manifold fails at isolated points.

6. Application to Lobachevskiian Geometry

We prove Theorem 1, formulated at the beginning of the paper. We consider a fundamental polyhedron of a group generated by reflections in a Lobachevskii space. It is known that if such a polyhedron has finite volume, then it is simple at the edges and *almost simple*, i.e., near each vertex is a cone over a product of simplices (the simplicity can fail only at infinitely distant vertices of the polyhedron). We say that Nikulin's estimate is valid for a polyhedron to dimension m, if the estimate of Nikulin's lemma holds for any l, k such that $0 \leqslant l < k \leqslant m$. Using an idea of V. V. Nikulin and É. B. Vinberg, M. N. Prokhorov proved the following theorem about a year ago [5].

Prokhorov's Theorem. In a Lobachevskii space of dimension >995 there do not exist discrete groups generated by reflections with fundamental polyhedron of finite volume, for which Nikulin's estimate is valid to dimension 4.

M. N. Prokhorov was also able to prove that any almost simple polyhedron (for $n \geqslant 5$) satisfies Nikulin's estimate to dimension 3, but his method of proof does not carry to higher dimensions. Nikulin's lemma and Theorem 10 are equally applicable in Euclidean geometry and Lobachevskiian geometry. This fact follows from the existence of Klein's model of Lobachevskiian geometry. Combining Prokhorov's theorem with Theorem 10, we get the proof of Theorem 1.

Remark. Theorem 10 has an elementary proof for almost simple polyhedra, since for a product of simplices it is obvious that the coefficients of the H-polynomial increases to the middle (for general simple polyhedra an elementary proof of this face is still unknown).

LITERATURE CITED

1. V. V. Nikulin, "Arithmetic groups generated by reflections in Lobachevskii spaces," Izv. Akad. Nauk SSSR, Ser. Mat., 44, 634-669 (1980).
2. V. V. Nikulin, "Classification of arithmetic groups generated by reflections in Lobachevskii spaces," Izv. Akad. Nauk SSSR, Ser. Mat., 45, 113-142 (1981).
3. É. B. Vinberg, "Nonexistence of crystallographic groups of reflections in Lobachevskii spaces of large dimension," Funkts. Anal. Prilozhen., 15, No. 2, 67-68 (1981).
4. É. B. Vinberg, "Nonexistence of crystallographic groups of reflections in Lobachevskii spaces of large dimension," Tr. MMO, 47, 68-102 (1984).
5. M. N. Prokhorov, "Nonexistence of discrete groups of reflections with noncompact fundamental polyhedron of finite volume in Lobachevskii spaces of large dimension," Izv. Akad. Nauk SSSR, Ser. Mat., 50, No. 2, 320-332 (1986).
6. P. McMullen, "The number of faces of simplicial polytopes," Israel J. Math., 9, 559-570 (1971).
7. L. J. Billera and C. W. Lee, "Sufficiency of McMullen's conditions for f-vectors of simplicial convex polytopes," Bull. Am. Math. Soc. (New Series), 2, 181-185 (1980).
8. R. Stanley, "The number of faces of a simplicial convex polytope," Adv. Math., 35, 236-238 (1980).
9. D. M. Y. Sommerville, "The relations connecting the angle-sums and volume of a polytope in space of n dimensions," Proc. R. Soc. London, Ser. A, 115, 103-119 (1927).
10. G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat, "Toroidal embedding. I," Lect. Notes Math., 339 (1972).
11. V. I. Danilov, "Geometry of toroidal manifolds," Usp. Mat. Nauk, 33, No. 2 (1978).
12. M. F. Atiyah, "Convexity and Commuting Hamiltonians," Bull. London Math. Soc., 14, 1-15 (1982).
13. V. Guillemin and S. Sternberg, "Convexity properties of the moment mapping," Invent. Math., 67, 491-513 (1982).
14. J. H. M. Steenbrink, "Mixed Hodge structure on the vanishing cohomology," Math. Institut Amsterdam (1976).