SINGULARITIES OF FUNCTION, WAVE FRONTS, CAUSICS AND MULTIDIMENSIONAL INTEGRALS

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§3 The Geometry of Formulas

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The number of integer points lying within a polyhedron, the volume of a polyhedron — these and other geometric quantities are encountered in answering various questions of the theory of singularities, in algebra and analysis. In this section we present examples of this kind.

3.1 The Newton Polyhedra. The Newton polyhedron of a polynomial which depends on several variables is the convex hull of the powers of the monomials appearing in the polynomial with nonzero coefficients. The Newton polyhedron generalizes the notion of degree and plays an analogous role. It is well known that the number of complex roots of a system of n equations of identical degree m in n unknowns is the same for nearly all values of the coefficients, and is equal to m^n (Bézout's theorem). Similarly, the number of complex roots of a system of n equations in n unknowns with the same Newton polyhedron is the same for nearly all values of the coefficients and is equal to the volume of the Newton polyhedron, multiplied by $n!$ (Kushnirenko's theorem, see 3.1.1).

The level line of a polynomial in two complex variables is a Riemann surface. For nearly all polynomials of fixed degree n, the topology of this surface (the number of handles g) is expressed in terms of its degree, and does not depend on the values of the coefficients of the polynomial: $g = (nl)(n2)/2$. In the more general case, where instead of polynomials of fixed degree we consider polynomials with fixed Newton polyhedra, all the discrete characteristics of the manifold of zeros of the polynomial (or several polynomials) are expressed in terms of the geometry of the Newton polyhedra. Among these discrete characteristics are the number of solutions of a system of n equations in n unknowns, the Euler characteristic, the arithmetic and geometric genus of complete intersections, and the Hodge number of a mixed Hodge structure on the cohomologies of complete intersections.

The Newton polyhedron is defined not only for polynomials but also for germs of analytic functions. For germs of analytic functions in general position, with given Newton polyhedra, one can calculate the multiplicity of the zero solution of a system of analytic equations, the Milnor number and ξ function of the monodromy operator, the asymptotics of oscillatory integrals (see section 2), the Hodge number of the mixed Hodge structure in vanishing cohomologies, and in the two-dimensional

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and multidimensional quasihomogeneous cases, one can calculate the modality of the germ of the function.

In the answers one meets quantities characterizing both the sizes of the polyhedra (volume, number of integer points lying inside the polyhedron) as well as their combinatorics (the number of faces of various dimensions, numerical characteristics of their contacts).

In terms of the Newton polyhedron one can construct explicitly the compactification of complete intersections, and the resolution of singularities by means of a suitable toric manifold.

Thus, the Newton polyhedra connect algebraic geometry and the theory of singularities to the geometry of convex polyhedra. This connection is useful in both directions. On the one hand, explicit answers are given to problems of algebra and the theory of singularities in terms of the geometry of polyhedra. We note in this connection that even the volume of the convex hull of a system of points is a very complicated function of their coordinates. Therefore, the formulation of answers in numerical terms is so opaque that without knowing their geometric interpretation no progress is possible. On the other hand, algebraic theorems of general character (the Hodge theorem on the index of an algebraic surface, the Riemann–Roch theorem) give significant information about the geometry of polyhedra. In this way one obtains, for example, a simple proof of the Aleksandrov–Fenchel inequalities in the geometry of convex bodies.

The Newton polyhedra are also met in the theory of numbers (Ref. 17) in real analysis (Ref. 29), in the geometry of exponential sums (Refs. 32, 60), in the theory of differential equations (Refs. 18, 19, 36). In this paragraph we present formulations of some theorems about Newton polyhedra. More details can be found in Refs. 4, 7, 15, 12–13, 15–29, 32–34, 41, 58, 60–61.

3.1.1 The Number of Roofs of a System of Equations with a Given Newton Polyhedron. According to Bézout's theorem, the number of nonzero roots of the equation $f(z) = 0$ is equal to the difference between the highest and the lowest powers of the monomials appearing in the polynomial f. This difference is the volume of the Newton polyhedron (in the present case, the length of a segment). In the following two paragraphs we present the generalizations of this theorem to the case of arbitrary Newton polyhedra.

Let us start with definitions. A *monomial* in n complex variables is a product of the coordinates to integer (possibly negative) powers. Each monomial is associated with its *degree*, an integer vector, lying in n-dimensional real space, whose components are equal to the powers with which the coordinate functions enter in the monomial. A *Laurent polynomial* is a linear combination of monomials. The *support of the Laurent polynomial* is the set of powers of monomials entering in the Laurent polynomial with nonzero coefficients. (The Laurent polynomial is an ordinary polynomial if its support lies in the positive octant.) The *Newton polyhedron* of the Laurent polynomial is the convex hull of its support. It is much more convenient to consider the Laurent polynomials not in $\mathbb{C}^n$ but in the $(\mathbb{C}\setminus 0)^n$-dimensional complex space, from which all the coordinate planes have been eliminated. With each face Γ of the Newton polyhedron of the Laurent polynomial f we associate a new Laurent polynomial, which is called the *restriction of the polynomial to the face*, denoted by f^Γ and defined as follows: only those monomials appear in f^Γ that have powers lying in the face Γ, with the same coefficients that they have in f.

Now let us consider a system of n Laurent equations $f_1 = \cdots = f_n = 0$ in $(\mathbb{C}\setminus 0)^n$ with a common Newton polyhedron. The restricted system, $f_1^\Gamma = \cdots = f_n^\Gamma = 0$ corresponding to each face Γ of the polyhedron. The restricted system actually depends on a smaller number of variables, and in the case of general position is incompatible in $(\mathbb{C}\setminus 0)^n$. We say that a system of n equations in n unknowns with a common Newton polyhedron is *regular* if all the restrictions of this system are incompatible in $(\mathbb{C}\setminus 0)^n$. The following theorem of Kushnirenko holds: The number of solutions in $(\mathbb{C}\setminus 0)^n$, counted with their multiplicities, of a regular system of n equations in n unknowns with a common Newton polyhedron is equal to the volume of the Newton polyhedron multiplied by $n!$.

Example: The Newton polyhedron of the polynomial of degree m in n unknowns is the simplex $0 \leq x_1, \ldots, 0 \leq x_n, \sum x_i \leq m$ (we assume that the polynomial contains all monomials of degree $\leq m$). The volume of such a simplex is $m^n/n!$. The number of roots of the total system of n equations of degree m in n unknowns, according to Kushnirenko's theorem, is equal to m^n. This answer agrees with Bezout's theorem. If the polynomial does not contain all monomials of degree less than or equal to m, then the Newton polyhedron can be smaller than the simplex, so the number of solutions, calculated from Kushnirenko's theorem, can be smaller than the number m^n calculated from Bezout's theorem. Because of the absence of the monomials, certain infinitely distant points may be roots of the system of equations. Bezout's theorem, which calculates the number of roots of the system in projective space, takes into account these parasitic roots, while Kushnirenko's theorem does not.

Remark: The proof of Kushnirenko's theorem is found in Ref 41. Another proof can be extracted from the recent theorem of Atiyah on symplectic actions of tori (Ref. 62). We associate with each n-dimensional, integer-valued, convex polyhedron a symplectic "Veronese manifold" and a set of n commuting Hamiltonians on it, for each of which the motion is periodic with a period independent of the trajectory. The "moment mapping" corresponding to these Hamiltonians, according to Atiyah's theorem, maps the Veronese manifold on some convex polyhedron. It is not difficult to calculate that this polyhedron coincides with the original polyhedron. The calculation of the number of roots of a typical system of equations with a given Newton polyhedron reduces to determining the volume of the Veronese manifold, which is easily done with the help of Atiyah's theorem: This volume is proportional to the volume of the polyhedron. The Veronese manifold is constructed from the convex integer polyhedron as follows. Consider the projective "space of monomials" $\mathbb{C}P^{N-1}$, the number of whose homogeneous coordinates is equal to the number N of integer points lying inside and on the faces of Δ. By a Δ-*Veronese manifold* we mean the image of the mapping $(\mathbb{C}\setminus 0)^n \to \mathbb{C}P^{N-1}$, which associates the point z of $(\mathbb{C}\setminus 0)^n$ with the point of projective space whose homogeneous coordinates coincide with the values at the point z of monomials whose degrees lie within Δ and on its faces. Since the Δ-Veronese manifold is algebraic, the numbers of points of its intersection with the planes of complementary dimension are the same for almost all planes. According to the Buffon-Crofton formula this number is proportional to the volume of the manifold. But a plane of codimension n in the space of monomials $\mathbb{C}P^{N-1}$ corresponds to a system of n equations with the Newton polyhedron Δ. The number of points of intersection of a plane with the Δ-Veronese manifold is equal to the number of solutions of the system. Therefore, the number of solutions of the system is proportional to the volume of the Veronese manifold.

Atiyah's theorem is used to calculate this volume. A real n-dimensional torus acts in the space $(\mathbb{C} \setminus 0)^n$: each of the n coordinates can be multiplied by a number with absolute value equal to one. This action of the torus on $(\mathbb{C} \setminus 0)^n$ could be carried over to the Δ-Veronese manifold and is symplectic there. Thus, Atiyah's theorem is applicable; it follows from this theorem that the volume of the Δ-Veronese manifold is proportional to the volume of the original Newton polyhedron Δ.

3.1.2 How Does One Find the Number of Solutions of a System of n Equations in n Unknowns with Different Newton Polyhedra?. Here is the answer to this question for a system in general position with fixed Newton polyhedra: the number of solutions not lying on the coordinate planes is equal to the mixed volume of the Newton polyhedra, multiplied by n! Below we shall give the definition of mixed volume and describe explicitly the conditions for degeneracy.

The *Minkowski sum* of two subsets of a linear space is the set of sums of all pairs of vectors, in which one vector of the pair lies in one subset and the second vector in the other. The product of a subset and a number can be determined in a similar manner. The Minkowski sum of convex bodies (convex polyhedra, convex polyhedra with vertices at integer points) is a convex body (convex polyhedron, convex polyhedron with vertices at integer points). The following theorem holds:

Minkowski theorem. *The volume of a body which is a linear combination with positive coefficients of fixed convex bodies lying in $\mathbb{R}^n$ is a homogeneous polynomial of degree n in the coefficients of the linear combination.*

Definition. The *mixed volume* $V(\Delta_1, \dots, \Delta_n)$ *of the convex bodies* $\Delta_1, \dots, \Delta_n$ in $\mathbb{R}^n$ is the coefficient in the polynomial $V(\lambda_1\Delta_1 + \dots, \lambda_n\Delta_n)$ of $\lambda_1 \times \dots \times \lambda_n$ divided by $n!$ (here $V(\Delta)$ is the volume of the body Δ).

The mixed volume of n identical bodies is equal to the volume of any one of them. The mixed volume of n bodies is expressed in terms of the usual volumes of their sums in the same way as the product of n numbers is expressed in terms of the n-th powers of their sums. For example, for $n = 2$,

$$ab = \frac{1}{2}[(a+b)^2 - a^2 - b^2]$$

$$V(\Delta_1, \Delta_2) = \frac{1}{2}[V(\Delta_1 + \Delta_2) - V(\Delta_1) - V(\Delta_2)].$$

Similarly, for $n = 3$,

$$V(\Delta_1, \Delta_2, \Delta_3) = \frac{1}{3!}\left[V(\Delta_1 + \Delta_2 + \Delta_3) - \sum_{i<j} V(\Delta_i + \Delta_j) + \sum V(\Delta_i)\right].$$

Example: Suppose that Δ_1 is the rectangle $0 \le x \le a$, $0 \le y \le b$ and Δ_2 is the rectangle $0 \le x \le c$, $0 \le y \le d$. The Minkowski sum $\Delta_1 + \Delta_2$ is the rectangle $0 \le x \le a + c$, $0 \le y \le b + d$. The mixed volume $V(\Delta_1, \Delta_2)$ is equal to $ad + bc$. The number $ad + bc$ is the permanent of the matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ (the definition of the permanent differs from that of the determinant only in that all the terms in the permanent have a plus sign). In the multidimensional case the mixed volume of n parallelepipeds with sides parallel to the coordinate axes is also equal to the permanent of the corresponding matrix.

Let us consider a system of n Laurent equations $f_1 = \dots = f_n = 0$ with Newton polyhedra $\Delta_1, \dots, \Delta_n$. Below we define the regularity condition for such systems.

Bernshtein's theorem. *The number of solutions in $(\mathbb{C}\setminus 0)^n$ (which take account of the multiplicity) of a regular system of n equations in n unknowns, is equal to the mixed volume of the Newton polyhedra of the equations of the system, multiplied by $n!$.*

Example: The number of roots of a general system of polynomial equations, in which the i-th variable enters in the j-th equation with a power no higher than a_{ij} is equal to the permanent of the matrix (a_{ij}), multiplied by $n!$.

Kushnirenko's theorem coincides with Bernshtein's theorem for equations with identical Newton polyhedra. We now proceed to the definition of a *regular system of equations*. We first define the truncations of a system associated with a function ξ. We take an arbitrary linear function ξ on the space $\mathbb{R}^n$ in which the Newton polyhedra lie. We denote by f^ξ the restriction of the Laurent polynomial f to that face of its Newton polyhedron on which the linear function ξ takes its maximum value. We associate the restricted system $f_1^\xi = \cdots = f_n^\xi = 0$ with the system of equations $f_1 = \cdots = f_n = 0$ and the linear function ξ. For a nonzero function ξ the restricted system actually depends on a smaller number of variables. Therefore, in the case of general position such a system is inconsistent in $(\mathbb{C}\setminus 0)^n$. A given system of equations has only a finite number of restricted systems (if the polyhedra of all the equations coincide, then the truncations correspond to the faces of the common polyhedron). A system of n equations in n unknowns is said to be regular if its restrictions for all nonzero functions ξ are inconsistent in $(\mathbb{C}\setminus 0)^n$. It is just such a system to which Bernshtein's theorem is applicable.

We note that if for each nonzero linear function ξ, the maximum in one of the polyhedra is attained at a vertex then the regularity conditions are automatically satisfied. [In this case the truncated system contains an equation which is contained in a monomial set equal to zero; this equation has no solutions in $(\mathbb{C}\setminus 0)^n$.] For example, in the case of $n = 2$, the regularity conditions are satisfied automatically, if the two Newton polyhedra on the real plane have no parallel sides.

3.1.3. Newton Polyhedron of the Germ of an Analytic Function. In this paragraph we discuss the calculation of the multiplicity of a root of a system of equations, the Milnor number, and the modality of a function in two variables, in terms of Newton polyhedra.

We begin with definitions. To each power series in n variables we associate its support, the set of powers of monomials appearing in the series with nonzero coefficients. To each point of the support we associate an octant with vertex at that point, and consisting of all points for which each coordinate is not smaller than the corresponding coordinate of the vertex. The *Newton polyhedron* of the series is the convex hull of the union of all octants with vertices on the support of the series. The *Newton diagram of the series* is the union of compact faces of the Newton polyhedron. A Newton polyhedron of the series is said to be *suitable* if it intersects all the coordinate axes.

By the *volume $V(\Delta)$ of a suitable polyhedron* we mean the volume of the (non-convex) region between the origin and the faces of the polyhedron in the positive octant $\mathbb{R}^n_+$. Minkowski's theorem holds for the volumes of suitable polyhedra: the volume of a linear combination, with positive coefficients, of suitable polyhedra (just like the usual volume of polyhedra) is a polynomial in the coefficients of the linear combination. Therefore, we can define a *mixed volume for suitable polyhedra*, which is a verbatim repetition of the usual definition of mixed volume (see par. 3.1.2). For example, in the plane the mixed volume $V(\Delta_1, \Delta_2)$ of two suitable polyhedra Δ_1, Δ_2 is defined by the formula

$$2V(\Delta_1, \Delta_2) = V(\Delta_1 + \Delta_2) - V\Delta_1) - V(\Delta_2)$$

The following local variant of Bernshtein's theorem (see par. 3.1.2) holds.

Theorem. *The multiplicity of the origin as of a regular solution of a system of n analytic equations of n unknowns with suitable Newton polyhedra is equal to the mixed volume of the suitable Newton polyhedra, multiplied by n!*

Example: The multiplicity of the origin as of a regular solution of the system $x_1^a + x_2^b = x_1^c + x_2^d = 0$ with positive powers a, b, c, d, with $ad \neq bc$, is equal to $\min(ad, bc)$.

We give a precise definition of regularity of the origin. The linear function ξ is said to be negative if all its values in the positive octant are negative (i.e., $\xi = \sum a_k, x_k,\ a_k < 0$). The restriction f^ξ of the analytic function f to negative functions ξ is its truncation to that face of the Newton polyhedron at which the function ξ attains its maximum. A null solution of a system of n analytic equations $f_1 = \dots = f_n = 0$ with suitable Newton polyhedra at the origin is said to be *regular* if the restricted of the system, $f_1^\xi = \dots = f_n^\xi = 0$, is inconsistent in $(\mathbb{C} \setminus 0)^n$ for all negative functions ξ.

We now give the formula for the Milnor number μ, of the germ of an analytic function. First, we shall give several definitions and notations. We say that the germ of an analytic function f with suitable Newton polyhedron Δ is Δ-*nondegenerate* if, for any compact face Γ of the Newton polyhedron, the system $z_1(\dfrac{\partial f^\Gamma}{\partial z_1}) = \dots = z_n(\dfrac{\partial f^\Gamma}{\partial z_n}) = 0$ is inconsistent in $(\mathbb{C} \setminus 0)^n$ (here f^Γ is the truncation of f to the face Γ). We give some geometric notations: for a suitable Newton polyhedron $\Delta \subset \mathbb{R}^n$, we denote by Δ^I its intersection with the coordinate plane $\mathbb{R}^I$ in $\mathbb{R}^n$, by $d(I)$ the dimension of this coordinate plane and by $V(\Delta^I)$ the $d(I)$-dimensional volume of the (nonconvex) region in the positive octant $\mathbb{R}_+^I$ between the origin and the boundary of Δ^I.

Theorem. *For a Δ-nondegenerate function f with a suitable Newton polyhedron Δ, the Milnor number is*

$$\sum d(I)!(-1)^{n-d(I)} V(\Delta^I) + (-1)^n$$

where the summation is taken over all intersections Δ^I of the suitable Newton polyhedron with the coordinate planes.

Example: For almost all functions of two variables with a given suitable Newton polyhedron, $\mu = 2Sab + 1$, where S is the area under the polyhedron, a and b are coordinates of the points of the polyhedron on the axes (see the figure).

The *modality* $m(f)$ of a function f is the dimension of the space of orbits of the group of diffeomorphisms in the space of convergent power series which has no free and no linear terms in the neighborhood of the orbit of the point f.

Theorem (Kushnirenko). *The modality of a Δ-nondegenerate function of two variables with suitable Newton polyhedron is equal to the number of integer points lying on the Newton diagram and below it, for which each coordinate is not less than two.*

Calculation of the Milnor number: $\mu = 2S - a - b + 1 = 24 - 5 - 7 + 1 = 13$

3.1.4 Complete Intersections. Consider in $(\mathbb{C} \setminus 0)^n$ a system of k Laurent equations $f_1 = \dots = f_k = 0$ with Newton polyhedra $\Delta_1, \dots, \Delta_k$. For Laurent polynomials in general position with given Newton polyhedra, the discrrete invariants of the set f solutions are identical and are expressed in terms of the polyhedra. Below we give an explicit description of the conditions for nondegeneracy and describe the Euler characteristic and the omotopy of the complete intersection.

Definition. A system $f_1 = \dots = f_k = 0$ is said to be *nondegenerate for its Newton polyhedra* if for any linear function ξ the following (ξ) condition is satisfied: for any solution z of the truncated system $f_1^\xi = \dots = f_k^\xi = 0$, lying in $(\mathbb{C} \setminus 0)^n$ the differentials df_i^ξ are linearly independent in the tangent space at the point z.

Each of the (ξ) conditions is satisfied for almost all Laurent polynomials with fixed polyhedra (this is the Sard–Bertini theorem).

Although formally the nondegeneracy conditions are described for any linear function ξ, actually the different conditions are finite in number (there exist only a finite number of different truncated systems). Therefore the nondegeneracy conditions are satisfied almost everywhere. Among the nondegeneracy conditions there is the (0) condition (for null function ξ), according to which the system $f_1 = \dots = f_k = 0$ determines a nonsingular $(n-k)$-dimensional manifold in $(\mathbb{C} \setminus 0)^n$. We note that for $k = n$ class of regular systems is broader than class of nondegenerate systems: for regular systems the (0) condition does not have to be satisfied (regular systems can have multiple roots).

Theorem. *The Euler characteristic of the nondegenerate complete intersection $f_1 = \dots = f_k = 0$ in $(\mathbb{C} \setminus 0)^n$, $(k \leq n)$, with the Newton polyhedra $\Delta_1, \dots, \Delta_k$ is equal to $(-1)^{n-k} n! \sum V(\Delta_1, \dots, \Delta_k, \Delta_{i_1}, \dots \Delta_{i_{n-k}})$, where the sum is taken over all sets $1 = i_1 \leq \dots \leq i_{n-k} \leq k$.*

We note one special case of this theorem.

Corollary. *The Euler characteristic of a hypersurface in $(\mathbb{C} \setminus 0)^n$ defined by nondegenerate equation with fixed Newton polyhedron is equal to the volume of the Newton polyhedron multiplied by $(-1)^{n-1} n!$.*

Anover consequence of this theorem iz Bernshtein's theorem (see par. 3.1.2): for $k = n$, the nondegenerate complete intersections consist of points, and the

Euler characteristic is equel to the number of points (strictly speaking, Bernshtein's theorem is slightly stronge than tis corollary, since it is applicable to a degenerate regular system).

Theorem (on homotopy type). *Suppose that the Newton polyhedra Δ_i have a total dimension* $\dim \Delta_i = n$ *and the complete intersection* $f_i = \cdots = f_k = 0$ *in* $\mathbb{C}^n$ *is nondegenerate. Then for* $k < n$ *the complete intersection is connected and has homotopy type of a bouquet of* $(n-k)$*-dimenrional spheres.??????????*

Corollary. *The cohomology groups of a nondegenerate complete intersection with Newton polyhedra of complete dimension are different from zero only in dimensions* nk *and* 0*, where for* $n > k$ *the zero-measure cohomology group is one-dimensional.*

3.1.5 Genus of Complete Intersections. The formulas given below for the genus of complete intersections are generalizations of the following formulas for Abelian and elliptic integrals. Let us consider a Riemann surface $y^2 = P_3(x)$ (the complex phase curve of motion of a point in a field with a cubic potential). This Riemann surface, which is diffeomorphic to the torus, exhibits a single, everywhere holomorphic, differential form dx/y (the differential of the time of motion along the phase curve). In the case of a potential of degree n, the curve $y^2 = P_n(x)$ is diffeomorphic to the sphere with g handles, where g is connected with n either by the formula $n = 2g+1$, or the formula $n = 2g+2$ (depending on the parity of n). The basis of holomorphic forms in this case is given by g forms of the type $x^m dx/y$, $0 \leq m < g$. The number g is the genus of the curve. The Newton polyhedron of the curve $y^2 == P_n(x)$ is a triangle with vertices $(0,0)$, $(0,2)$, and $(n,0)$. There are exactly g points with integer coordinates strictly inside this triangle. In terms of these points, one can give a basis for the space of holomorphic forms: the point $(1,a)$, which lies inside the triangle, corresponds to the form $x^{a-1}dx/y$. We give below the generalization of this procedure for constructing a basis of holomorphic forms for the multi-dimensional case.

The nondegenerate complete intersections $f_1 = \cdots = f_k = 0$, where the f_i, are Laurent polynomials, are smooth algebraic affine manifolds. In the cohomologies of such manifolds there is an additional structure, namely, the mixed Hodge structure. The discrete invariants of such a structure are calculated in terms of the Newton polyhedra. We consider only the calculation of the arithmetic and geometric genus which are invariants of this kind.

We first recall some definitons and general statements. Suppose that Y is a nonsingular (possibly noncompact) algebraic manifold. The set of holomorphic p-forms on Y, which extend holomorphically to any nonsingular algebraic compactification, is automatically closed, and it realizes the zero class of homology of the manifold Y only if it is equal to zero. Forms of this kind form a subspace in the p-dimensional cohomologies of the manifold Y. We denote the dimension of this subspace by $h^{p,0}(Y)$. The *arithmetic genus* of the manifold Y is the alternating sum $\sum(-1)^p h^{p,0}(Y)$ of the numbers $h^{p,0}(Y)$. The *geometric genus* of the manifold Y is the number $h^{n,0}(Y)$, where n is the complex dimension of the manifold Y.

Now we turn to the Newton polyhedron. We shall use the *characteristic* $B(\Delta)$ *of integer polyhedra*. Here is its definition. Suppose that Δ is a q-dimensional polyhedron with vertices at integer points, lying in $\mathbb{R}^n$ and $\mathbb{R}^q$ is a q-dimensional subspace, containing Δ. The number $B(\Delta)$ is defined as the number of integer points lying strictly within he polyhedron Δ (in the geometry of the subspace $\mathbb{R}^q$), multiplied by $(-1)^q$.

Theorem. *The arithmetic genus of the nondegenerate complete intersection* $f_1 = \cdots = f_k = 0$ *in* $(\mathbb{C} \setminus 0)^n$, $(k \leq n)$, *with the Newton polyhedra* $\Delta_1, \ldots, \Delta_k$ *is*

$$1 - \sum B(\Delta_i) + \sum_{j>i} B(\Delta_i + \Delta_j) - \cdots + (-1)^k B(\Delta_1 + \cdots + \Delta_k).$$

Corollary. *The geometric genus of the nondegenerate complete intersection (for* $k < n$*) with the polyhedra of full dimensionality is*

$$(-1)^{n-k} \left(- \sum B(\Delta_i) + \sum_{j>i} B(\Delta_i + \Delta_j) - \cdots + (-1)^k B(\Delta_1 + \cdots + \Delta_k) \right).$$

Proof of the corollary. According to the theorem on the homotopic type of the complete intersection (see par. 3.1.4), $h^{0,0} = 1$ and $h^{p,0} = 0$ for $0 < p < nk$. The corollary now follows from the calculation of the arithmetic genus.

We now give a complete discription of the holomorphic forms of highest dimension which could be extended holomorphically to the compactification, for the case of a nondegenerate hypersurface $f = 0$ in $(\mathbb{C} \setminus 0)^n$ with the Newton polyhedron Δ of complete dimension. For each integer point, lying strictly within the Newton polyhedron Δ, we denote by ω_a the n-form on the hypersurface $f = 0$, defined by the formula

$$\omega_a = z_1^{a_1} \times \cdots \times z_n^{a_n} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n} / df; \quad a = a_1, \ldots, a_n.$$

Theorem. *The forms* ω_a *lie in the space of forms extendable holomorphically on the compactification of the manifold* $f = 0$*; they are linearly independent and generate that space. In particular, the geometric genus of the hypersurface is equal to* $|B(\Delta)|$.

Example: Consider a curve in the plane determined by the equation $y^2 = P_n(x)$. The interior monomials for the Newton polyhedron of this curve have the form yx^a, where $1 \leq a < n/2$. The forms ω_a corresponding to the monomials coincide [on the curve $y^2 = P_n(x)$] with the forms $x^{a-1} dx/2y$. Thus, for the curve $y^2 = P_n(x)$ we get the usual description of all Abelian differentials.

3.1.6 Total Intersections in $\mathbb{C}^n$*.* It is customary to consider systems of equations not in $(\mathbb{C} \setminus 0)^n$, but in the usual complex space $\mathbb{C}^n$. Calculations with Newton polyhedra also are carried out in this situation. The answers here are more messy. As an example we consider the calculation of the Euler characteristic of a hypersurface in $\mathbb{C}^n$. Suppose that f is a polynomial in n complex variables with a nonzero free term and Newton polyhedron Δ. We introduce the following notation: Δ^I is the intersection of the Newton polyhedron Δ with the coordinate plane $\mathbb{R}^I$ in $\mathbb{R}^n$, $d(I)$ is the dimension of this plane, and $V(\Delta^I)$ is the $d(I)$-dimensional volume of the polyhedron Δ^I.

Theorem. *Let* f *be a nondegenerate polynomial with the Newton polyhedron* Δ *which has a nonzero free term. Then* $f = 0$ *is a nonsingular hypersurface in* $\mathbb{C}^n$, *intersecting transversally all the coordinate planes in* $\mathbb{C}^n$. *The Euler characteristic* *of this hypersurface is equal to* $\sum(-1)^{d(I)-1}d(I)!V(\Delta^I)$, *where the summation runs over all intersections of the Newton polyhedron* Δ^I *with the (nonzero) coordinate planes.*

This theorem follows from the calculation of the Euler characteristic in $(\mathbb{C}\setminus 0)^n$ (see par. 3.1.4) and from the additivity of the Euler characteristic. We also note that the formula given for a hypersurface is analogous to the formula for the Milnor number (see par. 3.1.3).

3.2 Index of a Vector Field. The investigation of the topology of real algebraic curves, surfaces, etc., is a necessary stage in the study of singularities in analysis (the Taylor expansion approximates all objects by algebraic ones, and the topology of these objects has a decisive influence over the topology of the analytic objects).

Unfortunately, very little is known about the topology of real algebraic manifolds (even curves) of high degree. Algebraic curves of degree 3 and 4 were studied by Newton and Descartes, but what a plane algebraic curve of degree 8 might look like is not known even today (a curve of sixth degree can't have more than 11 ovals, and in that case only one of them contains other ovals inside it, namely, 1, 5 or 9 others — this was discovered in 1970).

The theory of singularities enables one to give some restrictions on the topology of real algebraic manifolds. For example, from the expression for the Poincare index of a singular point of the vector field in terms of the local algebra of the singular point there follows the inequality of Petrovskii–Oleinik for the Euler characteristic of algebraic manifolds of fixed degree.

3.2.1 The Index of a Homogeneous Singular Point. What values can the index of an isolated singular point of a vector field in $\mathbb{R}^n$, whose components are homogeneous polynomials of degree $m_1,\dots,m_n$, have? Theorem 1 formulated below gives a complete answer to this question.

We introduce the following notations, $m = m_1\dots,m_n$ is a set of natural numbers; $\Delta(m)$ is a parallelepiped in $\mathbb{R}^n$, defined by the inequalities $0 \le y_1 \le m_1 - 1,\dots,0 \le y_n \le m_n 1$; $\mu_1 = m_1\dots m_n$ is the number of integer points in the parallelepiped $\Delta(m)$; $\Pi(m)$ is the number of integer points in the central section $y_1+\dots+y_n = \frac{1}{2}(m_1+\dots+m_n n)$ of the parallelepiped $\Delta(m)$.

Theorem 1 (Refs. 33 and 34)**.** *The index* ind *of an isolated singular point* 0 *of a vector field in* $\mathbb{R}^n$*, whose components are homogeneous polynomials of degree* $m_1,\dots,m_n$*, satisfies the inequality* $|\mathrm{ind}| \le \Pi(m)$ *and the congruence* $\mathrm{ind} \equiv \mu \mod 2$. *There are no other restrictions on the index.*

The proof of the estimate of the index in Theorem 1 is obtained from the signature formula for the index (see par. 3.2.5).

We note that the algebraic formula for the number $\Pi(m)$ is no means simple. Just the simple geometric interpretation of this number enables one to present examples of vector fields with all indices not forbidden by Theorem 1 (Ref. 33).

3.2.2 Petrovskii–Oleinik Inequalities. Consider a real nonsingular projective hypersurface A of degree $m+1$, given by a homogeneous polynomial f in n variables.

The Petrovskii–Oleinik inequalities consist of the following:

$$|\chi(A)1| \le N_n(m) \quad \text{if } n \text{ is even;}$$
$$|\chi(B_+) - \chi(B_-) \le N_n(m) \quad \text{if } n \text{ and } m \text{ are odd.}$$

Here χ is the Euler characteristic, B_+ and B_- are parts of $\mathbb{R}P^{n-1}$ given by the conditions $f \geq 0$, $f \leq 0$, respectively, while the number $N_n(m)$ is the number of integer points in the central section of the cube with side m [i.e., $N_n(m) = \Pi(\underbrace{m, \ldots, m}_{n \text{ times}})$].

It is not difficult to show that both for even and odd n the left sides of the Petrovskii–Oleinik inequalities are equal in modulus to the index of the gradient of the homogeneous polynomial f giving the hypersurface. Thus, the Petrovskii–Oleinik inequalities represent a special case of the inequality $|\text{ind}| \leq \Pi(m)$ of Theorem 1: this inequality must be applied to the index of the gradient vector field of the homogeneous polynomial of degree $m+1$.

The Petrovskii–Oleinik inequalities were discovered in 1949 (Ref. 35). The relationship between these inequalities and the estimate of the index of a vector field was established in 1978 (Ref. 34).

3.2.3 Index of a Polynomial Field. What values can the total index of singular points of a vector field in the region $\mathbb{R}^n$ have, if the region is defined by a polynomial inequality $P_0 < 0$, and if we know the degrees of the components of the field and the degree of the polynomial P_0? In this paragraph we formulate a complete answer to this question.

We denote by ind the sum of the indices of all the singular points of the field V in $\mathbb{R}^n$, and by ind^+ and ind^- the sums of the indices of all the points of V in the regions $P_0 > 0$ and $P_0 < 0$. We shall say that the pair V, P_0 has a degree not exceeding (equal to) m, m_0, where $m = m_1, \ldots, m_n$, if the degree of the i-th component of the field does not exceed (is equal to) m_i, and the degree of the inequality $P_0 < 0$ does not exceed (is equal to) m_0. We shall say that the pair V, P_0 is *nondegenerate* if, first, the real hypersurface $P_0 = 0$ does not pass through singular points of the field V and if, secondly, the real singular points of the field V have finite multiplicity and lie "in a finite part of the space $\mathbb{R}^n$" (the last condition means that the system $x_0 = \overline{P}_1 = \cdots = \overline{P}_n = 0$ has only the zero solution. Here the $\overline{P}_i$ are homogeneous polynomials of degree m_i in $n+1$ variables, equal to P_i when $x_0 \equiv 1$.)

We supplement the geometric definitions of par. 3.2.1: $\Pi(m, m_0)$ is the number of integer points of the parallelepiped $\Delta(m)$ satisfying the inequalities

$$\frac{1}{2}(m_1 + \cdots + m_n - n - m_0) \leq y_1 + \cdots + y_n \leq \frac{1}{2}(m_1 + \cdots + m_n - n + m_0)$$

$O(m, m_0)$ is the number of integer points of the parallelepiped satisfying the inqualities

$$\frac{1}{2}(m_1 + \cdots + m_n - n - m_0) \leq y_1 + \cdots + y_n \leq \frac{1}{2}(m_1 + \cdots + m_n - n)$$

We note that $O(m, m_0) = \frac{1}{2}(\Pi(m, m_0) + \Pi(m))$ and that $\Pi(m) \equiv \Pi(m, m_0) \equiv \mu(mod 2)$.

Theorem 2 (Ref. 33)**.** *For the nondegenerate pair V, P_0 with degrees m, m_0 the numbers $a = \text{ind}$, $b = \text{ind}^+\text{ind}^-$ and $c = \text{ind}^+$ satisfy the inequalities $|a| \leq \Pi(m)$, $|b| \leq \Pi(m, m_0)$, $|c| \leq O(m, m_0)$ and the congruences $a \equiv b \equiv c \equiv \mu \mod 2$. Conversely, for any number a (number b, number c) satisfying these restrictions, there* *exists a nondegenerate pair* V, P_0 *of degrees* m, m_0 *for which* ind $= a$ *(*$\mathrm{ind}^+\mathrm{ind}^- = b$, $\mathrm{ind}^+ = c$*).*

We note that Theorem 1 is a special case of Theorem 2 (the case where the field V has homogeneous components and a single singular point at the origin).

We can also evaluate the index for vector fields V with "infinitely remote singular points". The number ind^+ is defined if the region $P_0 > 0$ contains only isolated singular points of the field V. The number ind is defined if all the singular points of the field V are isolated.

Theorem 3 (Ref. 33)**.** *Suppose that for the pair* V, P_0 *of degrees not exceeding* m, m_0*, the number* ind^+ *is defined. Then if* $m_0 + \cdots + m_n \equiv n(\mod 2)$ *then the modulus of the number* ind^+ *does not exceed* $O(m, m_0)$*. In this case no other restrictions exist on the number* ind^+*. If* $m_0 + \cdots + m_n \not\equiv n(\mod 2)$ *and* m_0 *is odd, then the modulus of the number* ind^+ *does not exceed* $O(m, m_0 + 1)$*. In this case there exists a pair* V, P_0 *with the extremal* $\mathrm{ind}^+ = \pm O(m, m_0 + 1)$*.*

Corollary. *Suppose that* V *is a vector field of degree not exceeding* $m = m_1, \ldots, m_n$ *with isolated singular points. Then, for* $m_1 + \cdots + m_n \equiv n(\mod 2)$ *the estimate* $|\mathrm{ind}| \leq \Pi(m)$ *holds, while for* $m_1 + \cdots + m_n \not\equiv n(\mod 2)$ *we have the estimate* $|\mathrm{ind}| \leq O(m, 1)$*. Both are exact.*

3.2.4 Inverse Jacobian Theorem. To study a multiple singular point one uses a small deformation to dissociate it into nonmultiple singular points. The key information about such a dissociation is provided by the theorem on the inverse Jacobian, which is also of intrinsic interest. The inverse Jacobian theorem is applied, for example, in the proof of the signature formula for the index (see par. 3.2.5). We give a formulation of this theorem and derive the classic Euler–Jacobi classification formula from it.

Suppose that a system of n holomorphic equations in n complex unknowns depends on parameters, where for general values of the parameters it has only nonmultiple roots, while for exceptional values of the parameters (which form, in general, a hypersurface in parameter space) certain roots merge and become multiple. At a multiple root the Jacobian of the system is equal to zero, so when the nonmultiple roots of the system coalesce, the Jacobian of the system tends to zero at the coalescing roots. The reciprocal of the Jacobian is defined only at nonmultiple roots, and for coalescing roots tends to infinity. It turns out, however, that when one sums the reciprocals of the Jacobian over all the roots of the system, all the infinities cancel out, and the sum remains finite. We give an exact formulation of this theorem. Suppose that $U \subseteq \mathbb{C}^n$ is a domain in $\mathbb{C}^n$, $f\colon U \to \mathbb{C}^n$ is a holomorphic mapping, $\mathcal{J} = \det(\dfrac{\partial f}{\partial x})$ is the Jacobian of this mapping, V is a region in the image space that does not intersect the image of the boundary of the region U, and $h\colon U \to \mathbb{C}$ is a holomorphic weight function.

Theorem (about the inverted Jacobian)**.** *The function*

$$\varphi(a) = \sum_{f(x)=a} h(x)\mathcal{J}^{-1}(x) \tag{1}$$

defined on regular values $a \in V$ *of the mapping* f*, is extendable holomorphically to the whole region* V *[the summation in (1) is taken over all roots of the system* $f(x) = a$ *lying in the region* U*].*

One of the corollaries of the inverted Jacobian theorem is the *Euler–Jacobi formula*. We recall this formula and give its derivation from the theorem.

Suppose that for the system

$$P_1 = \cdots = P_n = 0 \tag{2}$$

of n polynomial equations of degrees $m_1, \ldots, m_n$ in n complex unknowns, all the roots are nonmultiple and "lie in the finite part" of the complex space. Suppose that h is an arbitrary polynomial in n variables, whose degree is less than the degree of the Jacobian $\mathcal{J}$ of the system (i.e., less than $m_1 + \cdots + m_n - n$). In this case the Euler–Jacobi formula

$$\rho = \sum_{P(x)=0} h(x)\mathcal{J}^{-1}(x) \tag{3}$$

holds [where the summation goes over all roots of the system (2)].

The proof is obtained by applying Theorem 1 to a special mapping of the $(n+l)$-dimensional space into itself with a special weight function. The first n components of this mapping are homogeneous polynomials in the $n+1$ variables, which coincide with the polynomials of the system (2) on the hyperplane $x_{n+1} = 1$, the last component is the coordinate function x_{n+1}. The weight function is a homogeneous polynomial coinciding with the polynomial h on the hyperplane $x_{n+1} = 1$.

The inverse image of the point $(0, \varepsilon)$ of this mapping consists of points $(\varepsilon x, \varepsilon)$, where x is a root of the system (2). A calculation shows that

$$\varphi(0, \varepsilon) = \rho\varepsilon^{p-(\sum m_i - n)}$$

where φ is the function defined by formula (1) for the mapping and for the weight function constructed above, ρ is a number defined by formula (3), and p is the degree of the weight function. As ε tends to zero, the function φ must remain bounded. If the degree of the weight polynomial is less than the degree of the Jacobian, then this is possible only if $\rho = 0$. Thus the formula of Euler–Jacobi is proved. We note that this formula is used in estimating the index of a polynomial vector field (see par. 3.2.3) and in the Petrovskii–Oleinik proof of their inequalities (Ref. 35).

3.2.5 General Theorems about the Index. In this paragraph we give a formulation of the signature formula for the index and prove the estimate of the index of a singular point of a field with homogeneous components from Theorem 1 of par. 3.2.1.

Suppose that $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is a mapping with finite multiplicity and Q is the local algebra of this mapping. We denote by $\mathcal{J}$ the image of the Jacobian of the mapping f in the local algebra Q.

Assertion 1. *The image of the Jacobian $\mathcal{J}$ in the local algebra is not equal to zero.*

Suppose that l is an arbitrary linear real-valued function on the local algebra Q. We define a bilinear form B_l on the local algebra by the formula $B_l(a, b) = l(a \cdot b)$.

Theorem 1 [signature formula for the index (Refs. 54 and 55)]**.** *For any linear function l that is positive on the image of the Jacobian $\mathcal{J}$, the signature of the bilinear form B_l is equal to the index of the singular point* 0 *of the mapping f.*

Corollary. *If in the μ-dimensional local algebra Q of the mapping f there exists an m-dimensional linear subspace L, the product of any two of whose elements is zero, then the modulus of the index of the point* 0 *does not exceed $\mu - 2m$.*

Proof of the Corollary. The space L is a zero space for the bilinear form B_l. The modulus of the signature of the quadratic form on a μ-dimensional space having a zero m-dimensional subspace does not exceed $\mu - 2m$.

Let us now prove the estimate of the index of a singular point with homogeneous components $m_1, \dots, m_n$ given in par. 3.2.1.

As the generators of the local algebra of this singular point we may take monomials whose degrees in the i-th variable are strictly less than m_i for any $i \leq n$. Such monomials are in one-to-one correspondence with the integer points of the parallelepiped $\Delta(m)$ (see par. 3.2.1). Consider the R-linear subspace L, which is spanned by the monomials whose degrees are greater than half their possible maximum $[d > \frac{1}{2}(\sum m_1 - n)]$. The product of any two monomials of this subspace is equal to zero in the local algebra.

Consequently, $|\mathrm{ind}| \leq \mu - 2 \dim_R \overline{L}$.

The right side of this inequality is equal to the number of base monomials of average degree, i.e., is equal to $\Pi(m)$ (see par. 3.2.1).

The estimate of Theorem 1 is proven. In conclusion, we give two general estimates of the index.

Theorem 2. *The index of a singular point of multiplicity μ in an n-dimensional space does not exceed $\mu^{1-1/n}$.*

This theorem is derived from the signature formula using the Teissier inequalities (Ref. 59).

Theorem 3 (Ref. 34)**.** *The modulus of a singular point of a gradient vector field in an even number of variables does not exceed the number $h_1^{n/2,n/2}$.*

The number $h_1^{n/2,n/2}$ in the theorem is one of the Hodge–Steenbrink numbers (Ref. 34), which characterizes the complex geometry of the germ of the hypersurface $f = 0$. For the gradient of a homogeneous function f the estimate of the theorem coincides with the estimate of Theorem 1 (see par. 3.2.1).

3.3 Few-Term Equations. The topology of objects given by algebraic equations (real algebraic curves, surfaces, singularities, etc.) becomes complicated rapidly as the degree of the equation increases. It has become clear recently that the complexity of the topology depends on the number of monomials they contain, rather than on the degree of the equations: the theorems formulated below estimate the complexity of the topology of geometric objects in terms of how cumbersome their equations are.

3.3.1. Real Few-Term Equations. The following theorem of Descartes is well-known. The number of positive roots of a polynomial in one real variable does not exceed the number of changes of sign in the sequence of its coefficients (zero. coefficients are dropped from the sequence).

Corollary (Descartes' estimate)**.** *The number of positive roots of a polynomial is less than the number of terms in it.*

A. G. Kushnirenko suggested that polynomials with a small number of terms be called fewnomials. Descartes' estimate shows that irrespective of the power of a few-term polynomial (which may be arbitrarily large), it has few positive roots.

Theorem 1 (Ref. 37). *The number of nondegenerate real roots of a system of n fewnomial equations in n unknowns, containing no more than k monomials (independent of the degree of the fewnomials) is estimated from above by a certain function φ_1 of n and k.*

Theorem 2 (Ref. 37). *The sum of the Betti numbers of the $(n-m)$-dimensional real manifold, determined by a nondegenerate system of m fewnomial equations, containing no more than k monomials, is estimated from above by some function φ_2 of n and k.*

We give the familar estimate for the function $\varphi_1 : \varphi_1(n,k) < (n+1)^k 2^{[k(k+1)/2]+n}$. This estimate is still far from exact. According to an unproved (but irrefutable) conjecture of A. G. Kushnirenko, an exact estimate of the number of solutions in the positive octant is

$$\Pi_{1 \geq i \geq n}(k_i - 1)$$

where k, is the number of monomials appearing in the i-th equation.

3.3.2 Complex Few-Term Polynomials. The complex roots of the simplest two-term equation $x^N - 1 = 0$, with increasing N, are uniformly distributed in argument. The theorem formulated below shows that the roots of any nondegenerate system ot fewnomial cquaiions are also uniformly distributed in argument.

Let us consider a regular system of equations $f_1 = \cdots = f_n = 0$ in $(\mathbb{C} \setminus 0)^n$, with fixed Newton polyhedra $\Delta_1, \ldots, \Delta_n$ (see par. 3.1.2). In writing these equations we encounter only k monomials (i.e., the union of the supports of these equations contains only k points). We denote by $N(f,G)$ the number of solutions of this system, whose arguments lie in a given domain G of the torus $T = \{\varphi_1, \ldots, \varphi_n\}$ mod 2Π. For the number $\Pi(\Delta^*, \partial G)$ defined below, which depends only on the geometry of the Newton polyhedra and on the domain G, the following theorem is valid.

Theorem (Ref. 60). *There exists a function φ of n and k such that for every regular system of n equations with k monomials the relation*

$$\left| N(f,G) - \frac{n!}{(2\pi)^n} V(G) V(\Delta_1, \ldots, \Delta_n) \right| \leq \varphi(n,k) \Pi(\Delta^*, \partial G)$$

is valid. Here $V(G)$ is the volume of the region $G \subset T^n$, and $V(\Delta_1, \ldots, \Delta_n))$ is the mixed volume of the Newton polyhedra of the system.

We give the definition of the number $\Pi(\Delta^*, \partial G)$. Suppose that Δ^* is the domain in $\mathbb{R}^n$ defined by the inequalities $\{\varphi \in \Delta^* || m\varphi| < \pi/2$ for all integer-valued vectors m, lying in the union of the supports of the Laurent polynomials $f_1, \ldots, f_n\}$. The number $\Pi(\Delta^*, \partial G)$ is defined as the minimum number of domains, which are equal to within a parallel displacement of the region Δ^*, necessary to cover the boundary ∂G of the region $G \subset T^n$.

Let us consider some special cases of the theorem: (1) the domain G coincides with the torus T^n. In this case $V(G) = (2\pi)^n$, $\Pi(\Delta^*, \partial G) = 0$, and the theorem coincides exactly with Bernshtein's theorem (see par. 3.1.2); (2) the domain G

contracts to a point $0 \in T^n$. In this case, $V(G) \to 0$, $\Pi(\Delta^*, \partial G) = 1$, and the theorem coincides exactly with Theorem 1 (par. 3.3.1); (3) we simultaneously increase the polyhedra Δ_i (without overlapping them), without increasing the number of monomials k and without changing the domain G. In this case the mixed volume $V(\Delta_1, \ldots, \Delta_n)$ is of the order of the n-th power of the size of the polyhedra Δ_i, while the number $\Pi(\Delta^*, \partial G)$ is of the order of the $(n-l)$-power of the size. In this case the theorem shows that the roots are distributed uniformly in argument.

3.3.3 Generalizations. The idea of few-term polynomials consists in the fact that the intersections of real level lines of "sufficiently simple" functions should be "sufficiently simple." At present, this notion has a basis not only for few-term polynomials, but also for real Liouville functions (Refs. 38 and 61), and for the even wider class of real, transcendent Pfaffian functions (Refs. 37 and 60). The class of Pfaffian functions is sufficiently wide: for example, if $f(x, y)$ is a Pfaffian function, then a solution of the differential equation $y' = f(x, y)$ is a Pfaffian function also. The construction of these functions is based on the fact that the solutions of the first-order equations do not oscillate (in contrast to the solutions of second-order equations such as $y'' = -y$).

The theorem on complex fewnomials has also been generalized (Ref. 60): This generalization allows one to use the theorem in estimating the number of zeros of the linear combinations of exponentials e^{ax}, where a is a real vector.

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