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# ON A CLASS OF SYSTEMS OF TRANSCENDENTAL EQUATIONS

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Bézout's theorem [1] evaluates the number of complex roots of a system of $n$ polynomial equations in $n$ unknowns in terms of the degrees of the polynomials. In this paper a fairly extensive class of systems of real transcendental equations is considered. As a rule, the number of complex roots of such systems is infinite. We estimate the number of real roots of such systems in terms of their complexity (vide infra for the definition of this term). As a corollary we get a new estimate of the number of real roots of a polynomial system. These results were the subject of a talk at a meeting of the Moscow Mathematical Society on October 9, 1979.

We say that the analytic functions $f_1, \dots, f_k$ on $\mathbf{R}^n$ form a *Pfaffian chain* (*P-chain*) *of length* $k$ if all partial derivatives of any $f_j$ in the chain are expressible as polynomials of the first $j$ functions of the chain and the coordinate functions in $\mathbf{R}^n$. In other words, for all $1 \leqslant i \leqslant n$ and all $1 \leqslant j \leqslant k$ there exist polynomials $P_{ij}$ such that

$$(\partial f_j/\partial x_i)(x) = P_{ij}(x, u_1, u_2, \dots, u_j),$$

where $x = x_1, \dots, x_n$ and $u_l = f_l(x)$ for $1 \leqslant l \leqslant j$. A *P-system* in $\mathbf{R}^n$ is any system $Q_1 = \cdots = Q_m = 0$ of equations in which the $Q_p$ are polynomials of coordinate functions in $\mathbf{R}^n$ and functions of a P-chain. The *complexity* of a P-system is the following collection of numbers: $n$, the length $k$ of the P-chain, and the degrees of the polynomials $Q_p$ and $P_{ij}$.

THEOREM 1. *The number of nondegenerate roots of a P-system consisting of n equations in* $\mathbf{R}^n$ *is finite and bounded from above by an explicitly given function of the complexity of the P-system.*

The proof is based on Theorem 2, given below.

We say that the *upper number of transformations* (u.n.t.) of a smooth mapping of manifolds of the same dimension *does not exceed* $N$ if every point of the image manifold has at most $N$ nondegenerate preimages.

Suppose that $\Gamma$ is a compact curve and $\xi$ is a vector field on $\Gamma$ that does not vanish anywhere. Furthermore, suppose that $g$ is a function on $\Gamma$ with nondegenerate zeros and $\hat{j}$ is a function on $\Gamma$ which coincides with the derivative $j = g'_\xi$ at the zeros of $g$ (i.e. if $g(a) = 0$ then $0 \neq g'_\xi(a) = \hat{j}(a)$). The following variant of Rolle's lemma holds:

PROPOSITION 1. *Suppose the u.n.t. of* $j$ *does not exceed* $N$. *Then* $g$ *has at most* $N$ *zeros.*

PROOF. Let $a$ and $b$ be zeros of $g$ which are neighbors in the sense of orientation determined by the field $\xi$. Then the numbers $g'_\xi(a)$ and $g'_\xi(b)$ have different signs. Let $\epsilon$ be

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the least value taken by $|\hat{j}|$ at the zeros of $g$. Then $\hat{j}$ takes all the values between $-\epsilon$ and $\epsilon$ on the interval $(a, b)$. The proposition readily follows.

Let $G$: $\mathbf{R}^{n+1} \longrightarrow \mathbf{R}$ be a smooth function with a nondegenerate zero level surface $M^n$, let $F$: $\mathbf{R}^{n+1} \longrightarrow \mathbf{R}^n$ be a proper smooth mapping, and let $\widetilde{F}$: $M^n \longrightarrow \mathbf{R}^n$ be its restriction to $M^n$. Further, let $\hat{J}$ be any smooth function on $\mathbf{R}^{n+1}$ which coincides on $M^n$ with the Jacobian $J$ of the mapping $(F, G)$: $\mathbf{R}^{n+1} \longrightarrow \mathbf{R}^n \times \mathbf{R}$. Under these conditions we have

THEOREM 2. *Let the u.n.t. of the mapping* $(F, \hat{J})$: $\mathbf{R}^{n+1} \longrightarrow \mathbf{R}^n \times \mathbf{R}$ *not exceed* $N$. *Then the u.n.t. of* $\widetilde{F}$: $M^n \longrightarrow \mathbf{R}^n$ *is* $\leqslant N$ *as well.*

PROOF. Let $\Gamma_a$ denote the preimage of a regular value $a$ of $F$. The smooth curve $\Gamma_a$ is compact since $F$ is proper. Let $\xi$ denote the vector field in $\mathbf{R}^{n+1}$ defined as follows: for every function $\Phi$ which is smooth on $\mathbf{R}^{n+1}$ its derivative $\Phi'_\xi$ along the field $\xi$ coincides with the Jacobian of the mapping $(F, \Phi)$: $\mathbf{R}^{n+1} \longrightarrow \mathbf{R}^n \times \mathbf{R}$. The curve $\Gamma_a$ is tangential for the field $\xi$ (since $F'_{i\xi} = 0$), and $\xi$ cannot have zero vlaues on $\Gamma_a$. Let $g$, $j$, and $\hat{j}$ denote the restrictions of $G$, $J$, and $\hat{J}$ to $\Gamma_a$. Then $g'_\xi = j$; hence $g'_\xi = \hat{j}$ at zeros of $g$. Let $a \in \mathbf{R}^n$ be a regular value for the mapping $\widetilde{F}$ as well. Then the derivative $g'_\xi$ is not zero at the zeros of $g$. Applying Proposition 1 we see that the number of preimages of $a$ under $\widetilde{F}$ does not exceed $N$. It follows from the implicit function theorem that the subset consisting of the points having not less than $N + 1$ nondegenerate preimages is open in $\mathbf{R}^{n+1}$. To complete the proof one must use Sard's theorem.

In Theorem 2 one can omit the requirement that $F$: $\mathbf{R}^{n+1} \longrightarrow \mathbf{R}^n$ be a proper mapping. Suppose that, for every regular value $a$ of $F$, the curve $F^{-1}(a)$ has at most $q$ noncompact components. Then we have

THEOREM 2$'$. *Suppose the u.n.t. of the mapping* $(F, \hat{J})$: $\mathbf{R}^{n+1} \longrightarrow \mathbf{R}^n \times \mathbf{R}$ *does not exceed* $N$. *Then the u.n.t. of* $\widetilde{F}$: $M^n \longrightarrow \mathbf{R}^n$ *is at most* $N + q$.

The proof repeats that of Theorem 2 almost verbatim.

Consider the proof of Theorem 1. We can always suppose that among the equations of the P-system there is an equation $Q = 0$ such that the corresponding polynomial determines a proper mapping $Q$: $\mathbf{R}^n \times \mathbf{R}^k \longrightarrow \mathbf{R}$. Indeed, if there is no such equation, then one can add a new unknown $x_0$ and a new equation $x_0 + \Sigma x_i^2 + \Sigma u_l^2 - R^2 = 0$ to the system. The complexity of the new system is expressible in terms of the complexity of the old system; it does not depend on $R$. The number of nondegenerate roots of the new system is twice the number of those of the old system which lie in the domain $\Sigma x_i^2 + \Sigma u_l^2 < R^2$.

Now we carry out an induction on the length of the P-chain. For P-systems with a P-chain of length 0 Theorem 1 follows from Bézout's theorem. Together with a P-system $Q_1 = \cdots = Q_n = 0$ in $\mathbf{R}^n$ with a P-chain of length $k$, consider an equivalent system of equations $F_1 = \cdots = F_n = G = 0$ in $\mathbf{R}^{n+1}$ with coordinates $(x_1, \ldots, x_n, v) = (x, v)$, where $F_j$ is the function whose value at $(x, v)$ is equal to the value of $Q_i(x, u_1, \ldots, u_k)$ at the point $(x, u_1 = f_1(x), \ldots, u_{k-1} = f_{k-1}(x), u_k = v)$, and $G(x, v) = f_k(x) - v$. It follows from the definition of a P-chain that all partial derivatives of the functions $F_j$ and $G$ are polynomially expressible via the coordinate functions $x$, $v$ and the functions $f_l$ from the P-chain. The function $f_k(x)$ coincides with $v$ on the surface $G = 0$. Therefore on this surface the Jacobian of the mapping $(F, G)$: $\mathbf{R}^{n+1} \longrightarrow \mathbf{R}^n \times \mathbf{R}$ is polynomially expressible in terms of $x_1, \ldots, x_n$, $v$ and $f_1, \ldots, f_n$, and the degree of the polynomial $\hat{J}$ which determines this mapping can be explicitly estimated in terms of the complexity of the original P-system. Theorem 2 lets us estimate the number of nondegenerate roots of the original P-system in terms of the u.n.t. of the mapping $(F, \hat{J})$: $\mathbf{R}^{n+1} \longrightarrow \mathbf{R}^n \times \mathbf{R}$, i.e. in terms of the maximum number of nondegenerate roots of the systems $F = a$, $\hat{J} = b$ with arbitrary right parts. Even though these P-systems have more unknowns, their P-chain $f_1, \ldots, f_{k-1}$ has a smaller length, and their complexity can be explicitly estimated in terms of the complexity of the original P-system. Theorem 1 is proved.

Now we outline a longer proof of Theorem 1 which produces a somewhat better bound for the number of roots. In our induction proof we can use Theorem 2′ instead of Theorem 2 without adding the compactifying equation to our system. The number of noncompact components of the curve $F^{-1}(a) \subset \mathbf{R}^{n+1}$ is estimated from the following statement: *If the number of transversal intersections of the curve with any hypersurface is at most $q$, then the curve has at most $q$ noncompact components.*

Consider applications of Theorem 1. Exponents of $k$ different linear functions in $\mathbf{R}^n$ form a P-chain. Applying Theorem 1 to P-systems with such a P-chain, we see that the number of nondegenerate roots of a system of quasipolynomials is finite and can be expressed explicitly in terms of the degrees of the quasipolynomials, the dimension $n$ and the number $k$ of exponents.

THEOREM 3 (on polynomials with few terms). *The number of nondegenerate roots of a polynomial system $P_1 = \cdots = P_n = 0$ lying in the positive octant $\mathbf{R}^n_+$ does not exceed $(n+2)^k \cdot 2^{k(k+1)/2}$, where $k$ is the number of different monomials which occur with a nonzero coefficient in a polynomial $P_j$.*

PROOF. The change of variables $x_i = \exp(y_i)$ transforms the original system into a system of quasipolynomials with $k$ exponents. Our bound is produced by the induction algorithm described in the proof of Theorem 1.

Apparently the bound given by Theorem 3 is considerably overstated. According to a conjecture of Kušnirenko [2] it can be lowered to $\Pi_1^n(k_i - 1)$, where $k_j$ is the number of monomials occurring in $P_j$. For $n = 1$ this follows from Descartes' rule. Our results originated from unsuccessful attempts to prove Kušnirenko's conjecture for $n > 1$.

Two similar theorems may be deduced from Theorem 1:

THEOREM 4. *Let a set $X \subseteq \mathbf{R}^n$ be defined by a P-system of $m$ equations. Then the following assertions are true*:

(1) *The number of connected components of $X$ is finite.*

(2) *If the system is nondegenerate then the sum of the Betti numbers of the $(n - m)$-dimensional manifold $X$ is finite.*

*Moreover, the number of connected components and the sum of the Betti numbers are bounded from above by certain explicit functions of complexity of the P-system.*

THEOREM 5. *Let an algebraic set $X \subseteq \mathbf{R}^n$ be defined by a system of $m$ polynomial equations. Let $k$ be the number of different monomials occurring with a nonzero coefficient in a polynomial from the system. Then the number of connected components of $X$ and (in the nondegenerate case) the sum of the Betti numbers of the smooth $(n - m)$-dimensional manifold $X$ are bounded from above by certain explicitly given functions of $n$ and $k$.*

The class of P-systems can be extended. Define the *class of P-functions* as the minimal class of functions containing all polynomials in any number of unknowns and closed under superpositions and solutions of Pfaffian equations (i.e., if an analytic function $f(x)$ satisfies the equation $df = \Sigma F_i(x, f)dx_i$, where all the $F_i$ are P-functions, then $f$ is a P-function). For systems of equations $f_1 = \cdots = f_m = 0$ with P-functions $f_j$ one can define the concept of complexity. Theorems 1 and 4 can be extended to such systems.

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