RECTIFICATION OF CIRCLES

A. G. Khovanskii UDC 513.27+513.73

In this note we describe diffeomorphisms of regions in the plane which take all lines into lines and circles. Diffeomorphisms of this type are useful in nomography (cf. [1, 2]). I am grateful to G. S. Khovanskii for interesting me in nomography.

A set of curves in the plane is called rectifiable near the point a if there exists a neighborhood U of a and a diffeomorphism of U taking the curves in the set (more precisely, the portions of the curves contained in the region U) into lines (more precisely, into portions of lines lying in the image of the region U). A bundle of curves with center at the point a is any set of curves passing through a. A bundle is called simple if curves in the bundle having identical tangents at the point a coincide identically in some neighborhood U of a. If a bundle of curves with center at the point a is rectifiable near a then the bundle is simple. We are interested in the behavior of the curves in a rectifiable bundle near the center. We identify curves which coincide identically in some neighborhood of a. The curves 1_α of a bundle will be regarded as the graphs of functions $y_\alpha = y_\alpha(x)$. The parameter α for the curves 1_α in a simple bundle can be taken to be the tangent of the angle of inclination of the tangent to the curve 1_α at the point a.

THEOREM 1. Assume that a simple bundle of lines $y_k(x)$ subject to the conditions $y_k(0) = 0$, $y'_k(0) = k$, is locally rectifiable near the point (0, 0) by means of a class C^{m+1} diffeomorphism. Then for every i, $1 < i \leq m$, there exists a polynomial P_i of degree $\leq 2i - 1$ such that $y_k^{(j)}(0) = P_i(k)$.

For the proof we will need an easily verified lemma, which is a sharpening of the implicit function theorem. Consider an equation $F(x, y(x)) = 0$ in which $F(x, y)$ is a C^{m+1} function. Let $F(0, 0) = 0$ and $(\partial F/\partial y)(0, 0) \neq 0$. By the implicit function theorem, the equation $F(x, y(x)) = 0$ can be solved locally for the function $y(x)$, and $y(x)$ is also of smoothness class C^m. Let $F(x, y) = \sum_{p+q\leqslant m} a_{p,q}x^p y^q + \ldots$ and $y(x) = \sum_{i\leqslant m} \psi_i x^i + \ldots$ be partial sums for the Taylor series for the functions F and y.

LEMMA 1. The coefficient of ψ_i is equal to some polynomial of degree $2i - 1$ in the coefficients $a_{p,q}$ (where $p + q \leq i$) divided by $a_{0,1}^{2i-1}$.

We continue with the proof of Theorem 1. Consider a diffeomorphism π rectifying a bundle of curves $y_k(x)$. Let A be an arbitrary nonsingular linear mapping of the plane. The diffeomorphism $A \circ \pi$ also rectifies the bundle $y_k(x)$. By a suitable choice of A we can arrange that the rectifying diffeomorphism has the identity differential at the point (0, 0). The rectifying diffeomorphism is now given by the formulas $u = f(x, y)$, $v = g(x, y)$, where $f = x + 0(|x| + |y|)$, $g = y + 0(|x| + |y|)$. In the (u, v) plane the bundle of curves $y_k(x)$ is given by the equations $v = ku$. Consequently the functions $y_k(x)$ are given by the equations $F_k(x, y) = 0$, where $F_k = g(x, y) - kf(x, y)$. The coefficients $a_{p,q}$ in the Taylor series of the function F_k depend linearly on k and the coefficient $a_{0,1} = 1$. Theorem 1 now follows from Lemma 1.

Remark 1. Assume that for every point $b \neq a$ in the region U there exists exactly one line passing through b which belongs to a simple bundle of lines with center a, and assume that this line depends smoothly on the point b. It is easily seen that every bundle with this property can be rectified by a diffeomorphism of the region U with is smooth everywhere except at a. The proof of Theorem 1 is based on the smoothness of the rectifying transformation at the point a.

Remark 2. According to Theorem 1, in a rectifiable bundle $y_k(x) = kx + \ldots + P_m(k)x^m + \ldots$ the functions $P_i(k)$ are polynomials of degree $\leq 2i - 1$. Calculations show that the coefficients of these polynomials satisfy certain relations, e.g., the coefficient $b_{i,2i-1}$ of $P_i(k)$ multiplying the power k^{2i-1} is equal to $\frac{2(2i-3)!}{i!(i-2)!}a^{i-1}$, where $a = b_{2,3}$ is the coefficient of k^3 in the polynomial $P_2(k)$. For large m there exist many other relations among the coefficients. However, for m = 2, 3 this is not so – for m = 2, 3 the bundle $y_k(x) = kx + \ldots + P_m(k)x^m + \ldots$ can be brought by means of a smooth change of coordinates into the form $y_k(x) = kx + O(x^m)$

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if and only if the functions $P_i(k)$ are polynomials of degree $\leq 2i-1$ (here $1 \leq i \leq m$) with highest coefficients satisfying the relation written out above (for $m = 3$ there is the single relation $b_{3,5} = 2a^2$ while for $m = 2$ there is no relation).

COROLLARY 1. The centers of curvature corresponding to the point (0, 0) of a rectifiable bundle $y = y_k(x)$ lie on a cubic $x^2 + y^2 = \varphi(x, y)$, where $\varphi(x, y)$ is some homogeneous polynomial of degree 3.

Indeed, by Theorem 1 if $y_k(x)$ is a rectifiable bundle of lines satisfying the conditions $y_k(0) = 0$, $y_k'(0) = k$, then there exists a polynomial $P_2(k)$ of degree 3 such that $y_k''(0) = P_2(k)$. The center of curvature of the curve $y_k(x)$ corresponding to the point (0, 0) lies on the line

$$x^2 + y^2 = 2y^3 P_2(-x/y) = \Phi(x, y).$$

We stop to discuss a corollary of Theorem 1 which is of a general character. The equation $F(x, y, a, b) = 0$ defines (when the usual solvability conditions are satisfied) a two-parameter family of lines $y = y(x, a, b)$ in the (x, y) plane and a two-parameter family of lines $b = b(a, x, y)$ in the (a, b) plane.

COROLLARY 2. If the family of lines $y = y(x, a, b)$ is locally rectifiable, then: 1) the family $y = y(x, a, b)$ satisfies the equation $y'' = L(x, y, y')$ in which $L(x, y, y')$ is a cubic polynomial in y'; 2) the family $b = b(x, a, b)$ satisfies the equation $b'' = M(a, b, b')$, where $M(a, b, b')$ is a cubic polynomial in b'.

Indeed by Theorem 1, in a rectifiable bundle the second derivative is a cubic polynomial in the first derivative [the coefficients of this polynomial depend on the center (x, y) of the bundle]. Condition 1 follows from this. Furthermore, the rectifiability of the family $y = y(x, a, b)$ is easily seen to imply the rectifiability of the family $b = b(a, x, y)$. The proof of condition 2) therefore also reduces to Theorem 1.

Remark 3. The assertion of Corollary 2 is not new. Another proof can be found in [3, pp. 46-56] (cf. also [4]). It is also stated in [3] that conditions 1) and 2) are sufficient for rectifiability of a family.

THEOREM 2. A simple bundle of circles containing at least eight circles is locally rectifiable if and only if all the circles in the bundle pass through a single point (distinct from the center of the bundle).

Proof. The equation for a simple bundle of circles with center at (0, 0) has the form $y = kx + A(x^2 + y^2)$, where $A = A(k)$ is some function of the parameter k. We show that rectifiability of the bundle is equivalent to linearity of the function $A(k)$. Upon solving the equations for the circles in the bundle up to terms of third order of smallness, we obtain $y_k(x) = kx + \psi_2(k)x^2 + \psi_3(k)x^3 + \ldots$, where $\psi_2(k) = A(k)(1 + k^2)$ and $\psi_3(k) = A^2(k)k(1 + k^2)$. These equalities imply that $2\psi_2^2 k = \psi_3(1 + k^2)$. By Theorem 2 the functions $\psi_2(k)$ and $\psi_3(k)$ are polynomials of third and fifth degree in k. The equality $2\psi_2^2 k = \psi_3(1 + k^2)$ is satisfied for all values of k corresponding to circles in the bundle, i.e., by at least eight values of k. Polynomials of degree seven which coincide at eight points coincide identically. The equality $2\psi_2^2 k \equiv \psi_3(1 + k^2)$ implies that the polynomial $\psi_2(k)$ is divisible by $1 + k^2$. Since $\varphi_2(k) = A(k)(1 + k^2)$, $A(k)$ is a linear function. Thus the equation for a rectifiable bundle of circles necessarily has the form $S_1 + kS_2 = 0$, where $S_1 = 0$ and $S_2 = 0$ are the equations for certain nontangent circles passing through (0, 0). We denote by b the second point of intersection of the circles $S_1 = 0$ and $S_2 = 0$. All the circles in the bundle $S_1 + kS_2 = 0$ pass through the point b. In order to rectify such a bundle of circles it suffices to take the point b to infinity via a conformal transformation. Theorem 2 is proved.

Remark 4. Theorem 2 remains valid for a simple bundle containing at least seven circles. Indeed, by Remark 2 the highest coefficients of the polynomials $2\psi_2^2 k$ and $\psi_3(1 + k^2)$ of degree seven coincide. Therefore, equality of these polynomials at seven points implies that they are identically equal.

Remark 5. The problem of the rectifiability of a simple bundle $y_k(x) = \psi_1(k)x + \ldots + \psi_l(k)x^l + \ldots$, $\psi_1(k) = k$ containing m lines, $k = \{k_j\}$, $1 \leq j \leq m$, admits an algebraic solution: in order for the bundle to be rectifiable, it is necessary and sufficient that certain algebraic relations among the numbers $\psi_i(k_j)$, $1 \leq i \leq m-3$, $1 \leq j \leq m$, be satisfied. Thus, in order for a simple bundle consisting of six lines to be rectifiable, it is necessary and sufficient that the following conditions hold: 1) there exists a polynomial $P_2(k)$ of degree 3 such that $P_2(k_j) = \psi_2(k_j)$; 2) there exists a polynomial $P_3(k_j)$ of degree 5 such that $P_3(k_j) = \psi_3(k_j)$; 3) the highest coefficients a and b of P_2 and P_3 satisfy $b = 2a^2$. In order for a simple bundle of five lines to be rectifiable, it is necessary and sufficient that condition 1) hold. A simple bundle containing four lines is always rectifiable.

We turn to two-parameter families of circles. We first give some definitions. The space 0 of equations of circles is the space of nonzero polynomials S of the form $S = a(x^2 + y^2) + bx + cy + d$ defined up to a factor. The space 0 is the projective space RP^3. A projective subspace L of 0 of dimension k (k = 1, 2) is called a k-dimensional linear system of circles. It is known from the geometry of circles (see, e.g., [5]) that up to a conformal transformation of the (x, y) plane there exist only three distinct two-dimensional linear systems of circles – the linear system of all circles orthogonal, respectively, to a fixed circle of positive, zero, and negative radius (these three systems are closely related to the three geometries of Lobachevskii, Euclid, and Riemann). Three planes L can be defined in the space 0 of polynomials $S = a(x^2 + y^2) + bx + cy + d$ which correspond to the three conformally inequivalent linear systems, e.g., by the equations $a = d$, $a = 0$, and $a = -d$. A two-dimensional family of circles N is any set of circles the equations of which lie in some two-dimensional linear system L(N) but not in any one-dimensional linear system. A characteristic map of a two-dimensional family N is a map $\varphi: R^2 \to RP^2$ defined by the formula $\varphi(x, y) = S_1(x, y) : S_2(x, y) : S_3(x, y)$, where S_1, S_2, S_3 are any three independent polynomials in the plane L(N). A characteristic map φ depends on the choice of the polynomials $S_i \in L(N)$ and is therefore defined up to a projective transformation. The singular points (x, y) of a characteristic map φ will be called the singular points of the two-dimensional family of circles. The singular points of the three linear systems of circles indicated above consist, respectively, of the points on the circle $x^2 + y^2 = 1$, the point (0, 0), and the empty set. We give one more definition. A family of lines in a region U is said to be representative if a curve in the family emanates from every point $p \in U$ in every direction contained within some cone K_p (it is assumed that the cone K_p depends continuously on the point p and has a nonzero apex angle).

THEOREM 3. 1) A representative family of circles in a neighborhood of the point p is rectifiable if and only if it is two-dimensional and p is a nonsingular point. 2) Every rectifying transformation of a two-dimensional representative family of circles coincides with a characteristic map of the family.

Proof. Assume that the representative family of circles is rectifiable. Let $S_1 = 0$ be the equation for some circle in the family passing through the point p with a tangent lying inside K_p. Let a and b be two points on the circle $S_1 = 0$ lying close to p but on different sides. The circles in the family passing through the points a and b form rectifiable bundles. Hence by Theorem 2 their equations have the form $S_1 + \alpha S_2 = 0$ and $S_1 + \beta S_3 = 0$.

For α and β of small absolute value, the circles $S_1 + \alpha S_2 = 0$ and $S_1 + \beta S_3 = 0$ are contained inside the cones K_a and K_b and therefore belong to our family of circles. Through each point p_1 close to p there pass circles in the family of the form $S_1 + \alpha S_2 = 0$ and $S_1 + \beta S_3 = 0$. By Theorem 2 all circles in the family passing through the point p_1 have the form $A(S_1 + \alpha S_2) + B(S_1 + \beta S_3) = 0$. Thus, the equations for all the circles in our rectifiable family lie in a plane $L \subset 0$, containing the equations $S_1 = 0$, $S_2 = 0$, and $S_3 = 0$. Furthermore, it is easily seen that near a singular point of the two-dimensional family there does not exist any representative rectifiable subfamily (this is verified separately for the three linear systems of circles). Near a nonsingular point of the family, the family is rectified by the characteristic transformation $\varphi(x, y) = S_1(x, y) : S_2(x, y) : S_3(x, y)$. It remains for us to show that up to projective transformations there exist no other rectifying maps. This follows immediately from Lemma 2, which follows (cf. [6]).

LEMMA 2. A local diffeomorphism of the plane which takes a representative family of lines into lines is projective.

The theorem is proved.

Let us say that a local diffeomorphism $\varphi: R^2 \to R^2$ rounds lines if the inverse image of every line under φ is either a line or a circle.

THEOREM 4. Up to a projective transformation of the plane of the image and a conformal transformation of the plane of the inverse image, there exist exactly three local diffeomorphisms which round lines. They are given by:

1) $\varphi(x, y) = x : y : 1 + (x^2 + y^2)$,

2) $\varphi(x, y) = x : y : 1$,

3) $\varphi(x, y) = x : y : 1 - (x^2 + y^2)$.

Proof. The inverse images of lines under a rounding diffeomorphism form a representative rectifiable family of circles. By Theorem 3, every representative family is two-dimensional. The diffeomorphisms 1)-3) are characteristic maps of the three conformally distinct two-dimensional linear systems of circles.

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