REAL LIOUVILLE FUNCTIONS

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Real rational functions $f(x)$ possess the following finiteness property: every equation $f(x) = a$ has only finitely many solutions. We show that an analogous property is possessed by all real Liouville functions.

Let U be a finite or infinite interval on the real line $\mathbf{R}^1$. We introduce an auxiliary definition. We say that f is a ΦU-function if: 1) f is defined and analytic in the region $U \setminus O(f)$ where $O(f)$ is a finite set; 2) f has a finite number of discrete zeros in $U \setminus O(f)$. On each interval of analyticity, a function either has discrete zeros or else is identically zero. Therefore, the set of zeros of every ΦU-function consists of finitely many points and a finite number of intervals. The restriction of a ΦU-function to an interval $J \subseteq U$ is a ΦJ-function. Let the interval U be a union of finitely many intervals J_i and a finite number of points. If the restriction of a function f to every J_i is a ΦJ_i-function, then f is a ΦU-function. A product of ΦU-functions is a ΦU-function. If a ΦU-function f has no zero intervals, then f^{-1} is defined and is a ΦU-function. By an integral and exponential integral of a function f, we mean any solutions of the equations $y' = f$ and $y' = fy$ analytic at the points of analyticity of f. For every ΦU-function an integral and exponential integral exist but are not uniquely defined – on every interval of analyticity of f, arbitrary constants can be added to the integral, and the exponential integral can be multiplied by arbitrary constant. An integral and exponential integral of a ΦU-function are ΦU-functions. Indeed, by Rolle's theorem the number of discrete zeros of an integral of f on each interval where f is analytic is at most one greater than the number of discrete zeros of f on the same interval. An exponential integral has no discrete zeros on intervals where f is analytic. Sums of ΦU-functions and the derivative of a ΦU-function may fail to be ΦU-functions.

Definition. A differential ring A consisting of functions in a domain U with the usual differentiation is said to have the finiteness property or be a ΦU-ring if A consists only of ΦU-functions.

Rings of polynomials and rational functions give examples of $\Phi\mathbf{R}^1$-rings. The restrictions of functions in a ΦU-ring A to a smaller interval $J \subseteq U$ form a ΦJ-ring. We denote this ring by A(J). Let y be a function. The extension A[y] of the differential ring A by the element y is the smallest differential ring containing A and y. The ring A[y] consists of polynomials with coefficients in A in the function y and all its derivatives.

THEOREM. Let the ring A have the finiteness property. In the following cases the extension A[y] also has the finiteness property: I) y is invertible over A, i.e., $y = f^{-1}$, where $f \in A$; II) y is an integral over A, i.e., $y' = f$, where $f \in A$; III) y is an exponential integral over A, i.e., $y' = fy$, where $f \in A$.

Proof. I) Let $y = f^{-1}$, where $f \in A$. The ring A[y] is formed by elements of type py^n where $p \in A$. The function py^n is a product of ΦU-functions and hence is a ΦU-function. II) Let $y' = f$ where $f \in A$. The ring A[y] consists of all polynomials P in y with coefficients in A. Assume by induction that every polynomial of degree $<k$ in every integral of y over every ΦV-ring B is a ΦV-function. Consider a polynomial $P = f_k y^k + \ldots + f_0$ of degree k where the $f_i \in B$. Let $J_1, \ldots, J_l$ be the intervals on which f_k is analytic and identically equal to zero, and let $I_1, \ldots, I_m$ be the remaining intervals on which f_k is analytic. The restriction of the function P to the interval J_i is a polynomial of degree $<k$ of an integral over the ΦJ_i-ring $B(J_i)$. By induction, the restriction of P is a ΦJ_i-function. We now restrict P to the interval I_i and consider it as a polynomial over the ring $B(I_i)[f_k^{-1}]$. By what has been proved, this ring is a ΦI_i-ring. The element $Z = Pf_k^{-1}$ satisfies the equation $Z = y^k + a_{k-1}y^{k-1} + \ldots + a_0$ over this ring, where $a_i = f_i f_k^{-1}$. We put $L = Z'$. Then $L = b_{k-1}y^{k-1} + \ldots + b_0$, where $b_i = a_i' + (i+1)a_{i+1}f$. By the induction assumption (applied to the ring $B(I_i)[f_k^{-1}]$) the polynomial L of degree $<k$ is a ΦI_i-function. Since it is an integral of the ΦI_i-function L, Z is a ΦI_i-function. Next, P is a ΦI_i-function since it is the product of the ΦI_i-functions Z and f_k. Thus the restriction of the function P to all the intervals $I_1, \ldots, I_l$ and $J_1, \ldots, J_m$ has the finiteness property. Therefore, P has the finiteness property on the interval U. The proof by induction is complete. III) Now let $y' = fy$ where $f \in A$. This case is analogous to the preceding one. Induction on the degree k shows that every polynomial $P = f_k y^k + \ldots + f_0$ is a ΦU-func-

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tion. In order to show this, we consider the functions Z and L = Z' (in the appropriate rings), where $Z = Pf_0^{-1} = a_k y^k + \ldots + 1$, $a_i = f_i f_0^{-1}$. The function L is equal to $y(b_k y^{k-1} + \ldots + b_1)$, where $b_i = a_i' + i a_i f$. The polynomial Ly^{-1} has degree < k and y is a ΦU-function since it is an exponential integral of f. This makes it possible to carry out the inductive step. The theorem is proved.

A ring $B \supseteq A$ is called a Liouville extension of A if there exists a chain of rings $A = A_0 \subseteq \ldots \subseteq A_n = B$ in which each ring A_{i+1} is obtained from A_i by adjoining an inverse element over A_i, an integral over A_i, or an exponential integral over A_i. A function f is called a real Liouville function if it lies in some Liouville extension of the ring of real constants. Examples of Liouville functions are the rational functions, e^x, $\ln|x|$, $|x|^\alpha$, arctan x. The class of real Liouville functions is closed under superposition of arithmetic operations, integration, and exponentiation.

COROLLARY. A Liouville function can be characterized by its complexity, i.e., by the number of arithmetic, integration, and exponentiation operations required to obtain it from the constants. It follows from our arguments that the number of discrete zeros of a Liouville function is estimated from above in terms of some function of its complexity. In other words, a Liouville function defined by a simple formula has few zeros. It would be of interest to obtain more precise estimates of this type.

Remark 2. The function cos x is a Liouville function over the field of complex constants **C**: $2\cos x = e^{ix} + e^{-ix}$, i.e., cos x lies in an extension of the ring **C** by the element e^{ix} satisfying the equation $y' = iy$. We note that complex Liouville functions also have special geometric properties (cf. [1]): the set of singularities of such functions in the complex plane is at most countable and the monodromy group is solvable.

Remark 3. The fact that cos x is not Liouville over the reals can evidently also be explained from the viewpoint of differential Galois theory [2]. The Galois group of the equation $y'' + y = 0$ over the field **R** is a circle. This circle has a normal tower of subgroups with quotient groups isomorphic either to the additive or the multiplicative group of the field **R**. In order to completely justify this explanation, it is necessary to modify the differential Galois theory somewhat – the theory is usually constructed for differential fields with an algebraically closed field of constants.

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AUTOMORPHISM OF VON NEUMANN ALGEBRAS AND APPROXIMATIVELY FINITE TYPE III_1 FACTORS WITH AN ALMOST-PERIODIC WEIGHT

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1. In this note we state some properties of automorphisms of factors which are then used to describe approximately finite (a.f.) factors M of type III_1 which possess a Γ-almost-periodic weight φ, where Γ is a countable subgroup of R_+^* (cf. Proposition 1.1 in [1]). In fact, a sketch is given of the proof of the following theorem.

THEOREM 1.1. If M is an a.f. type III_1 factor admitting a faithful normal (f.n.) semifinite Γ-almost-periodic weight, then $M \sim R_\infty$, where R_∞ is an Araki–Woods factor of type III_1 (cf. [2]).

By Lemma 4.9 in [1], every such algebra M can be represented as a crossed product M = R(N, Γ) of a type II_∞ algebra N with an f.n. semifinite trace τ by a group Γ of automorphisms of N with generators θ_i (1 ≤

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