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NEWTON POLYHEDRA AND THE EULER-JACOBI FORMULA

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The Euler–Jacobi formula [1] is valid for non-degenerate systems of polynomials of fixed degrees. Here we give a generalization of this formula, which is valid for non-degenerate systems of polynomials with fixed Newton polyhedra. I am grateful to V. I. Arnol'd for posing the problem and for his interest.

1. General Lemmas. Let M be an n-dimensional compact complex analytic manifold and $D_1, \ldots, D_n$ non-singular transversal analytic hypersurfaces in M. Let $M_0 = M \setminus D_1 \cup \ldots \cup D_n$, $M_1 = D_1 \setminus D_2 \cup \ldots \cup D_n, \ldots, M_n = D_1 \cap \ldots \cap D_n$. The set M_n consists of separate points a_k, $M_n = \{a_k\}$ $(k = 1, \ldots, N)$. Let $T_1, \ldots, T_N$ be the real n-dimensional tori in M_0 that "run around" all the surfaces D about the points $a_1, \ldots, a_N$. More precisely, let $T_k = \delta a_k$, where δ is the Leray complex coboundary (see [2], p. 57).

LEMMA 1. *The cycle $T_1 + \ldots + T_N$ is homologous to zero in M_0.*

PROOF. Let $H_*(M_n) \xrightarrow{\delta_n} H_*(M_{n-1}) \to \ldots \xrightarrow{\delta_1} H_*(M_0)$ be the Leray coboundary sequence, and $\delta = \delta_1 \circ \ldots \circ \delta_n$. Let $\gamma_1, \ldots, \gamma_N$ be real curves "running around" the points $a_1, \ldots, a_N$ on the complex curve $\overline{M}_{n-1} = D_1 \cap \ldots \cap D_{n-1}$. More precisely, let $\gamma_k = \delta_n a_k$. The cycle $\gamma_1 + \ldots + \gamma_N$ is homologous to zero in $\overline{M}_{n-1}$. For it bounds the film that is obtained from M_{n-1} after rejecting the discs B_k with boundaries γ_k. This completes the proof of the lemma, since $\Sigma T_k = \delta \Sigma a_k = \delta_1 \circ \ldots \circ \delta_{n-1} \Sigma \gamma_k = 0$.

Let $z = z_1, \ldots, z_n$ be local coordinates on M about the point a_k and $P_1 = 0, \ldots, P_n = 0$ the local equations of $D_1, \ldots, D_n$ about this point. We denote by $\partial P/\partial z$ the determinant of the corresponding Jacobian matrix. We consider the meromorphic form $\omega = f/(P_1 \cdot \ldots \cdot P_n)\, dz_1 \wedge \ldots \wedge dz_n$, where f is a holomorphic function in a neighbourhood of a_k.

LEMMA 2. $\left(\frac{1}{2\pi i}\right)^n \int\limits_{T_k} \omega = \left(f \Big/ \frac{\partial P}{\partial z}\right)\Big|_{a_k}$.

Lemma 2 is called the complex residue formula. It is proved by applying Cauchy's residue formula n times.

2. **Theorem.** Let $P_1, \ldots, P_n$ be a non-degenerate system of Laurent polynomials with Newton polyhedra $\Delta_1, \ldots, \Delta_n$ (see [3]). Let Q be an arbitrary Laurent polynomial whose Newton polyhedron $\Delta(Q)$ lies strictly inside $\Delta_1 + \ldots + \Delta_n$, $\Delta(Q) < \Delta_1 + \ldots + \Delta_n$.

THEOREM (the generalized Euler–Jacobi formula). *The sum $\sum\limits_{\{a_k\}} (Q/z_1 \cdot \ldots \cdot z_n \cdot \partial P/\partial z)|_{a_k}$ is zero. The summation is over the set $\{a_k\}$ of roots of the system of equations $P_1 = \ldots = P_n = 0$ in $(C \setminus 0)^n$ (that is, $z_1 \neq 0, \ldots, z_n \neq 0$).*

PROOF. We consider the toric compactification M of $(C \setminus 0)^n$, which is complete enough for $\Delta_1, \ldots, \Delta_n$ (see [3]). Let $D_1, \ldots, D_n$ be the closures in M of the hypersurfaces in $(C \setminus 0)^n$ given by the equations $P_1 = 0, \ldots, P_n = 0$. We extend to M the meromorphic form ω defined in $(C \setminus 0)^n$ by the formula $\omega = \frac{Q}{P_1 \cdot \ldots \cdot P_n} \cdot \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$. It is not difficult to show that ω *is regular outside* $D_1, \ldots, D_n$. Consequently, $\Sigma \int\limits_{T_k} \omega = 0$, since by Lemma 1 the cycle $T_1 + \ldots + T_n$ is homologous to zero in $M_0 = M \setminus D_1 \cup \ldots \cup D_n$. Moreover, by Lemma 2, $\sum\limits_{\{a_k\}} \int\limits_{T_k} \omega = \Sigma\ (Q/z_1 \cdot \ldots \cdot z_n \cdot \partial P/\partial z)|_{a_k}$.

COROLLARY (the Euler–Jacobi formula). *Let $P_1, \ldots, P_n$ be a general system of polynomials of degree $m_1, \ldots, m_n$, and $\widetilde{Q}$ any polynomial of degree less than $\Sigma(m_i - 1)$. Then* $\Sigma\ (\widetilde{Q}/\partial P/\partial z)|_{a_k} = 0$.

PROOF. If no roots lie on the coordinate planes, the corollary is obtained by direct application of the theorem for $Q = z_1 \cdot \ldots \cdot z_n \cdot \widetilde{Q}$. It is easy to get rid of the additional restriction by a small change in the coefficients of the system of equations $P_1 = \ldots = P_n = 0$.

3. **Remarks.** We note that in the case of polyhedra $\Delta_i = \Delta(P_i)$ of full dimension the theorem does not admit any improvement: in this case *any function f on the roots $\{a_k\}$ subject to the generalized Euler–Jacobi condition $\Sigma f(a_k) = 0$ can be obtained as a residue of some form*

$\dfrac{Q}{P_1 \cdot \ldots \cdot P_n} \cdot \dfrac{dz_1}{z_1} \wedge \ldots \wedge \dfrac{dz_n}{z_n}$, *where $\Delta(Q) < \Delta_1 + \ldots + \Delta_n$*. This assertion follows easily from the cohomology calculations of [4]. We note that the case of zero-dimensional complete intersections is exceptional: for complete intersections $P_1 = \ldots = P_m = 0$ of positive dimension $(m < n)$ *any* holomorphic form of higher degree can be obtained as a residue of some form

$$\frac{Q}{P_1 \cdot \ldots \cdot P_m} \cdot \frac{dz_1}{z_1} \wedge \ldots \wedge \frac{dz_n}{z_n}, \quad \text{where } \Delta(Q) < \Delta_1 + \ldots + \Delta_m \text{ [3]}.$$

In conclusion we mention that the Euler–Jacobi formula is applied in real algebraic geometry [5]. The generalized formula undoubtedly has a similar application.

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