POINT-SET TOPOLOGY

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About these notes

Warning! The notes you have in your hands are not a textbook, but an outline for a Moore method course on point-set topology. It contains definitions and theorems, exercises and questions, along with a few remarks that provide motivation and context. However, there are no proofs and few explanations – your job is to discover these for yourself. Even more so than with the average math textbook, **you cannot simply read this text!** Rather, treat it as a set of guide posts to help you find your own way through a rugged but beautiful terrain. You are not the first to visit this magnificent landscape, but you can still experience the thrill of exploring it for yourself. It's like hiking up to a mountain summit instead of taking the tour bus.

You are not setting out on this trek alone: you have your classmates to help you. In class, students will take turns presenting proofs of statements and critiquing each other's arguments. What one person is confused about, the class as a whole should be able to unravel (with only minimal participation by me). For this system to work, it is crucial that you treat your fellow students with sensitivity and respect – but don't let them get away with a bad argument! And when you are the presenter yourself, do not be afraid of getting stuck or of making a mistake; it's no big deal. It happens to everyone, and it is the only way to make progress in mathematics.

Your homework in this course will consist of preparing proofs, examples, explanations, etc., to present during the following class period. **You are expected to work on this course daily outside of class hours. I will be available daily at TAU, which is a good time for group work and for asking me questions.** You are encouraged to work on this with other students, but please do not consult any books or online resources. You are responsible for the proof of every numbered statement in the text, except for definitions and statements marked with "$\flat$". In $\flat$-statements, the proofs are relatively routine. They may be tedious sometimes, so it may not be worth your while to write out the entire proof in detail. (However, if you are not sure how the proof would go, you are encouraged to do some part of it, or to try a few examples of your own devising, to help you understand the statement; of course, this is always a good idea!) Statements labelled "Challenge" are particularly difficult. Statements labelled "Fuzzy" are vaguely-phrased questions that hint at an interesting notion or result; they are an opportunity for you to unleash your creativity and come up with your own conjectures.

Acknowledgment: This introduction is plagiarized from a Moore Method course taught by Mira Bernstein. These notes are based on a (less ambitious) course I lead at the University of Victoria with assistance from Jason Siefken.

1 The definition of topology

We are about to introduce the main objects of this course: topologies. The definition may appear at first random and capricious, and you may wonder why we should care about it. Later on, we will motivate where this definition comes from and why it is a useful one. For now, we want to concentrate simply on getting familiar with the concept.

Definition 1.1. Let X be a set. A *topology* on X is a family τ of subsets of X which satisfies three properties (spelled out below). We will say that a subset of X is an open set iff it is an element of τ. The three properties are:

- (T1) The total set and the empty set are open sets.
  Equivalently, $X \in \tau$ and $\emptyset \in \tau$.
- (T2) The intersection of any two open sets is an open set.
  Equivalently, if $A, B \in \tau$ then $A \cap B \in \tau$.
- (T3) The union of open sets (no matter how many, including infinitely many) is an open set.
  Equivalently, if I is a set of indices and $A_i \in \tau$ for all $i \in I$ then $\bigcup_{i \in I} A_i \in \tau$.
  Or, equivalently, for any $\sigma \subseteq \tau$, $\bigcup_{B \in \sigma} B \in \tau$

A topological space is a pair (X, τ) where X is a set and τ is a topology on X.

Exercise 1.2. ($\flat$) Among the following, some are topologies on the set $\mathbb{Z}$ and some are not. Which ones are? If an example is not a topology, but you can modify it slightly to make it into a topology, do so. If an example is a topology and you can generalize it into more examples, do so.

(a) $\tau = \{V \subseteq \mathbb{Z} \mid 0 \in V\}$. In words, a set is open iff it contains 0.

(b) $\tau = \{V \subseteq \mathbb{Z} \mid 0 \notin V\}$. In words, a set is open iff it does not contain 0.

(c) $\tau = \{V \subseteq \mathbb{Z} \mid 0 \in V \text{ and } 1 \in V\}$.

(d) $\tau = \{V \subseteq \mathbb{Z} \mid 0 \in V \text{ or } 1 \in V\}$.

(e) $\tau = \{V \subseteq \mathbb{Z} \mid V \text{ is finite}\}$.

(f) $\tau = \{V \subseteq \mathbb{Z} \mid V \text{ is infinite}\}$.

Exercise 1.3. Among the following, which ones are topologies on the set $\mathbb{R}$ and which ones are not?

(a) $\tau = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$

(b) $\tau = \{[a, \infty) \mid a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$

Exercise 1.4. Let X be any set.

(a) What is the topology on X that has the most open sets? This is called the *discrete* topology on X.

(b) What is the topology on X that has the least open sets? This is called the *indiscrete* topology on X.

Exercise 1.5. Let X be an arbitrary set. Which ones of the following are topologies?

(a) The *cofinite* topology: A set $V \subseteq X$ is open iff $[X \setminus V$ is finite or $V = \emptyset]$.

(b) The *coinfinite* topology: A set $V \subseteq X$ is open iff $[X \setminus V$ is infinite or $V = \emptyset$ or $V = X]$.

(c) The *cocountable* topology: A set $V \subseteq X$ is open iff $[X \setminus V$ is countable or $V = \emptyset]$.

Note: "Countable" includes " finite" as a particular case.

Exercise 1.6. Let (X, τ) be a topological space. Prove each of the following statements true or false.

(a) The intersection of any three open sets is open.

(b) The intersection of finitely many open sets is open.

(c) The intersection of open sets is open.

Definition 1.7. Let $x \in \mathbb{R}^N$ and let $\varepsilon > 0$. The *ball* centered at x with radius ε is

$$B_\varepsilon(x) := \{y \in \mathbb{R}^N \mid d(y, x) < \varepsilon\}.$$

where $d(y, x)$ is the Euclidean distance between the points x and y.

Exercise 1.8. ($\flat$) Describe geometrically what a ball is in $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$.

Definition 1.9. We define the *standard topology* or the *usual topology* on $\mathbb{R}^N$ as follows. Let $V \subseteq \mathbb{R}^N$. We say that V is *open* (in this topology) iff the following property is true: "For every $x \in V$, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq V$". When we refer to a topology on $\mathbb{R}^N$ or to open sets on $\mathbb{R}^N$ without specifying which topology, we mean the standard one. This is the topology one often uses in analysis.

Exercise 1.10. Prove that the topology in Definition 1.9 is actually a topology.

Examples 1.11. Show which ones of the following examples are open according to Definition 1.9:

(a) ($\flat$) The set $\{1\}$ in $\mathbb{R}$

(b) ($\flat$) The interval $(2, 5)$ in $\mathbb{R}$

(c) The ball $B_\varepsilon(x)$ in $\mathbb{R}^N$ for any $x \in \mathbb{R}^N$ and any $\varepsilon > 0$.

(d) ($\flat$) The interval $[0, 1)$ in $\mathbb{R}$

(e) ($\flat$) The set $\{(x, y) \in \mathbb{R}^2 \mid x > y\}$ in $\mathbb{R}^2$

Exercise 1.12. Find all the topologies on the set $X = \{0, 1, 2\}$

Fuzzy 1.13. Look back at your answer to Exercise 1.12. Some of those topologies are very similar. One could even say that they are practically "the same topology" with different names. Come up with a definition of what *practically the same topology* could mean. Also, come up with a better name. With this definition, how many essentially different topologies are there on $\{0, 1, 2\}$?

2 Sequences and limits

In this chapter we will talk about limits and accumulation points of sequences in any topological space. Whenever we have a topology, we have a notion of limit, even if there is not a distance or a notion of "being close". Challenge 2.16 at the end of the chapter is the first surprise of the course and it illustrates how sequences do not quite behave in general the way you have gotten used to. This will be one *leitmotif* of this course.

Definition 2.1. Let X be a set. A *sequence* in X is a map $x : \mathbb{N} \to X$. Notice that in this course we will include 0 in $\mathbb{N}$. As notation, we often write x_n instead of $x(n)$ for an element in X. We may also write (x_n) or $(x_n)_{n\in\mathbb{N}}$ or $(x_n)_{n=0}^{\infty}$ to refer to the whole sequence.

Definition 2.2. Let $P(n)$ be a statement that depends on a natural number $n \in \mathbb{N}$. We say that "$P(n)$ is eventually true for all n" if there exists $n_0 \in N$ such that $P(n)$ is true for all $n \geq n_0$. If there is no ambiguity, we will say simply that "$P(n)$ is eventually true".

Definition 2.3. Let (X, τ) be a topological space. Let (x_n) be a sequence in X. Let $a \in X$. We say that a is a *limit* of the sequence when the following statement is true: "If $V \subseteq X$ is an open set such that $a \in V$, then $x_n \in V$ eventually for all n." In this case we say that the sequence *converges* to a. In words, this means that every open set containing a has to contain all the sequence, except for the first few terms.

We say that a sequence is *convergent* if it has at least one limit.

Exercise 2.4. Consider the sequence $0, 1, 0, 2, 0, 3, 0, 4, 0, 5, 0, 6, \ldots$ on the set $X = \mathbb{R}$. For each of the following topologies, find all of its limits.

1. the discrete topology,
2. the indiscrete topology,
3. the cofinite topology,
4. ($\flat$) the cocountable topology,
5. ($\flat$) the topology in Exercise 1.3,
6. the standard topology.

Lemma 2.5. Let (X, τ) be a topological space. Let (x_n) be a sequence in X. Let $a \in X$. Prove that the following two statements are equivalent:

1. "For every open set $V \subseteq X$ such that $a \in V$ and for every $n_0 \in \mathbb{N}$, there exists $n \geq n_0$ such that $x_n \in V$."
2. "For every open set $V \subseteq X$ such that $a \in V$, there are infinitely many $n \in \mathbb{N}$ such that $x_n \in V$". In words, every open set containing a contains infinitely many terms of the sequence.

Definition 2.6. In the situation of Lemma 2.5, when the two equivalent conditions are satisfied, we say that a is an *accumulation point of the sequence.*

Proposition 2.7. (♭) Every limit of a sequence is also an accumulation point.

Exercise 2.8. Repeat Exercise 2.4, but this time find all the accumulation points instead of all the limits.

Fuzzy 2.9. Look back at Exercise 2.4. It included some examples where a sequence has more than one limit. Think of the discrete and indiscrete case; if a topology has more open sets, are sequences more or less likely to have multiple limits? Try to prove that in $\mathbb{R}$ with the usual topology, no sequence can have more than one limit. Which other topologies satisfy that? Can you come up with a necessary condition or a sufficient condition for a topology not to have sequences with multiple limits?

Exercise 2.10. Let $\mathcal{C}$ be the set of students who came to class today. We define a topology on $\mathcal{C}$ as follows. Given $V \subseteq \mathcal{C}$, we say that V is open iff it satisfies the following property:

"If $x \in V$ and y is sitting immediately to the left of x, then $y \in V$."

1. Prove that this is actually a topology.
2. Consider the sequence Aaron, Cindy, Aaron, Cindy, Aaron, Cindy, ... Find all its limits.
3. Find all the accumulation points of the same sequence.

Exercise 2.11. Consider the set $\mathbb{Z}$ with the cofinite topology. Find an example of a sequence such that (or prove that such an example does not exist):

1. it has more than one limit,
2. it has exactly one limit and exactly one accumulation point,
3. it has exactly one limit and it has more than one accumulation point,
4. it has no limits and no accumulation points,
5. it has no limits and it has exactly one accumulation point,
6. it has no limits and it has more than one accumulation point.

Exercise 2.12. Describe all the convergent sequences in $\mathbb{R}$ with the cocountable topology.

Definition 2.13. Let (X, τ) be a topological space. Let $x : \mathbb{N} \to X$ be a sequence. A *subsequence* of x is a sequence of the form $x \circ \lambda$ where $\lambda : \mathbb{N} \to \mathbb{N}$ is a strictly increasing map. As notation, if we write $x_n := x(n)$ for each $n \in \mathbb{N}$ and $n_k := \lambda(k)$ for each $k \in N$, we will often write that $(x_{n_k})_k$ is a subsequence of $(x_n)_n$. If there is no danger of ambiguity, we may write simply that (x_{n_k}) is a subsequence of (x_n).

Proposition 2.14. Let (x_n) be a sequence in the topological space (X, τ) and let $a \in X$. Assume that a is the limit of a subsequence of (x_n). Prove that a is an accumulation point of (x_n).

Proposition 2.15. Let (x_n) be a sequence in $\mathbb{R}$ with the standard topology. Let $a \in X$. Assume that a is an accumulation point of (x_n). Prove that a is the limit of some subsequence of (x_n).

Challenge 2.16. Give an example that shows that Proposition 2.15 may fail if we use an arbitrary topological space.

3 Closed sets

You know what an open set; now you are ready to learn what a closed set is. Just be careful and remember that closed does not equal "not open" (see Video 3.5). Closed sets give us an alternative way to think of a topology without mentioning open sets (see Theorem 3.8 and Note 3.9). Finally, you should look at results 3.13–3.14–3.15 in parallel with 2.14–2.15–2.16. Notice the pattern.

Definition 3.1. Let (X, τ) be a topological space. Let $A \subseteq X$. We say that A is *closed* when $X \setminus A$ is open.

Exercise 3.2. (♭) Which ones of the following sets are closed in $\mathbb{R}$ with the standard topology?

- $A = [0, 1]$,
- $B = \{0\}$,
- $C = (0, 1]$,
- $D = [0, \infty)$,
- $E = \{\frac{1}{n} \mid n \in \mathbb{N} \setminus \{0\}\}$
- $F = E \cup \{0\}$

Exercise 3.3. For each of the following topological spaces (X, τ), give examples of subsets $A_1, A_2, A_3, A_4 \subseteq X$ such that A_1 is open but not closed, A_2 is closed but not open, A_3 is both open and closed, and A_4 is neither open nor closed (or prove that such subsets do not exist). You may not use X or $\emptyset$ as any of your subsets.

1. X is the topological space of Exercise 3.10 in your class notes (the topology on the students in the class).
2. $X = \mathbb{Z}$ with the cofinite topology.

Definition 3.4. Let (X, τ) be a topological space. A set $A \subseteq X$ is called *clopen* if it is both open and closed.

Video 3.5. If Definition 3.4 makes you uncomfortable, you are not alone. Watch the following video: `http://youtu.be/SyD4p8_y8Kw`
Warning: The video contains profanity and dark humour. If you think such things may offend you, please do not watch the video.

Theorem 3.6. Let (X, τ) be a topological space. Let $\mathcal{F}$ denote the collection of closed sets. Then the following properties are true:

- (C1) $X \in \mathcal{F}$ and $\emptyset \in \mathcal{F}$.
  In words, the total set and the empty set are closed sets.
- (C2) If $A, B \in \mathcal{F}$ then $A \cup B \in \mathcal{F}$.
  In words, the union of any two closed sets is a closed set.

- (C3) For any $\Lambda \subseteq \mathcal{F}$, $\bigcap_{B \in \Lambda} B \in \mathcal{F}$.
  In words, the intersection of closed sets (no matter how many, including infinitely many) is a closed set.

Exercise 3.7. Show with an example in the standard topology that the arbitrary union of closed sets may not be closed.

Theorem 3.8. Let X be a set. Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a family of subsets satisfies conditions (C1), (C2), and (C3) from Theorem 3.6. Then there exists a unique topology on X that has $\mathcal{F}$ as the family of closed sets.

Note 3.9. Theorem 3.8 is very important and it is easy to miss the deep implications it has. Until now, whenever we wanted to define a topology on a set, we would define the family of open sets. We could define the open sets to be any family we wanted, as long as they satisfied conditions (T1), (T2), (T3). Theorem 3.8 says that, instead, we may chose to define a topology by saying who the closed set are, and that we can choose any family of subsets to be the closed sets as long as they satisfy (C1), (C2), (C3). If they do, we do not need to worry about what the open sets are or about checking that they satisfy (T1), (T2), (T3). It will come for free. The following example shows that some topologies are more naturally defined by saying what the closed sets are than by saying what the open sets are.

Exercise 3.10. ($\flat$) Let X be a set. We are going to define two topologies on X. You already proved that they were topologies (back in Exercise 1.5). Show again that they are topologies, but this time using Theorem 3.8 and Note 3.9. Notice that the proofs are now shorter and more natural.

1. The *cofinite* topology on X is the topology where the closed sets are the finite sets and X.
2. The *cocountable* topology on X is the topology where the closed sets are the countable sets and X.

Definition 3.11. Let (X, τ) be a topological space. Let $A \subseteq X$. We say that A is *sequentially closed* when it satisfies the following property:

"Let (x_n) be a sequence in X and let $a \in X$ be a limit of the sequence.
Assume that $x_n \in A$ for all $n \in \mathbb{N}$. Then $a \in A$."

Exercise 3.12. Prove that $(0, 1]$ is not sequentially closed in $\mathbb{R}$ with the standard topology. Prove it directly from Definition 3.11, without using Proposition 3.13 below.

Proposition 3.13. Let (X, τ) be a topological space and let $A \subseteq X$. Prove that if A is closed, then A is sequentially closed.

Proposition 3.14. Consider $\mathbb{R}$ with the standard topology. Let $A \subseteq \mathbb{R}$. Prove that if A is sequentially closed, then A is closed.

Challenge 3.15. Give an example of a topological space (X, τ) and a subset $A \subseteq X$ such that A is sequentially closed but A is not closed.

4 Neighborhoods

There are a few things that are easier to do in the standard topology because it has balls. Topologies in general do not have balls, but they do have *neighborhoods*. To a certain extent, neighborhoods are like balls. Let's get acquainted.

Challenge 4.6 below is very important. If you read it next to Theorem 3.8 you will see one of the themes of this course: open sets are not the only way to define a topology. If you do not find Exercise 4.7 below at least a bit cute, you may be Vulcan. Also, Exercise 4.8 is what we call in Spanish "*matar moscas a cañonazos*".

Definition 4.1. Let (X, τ) be a topological space. Let $A \subseteq X$. Let $x \in X$. We say that A is an *open neighborhood of* x when A is open and $x \in A$.

Definition 4.2. Let (X, τ) be a topological space. Let $x \in X$. A *basis of open neighborhoods of* x in (X, τ) is a family of open sets $\mathcal{B}_x \subseteq \tau$ such that

- W is an open neighborhood of x for every $W \in \mathcal{B}_x$.
- If V is an open neighborhood of x, then there exists $W \in \mathcal{B}_x$ such that $W \subseteq V$.

Exercise 4.3. Let $x \in \mathbb{R}$. Among the following families of sets, which ones are bases of open neighborhoods of x in $\mathbb{R}$ with the standard topology?
For this exercise, do not write lengthy proofs. Just answer "yes" or "no" for each candidate, and, if needed, give a one-line explanation at most.

1. $\{ (x-\varepsilon, x+\varepsilon) \mid \varepsilon > 0\}$
2. $\{ (x-1, x+\varepsilon) \mid \varepsilon > 0\}$
3. $\{ [x-\varepsilon, x+\varepsilon] \mid \varepsilon > 0\}$
4. $\{ (x-\varepsilon, x+2\varepsilon) \mid \varepsilon > 0\}$
5. $\{ (x-\frac{1}{n}, x+\frac{1}{n}) \mid n \in \mathbb{Z}^+\}$
6. $\{ (x-\frac{1}{n}, x+\frac{1}{n}) \mid n \in \mathbb{Z}^+, n > 100, n \text{ is odd}\}$
7. $\{ (x-\varepsilon, x+\varepsilon) \cup (x+2\varepsilon, x+3\varepsilon) \mid \varepsilon > 0\}$
8. $\{ (a, b) \mid a < x < b\}$
9. $\{V \subseteq \mathbb{R} \mid V \text{ is open and } x \in V\}$

Exercise 4.4. For each one of the following topological spaces, come up with a basis of open neighborhoods of each point which is as "small" as you can make it.

1. A discrete topological space.
2. An indiscrete topological space.

3. The set of students in class with the topology of Exercise 2.10.
4. The cofinite topology on $\mathbb{Z}$.

Exercise 4.5. Let (X, τ) be a topological space. Let $a \in X$ and let $\mathcal{B}_a$ be a basis of open neighborhoods of a in (X, τ). Write a condition equivalent to "a is a limit of the sequence (x_n) in the topological space (X, τ)" which uses the basis of open neighborhoods instead of open sets in general. Then prove they are equivalent.

Challenge 4.6. So far we know how to define a topology by saying who the open sets are, or by saying who the closed sets are. Find out a way to define a topology by saying who the neighborhoods of each point are. Specifically, list a set of axioms such as:

- If (X, τ) is a topological space, and $\mathcal{B}_x$ is a basis of open neighborhoods for x for every $x \in X$, then the families $\mathcal{B}_x$ satisfies the set of axioms.
- Let X be a set (not a topological space). Assume that for every $x \in X$ we have a family $\mathcal{B}_x \subseteq \mathcal{P}(X)$ and that they satisfy the axioms. Then there exists a unique topology on X which has $\mathcal{B}_x$ as a basis of neighborhoods of x for all $x \in X$.

  Note: The existence part of the proof is tricky, and you may think you are done before you are. First, you need to define a topology. Then, you need to check it *is* a topology. Finally, you need to prove that this topology does have the original families $\mathcal{B}_x$ as bases of open neighborhoods.

Once we have this theorem, we may choose to describe a topology by giving a basis of open neighborhoods of each point in X instead of by describing the open sets. We are allowed to choose these bases any way we want as long as they satisfy the axioms. If you think about the definition of the standard topology in $\mathbb{R}^N$, you will notice that it makes more sense to define it in terms of open neighborhoods than in terms of open sets, just like it makes more sense to definite the cofinite topology in terms of closed sets than in terms of open sets.

Exercise 4.7. In you have taken a calculus or analysis class, you may have learned the definition of three "different" concepts. Given a sequence (x_n) in $\mathbb{R}$, you may have defined the concepts

$$\lim_{n\to\infty} x_n = a \in \mathbb{R}, \qquad \lim_{n\to\infty} x_n = \infty, \qquad \lim_{n\to\infty} x_n = -\infty,$$

These three concepts are not so different if we look at them from the lens of topology!

Consider the set $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty, -\infty\}$. Define a topology on $\overline{\mathbb{R}}$ such that the three notions of limit are all particular cases of the topological definition of limit.

Exercise 4.8. We are going to prove the infinitude of primes using topology!

Given $a, b \in \mathbb{Z}$, let us define the following set of integers: $S_{a,b} := \{a + nb \mid n \in \mathbb{Z}\}$.

1. Prove that there exists a topology τ_p on $\mathbb{Z}$ that has $\{S_{a,b} \mid b \neq 0\}$ as a basis of open neighborhoods of a for every $a \in \mathbb{Z}$.
2. Prove that for every $a, b \in \mathbb{Z}$, $b \neq 0$, $S_{a,b}$ is clopen on $(\mathbb{Z}, \tau_p)$.
3. Note that
$$\mathbb{Z} \setminus \{1, -1\} = \bigcup_{\text{primes } p} S_{0,p}$$
and that $\{1, -1\}$ is not open. Now, assume there are finitely many primes, and get a contradiction.

5 Continuous maps

Whenever we study a new structure in mathematics, we also want to study maps between objects having that structure. For example, we study vector spaces together with linear maps; or we study groups together with group homomorphisms. In this chapter we will introduce the "good" kind of maps between topological spaces. The definition may not be the one you might have guessed, but hopefully the various results in this section will persuade you that it is the right notion to study. Finally, you should look at 5.8–5.9 in parallel with 2.14–2.15–2.16 and with 2.9 and with 3.13–3.14–3.15. Do you see a theme?

Definition 5.1. Let X and Y be sets. A *map* or *function* from X to Y is a rule that associates to each element $x \in X$ an element $f(x) \in Y$. As notation, we write $f : X \to Y$ to mean that f is a map from X to Y.

More formally, a map f from X to Y is a set $S \subseteq X \times Y$ such that for every $x \in X$ there exists a unique $y \in Y$ such that $(x, y) \in S$. In that case, we write $y = f(x)$. We will normally not think in this more formal way (leave it for your logic classes) and instead we will think of a map in the more classic way above.

Notice that in this course we will use the terms "map" and "function" as synonyms. This may differ from the conventions you have used in other courses.

Definition 5.2. Let X and Y be sets. Let $f : X \to Y$ be a map.

- Let $A \subseteq X$. The *image* of A is $f(A) := \{f(x) \mid x \in A\}$. Notice that $f(A) \subseteq Y$.
- Let $B \subseteq Y$. The *preimage* of B is $f^{-1}(B) := \{x \in X \mid f(x) \in B\}$. Notice that $f^{-1}(B) \subseteq X$. The map f does not have to be invertible for us to be able to write $f^{-1}(B)$.

  Notice that if f is invertible, then the notation $f^{-1}(B)$ might mean two things: the image of B by the map f^{-1} or the preimage of B by the map f. Luckily, the two things are the same.

Exercise 5.3. ($\flat$) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Calculate $f^{-1}(\{1\})$, $f^{-1}((0, 3))$, and $f^{-1}((-2, 1))$.

Exercise 5.4. Let $f : X \to Y$ be a map between two sets. Prove or disprove:

1. $f^{-1}(f(A)) = A$ for every $A \subseteq X$.
2. $f(f^{-1}(B)) = B$ for every $B \subseteq Y$.

If any of them is false, is there an extra condition that will make it true?

Definition 5.5. Let (X, τ) and (Y, σ) be topological spaces. Let $f : X \to Y$ be a map.

- We say that f is a *continuous* map when it satisfies the following property: "For every $W \in \sigma$, $f^{-1}(W) \in \tau$". In words, the preimage of every open set is open.
- We say that f is a *coconut* map when the preimage of every closed set is closed.

- We say that f is *sequentially continuous* when it satisfies the following property: "If (x_n) is a sequence in (X, τ) that converges to $a \in X$, then the sequence $f(x_n)$ converges to $f(a)$ in (Y, σ)". In words, the image of a convergent sequence converges to the images of the limits.

Exercise 5.6. Let $\mathcal{C}$ be the topological space of Exercise 2.10. Build a non-constant, continuous map from $\mathbb{R}$ with the standard topology to $\mathcal{C}$. You get to choose who are the students in class today and how they are sitting.

Lemma 5.7. ($\flat$)

1. The composition of two continuous maps is a continuous map.
2. The identity map (from a topological space to itself) is a continuous map.
3. Let (X, τ) and (Y, σ) be two topological spaces. Let $b \in Y$. Then the constant map $f : X \to Y$ defined by $f(x) = b$ for all $x \in X$ is a continuous map.

Challenge 5.8. In general, what is the relation between the three concepts in Definition 5.5? In other words, which one(s) implies which one(s)?

Theorem 5.9. Let $f : \mathbb{R}^N \to \mathbb{R}^M$ be a map, where we are using the standard topology in both $\mathbb{R}^N$ and $\mathbb{R}^M$. Prove that the four following conditions are equivalent:

1. For every $a \in \mathbb{R}^N$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x, a) < \delta \implies d(f(x), f(a)) < \varepsilon$.
2. For every $a \in \mathbb{R}^N$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_\delta(a)) \subseteq B_\varepsilon(f(a))$.
3. f is a continuous map.
4. f is a sequentially continuous map.

Note 5.10. Condition 1 in Theorem 5.9 is the one you may see as the definition of continuous function in an analysis course. Notice that our definition is much simpler, since it makes no mention of epsilons. Now that you know that the conditions are equivalent, you may use without proof the fact that polynomial functions from $\mathbb{R}^N$ to $\mathbb{R}^M$ are continuous. We leave the proof of that to your analysis course, or you can attempt it as an exercise.

Fuzzy 5.11. We defined continuous as "the preimage of every open set is open". There is a related concept: a map is called an *open map* when the image of every open set is open. Looking back at Lemma 5.7 and at Challenge 5.8, why is the notion of open map less useful than the notion of continuous map?

Exercise 5.12. Use our definition of continuity, together with Theorem 5.9 and Note 5.10, to prove that each of the following is an open set. Notice that without our definition of continuity, this would be much messier.

1. The set $A = \{(x, y) \in \mathbb{R}^2 \mid x > y\}$ in $\mathbb{R}^2$
2. The set $B = \{(x, y, z, w) \in \mathbb{R}^4 \mid xy^2 + z > 3, wyz + 2tz^2 < z^5 - 1, y \neq 3\}$ in $\mathbb{R}^4$

Exercise 5.13. Let n be a positive integer. We can think of $\mathbb{R}^{n^2}$ as the set of n-by-n matrices. Let $C \subseteq \mathbb{R}^{n^2}$ be the set of invertible matrices. Prove that C is open in $\mathbb{R}^{n^2}$ with the standard topology.

6 The subspace topology

If we have a topology on X and $A \subseteq X$, how can we make a topology on A?

Definition 6.1. Let τ_1 and τ_2 be two topologies on a set X. Assume that $\tau_1 \subseteq \tau_2$. We say that τ_1 is *coarser than or equal to* τ_2 and that τ_2 is *finer than or equal to* τ_1.

Exercise 6.2. Let (X, τ) be a topological space. Let $A \subseteq X$. Find all the topologies on A that make the inclusion map $\iota : A \to X$ continuous. The inclusion map is defined by $\iota(x) = x$ for all $x \in A$.

Exercise 6.3. Let (X, τ) be a topological space. Let $A \subseteq X$. Find all the topologies on A that satisfy the following property. For every (Y, σ) topological space and for every $f : X \to Y$ continuous map, the *restriction map* $f|_A : A \to Y$ is continuous. The restriction map is defined by $f|_A(x) = f(x)$ for all $x \in A$.

Exercise 6.4. Let (X, τ) be a topological space. Let $A \subseteq X$. Find all the topologies on A that satisfy the following property. For every sequence (x_n) and for every $a \in A$, if (x_n) converges to a in (X, τ), then (x_n) converges to a in (A, τ_A)

Definition 6.5. Let (X, τ) be a topological space and let $A \subseteq X$. We define the *topology inherited by A from τ* or the *topology induced by τ on A* as ... [You fill in the blanks, based on your answer to Exercises 6.2, 6.3, 6.4]. We also refer to τ_A as the *subspace topology.*

Notes 6.6. In the situation of Definition 6.5, if $V \subseteq A$, sometimes we say that "V is open in A" to mean $V \in \tau_A$ and that "V is open in X" to mean $V \in \tau$".

In general, when (X, τ) is a topological space and $A \subseteq X$, if we are referring to A as a topological space and we do not say specifically with which topology, we mean (A, τ_A).

Exercise 6.7. Using the standard topology, is $[0, 1)$ open in $\mathbb{R}$? Is $[0, 1)$ open in $[0, \infty)$? Is $[0, 1)$ open in $(-\infty, 1]$?

Exercise 6.8. Let (X, τ) be a topological space and let $B \subseteq A \subseteq X$. There are two topologies in B: the topology inherited by B from τ, and the topology inherited by B from the topology inherited by A from τ. What is the relation between these two topologies?

7 Homeomorphisms

You came up with your own definitions of "is the same topological space as" in Chapter 1. It is time to briefly review this concept.

Discussion 7.1. Let X and Y be two sets. Let $f : X \to Y$ be a map. We know f induces maps $F : \mathcal{P}(X) \to \mathcal{P}(Y)$ and $G : \mathcal{P}(Y) \to \mathcal{P}(X)$ defined by

- $F(A) := \{f(x) \mid x \in A\}$ for all $A \in \mathcal{P}(X)$, and
- $G(B) := \{x \in X \mid f(x) \in B\}$ for all $B \in \mathcal{P}(Y)$.

This agrees with what we called the image or the preimage of a subset in Definition 5.2. Notice that in Definition 5.2 we abused notation and simply called $F = f$ and $G = f^{-1}$, but just for this discussion we are going to be ultracareful and use different letters. After this discussion we will go back to the notation of Definition 5.2.

Now assume that f was a bijection to begin with. Then F and G are also bijections and, moreover, $F^{-1} = G$.

Definition 7.2. Let (X, τ) and (Y, σ) be topological spaces. A *homeomorphism* between them is a bijection $f : X \to Y$ satisfying the following property:

"For every $U \subseteq X$, U is open in X iff $f(U)$ is open in Y."

We say that (X, τ) and (Y, σ) are *homeomorphic* if there exists a homeomorphism between them. In that case, we write $(X, \tau) \cong (Y, \sigma)$. When the topologies are understood, we may abuse notation and simply write $X \cong Y$.

Theorem 7.3. ($\flat$) Let $f : X \to Y$ be a map between two topological spaces (X, τ) and (Y, σ). Then the following are equivalent:

1. f is a homeomorphism,
2. f is bijective, f is continuous, and f^{-1} is continuous.
3. f is bijective, and the bijection $F : \mathcal{P}(X) \to \mathcal{P}(Y)$ in 7.1 induces another bijection between τ and σ.

Lemma 7.4. ($\flat$) Being homeomorphic is an equivalence relation among topological spaces.

Exercise 7.5. Prove that $(0, 1)$, $(0, 2)$ and $\mathbb{R}$ are all homeomorphic (with the standard topology)

Exercise 7.6. ($\flat$) Are $X = \mathbb{R}$ and $Y = \mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$ homeomorphic with the standard topology.

Exercise 7.7. Consider the following five topologies on $\mathbb{R}$: discrete, indiscrete, standard, cocountable, and the topology from Exercise 1.3. They give us five topological spaces. Prove that no two of them are homeomorphic. Notice that it is not enough to prove that the identity from $\mathbb{R}$ to $\mathbb{R}$ with different topologies is not a homeomorphism. There could be a different bijection from $\mathbb{R}$ to $\mathbb{R}$ that does the trick.

Challenge 7.8. Consider the topological spaces $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, $[0, 1]$, and $[0, 1)$ with the standard topology. They are all pairwise non-homeomorphic. How much of that can you prove?

8 Some additional things that happened in class

This section contains some definitions that we introduced in class while solving other problems, some results that we proved, and some challenges we have not been able to solve yet.

Definition 8.1. Let (X, τ) be a topological space.

- (X, τ) is T_0 when for every distinct $x, y \in X$,

  [$\exists U$ open such that $x \in U$ and $y \notin U$] OR [$\exists V$ open such that $x \notin V$ and $y \in U$]

- (X, τ) is T_1 when for every distinct $x, y \in X$,

  [$\exists U$ open such that $x \in U$ and $y \notin U$] AND [$\exists V$ open such that $x \notin V$ and $y \in U$]

- (X, τ) is T_2 (or *Haussdorf*) when for every distinct $x, y \in X$,

  $\exists U, V$ open sets such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

Definition 8.2. A topological space (X, τ) is called *first countable* when every point $x \in X$ has a countable basis of open neighborhoods.

Lemma 8.3. Let (X, τ) be a first countable topological space. Then for every $A \subseteq X$, A is closed iff A is sequentially closed.

Challenge 8.4. Is there a non-first countable topological space where closed is the same as sequentially closed?

Note 8.5. Our answer to Challenge 4.6

Let X be a set. For each $x \in X$, let $\mathcal{B}_x \subseteq \mathcal{P}(X)$. Assume the following four conditions are true:

- (B1) Let $x \in X$. Then $\mathcal{B}_x \neq \emptyset$.
- (B2) Let $x \in X$ and let $V \in \mathcal{B}_x$. Then $x \in V$.
- (B3) Let $x \in X$ and let $V_1, V_2 \in \mathcal{B}_x$. Then there exists $V_3 \in \mathcal{B}_x$ such that $V_3 \subseteq V_1 \cap V_2$.
- (B4) Let $x \in X$ and let $V \in \mathcal{B}_x$. Then for every $y \in V$, there exists $W \in \mathcal{B}_y$ such that $W \subseteq V$.

We define $\tau \subseteq \mathcal{P}(X)$ by

$$\tau := \{V \subseteq X \mid \forall x \in V \; \exists B \in \mathcal{B}_x \text{ s.t. } B \subseteq V\}$$

Then

1. τ is a topology on X.
2. For every $x \in X$, $\mathcal{B}_x$ is a basis of open neighborhoods for x on (X, τ)
3. τ is the only topology on X for which this is true

9 Compactness

9.1 Motivation

Recall 9.1. Here are some definitions and results that you learned or will learn in a calculus or analysis class. Let $A \subseteq \mathbb{R}$.

- An *upper bound* of A is an element $u \in \mathbb{R}$ such that $x \leq u$ for every $x \in A$. The set A is *bounded above* if it has an upper bound. We define "lower bound" and "bounded below" similarly. We say that A is *bounded* when it is both bounded above and bounded below.
- Assume A is bounded above. Then there exists a unique element $s \in \mathbb{R}$ which is the *least upper bound* of A. This means that 1) s is an upper bound of A, and 2) if t is another upper bound of A, then $s \leq t$. We say that s is the *supremum* of A, and we write $s = \sup A$. We define *greatest lower bound* or *infimum* similarly.
- Assume that A is bounded above and that s is the supremum of A. If, in addition, $s \in A$, then we say that s is the maximum of A and we write $s = \max A$.

Definition 9.2. Let X be a set and let $f : X \to \mathbb{R}$ be a map. We say that f *reaches a maximum* at $c \in X$ when $f(x) \leq f(c)$ for every $x \in X$. This means two things:

1. The image set $f(X)$ is bounded above.
2. The supremum of the set $f(X)$ is actually a maximum.

Note 9.3. Throughout this chapter, whenever we look at a subset of $\mathbb{R}^N$ as a topological space, we mean with the standard topology (unless we say otherwise).

Definition 9.4. Let (X, τ) be a topological space. We say that (X, τ) satisfies the EVP ("Extreme Value Property") when every continuous map $f : X \to \mathbb{R}$ reaches a maximum.

Discussion 9.5. Our motivation in this chapter is the following question: Which topological spaces satisfy the EVP?

If you have taken a calculus class, you learned (with or without proof) that $[a, b]$ satisfies the EVP. What is special about $[a, b]$? First, come up with examples that show that the intervals (a, b) or $[a, \infty)$ do not satisfy the EVP. Among all the real intervals, the ones of the form $[a, b]$ are *closed and bounded*. So maybe we just need to look at closed and bounded topological spaces? No, we cannot do that. "Being closed" is a relative property—for example, $(0, 1)$ is closed in $(0, 1)$, even though $(0, 1)$ is not closed in $\mathbb{R}$. We want a property that is inherent about $[a, b]$ as a topological space by itself, not a property that depends on looking at $[a, b]$ as a subset of $\mathbb{R}$. "Being bounded" is even worse, as in arbitrary sets we cannot talk about "bounded" or "unbounded".

In the next exercises, we are going to find two properties of $[a, b]$ that we can extend to arbitrary topological spaces, each one sufficient by itself to prove the EVP

9.2 Sequential compactness

Definition 9.6. Let (X, τ) be a topological space. We say that (X, τ) is *sequentially compact* when every sequence in X has a convergent subsequence.

Examples 9.7.

1. Show that $(0, 1)$ and $\mathbb{R}$ are not sequentially compact.
2. Show that $[0, 1]$ is sequentially compact. Be careful with your proof. If your proof appears to work as well on $[a, b]$ as on (a, b) then you are missing a part of it.
3. Is $\mathbb{Z}$ with the cofinite topology sequentially compact?

Theorem 9.8. Let (X, τ) be a sequentially compact topological space. Let $f : X \to \mathbb{R}$ be a sequentially continuous map. Then f reaches a maximum.

Corollary 9.9. ($\flat$) Every sequentially compact topological space satisfies the EVP.

Discussion 9.10. We now have a way to prove that a topological space satisfies the EVP. We do not want to stop here, however. We want to improve on the definition of "sequentially compact". Let's compare with some other concepts we understand well:

- Back in Chapter 5, we had the notion of continuous map (in terms of open sets) and the related notion of sequentially continuous map (in terms of sequences). They are equivalent for the standard topology, but not for arbitrary topological space. In general, the notion of continuous is more useful.
- Similarly, in Chapter 3, we had the notion of closed set (in terms of open sets) and the related notion of sequentially closed set (in terms of sequences). They are equivalent for the standard topology, but not for arbitrary topological spaces. In general, the notion of closed set is more useful.

So, now we want to find a new notion which, unlike sequential compactness, is written only in terms of open sets without using sequences. We want this new notion to still allow us to prove the EVP. This is going to require a bit of preparation. Be patient as you unravel the new definitions.

9.3 The definition of compactness

Definition 9.11. Let (X, τ) be a topological space. Let $A \subseteq X$.

- An *open cover of A in (X, τ)* is a family of open sets $\mathcal{C} \subseteq \tau$ such that $A \subseteq \bigcup_{V \in \mathcal{C}} V$.

  A particularly important case is when $A = X$. In that case, we may refer to $\mathcal{C}$ simply as an *open cover of* (X, τ).
- A finite open cover is an open cover $\mathcal{C}$ such that $\mathcal{C}$ has finitely many elements.
- A *subcover* of an open cover $\mathcal{C}$ is another open cover $\mathcal{C}_1$ such that $\mathcal{C}_1 \subseteq \mathcal{C}$.

Definition 9.12. Let (X, τ) be a topological space. Let $A \subseteq X$. We say that A is *compact in* (X, τ) when every open cover of A in (X, τ) has a finite subcover.

We say that the topological space (X, τ) is *compact* when X is compact in (X, τ).

Examples 9.13. Which ones of the following are compact?

1. $\mathbb{R}$.
2. $(0,1)$ in $\mathbb{R}$.
3. A discrete topological space.
4. An indiscrete topological space.
5. $\mathbb{Z}$ with the cofinite topology.

Lemma 9.14. Let (X, τ) be a topological space. Let $A \subseteq X$. Then A is compact in (X, τ) if and only if (A, τ_A) is compact.

Note 9.15. Lemma 9.14 says that "being compact" is an intrinsic property of a topological space by itself, and it does not depend on whether we regard it as a subset of something bigger. The same is not true of "being closed". For example, $(0,1)$ is not a closed subset of $\mathbb{R}$, but $(0,1)$ *is* a closed subset of $(0,1)$. On the other hand, $(0,1)$ is not compact in $\mathbb{R}$, and $(0,1)$ is also not compact in $(0,1)$.

Theorem 9.16. Let (X, τ) and (Y, σ) be topological spaces. Let $f : X \to Y$ be a continuous map. Let $A \subseteq X$. Assume that A is compact. Then $f(A)$ is also compact.

Corollary 9.17. Every compact topological space satisfies the EVP.

Joke 9.18. What is a *compact city*? A city that can be protected by finitely many guards, no matter how near-sighted each one of them is.

Discussion 9.19. Corollary 9.17 gives us a second possible way to prove the EVP for a given topological space. To conclude our mission, in the next few exercises, we are going to prove that $[a, b]$ is compact. In fact, we will do more than that: we will find a complete characterization of all compact subsets of $\mathbb{R}$. This characterization is the very important Heine-Borel Theorem. But first, a very serious challenge.

9.4 The serious challenge

Definition 9.20. Let (X, τ) be a topological space. We say that (X, τ) *has the Bolzano-Weierstrass property* when every sequence in X has an accumulation point.

Challenge 9.21. What is the relation between compact, sequentially compact, and having the Bolzano-Weierstrass property?

9.5 The Heine-Borel Theorem

Theorem 9.22. The interval $[0,1]$ with the standard topology is compact.

Corollary 9.23. Let $a, b \in \mathbb{R}$ such that $a < b$. Then $[a, b]$ is homeomorphic to $[0, 1]$, and hence it is compact.

Corollary 9.24. The topological spaces $\mathbb{R}$ and $[0, 1]$ are not homeomorphic.

Lemma 9.25. Let $A \subseteq \mathbb{R}$. Assume A is compact with the standard topology. Then A is bounded.

Lemma 9.26. Let (X, τ) be a compact topological space and let $A \subseteq X$. Assume A is closed in X. Then A is compact.

Lemma 9.27. Let (X, τ) be a Hausdorff topological space and let $A \subseteq X$. Assume A is compact. Then A is closed in X.

Theorem 9.28. [HEINE-BOREL]
Let $A \subseteq \mathbb{R}$. Then A is compact if and only if [A is closed in $\mathbb{R}$ and bounded].

10 Product topology

10.1 Finite product

Given two topological spaces (X, τ_X) and (Y, τ_Y), is there a "good" topology on $X \times Y$?

Notation 10.1. Let N be a positive integer. Let $X_1, \dots X_N$ be sets. We define their *Cartesian product* to be the set:

$$X_1 \times \ldots \times X_N \;=\; \prod_{i=1}^{N} X_i \;=\; \{(x_1, \ldots, x_N) \mid x_i \in X_i \text{ for } i = 1, \ldots N\}$$

There are N canonical maps, called the *projections*: for each i we define

$$\pi_i : X_1 \times \ldots \times X_N \to X_i$$

by $\pi_i(x_1, \ldots, x_N) = x_i$.

Let us call $X = X_1 \times \ldots \times X_N$. If x is an element of X, we may write $x_i = \pi_i(x)$.

Now let Z be another set and let $f : Z \to X$ be a function. For each i, we can define the function $f_i : Z \to X_i$ by the equation $f_i = \pi_i \circ f$. In other words, for every $z \in Z$, we have $f(z) = (f_1(z), \ldots, f_N(x))$. The functions $f_1, \ldots, f_N$ are called the *component functions* of f. Notice that f can be reconstructed from $f_1, \ldots, f_N$ and vice versa.

We have to be careful with sequences. Let $(x_n)_{n=1}^{\infty}$ be a sequence in X. This means that $x_n \in X$ for every n. Now we need a different way to denote the components of x_n! We may, for example, denote $x_n^{(i)} = \pi_i(x_n)$ for each $i = 1, \ldots N$ and for each $n \in \mathbb{N}$. This means that $(x_n^{(i)})_{n=1}^{\infty}$ is a sequence in X_i for each fixed i, whereas $(x_n^{(1)}, \ldots, x_n^{(N)})$ is an element of X for each fixed n. We will refer to $(x_n^{(i)})_{n=1}^{\infty}$ as the *component sequences* of $(x_n)_{n=1}^{\infty}$. Like for functions, sequences can be reconstructed from their component sequences and vice versa.

Exercise 10.2. Let $(X_1, \tau_1), \ldots, (X_N, \tau_N)$ be topological spaces. Let X be the set $X = X_1 \times \ldots \times X_N$. We want to define a topology on X that satisfies the following properties:

- The projection maps $\pi_1, \ldots, \pi_N$ are continuous.
- For every topological space (Z, σ) and for every map $f : Z \to X$, the map f is continuous if and only if the component maps $f_1, \ldots, f_N$ are continuous.

Find one such topology.

Definition 10.3. Let $(X_1, \tau_1), \ldots, (X_N, \tau_N)$ be topological spaces. We define the *product topology* on their cartesian set $X = X_1 \times \ldots \times X_N$ as ... [You fill in the blank]

Challenge 10.4. [**Universal property!** – This theorem is a bit difficult to parse because it is category theory in disguise. Take a deep breath.]

Let (Y, τ') be another topological space that satisfies the property in Exercise 10.2. This means that

- For each $i = 1, \ldots, N$ there is a continuous map $p_i : Y \to X_i$.

- For every topological space (Z, σ) and for every map $f : Z \to Y$, the map f is continuous if and only if the maps $p_1 \circ f, \ldots, p_N \circ f$ are continuous.

Then X with the product topology is homeomorphic to (Y, τ'). What is more, there exists a *unique* homeomorphism $h : X \to Y$ such that $\pi_i \circ h = p_i$ for all $i = 1, \ldots, N$.

This Theorem means that the property described in Exercise 10.2 is the *definition* of the product topology. Sometimes this theorem is summarized by saying that the product topology is *unique up to unique homeomorphism*.

Note 10.5. In the following, whenever we refer to a topology on the product of sets, we always mean the product topology unless otherwise noted.

Exercise 10.6. Let (X, τ_X) and (Y, τ_Y) be topological spaces. Which of the following statements are true or false?

1. A product of an open set in X and an open set in Y is open in $X \times Y$.
2. Every open set in $X \times Y$ is a product of an open set in X and an open set in Y.
3. A product of a closed set in X and a closed set in Y is closed in $X \times Y$.
4. The standard topology on $\mathbb{R}^2$ is the product of the standard topology on $\mathbb{R}$ with itself.
5. Let (x_n, y_n) be a sequence in $X \times Y$. The sequence (x_n, y_n) converges to (a, b) in $X \times Y$ iff [the sequence (x_n) converges to a in X and the sequence (y_n) converges to b in Y].
6. Let (x_n, y_n) be a sequence in $X \times Y$. The sequence (x_n, y_n) has (a, b) as an accumulation point in $X \times Y$ iff [a is an accumulation point of the sequence (x_n) in X and b is an accumulation point of the sequence (y_n) in Y].

Definition 10.7. Let $\mathbb{P}$ be a property that topological spaces may have. We say that $\mathbb{P}$ is *finitely-multiplicative* when the following is true: "If (X, τ_X) and (Y, τ_Y) are topological spaces that satisfy $\mathbb{P}$, then their product with the product topology also satisfies $\mathbb{P}$".

Exercise 10.8. Which of the following are finitely-multiplicative properties?

1. discrete
2. indiscrete
3. Haussdorff
4. first-countable
5. sequentially compact
6. compact

10.2 General products

Now we want to take the product of infinitely many topological spaces. Maybe even of uncountably infinitely many. Why? Because we want to prove the Axiom of Choice! *¡Agárrate los machos!* The notation is going to get messy.

Notation 10.9. Let I be a set of indices. For each $i \in I$, let X_i be a set. Their product set is defined as

$$\prod_{i\in I} X_i := \{f : I \to \bigcup_{i\in I} X_i \mid f(i) \in X_i \text{ for all } i \in I\}$$

Make sure you understand why this definition makes sense before moving on. In particular, when $I = \{1, \ldots, N\}$, this definition should agree with the definition of cartesian product of finitely many spaces.

A particular case of importance is when $X_i = Y$ for all $i \in I$. In this case we will write

$$Y^I = \prod_{i\in I} Y = \{f : I \to Y\}$$

Let's stay in the general case and let us define $X = \prod_{i\in I} X_i$. For each $i \in I$ there is a *projection map* $\pi_i : X \to X_i$ defined by $\pi_i(f) = f(i)$.

Now let Z be another set and let $g : Z \to X$ be a map. For each $i \in I$ we define the component map $g_i : Z \to X_i$ by $g_i = \pi_i \circ g$. As in the finite case, the map g can be recovered from its component maps $\{g_i \mid i \in I\}$ and vice versa.

Finally, let $(f_n)_{n=1}^{\infty}$ be a sequence on X. Then for each $i \in I$ we have a *component sequence* $(f_n(i))_{n=1}^{\infty}$ on X_i.

Challenge 10.10. Let I be a set of indices. For each $i \in I$, let (X_i, τ_i) be a topological space. Let $X = \prod_{i\in I} X_i$. We want to define a topology on X that satisfies the same property as in Exercise 10.2. Find this topology (yes, it is unique). Beware, as the topology is not the one you may be thinking of at first (see 10.11 below).

And yes, there is also an analogue of Challenge 10.4 here if you are up for it.

Definition 10.11. Let I be a set of indices. For each $i \in I$, let (X_i, τ_i) be a topological space. Let $X = \prod_{i\in I} X_i$.

- The *product topology* on X is the one you obtained in Challenge 10.10.
- Let $f \in X$. We define $\mathcal{B}_f := \{\prod_{i\in I} V_i \mid f(i) \in V_i \in \tau_i \text{ for all } i \in I\}$. Then there is a unique topology on X for which $\mathcal{B}_f$ is a basis of open neighborhoods for f for all $f \in X$. This is called the *box topology*.

Exercise 10.12. Continue with the same notation as in Definition 10.11. Let (f_n) be a sequence in X. Let $f \in X$. Prove that (f_n) converges to f in X with the product topology iff $[(f_n(i))$ converges to $f(i)$ in X_i for all $i \in I]$. Show with an example that the same statement is not true if we substitute the box topology for the product topology.

Definition 10.13. Let $\mathbb{P}$ be a property that topological spaces may have.

- We say that $\mathbb{P}$ is *countably-multiplicative* when the following is true: "The product topology of countably many topological spaces that are $\mathbb{P}$ is $\mathbb{P}$.
- We say that $\mathbb{P}$ is *multiplicative* when the following is true: "The product topology of (any amount of) topological spaces that are $\mathbb{P}$ is $\mathbb{P}$.

Exercise 10.14. Which ones of the following are countably-multiplicative properties? Which ones are multiplicative properties?

1. discrete
2. indiscrete
3. Haussdorff
4. first-countable
5. **(Challenge!)** sequentially compact
6. **(Challenge!)** compact

Note 10.15. The last part of the last question is not fair. You do not have the tools to prove it yet, and it is better to wait until we have introduces nets, filters, or waffles.

This is actually a very important result, called **Tychonoff's Theorem**: compact is a multiplicative property.

Exercise 10.16. Use Tychonoff Theorem to prove the Axiom of Choice.

The Axiom of Choice states that if I is a set of indices, and for every $i \in I$ we have a non empty set X_i, then the product set $\prod_{i \in I} X_i$ is also non-empty.

Hint: Assume X_i non-empty for each i. Define $Y_i = X_i \cap \{\heartsuit\}$ with topology $\{\emptyset, \{\heartsuit\}, Y_i\}$ on Y_i. Build a "cover" of $\prod_{i \in I} Y_i$ consisting of $\{\pi_i^{-1}(\heartsuit) \mid i \in I\}$. Then...

11 Waffles

11.1 The problems we want to fix

So far, we have learned various results about sequences which are not as we would like them to be. We are going to introduce a new object to replace sequences and fix these problems. Let's recall the problems first.

Problem 11.1. In a topological space, every closed subset is sequentially closed, but the converse is not true.

Problem 11.2. We can have to different topologies on the same set that have exactly the same convergence of sequences.

Problem 11.3. Every continuous map is sequentially continuous. The converse is not true.

Problem 11.4. In a Haussdorf topological space, every sequence has at most one limit. However, it is possible to have a non-Haussdorf topological space where every sequence has at most one limit.

Problem 11.5. Accumulation points of a sequence are not the same as limits of subsequences.

Problem 11.6. There is no relation between compact and sequentially compact.

11.2 The definition of waffle

Discussion 11.7. Let (X, τ) a topological space. Recall that a sequence in X is a map $x : \mathbb{N} \to X$. For every $n \in N$, define $F_n := \{m \in \mathbb{N} \mid m \geq n\}$. We say that a is a limit of the sequence when for every open neighbourhood V of a there exists $n \in N$ such that $x(F_n) \subseteq V$. Make sure you agree that this is the same definition of limit that we had learned earlier before you continue reading.

A way to interpret this is as follows. Consider the collection $\mathcal{M} := \{F_n \mid n \in \mathbb{N}\}$. We can think of $\mathcal{M}$ as the collection of "large" subsets of $\mathbb{N}$. Then, a sequence in X, though of as a map $x : \mathbb{N} \to X$, coverges to $a \in X$ iff every neighbourhood of a contains the image of some large subset of $\mathbb{N}$.

We are going to define waffles as a generalization of sequences when we substitute an arbitrary index set for $\mathbb{N}$ and we replace $\mathcal{M}$ by a collection of subsets that we will think of as "the large subsets". We will refer to a large subset as a "whale", and to the collection $\mathcal{M}$ as a "pob".

Definition 11.8. Let I be a set. A *pod* on I is a family of subsets $\mathcal{M} \subseteq \mathcal{P}(I)$ satisfying:

1. The pod is not empty, i.e. $\mathcal{M} \neq \emptyset$.
2. Every element of the pod is non-empty, i.e. $\emptyset \notin \mathcal{M}$.
3. For every $V_1, V_2 \in \mathcal{M}$, there exists $V_3 \in \mathcal{M}$ such that $V_3 \subseteq V_1 \cap V_2$.

We refer to the elements of $\mathcal{M}$ as *whales*.

Note 11.9. Whales are large, and a collection of whales is a pod. We will think of a pod on I as the collection of "large" subsets of I. In other contexts, what I am calling a pod is normally called a *filter basis*. The reason for this is that a filter basis generates a filter. We will not be needing filters in this course, however, so I will avoid introducing lots of unnecessary definitions, and we will stick with calling them pods.

Example 11.10. Which ones of the following are pods on $\mathbb{N}$?

(a) The family $\{F_n \mid n \in \mathbb{N}\}$, as defined in Discussion 11.7.

(b) The set of finite subsets of $\mathbb{N}$.

(c) The set of infinite subsets of $\mathbb{N}$.

(d) The set of cofinite subsets of $\mathbb{N}$.

Definition 11.11. A *partially order set* or *poset* is a pair $(I, \leq)$ where I is a set and $\leq$ is a binary relation on I satisfying:

- $\leq$ is reflexive: $a \leq a$ for all $a \in I$.
- $\leq$ is transitive: For all $a, b, c \in I$, if $a \leq b$ and $b \leq c$, then $a \leq c$.
- $\leq$ is antisymmetric: For all $a, b \in I$, if $a \leq b$ and $b \leq a$, then $a = b$.

Exercise 11.12. Let $(I, \leq)$ be a non-empty poset. For each $i \in I$ we define $F_i := \{j \in I \mid j \geq i\}$. Let us call $\mathcal{M} := \{F_i \mid i \in I\}$. In general, $\mathcal{M}$ is not a pod on I. Find a necessary and sufficient condition on the order $\leq$ for $\mathcal{M}$ to be a pod on I. Whenever this condition is satisfied, we say that $\leq$ is *cofinal*, we say that $(I, \leq)$ is an *oriented set*. In this situation, we will refer to $\mathcal{M}$ as the pod generated by the order $\leq$.

Example 11.13. Let (X, τ) be a topological space and let $a \in X$. Let $\mathcal{B}_a$ be a basis of open neighborhoods for a. Notice that $\supseteq$ is a partial order on $\mathcal{B}_a$. Under what circumstances is this order cofinal (and hence it generates a pod on $\mathcal{B}_a$?

In the future, by default, we will always think of any basis of open neighborhoods as equipped with this pod.

Definition 11.14. Let X be a set. A *waffle* on X is a triple $(I, \mathcal{M}, w)$, where

- I is a set.
- $\mathcal{M}$ is a pod on I.
- w is a map $w : I \to X$.

Definition 11.15. Let (X, τ) be a topological space. Let $(I, \mathcal{M}, w)$ be a waffle on X. Let $a \in X$.

- We say that a is a limit of the waffle when for every neighbourhood V of a there exists a whale $F \in \mathcal{M}$ such that $w(F) \subseteq V$.
- We say that a is an accumulation point of the waffle when $w(F) \cap V \neq \emptyset$ for every neighbourhood V of a and for every whale $F \in \mathcal{M}$.

Exercise 11.16.

1. Verify that every sequence is a waffle (using the pod in Example 11.10-a). Verify that the definition of limit is the same if we think of it as a sequence or as a waffle. Verify that the definition of accumulation point is the same whether we think of it as a sequence or as a waffle.

2. If we think of a sequence as a waffle using any of the other pods on $\mathbb{N}$ in Example 11.10 (only the ones which *are* actually pods), how do the notions of limit and accumulation point in these cases relate to the usual ones?

Lemma 11.17. Let (X, τ) be a topological space. Let $a \in X$. Let $\mathcal{B}_a$ be a basis of open neighborhoods for a. Let $\mathcal{M}$ be the pod defined in Example 11.13. For every $V \in \mathcal{B}_a$ choose any point $w(V) \in V$. This defines a map $w : \mathcal{B}_a \to X$. Then a is a limit of the waffle $(\mathcal{B}_a, \mathcal{M}, w)$.

11.3 The first few solutions!

Challenge 11.18. Define "*waffly closed*". Prove that in every topological space, waffly closed is the same as closed. This fixes Problem 11.1.

Challenge 11.19. Fix Problem 11.2 using waffles.

Challenge 11.20. Fix Problem 11.3 using waffles. You may have to define something first.

Challenge 11.21. Prove that a topological space is Haussdorff iff every waffle has at most one limit.

11.4 Subwaffles and ultrawaffles

Definition 11.22.

- Let I be a set. Let $\mathcal{M}$ be a pod on I. A *superpod* of $\mathcal{M}$ is another pod $\mathcal{M}'$ on I such that $\mathcal{M} \subseteq \mathcal{M}'$.
- The pod $\mathcal{M}$ is called an *ultrapod* when it does not have any superpods other than itself.
- A *subwaffle* of a waffle $(I, \mathcal{M}, w)$ is another waffle of the form $(I, \mathcal{M}', w)$ where $\mathcal{M}'$ is a superpod of $\mathcal{M}$.
- An *ultrawaffle* is a waffle whose pod is an ultrapod.

Exercise 11.23. Show that if we think of a sequence as a particular type of waffle, then every subsequence of it is a subwaffle. What is the superpod in this case?

Is every subwaffle of a sequence a subsequence?

Challenge 11.24. Show that for waffles, limit of a subwaffle and accumulation points are the same thing. This fixes Problem 11.5.

Challenge 11.25. (♯) Every pod has a superpod which is an ultrapod. This implies that every waffle has a subwaffle which is an ultrawaffle.

Note: This requires Zorn's Lemma.

11.5 Waffles and compactness

Theorem 11.26. Let (X, τ) be a topological space. The following are equivalent:

1. (X, τ) is compact.
2. Every waffle on X has an accumulation point.
3. Every waffle on X has a convergent subwaffle.
4. Every ultrawaffle on X is convergent.

Notice that this fixes Problem 11.6.

Theorem 11.27. Let $\mathcal{A}$ be an index set. For each $\alpha \in \mathcal{A}$, let (X_α, τ_α) be a topological space. Let $X = \prod_{\alpha \in A} X_\alpha$ with the product topology. Let $(I, \mathcal{M}, w)$ be a waffle on X. For each $\alpha \in \mathcal{A}$, let $\pi_\alpha : X \to X_\alpha$ be the corresponding projection map and let $w_\alpha = \pi_\alpha \circ w : I \to X_\alpha$. Then $(I, \mathcal{M}, w_\alpha)$ is a waffle on X_α.

In this situation, the waffle $(I, \mathcal{M}, w)$ is convergent on X if and only if $(I, \mathcal{M}, w_\alpha)$ is convergent on X_α for every $\alpha \in \mathcal{A}$.

Exercise 11.28. Use Theorems 11.26 and 11.27 to prove Tychonoff's Theorem.