MAT 137Y - Practice problems
Unit 13 - Series

1. **Definition of series.** We want to compute the sum of the series $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$.

   (a) Compute the first few partial sums. Looking at those values, guess a formula for the value of the k-th partial sum, and prove it by induction.

   (b) Use the definition of series to compute the value of $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$.

2. **Geometric series.** You have learned that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ if } |x| < 1$$

   and the series is divergent if $|x| \geq 1$. Calculate the following infinite sums:

   (a) $\sum_{n=0}^{\infty} (\ln 2)^n$ (b) $\sum_{n=0}^{\infty} (\ln 3)^n$ (c) $\sum_{n=0}^{\infty} \frac{(-1)^n}{e^{2n+3}}$ (d) $\sum_{n=m}^{\infty} x^n$

   (e) $\frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^4} + \frac{1}{5^5} + \frac{1}{5^7} + \frac{1}{5^8} + \frac{1}{5^{10}} + \frac{1}{5^{11}} + \dots$

   (f) $\frac{1}{2^{0.5}} + \frac{1}{2} + \frac{1}{2^{1.5}} - \frac{1}{2^2} + \frac{1}{2^{2.5}} + \frac{1}{2^3} + \frac{1}{2^{3.5}} - \frac{1}{2^4} + \frac{1}{2^{4.5}} + \frac{1}{2^5} + \frac{1}{2^{5.5}} - \frac{1}{2^6} + \dots$

3. **Telescopic series.** Calculate the value of the following infinite sums. In all cases, you can start by finding a formula for the k-th partial sum, and then taking the limit.

   (a) $\sum_{n=0}^{\infty} [\arctan n - \arctan(n+1)]$

   (b) $\sum_{n=1}^{\infty} \left[\ln \frac{n}{n+1}\right]$

   (c) $\sum_{n=1}^{\infty} \frac{1}{n^2+3n}$ *Hint:* $\frac{1}{n^2+3n} = \frac{A}{n} + \frac{B}{n+3}$

   (d) $\sum_{n=3}^{\infty} \frac{n+2}{n^3-n}$

4. **Infinite decimal expansions.** We can interpret any finite decimal expansion as a finite sum. For example:

$$2.13096 = 2 + \frac{1}{10} + \frac{3}{10^2} + \frac{0}{10^3} + \frac{9}{10^4} + \frac{6}{10^5}$$

   Similarly, we can interpret any infinite decimal expansion as an infinite series.

   Interpret the following numbers as series, and add up the series to calculate their value as fractions:

   (a) $0.99999\dots$ (b) $0.11111\dots$ (c) $0.252525\dots$ (d) $0.3121212\dots$

5. **The tail of a series.** Prove that the series $\sum_{n=0}^{\infty} a_n$ is convergent if and only if the series $\sum_{n=1}^{\infty} a_n$ is convergent. Moreover, prove that in that case $\left[\sum_{n=0}^{\infty} a_n\right] = a_0 + \left[\sum_{n=1}^{\infty} a_n\right]$.

   Note: This was stated as a Theorem without proof in Video 13.7. For your proof, imitate the proof of linearity in Video 13.6. You can do this just by using the definition of series and limit laws. There is no need to get messy with epsilons.

6. **Convergence tests.** Is each of the following series convergent or divergent?

   (a) $\sum_n^{\infty} \frac{1}{\arctan n}$

   (b) $\sum_n^{\infty} \frac{(-1)^n}{\arctan n}$

   (c) $\sum_n^{\infty} \frac{1}{2^{n/5}}$

   (d) $\sum_n^{\infty} \frac{1}{n^{2/5}}$

   (e) $\sum_n^{\infty} \frac{1}{n}$

   (f) $\sum_n^{\infty} \frac{(-1)^n}{n}$

   (g) $\sum_n^{\infty} \frac{1}{n^2}$

   (h) $\sum_n^{\infty} \frac{\sqrt{n^3+2n+1}}{n^4+\ln n+10}$

   (i) $\sum_n^{\infty} \frac{n^2+3n+1}{\sqrt{n^5+4n+11}}$

   (j) $\sum_n^{\infty} \frac{3+\cos^2(n^2+1)}{n^3}$

   (k) $\sum_n^{\infty} \frac{n}{n^2+1}$

   (l) $\sum_n^{\infty} \frac{(-1)^n\, n}{n^2+1}$

   (m) $\sum_n^{\infty} \frac{1}{(\ln n)^3}$

   (n) $\sum_n^{\infty} \frac{1}{n\,(\ln n)^2}$

   (o) $\sum_n^{\infty} \frac{1}{n\sqrt{\ln n}}$

   (p) $\sum_n^{\infty} \frac{\ln n}{n^{1.1}}$

   (q) $\sum_n^{\infty} \frac{n^2 n!}{(2n)!}$

   (r) $\sum_n^{\infty} \sin^2 \frac{1}{n}$

   (s) $\sum_n^{\infty} (-1)^n \sin \frac{1}{n}$

   (t) $\sum_n^{\infty} \frac{\sin n}{n^2}$

   (u) $\sum_n^{\infty} \frac{(n+3)2^n}{n!}$

   (v) $\sum_n^{\infty} \frac{(3n)!}{n!(2n)!}$

   (w) $\sum_n^{\infty} \frac{1}{n^n}$

   (x) $\sum_n^{\infty} \frac{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)} \cdot \frac{\pi^{n+1}}{e^{2n-1}}$

7. **Proofs with convergent tests.** Let $\sum_n^{\infty} a_n$ be a CONVERGENT series. Assume that for all $n \in \mathbb{N}$, $a_n > 0$. Decide whether each of the following series is convergent, divergent, or we do not have enough information to decide. If convergent or divergent, write a proof. If you think we do not have enough information to decide, show it by giving one convergent and one divergent examples:

   (a) $\sum_n^{\infty} \sin a_n$ (b) $\sum_n^{\infty} \cos a_n$ (c) $\sum_n^{\infty} \sqrt{a_n}$ (d) $\sum_n^{\infty} (a_n)^2$

8. **One more test.** In Video 13.18 you learned the statement of the Ratio Test. The video includes only an idea of the proof, without writing it formally with all the details.

   There is a sibling theorem to the Ratio Test:

   Theorem [Root Test]

   Let $\sum_{n}^{\infty} a_n$ be a series. Assume the limit $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists or is ∞.

   - IF $0 \leq L < 1$ THEN the series is ...
   - IF $L > 1$ THEN the series is ...

   Your goal is to figure out what the theorem says and why. Mimic the argument in the "Idea of the Proof" part of Video 13.18. Then, based on your heuristic argument, decide what the full statement of the Root Test and convince yourself that it is true. You do not need to write all the details of a full proof.

9. **Estimating an alternating series.**

   Estimate the value of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ with an error less than 0.01.

Bonus questions: harmonic sums, the Euler-Mascheroni constant, and non-commutative infinite sums

For every positive integer N we define the N-th *harmonic sum*:

$$H_N = \sum_{n=1}^{N} \frac{1}{n}.$$

As you know, thanks to the integral test, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ and the integral $\int_1^{\infty} \frac{1}{x}\, dx$ behave the same way: both are divergent. We can go one step further: the partial sums of the series and the "partial integrals" of the integral also "behave the same way". Specifically, you are going to prove the following theorem:

Theorem 1: There exists $\gamma \in \mathbb{R}$ and a sequence $\{\varepsilon_N\}_{N=1}^{\infty}$ such that

$$H_N \;=\; \ln N \;+ \gamma \;+\; \varepsilon_N, \qquad \text{and} \qquad \lim_{N\to\infty} \varepsilon_N = 0. \tag{1}$$

The number γ is called the *Euler-Mascheroni constant*. In other words:

$$\text{for large values of } N, \qquad H_N \;\approx\; \ln N + \gamma$$

As a (surprising) corollary of this theorem you will be able to prove that infinite sums are not commutative. More specifically, you will be able to prove that these two infinite sums

$$A \;=\; 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \ldots \tag{2}$$

$$B \;=\; 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} - \frac{1}{8} + \frac{1}{17} + \frac{1}{19} - \frac{1}{10} + \ldots \tag{3}$$

are convergent to different numbers: $A = \ln 2$ and $B = \frac{3}{2} \ln 2$.

10. **Proving Theorem 1.**

Let $f(x) = \frac{1}{x}$. For each integer $N \geq 2$, let us define

$$I_N \;=\; \int_1^N f(x)dx \;=\; \ln N, \qquad \text{and} \qquad S_N \;=\; \sum_{n=2}^{N} f(n) \;=\; H_N - 1.$$

(a) Draw a picture. Interpret I_N as an area. In the same picture, draw the Riemann sum corresponding to breaking $[1, N]$ into subintervals of length 1, and choosing the right-end point on each subintervals. What is the value of this Riemann sum?

(b) We now define the sequence $\{b_N\}_{N=2}^{\infty}$ by the formula $b_N = I_N - S_N$. Prove that the sequence $\{b_N\}_{N=2}^{\infty}$ is increasing.
Suggestion: Use your picture, interpret b_N as an area, and do a "proof by picture".

(c) Prove that $b_N \leq 1$ for every $N \geq 2$.
Suggestion: Do a "proof by picture" again.

(d) Prove that the sequence $\{b_N\}_{N=2}^{\infty}$ is convergent.
Hint: This should be very quick.

(e) Prove Theorem 1.
Hint: Show the sequence $\{H_N - \ln N\}_N^{\infty}$ is convergent. Call its limit γ.
Define $\varepsilon_N = H_N - \ln N - \gamma$.

11. **Non-commutative infinite sums!**

(a) Consider the following sums:

$$E_N = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \ldots + \frac{1}{2N}$$
$$O_N = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \ldots + \frac{1}{2N-1}$$

Find a formula for E_N and O_N in terms of H_N and H_{2N}.

(b) Look back at Equation (2). Interpret it as a series. Let us call A_N the N-th partial sum of this series (that is, the sum of the first N terms). With the help from your answer to Question 11a, find a formula for A_{2N} in terms of H_N and H_{2N}.

(c) Use Equation (1) in your answer to Question 11b to prove that $A = \lim_{N\to\infty} A_{2N} = \ln 2$.

(d) Do a similar calculation to prove that $B = \frac{3}{2} \ln 2$.

Some answers and hints

1. (a) $\displaystyle\sum_{n=1}^{k} \frac{n}{(n+1)!} = 1 - \frac{1}{(k+1)!}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1$

2. (a) $\dfrac{1}{1-\ln 2}$

   (b) Divergent

   (c) $\dfrac{1}{e(e^2+1)}$

   (d) $\dfrac{x^m}{1-x}$ if $|x| < 1$; divergent otherwise.

   (e) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{5^n} - \sum_{n=0}^{\infty} \frac{1}{5^{3n}} \;=\; \frac{5}{4} - \frac{125}{124} \;=\; \frac{15}{62}$

   (f) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\left(\sqrt{2}\right)^n} - 2\sum_{n=1}^{\infty} \frac{1}{4^n} \;=\; \frac{1}{\sqrt{2}-1} - \frac{2}{3} \;=\; \sqrt{2} + \frac{1}{3}$

3. (a) $-\pi/2$

   (b) $-\infty$

   (c) $11/18$

   (d) $7/12$

4. (a) $\displaystyle 0.99999... \;=\; \sum_{n=1}^{\infty} \frac{9}{10^n} \;=\; 1$

   (b) $\displaystyle 0.11111... \;=\; \sum_{n=1}^{\infty} \frac{1}{10^n} \;=\; \frac{1}{9}$

   (c) $\displaystyle 0.25252525... \;=\; \sum_{n=1}^{\infty} \frac{25}{10^{2n}} \;=\; \frac{25}{99}$

   (d) $\displaystyle 0.3121212... \;=\; \frac{3}{10} + \sum_{n=1}^{\infty} \frac{12}{10^{2n+1}} \;=\; \frac{103}{330}$

5. Imitate the proof of linearity in Video 13.6.

6. I am indicating one possible test that helps with each one, but there may be others.

   (a) Divergent (necessary condition)
   (b) Divergent (necessary condition)
   (c) Convergent (geometric)
   (d) Divergent (p-series)
   (e) Divergent (p-series)
   (f) Convergent (alternating series test)
   (g) Convergent (p-series)
   (h) Convergent (LCT)

   (i) Divergent (LCT)
   (j) Convergent (BCT)
   (k) Divergent (LCT)
   (l) Convergent (alternating series test)
   (m) Divergent (BCT or generalized LCT)
   (n) Convergent (integral test)
   (o) Divergent (integral test)
   (p) Convergent (BCT or generalized LCT)

   (q) Convergent (ratio test)
   (r) Convergent (LCT)
   (s) Covergent (alternating series test)
   (t) Convergent (absolute convergence test + BCT)
   (u) Convergent (ratio test)
   (v) Divergent (ratio test)
   (w) Convergent (BCT)
   (x) Convergent (ratio test)

7. (a) Convergent
   (b) Divergent
   (c) We do not have enough information to decide
   (d) Convergent

8. The conclusion is the same as in the Ratio Test:
   - If $0 \le L < 1$ then the series is convergent
   - If $L > 1$ then the series is divergent

9. The first partial sum that can be used as an estimate guaranteeing an error smaller than 0.01 is

$$S_4 = \sum_{n=1}^{4} \frac{(-1)^n}{n^3} \approx -0.896...$$

   The exact value is $-0.9015...$