MAT 137Y - Practice problems
Unit 7 - The definition of integral

1. Compute these sums

   (a) $\sum_{i=2}^{4} i^j$ (b) $\sum_{j=2}^{4} i^j$ (c) $\sum_{k=2}^{4} i^j$ (d) $\sum_{n=0}^{137} \sin\left(\frac{\pi n}{2}\right)$

2. For each positive integer N, let us define $H_N = \sum_{i=1}^{N} \frac{1}{i}$. This is a new sequence of terms (called the *harmonic sums*), that appears often in mathematics, just like $N!$, for example. Write each of the following sums in terms of harmonic sums:

   (a) $\sum_{i=10}^{20} \frac{1}{i}$

   (b) $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \ldots + \frac{1}{100}$

   (c) $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots + \frac{1}{99}$

   (d) $\sum_{i=1}^{100} \frac{(-1)^{i+1}}{i}$

3. Calculate the supremum, infimum, maximum, and minimum (if they exist) of the following sets:

   (a) $\{x \in \mathbb{R} \mid x^2 \leq 2\}$ (b) $\{x \in \mathbb{Q} \mid x^2 \leq 2\}$ (c) $\{x \in \mathbb{Z} \mid x^2 \leq 2\}$

4. Consider the function f defined by $f(x) = \tan x$. If possible, construct an interval I contained in the domain of f such that...

   (a) ... f has a maximum and a minimum on I.
   (b) ... f has a supremum but not a maximum on I.
   (c) ... f has no supremum and no infimum on I.
   (d) ... f has an infimum but not a supremum on I.
   (e) ... f has a minimum, but not an infimum on I.

5. Consider the function f defined by $f(x) = x$.

   (a) Find a partition P of $[0,1]$ such that $L_P(f) = 0.24$.
   Hint: You can do this with just 3 points in the partition.
   (b) Find the largest lower sum of f among all partitions with exactly 3 points.
   Is this number smaller or greater than the lower integral of f?
   (c) Prove that for every partition P of $[0,1]$, $L_P(f) + U_P(f) = 1$.

6. Let $a < b$. Let f be a bounded function on $[a,b]$. Given a partition P of $[a,b]$, let us call $\ell_P(f)$ the Riemann sum for f and P when we choose the left end-point for each subinterval.

(a) Find a simple property f must satisfy to guarantee that $L_P(f) = \ell_P(f)$.

(b) Find a simple property f must satisfy to guarantee that $U_P(f) = \ell_P(f)$.

7. Let $a < b$. Let f be a continuous function on $[a, b]$. Prove that there exists $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(t)dt.$$

Note: This result is sometimes called the "Integral Mean Value Theorem". The quantity $\frac{1}{b-a} \int_a^b f(t)dt$ represents the average value of the function f on $[a, b]$.

8. Consider the set $A = \left\{ \frac{1}{n} \ : \ n \in \mathbb{Z}, n > 0 \right\}$. We define the function f by the equation:

$$f(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

(a) Let P be any partition of $[0, 1]$. Compute $L_P(f)$.

(b) What is $\underline{I_0^1}(f)$?

(c) Construct a partition P of $[0, 1]$ such that $U_P(f) = 1$.

(d) Construct a partition P of $[0, 1]$ such that $U_P(f) < 1$.

(e) Construct a partition P of $[0, 1]$ with exactly 4 points (3 subintervals) such that $U_P(f) = 0.52$.

(f) Construct a partition P of $[0, 1]$ with exactly 4 points (3 subintervals) such that $U_P(f) = 0.5001$.

(g) Construct a partition P of $[0, 1]$ with exactly 6 points (5 subintervals) such that $U_P(f) = \frac{1}{3} + 0.0001$.

(h) Construct a partition P of $[0, 1]$ such that $U_P(f) < 0.1$.

(i) What is $\overline{I_0^1}(f)$? Prove it.

(j) Is f integrable on $[0, 1]$?

9. Imitate the example in Video 7.10 to compute $\int_1^2 x^2 dx$ using Riemann sums.

10. Let $a < b$. Let f be a bounded function on $[a, b]$. You are going to prove that[1] f is integrable on $[a, b]$ if and only if

$$\forall \varepsilon > 0, \text{ there exists a partition } P \text{ of } [a, b] \text{ such that } U_P(f) - L_P(f) < \varepsilon. \qquad (\star)$$

(a) Prove that if f satisfies $(\star)$, then f is integrable on $[a, b]$.

(b) Prove that if f is integrable on $[a, b]$ then f satisfies $(\star)$.

[1]This is sometimes called the "ε-characterization of integrability". It simplifies proofs about integrability. You will use it in MAT237.

Bonus problems: proving properties of the integral from the definition

In Video 7.11 you learned the basic properties of definite integrals. Like with any other concept, they must be proven directly from the definition. Since the definition is complex, so are the proofs. You will learn and practice writing these proofs in MAT237. For now, as a preview, you can prove one of them with this set of questions. Your goal is to prove:

> **Theorem 3**. Let $a < b < c$. Let f be a bounded function on $[a, c]$.
> IF f is integrable on $[a, b]$ and f is integrable on $[b, c]$,
> THEN f is integrable on $[a, c]$ and $\displaystyle\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx.$

11. Let $a < b < c$. Let f be a bounded function on $[a, c]$.

    (a) Prove that for every partition P_1 of $[a, b]$ and for every partition P_2 of $[b, c]$, $\overline{I_a^c}(f) \leq U_{P_1}(f) + U_{P_2}(f)$.
    Hint: Construct a partition P of $[a, c]$ from P_1 and P_2.

    (b) Use Question (11a) to prove that $\overline{I_a^c}(f) \leq \overline{I_a^b}(f) + \overline{I_b^c}(f)$.
    Note: If at this stage you think you have proven an equality instead of an inequality, then your proof is probably wrong. Make sure you understand why.

    (c) Repeat the previous two questions for lower integrals instead of upper integrals.

    (d) Use Questions (11b), and (11c) to prove Theorem 3.

Some answers and hints

1. (a) $2^j + 3^j + 4^j$ (b) $i^2 + i^3 + i^4$ (c) $3i^j$ (d) 1

2. (a) $H_{20} - H_9$ (b) $\frac{1}{2}H_{50}$ (c) $H_{100} - \frac{1}{2}H_{50}$ (d) $H_{100} - H_{50}$

3. (a) sup and max are $\sqrt{2}$; inf and min are $-\sqrt{2}$.
   (b) sup is $\sqrt{2}$; inf is $-\sqrt{2}$; there is no max or min.
   (c) sup and max are 1; inf and min are -1.

4. They are all possible, except for (e).

5. (a) $P = \{0, 0.4, 1\}$
   (b) 0.25.
   This must be less than (or equal to) the lower integral of f.
   (c) This can be done algebraically or geometrically (draw a picture!)

6. The answer has something to do with monotonicity.

7. Bound the quantity $\int_a^b f(t)dt$ above and below using the maximum and minimum of f on $[a, b]$. (Why can we do that?) Then use the IVT.

8. Do the questions in order. In particular, questions (8c) to (8h) will feel like a natural progression. Question (8h) will be much harder if you have not solved the previous questions first. Once you have the right partitions, you will know your answer is right.

   f is integrable on $[0, 1]$ and the integral is 0.

9. If you need them, you may use the identities

$$\sum_{i=0}^{N} i = \frac{N(N+1)}{2}, \qquad \sum_{i=0}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}, \qquad \sum_{i=0}^{N} i^3 = \frac{N^2(N+1)^2}{4}$$

   without proof. If you wish to prove them, use induction.

   The answer to the question is 7/3. If you make the same choice of partitions and of points on each subintervals as in the Video, you will get

$$\int_1^2 x^2 dx = \lim_{n\to\infty} \left[\sum_{i=1}^{n} \left(1 + \frac{i}{n}\right)^2 \cdot \frac{1}{n} \right] = \ldots = \frac{7}{3}$$

10. (a) Assume f satisfies ($\star$). Show that $\forall \varepsilon > 0$, $\overline{I_a^b} - \underline{I_a^b} < \varepsilon$. Then...
    (b) Assume f in integrable on $[a, b]$. Fix $\varepsilon > 0$. Conclude there exists a partition Q such that $U_Q(f) < \overline{I_a^b} + \frac{\varepsilon}{2}$ and a partition R such that $L_R(f) > \underline{I_a^b} - \frac{\varepsilon}{2}$. Then...