MAT 137Y: Calculus with proofs
Test 2 - Part B

1. Let f be a function with domain $\mathbb{R}$. Assume $f(0) = 0$ and f is differentiable at 0. Calculate

$$\lim_{x \to 0} \frac{f(x)}{\sqrt[3]{x}}$$

Hint: Use the definition of $f'(0)$.

Solution:

I will prove the limit is 0.

Since f is differentiable at 0, the limit

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x}$$

exists. Therefore, I can use the limit law for product as follows:

$$\lim_{x \to 0} \frac{f(x)}{\sqrt[3]{x}} = \lim_{x \to 0} \left[\frac{f(x)}{x} \cdot x^{2/3} \right] = \left[\lim_{x \to 0} \frac{f(x)}{x} \right] \cdot \left[\lim_{x \to 0} x^{2/3} \right] = f'(0) \cdot 0 = 0$$

Note: **You cannot use L'Hôpital's Rule in this problem!**

We only know that f is differentiable at 0 and nowhere else. Even if f were differentiable everywhere, we do not know whether f' is continuous at 0. If you used L'Hôpital's Rule in this question it shows that you are just pushing symbols without thinking or understanding, and in particular that you do not understand L'Hôpital's Rule or the what it means to be differentiable.

2. Let f be a function with domain $\mathbb{R}$. Assume f is differentiable and f' is continuous. Let $a < b$. Prove that there exists $M \in \mathbb{R}$ such that

$$\forall x, y \in [a, b], \quad |f(x) - f(y)| \leq M|x - y|$$

Hint: Do some rough work first (but do not submit it). Use MVT. Use EVT.

Solution:

> Important note on proof structure: Your proof must **begin** by constructing M. In addition, M can only depend on f, a, and b. If you choose a value of M that depends on other variables (such as x, y, or c), your "proof" does not make any sense. Review Unit 1, please.

STEP 1: Constructing M

First, we need to find a number $M \in \mathbb{R}$ that satisfies:

$$\forall x \in [a, b], \quad |f'(x)| \leq M. \tag{1}$$

There are at least two ways to do it:

- Method 1: Since the function f' is continuous, and the absolute value function is continuous, the function $|f'|$ is also continuous, because it is the composition of two continuous functions. We can apply the Extreme Value Theorem to $|f'|$ on the interval $[a.b]$. We conclude that $|f'|$ has a maximum on $[a, b]$. Call it M. This number M satisfies (1).
- Method 2: We apply the Extreme Value Theorem to f' on $[a, b]$. By assumption f' is continuous on $[a, b]$, so the hypotheses of the theorem are satisfied. Let M_1 and M_2 be the maximum and minimum values of f' on $[a, b]$, respectively. Then we have that

$$\forall x \in [a, b], \quad M_1 \leq f'(x) \leq M_2$$

  You may be tempted to think that we can take $M = M_2$, but that would not work, since f' may take negative values. Instead, we can conclude that

$$\forall x \in [a, b], \quad |f'(x)| \leq \max\{-M_1, M_2\}.$$

  This is because:

  - If, for a value of x, $f'(x) \geq 0$, then $M_2 \geq 0$ as well. Then the inequality

$$0 \leq f'(x) \leq M_2$$

    implies

$$|f'(x)| \leq M_2$$

– If, for a value of x, $f'(x) \leq 0$, then $M_1 \leq 0$ as well. Then the inequality

$$M_1 \leq f'(x) \leq 0$$

implies

$$|f'(x)| \leq -M_1$$

We take $M = \max\{-M_1, M_2\}$. We have proven that it satisfies (1).
Another option is to take $M = \max\{|M_1|, |M_2|\}$. It turns out to be the same.
Note: If you found this confusing, consider these two examples:

– If we know that $-4 \leq f'(x) \leq 8$, we can conclude that $|f'(x)| \leq 8$
– If we know that $-7 \leq f'(x) \leq 5$, we can conclude that $|f'(x)| \leq 7$

STEP 2: Proving that this value of M works

- Fix arbitrary numbers $x \in [a, b]$ and $y \in [a, b]$. We will show that $|f(x) - f(y)| \leq M|x - y|$.
- We may assume, without loss of generality, that $x < y$:
  – If $y < x$, the same argument will work with the roles of x and y reversed.
  – If $x = y$, the inequality (actually, an equality in this case) is immediately true and there is nothing to prove.
- We want to apply the MVT to f on $[x, y]$. Before doing that, we need to verify the hypotheses:
  – By assumption f is differentiable on $\mathbb{R}$ so f is also differentiable on the smaller interval (x_1, x_2).
  – Since f is differentiable on $\mathbb{R}$, f must also be continuous on $\mathbb{R}$. Therefore, f is also continuous on the smaller interval $[x_1, x_2]$.

  Thus, the hypotheses of the MVT are satisfied.
- By MVT, we know there exists $c \in (x, y)$ such that

$$\frac{f(y) - f(x)}{y - x} = f'(c).$$

  This implies that $f(y) - f(x) = f'(c)(y - x)$.
  Notice that $c \in (x, y) \subseteq [a, b]$. Hence, using (1), $|f'(c)| \leq M$.
- From here we can conclude

$$|f(x) - f(y)| = |f'(c)(x - y)| = |f'(c)||x - y| \leq M|x - y|,$$

This completes the proof. □