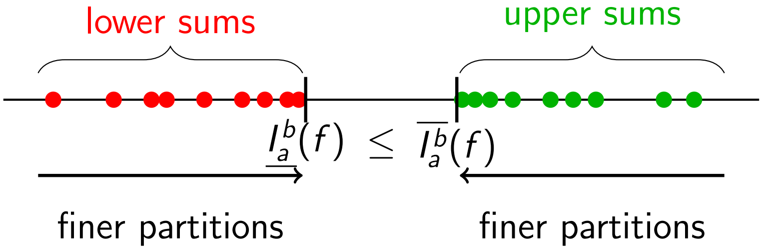
MAT137 - Calculus with proofs

- Today: Definition of the integral
- MONDAY: Examples and properties of the integral **(Videos 7.7, 7.8, 7.11)**



Equivalent definitions of supremum

Assume S is an upper bound of the set A.
Which of the following is equivalent to "S is the supremum of A"?

1. If R is an upper bound of A, then $S \leq R$.
2. $\forall R \geq S$, R is an upper bound of A.
3. $\forall R \leq S$, R is not an upper bound of A.
4. $\forall R < S$, R is not an upper bound of A.
5. $\forall R < S$, $\exists x \in A$ such that $R < x$.
6. $\forall R < S$, $\exists x \in A$ such that $R \leq x$.
7. $\forall R < S$, $\exists x \in A$ such that $R < x \leq S$.
8. $\forall R < S$, $\exists x \in A$ such that $R < x < S$.
9. $\forall \varepsilon > 0$, $\exists x \in A$ such that $S - \varepsilon < x$.
10. $\forall \varepsilon > 0$, $\exists x \in A$ such that $S - \varepsilon < x \leq S$.

Equivalent or not?

Let $A \subseteq \mathbb{R}$. Let $S \in \mathbb{R}$.

(A) $\exists x \in A$ such that $2 < x$
(B) $\exists x \in A$ such that $2 \leq x$

1. Does (A) imply (B)?
2. Does (B) imply (A)?

(C) $\forall R < S, \exists x \in A$ such that $R < x$
(D) $\forall R < S, \exists x \in A$ such that $R \leq x$

3. Does (C) imply (D)?
4. Does (D) imply (C)?

Warm up: partitions

Which ones are partitions of $[0, 2]$?

1. $[0, 2]$
2. $\{0.5,\ 1,\ 1.5\}$
3. $\{0,\ 2\}$
4. $\{1,\ 2\}$
5. $\{0,\ e,\ 2\}$
6. $\{0,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2\}$
7. $\left\{\dfrac{n}{n+1}\ :\ n \in \mathbb{N}\right\} \cup \{2\}$

Warm up: lower and upper sums

Let $f(x) = \sin x$.

Consider the partition $P = \{0, 1, 3\}$ of the interval $[0, 3]$.

Calculate $L_P(f)$ and $U_P(f)$.

Equations for lower and upper sums

Let f be a **decreasing**, bounded function on $[a, b]$.
Let $P = \{x_0, x_1, \ldots, x_N\}$ be a partition of $[a, b]$

Which ones are a valid equation for $L_P(f)$? For $U_P(f)$?

1. $\displaystyle\sum_{i=0}^{N} f(x_i)\,\Delta x_i$
2. $\displaystyle\sum_{i=1}^{N} f(x_i)\,\Delta x_i$
3. $\displaystyle\sum_{i=0}^{N-1} f(x_i)\,\Delta x_i$
4. $\displaystyle\sum_{i=1}^{N} f(x_{i+1})\,\Delta x_i$
5. $\displaystyle\sum_{i=1}^{N} f(x_{i-1})\,\Delta x_i$
6. $\displaystyle\sum_{i=0}^{N-1} f(x_i)\,\Delta x_{i+1}$

Recall: $\Delta x_i = x_i - x_{i-1}$.

Joining partitions

Assume

$$L_P(f) = 2, \quad U_P(f) = 6$$
$$L_Q(f) = 3, \quad U_Q(f) = 8$$

1. Is $P \subseteq Q$?
2. Is $Q \subseteq P$?
3. What can you say about $L_{P \cup Q}(f)$ and $U_{P \cup Q}(f)$?

A tricky question

Let f be a bounded function on $[a, b]$. Which statement is true?

1. There exists a partition P of $[a, b]$ such that
$$\underline{I_a^b}(f) = L_P(f) \qquad \text{and} \qquad \overline{I_a^b}(f) = U_P(f).$$
2. There exist partitions P and Q of $[a, b]$ such that
$$\underline{I_a^b}(f) = L_P(f) \qquad \text{and} \qquad \overline{I_a^b}(f) = U_Q(f).$$
3. There exists a partition P of $[a, b]$ such that
$$\underline{I_a^b}(f) = L_P(f).$$