- Test 1 opens on Friday, October 23. See details on course website.
- TODAY: Squeeze theorem and more proof with limits
- FRIDAY: Continuity (**Watch videos 2.14, 2.15**)

Limits involving $\sin(1/x)$

The limit $\lim_{x \to 0} \sin(1/x)$...

1. DNE because the function values oscillate around 0
2. DNE because $1/0$ is undefined
3. DNE because no matter how close x gets to 0, there are x's for which $\sin(1/x) = 1$, and some for which $\sin(1/x) = -1$
4. all of the above
5. is 0

The limit $\lim_{x \to 0} x^2 \sin(1/x)$...

1. DNE because the function values oscillate around 0
2. DNE because $1/0$ is undefined
3. DNE because no matter how close x gets to 0, there are x's for which $\sin(1/x) = 1$, and some for which $\sin(1/x) = -1$
4. is 0
5. is 1

A new squeeze

This is the Squeeze Theorem, as you know it:

The (classical) Squeeze Theorem

Let $a, L \in \mathbb{R}$.
Let f, g, and h be functions defined near a, except possibly at a.

IF
- For x close to a but not a, $h(x) \leq g(x) \leq f(x)$
- $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} h(x) = L$

THEN
- $\lim_{x \to a} g(x) = L$

Come up with a new version of the theorem about limits being infinity. (The conclusion should be $\lim_{x \to a} g(x) = \infty$.)

Hint: Draw a picture for the classical Squeeze Theorem. Then draw a picture for the new theorem.

A new theorem about products

We were trying to prove:

Theorem

Let $a \in \mathbb{R}$. Let f and g be functions with domain $\mathbb{R}$, except possibly a. Assume

- $\lim_{x \to a} f(x) = 0$, and
- g is bounded. This means that

$$\exists M > 0 \text{ s.t. } \forall x \neq a, |g(x)| \leq M.$$

THEN $\lim_{x \to a} [f(x)g(x)] = 0$

Proof feedback

1. Is the structure of the proof correct?
   (First fix ε, then choose δ, then ...)
2. Did you say exactly what δ is?
3. Is the proof self-contained?
   (I do not need to read the rough work)
4. Are all variables defined? In the right order?
5. Do all steps follow logically from what comes before?
   Do you start from what you know and prove what you have to prove?
6. Are you proving your conclusion or assuming it?

Critique this "proof" – #1

- WTS $\lim_{x \to a} [f(x)g(x)] = 0$. By definition, WTS:
$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies |f(x)g(x)| < \varepsilon$$
- Let $\varepsilon > 0$.
- Use the value $\frac{\varepsilon}{M}$ as "epsilon" in the definition of $\lim_{x \to a} f(x) = 0$
$$\exists \delta_1 \in \mathbb{R} \text{ s.t. } 0 < |x - a| < \delta_1 \implies |f(x)| < \frac{\varepsilon}{M}.$$
- Take $\delta = \delta_1$.
- Let $x \in \mathbb{R}$. Assume $0 < |x - a| < \delta$
- Since $\exists M > 0$ s.t. $\forall x \neq 0, |g(x)| \leq M$
$$|f(x)g(x)| < \frac{\varepsilon}{M} \cdot M = \varepsilon.$$

Critique this "proof" – #2

- Since g is bounded, $\exists M > 0 \quad \text{s.t.} \quad \forall x \neq 0, \; |g(x)| \leq M$

- Since $\lim_{x \to a} f(x) = 0$, there exists $\delta_1 > 0$ s.t.
  if $0 < |x - a| < \delta_1$, then $|f(x) - 0| = |f(x)| < \varepsilon_1 = \dfrac{\varepsilon}{M}$.

- $|f(x)g(x)| = |f(x)| \cdot |g(x)| \leq |f(x)| \cdot M < \varepsilon_1 \cdot M = \dfrac{\varepsilon}{M} \cdot M = \varepsilon$

- In summary, by setting $\delta = \min\{\delta_1\}$, we find that

  if $0 < |x - a| < \delta$, then $|f(x) \cdot g(x)| < \varepsilon$.

Critique this "proof" – #3

- WTS $\lim_{x \to a} [f(x)g(x)] = 0$:
  $$\forall \varepsilon > 0, \exists \delta > 0 \quad \text{s.t.} \quad 0 < |x - a| < \delta \implies |f(x)g(x)| < \varepsilon.$$
- We know $\lim_{x \to a} f(x) = 0$
  $$\forall \varepsilon_1 > 0, \exists \delta_1 > 0 \quad \text{s.t.} \quad 0 < |x - a| < \delta_1 \implies |f(x)| < \varepsilon_1.$$
- We know $\exists M > 0 \quad \text{s.t.} \quad \forall x \neq 0, \ |g(x)| \leq M.$
- $|f(x)g(x)| = |f(x)||g(x)| < \varepsilon_1 M$
- $\varepsilon = \varepsilon_1 M \implies \varepsilon_1 = \dfrac{\varepsilon}{M}$
- Take $\delta = \delta_1$

Let $a \in \mathbb{R}$

Let f and g be functions with domain $\mathbb{R}$, except possibly a.

Assume

(1) $\lim_{x \to a} f(x) = 0$, i.e., $\forall \varepsilon > 0, \exists \delta > 0$ s.t $0 < |x-a| < \delta \Rightarrow |f(x)| < \varepsilon$

(2) g is bounded, i.e., $\exists M > 0$ s.t. $\forall x \neq a$, $|g(x)| \leq M$.

WTS $\lim_{x \to a} [f(x)g(x)] = 0$, i.e.,

$\forall \varepsilon > 0, \exists \delta > 0$ S.t. $0 < |x-a| < \delta \Rightarrow |f(x)g(x)| < \varepsilon$

Let $\varepsilon > 0$

Since we know $\lim_{x \to a} f(x) = 0$ by assumption

if we take $\frac{\varepsilon}{M}$ as ε in the definition of $\lim_{x \to a} f(x) = 0$

there must exist δ_1 such that $0 < |x-a| < \delta_1 \Rightarrow |f(x)| < \frac{\varepsilon}{M}$

Take $\delta = \delta_1$

Let $x \in \mathbb{R}$

Assume $0 < |x-a| < \delta$

Then

$|f(x)| < \frac{\varepsilon}{M}$ (By assumption (1))

$|f(x)| \cdot |g(x)| < \frac{\varepsilon}{M} \cdot M$ (By assumption (2)).

$|f(x)g(x)| < \varepsilon$

$\square$