MAT137 - Calculus with proofs

- TODAY: More on the definition of limit
- MONDAY: Limit laws (**Watch videos 2.10, 2.11**)

Recall

We were trying to solve this question

Let $a \in \mathbb{R}$. Let f be a function defined at least on an interval around a, except possibly at a.

Write a formal definition for $\lim_{x \to a} f(x) = \infty$.

We were looking at these two answers:

- $\forall M > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies f(x) > M$
- $\forall M \in \mathbb{R}, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies f(x) > M$

M greater than what?

Let $a \in \mathbb{R}$. Let f be a function. Consider these statements:

$$\text{(A)} \quad 0 < |x - a| < 0.1 \implies f(x) > 4$$
$$\text{(B)} \quad 0 < |x - a| < 0.1 \implies f(x) > 2$$

1. Does (A) imply (B)?
2. Does (B) imply (A)?

$$\text{(C)} \quad \forall M > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies f(x) > M$$
$$\text{(D)} \quad \forall M > 5, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies f(x) > M$$

3. Does (C) imply (D)?
4. Does (D) imply (C)?

Infinite limits

Which ones are (equivalent to) the definition of $\lim_{x \to a} f(x) = \infty$?

1. $\forall M > 0, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies f(x) > M$
2. $\forall M > 5, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies f(x) > M$
3. $\forall M \in \mathbb{R}, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies f(x) > M$
4. $\forall M \in \mathbb{Z}, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies f(x) > M$
5. $\forall M \in \mathbb{R}, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies f(x) \geq M$
6. $\forall M < 9, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies f(x) > M$

Preparation: choosing deltas

1. Find one value of $\delta > 0$ such that
$$|x - 3| < \delta \implies |5x - 15| < 1.$$
2. Find *all* values of $\delta > 0$ such that
$$|x - 3| < \delta \implies |5x - 15| < 1.$$
3. Find *all* values of $\delta > 0$ such that
$$|x - 3| < \delta \implies |5x - 15| < 0.1.$$
4. Let us fix $\varepsilon > 0$. Find *all* values of $\delta > 0$ such that
$$|x - 3| < \delta \implies |5x - 15| < \varepsilon.$$

Your first $\varepsilon - \delta$ proof

Goal

We want to prove that

$$\lim_{x \to 3} (5x + 1) = 16 \tag{1}$$

directly from the definition.

1. Write down the formal definition of the statement (1).
2. Write down what the structure of the formal proof should be, without filling the details.
3. Write down a complete formal proof.

What is wrong with this “proof”?

Prove that

$$\lim_{x \to 3} (5x + 1) = 16$$

“Proof:”

Let $\varepsilon > 0$.
WTS $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$0 < |x - 3| < \delta \implies |(5x + 1) - (16)| < \varepsilon$$

$$|(5x + 1) - (16)| < \varepsilon \iff |5x + 15| < \varepsilon$$
$$\iff 5|x + 3| < \varepsilon \implies \delta = \frac{\varepsilon}{3}$$

□