MAT 137Y: Calculus with proofs
Assignment 6 - Solutions

The goal of this assignment is to prove the following result from the definition of integral:

Theorem 1: Let $a < b$. Let f and g be bounded functions on $[a, b]$. Let $h = f + g$.

- IF f and g are integrable on $[a, b]$
- THEN h is integrable on $[a, b]$ and

$$\int_a^b h(x)dx \;=\; \int_a^b f(x)dx \;+\; \int_a^b g(x)dx$$

We will break the proof into pieces and guide you through them. In mathematical terms, you will be proving a few "lemmas" before you prove Theorem 1. For all the questions, let $a < b$ and let f and g bounded functions on $[a, b]$. We won't repeat this every time. **Do not assume that any of the functions is integrable, unless specified**: many of the intermediate results hold for non-integrable functions as well. In many of the questions, you will need to use the results of one or various previous questions in your proof.

1. (a) It is NOT necessarily true that for every partition P of $[a, b]$

$$L_P(f) \ + \ L_P(g) \ = \ L_P(h). \tag{1}$$

Show it with a counterexample.

Solution:

- Let $a = 0$ and $b = 1$ so we are looking at the interval $[a, b] = [0, 1]$.
- Consider the partition $P = \{0, 1\}$. (In other words, we have one single subinterval.)
- Let f be defined by $f(x) = x$.
  f is increasing so it has a minimum at $x = 0$, which is also the infimum, which is $f(0) = 0$. Therefore, $L_P(f) = 0$
- Let g be defined by $g(x) = 1 - x$.
  f is decreasing so it has a minimum at $x = 1$, which is also the infimum, which is $g(1) = 0$. Therefore, $L_P(g) = 0$
- Notice that h is a constant function: $h(x) = 1$ for all x. Therefore $L_P(h) = 1$.
- This example shows that in general $L_P(f) + L_P(g) \neq L_P(h)$. They may be equal in some cases, but not always.

(b) However, if we turn the equality into an inequality in (1), then it becomes true. Which inequality? Prove it.

Solution: It is always true that

$$L_P(f) \;+\; L_P(g) \;\leq\; L_P(h).$$

To prove it, let $P = \{x_0, x_1, x_2, \dots, x_n\}$ be an arbitrary partition of $[a, b]$. I introduce some notation:

- Let m_i be the infimum of f on $[x_{i-1}, x_i]$.
- Let m_i' be the infimum of g on $[x_{i-1}, x_i]$.
- Let m_i'' be the infimum of h on $[x_{i-1}, x_i]$.
- Let $\Delta x_i = x_i - x_{i-1}$.

Let $x \in [x_{i-1}, x_i]$. Then, by definition of m_i and m_i' we have

$$\begin{aligned} m_i &\leq f(x) \\ m_i' &\leq g(x) \end{aligned}$$

Adding both inequalities I get

$$m_i + m_i' \leq f(x) + g(x) = h(x)$$

Since x was arbitrary, I have proven that the number $m_i + m_i'$ is a lower bound for h on $[x_{i-1}, x_i]$. Since m_i'' is the greatest lower bound for h on $[x_{i-1}, x_i]$, this means that

$$m_i + m_i' \leq m_i'' \tag{2}$$

Finally, I use the definition of lower sum, some algebra, and (2):

$$\begin{aligned} L_P(f) + L_P(g) \;&=\; \sum_{i=1}^{n} m_i \Delta x_i \;+\; \sum_{i=1}^{n} m_i' \Delta x_i \\ &=\; \sum_{i=1}^{n} (m_i + m_i')\, \Delta x_i \\ &\leq\; \sum_{i=1}^{n} m_i'' \Delta x_i \;=\; L_P(h) \end{aligned}$$

This is what I wanted to prove.

2. **[Do not submit]** Prove that for every $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$$\underline{I_a^b}(f) - \varepsilon \ < \ L_P(f).$$

Note: This is a very, very short proof if you understand the definition of lower integral as supremum. You may even have learned something similar in class. You do not need to submit your answer to this question, but we want to make sure you think about it before trying the harder (and related) next question.

Solution:

- By definition $\underline{I_a^b}(f)$ is the supremum (or lowest upper bound) of the set

$$\mathcal{S} = \{ \ L_P(f) \mid P \text{ is a partition of } [a, b] \ \}$$

- Let us fix $\varepsilon > 0$.
- Then $\underline{I_a^b}(f) - \varepsilon < \underline{I_a^b}(f)$. Therefore $\underline{I_a^b}(f) - \varepsilon$ is not an upper bound of the set $\mathcal{S}$.
- This means there exists a partition P of $[a, b]$ such that $\underline{I_a^b}(f) - \varepsilon < L_P(f)$. That's it.

3. Prove that for every $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$$\underline{I_a^b}(f) + \underline{I_a^b}(g) - \varepsilon < L_P(f) + L_P(g).$$

Hint: This will feel a bit like those "ε-δ" proofs you learned in Unit 2.

Solution:

- Let us fix $\varepsilon > 0$.
- $\underline{I_a^b}(f) - \varepsilon/2 < \underline{I_a^b}(f)$.
  Therefore, $\underline{I_a^b}(f) - \varepsilon/2$ is NOT an upper bound of the set of lower sums of f.
  This means there exists a partition P_1 of $[a, b]$ such that
  $$\underline{I_a^b}(f) - \varepsilon/2 < L_{P_1}(f) \tag{3}$$
- $\underline{I_a^b}(g) - \varepsilon/2 < \underline{I_a^b}(g)$.
  Therefore, $\underline{I_a^b}(g) - \varepsilon/2$ is NOT an upper bound of the set of lower sums of g.
  This means there exists a partition P_2 of $[a, b]$ such that
  $$\underline{I_a^b}(g) - \varepsilon/2 < L_{P_2}(g) \tag{4}$$
- Let $P = P_1 \cup P_2$. Then P is a partition of $[a, b]$ that satisfies $P_1 \subseteq P$ and $P_2 \subseteq P$. Therefore
  $$L_{P_1}(f) \leq L_P(f), \qquad L_{P_2}(g) \leq L_P(g) \tag{5}$$
- Combining inequalities (3), (4), and (5) we get
  $$\underline{I_a^b}(f) + \underline{I_a^b}(g) - \varepsilon \;=\; (\underline{I_a^b}(f) - \varepsilon/2) + (\underline{I_a^b}(g) - \varepsilon/2) \;<\; L_{P_1}(f) + L_{P_2}(g) \;\leq\; L_P(f) + L_P(g)$$
  and therefore
  $$\underline{I_a^b}(f) + \underline{I_a^b}(g) - \varepsilon \;<\; L_P(f) + L_P(g)$$
  which is what we needed to prove.

4. Prove that
$$\underline{I}_a^b(f) \ + \ \underline{I}_a^b(g) \ \leq \ \underline{I}_a^b(h).$$

Note: If, at this moment, you think you have proven a strict inequality instead of a non-strict inequality, then your argument is probably wrong.

Solution:

- Let us fix an arbitrary $\varepsilon > 0$.
- In Question 3 we proved there exists a partition P of $[a, b]$ such that
$$\underline{I}_a^b(f) + \underline{I}_a^b(g) - \varepsilon < L_P(f) + L_P(g) \tag{6}$$
- In Question 1 we proved that
$$L_P(f) + L_P(g) \leq L_P(h) \tag{7}$$
- In addition, by definition of lower integral
$$L_P(h) \leq \underline{I}_a^b(h) \tag{8}$$
- Putting (6), (7), and (8) together we get
$$\underline{I}_a^b(f) + \underline{I}_a^b(g) - \varepsilon \ < \ L_P(f) + L_P(g) \ \leq \ L_P(h) \leq \underline{I}_a^b(h)$$
and therefore
$$\underline{I}_a^b(f) + \underline{I}_a^b(g) - \varepsilon \ < \ \underline{I}_a^b(h)$$
- Notice that we had fixed an *arbitrary* $\varepsilon > 0$. In other words, we have proven that
$$\forall \varepsilon > 0, \quad \underline{I}_a^b(f) + \underline{I}_a^b(g) - \varepsilon \ < \ \underline{I}_a^b(h) \tag{9}$$
- (9) can be rewritten as
$$\forall \varepsilon > 0, \quad \underline{I}_a^b(f) + \underline{I}_a^b(g) - \underline{I}_a^b(h) < \varepsilon. \tag{10}$$
- Notice that for any real number $x \in \mathbb{R}$:[1]
$$\forall \varepsilon > 0, \ x < \varepsilon \quad \iff \quad x \leq 0.$$
Therefore, (10) implies
$$\underline{I}_a^b(f) + \underline{I}_a^b(g) - \underline{I}_a^b(h) \ \leq \ 0$$
which can be rewritten as
$$\underline{I}_a^b(f) + \underline{I}_a^b(g) \ \leq \ \underline{I}_a^b(h)$$
which is what I had to prove.

[1] If you want to prove this equivalence more rigorously, the $\Leftarrow$ direction is immediate to check, and the $\Rightarrow$ direction can be proven by contradiction.

5. This question is irrelevant to the proof of Theorem 1, but it is also interesting. Is it always true that

$$\underline{I}_a^b(f) \ + \ \underline{I}_a^b(g) \ = \ \underline{I}_a^b(h)?$$

Prove it.

Solution:

No, this is not always true.[2]

For a counterexample, let $[a, b] = [0, 1]$ and consider the functions f and g defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases} \qquad\qquad g(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}$$

- In Video 7.8 we proved that $\underline{I}_a^b(f) = 0$.
- The same argument shows that $\underline{I}_a^b(g) = 0$
- However, the function h is constant: $h(x) = 1$ for all x, and $\underline{I}_a^b(h) = 1$.
- This example shows that in general $\underline{I}_a^b(f) + \underline{I}_a^b(g) \neq \underline{I}_a^b(h)$. They may be equal in some cases, but not always.

[2]If we assume that f and g were integrable on $[a, b]$, then yes, it must be true. It is a direct consequence of Theorem 1.

6. **[Do not submit]** Repeat the steps from the previous questions (with upper rather than lower sums and integrals) to prove that

$$\overline{I_a^b}(h) \;\leq\; \overline{I_a^b}(f) \;+\; \overline{I_a^b}(g).$$

7. Prove Theorem 1.

Solution:

- In Question 4, I proved
$$\underline{I_a^b}(f) + \underline{I_a^b}(g) \;\leq\; \underline{I_a^b}(h)$$
- We know that
$$\underline{I_a^b}(h) \;\leq\; \overline{I_a^b}(h)$$
because every function satisfies this.
- In Question 6, I proved
$$\overline{I_a^b}(h) \;\leq\; \overline{I_a^b}(f) + \overline{I_a^b}(g)$$
- Putting the previous three inequalities together
$$\underline{I_a^b}(f) + \underline{I_a^b}(g) \;\leq\; \underline{I_a^b}(h) \;\leq\; \overline{I_a^b}(h) \;\leq\; \overline{I_a^b}(f) + \overline{I_a^b}(g) \tag{11}$$
- We are assuming that f and g are integrable on $[a, b]$. This means that $\underline{I_a^b}(f) = \overline{I_a^b}(f)$ and $\underline{I_a^b}(g) = \overline{I_a^b}(g)$. This means that the first and last terms in the chain of inequalities (11) are equal. Hence, all the steps in the chain are equalities!
$$\underline{I_a^b}(f) + \underline{I_a^b}(g) \;=\; \underline{I_a^b}(h) \;=\; \overline{I_a^b}(h) \;=\; \overline{I_a^b}(f) + \overline{I_a^b}(g)$$
- In particular $\underline{I_a^b}(h) = \overline{I_a^b}(h)$, so we have proven that h is integrable on $[a, b]$.
- Finally, now that we know that f, g, and h are integrable on $[a, b]$ the equation
$$\underline{I_a^b}(f) + \underline{I_a^b}(g) \;=\; \underline{I_a^b}(h)$$
becomes
$$\int_a^b f(x)dx \;+\; \int_a^b g(x)dx \;=\; \int_a^b h(x)dx$$
which completes the proof.