MAT 137Y: Calculus with proofs
Assignment 4 – Sample solutions

1. In this problem, we will only work with functions with domain $\mathbb{R}$ and codomain $\mathbb{R}$. Therefore, if we say that two functions f and g are equal ($f = g$), it means that

$$\forall x \in \mathbb{R},\ f(x) = g(x).$$

We need a new definition. We say that a function f is *faithful* when

"For every two functions g and h, $\quad f \circ g = f \circ h \implies g = h$."

Before beginning this question, we recall the definitions for a function f with domain $\mathbb{R}$ to be one-to-one.

$$\forall x_1, x_2 \in \mathbb{R}, \quad f(x_1) = f(x_2) \implies x_1 = x_2 \tag{1}$$

(a) Prove that if a function is one-to-one, then it is faithful.

Solution:

- Let f be a one-to-one function. We would like to prove that f is faithful.
- Let g, h be two functions. Assume that $f \circ g = f \circ h$. We want to show that $g = h$.
- Let $x \in \mathbb{R}$. We want to show that $g(x) = h(x)$.
- We know that $(f \circ g)(x) = (f \circ h)(x)$. Thus $f(g(x)) = f(h(x))$. We can regard $g(x)$ and $h(x)$ as two inputs to the function f. Since f is one-to-one, from (1), we conclude that $g(x) = h(x)$.

(b) Prove that if a function is NOT one-to-one, then it is NOT faithful.

Solution:

- Let f be a function which is not one-to-one. Negating (1), this means there exists $x_1, x_2 \in \mathbb{R}$, such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$.
- We wish to show that f is not faithful. In other words, we want to find two functions g and h such that $f \circ g = f \circ h$ but $g \neq h$.
- Define a function g to be the constant function with value x_1 (so $g(u) = x_1$ for all $u \in \mathbb{R}$) and h to be the constant function with value x_2 (so $h(u) = x_2$ for all $u \in \mathbb{R}$).
  - Then for all $u \in \mathbb{R}$, we have

$$f(g(u)) = f(x_1) = f(x_2) = f(h(u))$$

    Thus $f \circ g = f \circ h$.
  - On the other hand $g \neq h$ because, for example, $g(0) = x_1$, $h(0) = x_2$, and $x_1 \neq x_2$.

  That is what we wanted to prove.

2. Given two functions f and g, we say that g is a *quasi-inverse* of f when

> "There exists a non-empty, open interval I contained in the domain of f, such that the restriction of f to I is one-to-one, and g is the inverse of that restriction."

For example, arctan is a quasi-inverse of tan.

Construct a function f that satisfies all the following properties at once:

(a) The domain of f is $\mathbb{R}$.

(b) f is differentiable.

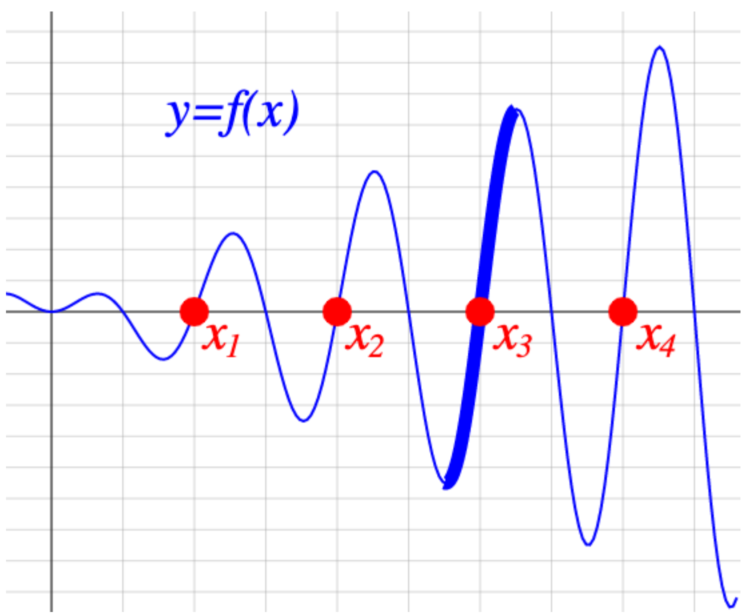
(c) For every $c > 0$ there exists a quasi-inverse g of f such that g is differentiable at 0 and and $0 < g'(0) < c$.

Solution: *In order to satisfy property (c) we need the graph of f to cross the x-axis infinitely many times and the slopes at these points should become arbitrarily large. There are many possible solutions, including* $f(x) = x \sin x$ *and* $f(x) = \sin(x^2)$. *I will use the former.*

Define the function $f(x) = x \sin x$.
The domain of f is $\mathbb{R}$, so (a) is satisfied.
Also, f is differentiable because it is a product of differentiable functions (so (b) is satisfied).
By the product rule:

$$f'(x) = x \cos x + \sin x.$$

Here is the graph of f:



For each positive integer n, define the point $x_n = 2\pi n$. These points are marked in red in the picture. Notice that they all satisfy

$$f(x_n) = 0, \qquad f'(x_n) = x_n = 2\pi n.$$

Each point x_n is inside an open interval I_n such that f is increasing (and hence one-to-one) on I_n. Specifically, $I_n = (2\pi n - \pi/2, 2\pi n + \pi/2)$. Therefore, the restriction of f to I_n is one-to-one. In the graph, I have marked the restriction of f to I_3 as an example.

Let g_n be the inverse of the restriction of f to I_n. By definition, g_n is a quasi-inverse of f. Since $f(x_n) = 0$, we get $g(0) = x_n$. In addition, using the Theorem about derivative of the inverse function (Video 4.4), we get

$$g_n'(0) \;=\; \frac{1}{f'(x_n)} = \frac{1}{2\pi n}.$$

We can now justify condition (c). Let $c > 0$. Pick a positive integer n such that $n > \dfrac{1}{2\pi c}$. This guarantees that $\dfrac{1}{2\pi n} < c$. Then g_n is a quasi-inverse of f such that $0 < g'(0) < c$.

3. In the videos we discussed how to define the number e, how to define exponentials and logarithms, and how to obtain formulas for their derivatives. In this problem you are going to get the same formulas in a different way.

We will assume that exponentials and logarithms are well-defined and continuous, and we will assume the common properties of exponentials and logarithms. However, we assume we still do not know anything about their derivatives – that is the point of this problem!

For this problem we will define the number e as this limit.

$$e = \lim_{x\to 0} (1+x)^{1/x}. \tag{2}$$

We should first prove that this limit exists, but we are going to skip that part. Let's just assume the limit exists and therefore this is a valid way to define the number e. Other than that, during this problem, make a particular effort to explain what you are doing: specifically mention any property, result, or identity that you use in any step.

(a) Prove that $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$.

Solution: We use Theorem 3 in Video 2.16:

- We know
$$e \;=\; \lim_{x\to 0} (1+x)^{1/x}.$$
- The function $\ln$ is continuous at e.
- Therefore
$$\ln e \;=\; \lim_{x\to 0} \ln\left((1+x)^{1/x}\right).$$

Then, using properties of logarithms:

$$1 \;=\; \ln e \;=\; \lim_{x\to 0} \ln\left((1+x)^{1/x}\right) \;=\; \lim_{x\to 0} \frac{\ln(1+x)}{x}$$

(b) Consider the function $L(x) = \ln x$. Prove that L is differentiable everywhere on its domain, and find a formula for its derivative.

Solution: Let $x > 0$. By the definition of derivative

$$L'(x) \;=\; \lim_{h\to 0} \frac{L(x+h) - L(x)}{h}$$

Then, using properties of logarithms:

$$\begin{aligned} L'(x) \;=\; \lim_{h\to 0} \frac{L(x+h) - L(x)}{h} \;&=\; \lim_{h\to 0} \frac{L\left(\frac{x+h}{h}\right)}{h} \;=\; \lim_{h\to 0} \frac{L\left(1+\frac{h}{x}\right)}{h} \\ &=\; \lim_{h\to 0} \left[\frac{1}{x} \cdot \frac{L\left(1+\frac{h}{x}\right)}{\frac{h}{x}} \right] \end{aligned} \tag{3}$$

Notice that x is fixed: it does not depend on h. For the purpose of the limit, it is a constant. It is also non-zero, and thus we could divide by it.

Next I want to use Theorem 2 from Video 2.16. Define the function f by the equation $f(h) = \frac{h}{x}$. Notice that when $h \neq 0$, $f(h) \neq 0$. Therefore, we can indeed use Theorem 2 from Video 2.16 to get:

$$\lim_{h\to 0} \frac{L(1+\frac{h}{x})}{\frac{h}{x}} \;=\; \lim_{h\to 0} \frac{\ln(1+f(h))}{f(h)} \;=\; \lim_{u\to 0} \frac{\ln(1+u)}{u}$$

Returning to (3), we conclude that

$$L'(x) \;=\; \frac{1}{x} \cdot \lim_{u\to 0} \frac{\ln(1+u)}{u} \;=\; \frac{1}{x} \cdot 1 \;=\; \frac{1}{x}.$$

Thus L is differentiable at x and $L'(x) = \frac{1}{x}$.

(c) Consider the function $E(x) = e^x$. Using the fact that E and L are inverses of each other, and now that you have a formula for L', obtain a formula for E'.

Solution:

- **Method 1:** Let $x \in \mathbb{R}$. Let $y = e^x$. Then $y > 0$ and $L(y) = x$. Notice that $L'(y) = \frac{1}{y} \neq 0$.

  We use the theorem about derivatives of inverse functions (Video 4.4):

$$E'(x) = \frac{1}{L'(y)} = y = e^x.$$

- **Method 2:** We know that for every $x \in \mathbb{R}$, $E(x) > 0$ and:

$$L(E(x)) = x$$

  Therefore

$$\frac{d}{dx}[L(E(x))] = \frac{d}{dx}[x]$$

  Using the Chain Rule:

$$L'(E(x)) \cdot E'(x) = 1$$

  Using our formula for L':

$$\frac{1}{E(x)} \cdot E'(x) = 1$$

  And finally

$$E'(x) = E(x) = e^x.$$

Note: Method 2 is simply repeating the derivation of the theorem we used in Method 1.

4. We define arccot as the inverse function of the restriction of cot to $(0, \pi)$.

   (a) *[Do not submit]* Sketch a graph of cot and convince yourself that this is the most reasonable choice to define arccot.

   (b) Obtain and prove a formula for $\frac{d}{dx} \operatorname{arccot} x$.

Solution:
Let $f(x) = \cot x$ and let $g(x) = \operatorname{arccot} x$.
The domain of arccot is $\mathbb{R}$.
We also know that $f'(x) = -\csc^2 x$.

- **Method 1:** Let $x \in \mathbb{R}$ and let $y = \operatorname{arccot} x$. We use the theorem about derivatives of inverse functions (Video 4.4):

$$g'(x) = \frac{1}{f'(y)} = \frac{1}{-\csc^2 y}.$$

  We use the trig identity $1 + \cot^2 y = \csc^2 y$ and therefore:

$$g'(x) = \frac{1}{-(1 + \cot^2 y)} = \frac{-1}{1 + x^2}$$

- **Method 2:** We know that for every $x \in \mathbb{R}$:

$$\cot(g(x)) = x$$

  Therefore

$$\frac{d}{dx}[\cot(g(x))] = \frac{d}{dx}[x]$$

  Using the Chain Rule:

$$-\csc^2(g(x)) \cdot g'(x) = 1$$

  And finally

$$g'(x) = \frac{-1}{\csc^2(g(x))} = \frac{-1}{1 + \cot^2(g(x))} = \frac{-1}{1 + x^2}$$

Note: Method 2 is simply repeating the derivation of the theorem we used in Method 1.

(c) The following "theorem" is not quite true as stated:

Flawed "Theorem": $\operatorname{arccot} x \;=\; \arctan \dfrac{1}{x}$

Fake "Proof":

$$\begin{aligned} \theta &= \operatorname{arccot} x \\ \cot\theta &= x \\ \tan\theta = \frac{1}{\cot\theta} &= \frac{1}{x} \\ \theta &= \arctan\frac{1}{x} \end{aligned}$$

□

Explain the problem with the statement of the theorem and the errors in the proof. Then fix them: correct the statement, and write a correct proof.

Solution:

- The statement is only correct for $x > 0$. For example, if $x = -1$, then $\operatorname{arccot} x = \frac{3\pi}{4}$, but $\arctan \frac{1}{x} = -\frac{\pi}{4}$.
- The flaw in the proof lies in the last step:
  $\tan\theta = \frac{1}{x}$ does not imply $\theta = \arctan\frac{1}{x}$ unless $\theta \in (-\pi/2, \pi/2)$.
  Equivalently, the identity $\arctan(\tan\theta) = \theta$ is only true when $\theta \in (-\pi/2, \pi/2)$.

Correct Theorem:

$$\operatorname{arccot} x \;=\; \begin{cases} \arctan\frac{1}{x} & \text{if } x > 0 \\ \pi \;+\; \arctan\frac{1}{x} & \text{if } x < 0 \end{cases} \tag{4}$$

Proof.

- Let $x \in \mathbb{R}$, assume that $x \neq 0$. Let $\theta = \operatorname{arccot} x$.

  By definition of arccot, we know that $\theta \in (0, \pi)$. Actually, we know more:
  - If $x > 0$, then $\theta \in (0, \pi/2)$
  - If $x < 0$, then $\theta \in (\pi/2, \pi)$
- Then
  $$\cot\theta = \cot(\operatorname{arccot} x) = x$$
  because this identity is true for all possible values of x.
- In addition
  $$\tan\theta = \frac{1}{\cot\theta} = \frac{1}{x}$$
  since $x \neq 0$.

- Remember that arctan is the inverse of the restriction of tan to $(-\pi/2, \pi/2)$. This means that if we find a value $\boxed{\alpha \in (\pi/2, \pi/2)}$ such that

$$\tan\alpha = \frac{1}{x},$$

then we can conclude

$$\alpha = \arctan\frac{1}{x}.$$

We need to break the proof into two cases.

  - **Case 1:** $x > 0$. In this case $\theta \in (0, \pi/2)$. Therefore

$$\tan\theta = \frac{1}{x}$$

does become

$$\theta = \arctan\frac{1}{x}.$$

  - **Case 2:** $x < 0$. In this case $\theta \in (\pi/2, \pi)$. However, we notice that $\theta - \pi \in (-\pi/2, 0)$ and

$$\tan(\theta - \pi) = \tan\theta = \frac{1}{x}$$

Therefore

$$\theta - \pi \;=\; \arctan\frac{1}{x}$$

and

$$\theta \;=\; \pi \;+\; \arctan\frac{1}{x}.$$

Putting both cases together, we have proven (4).

$\square$