# MAT 137Y: Calculus with proofs
# Assignment 2 - Sample solutions

**Question 1** *Sketch the graph of a function $h$ that satisfies all the following properties at once:*

(a) *The domain of $h$ is $\mathbb{R}$.*

(b) $\lim_{x\to 2} h(x) = 0$ *and* $\lim_{x\to 2} h(h(x)) = \infty$.

(c) $\lim_{x\to 4} h(x) = 0$ *and* $\lim_{x\to 4} h(h(x)) = -\infty$.

(d) $\lim_{x\to 0} h(h(x)) = 3$.

(e) $\lim_{x\to -3^+} h(x) = 0$ *and*
$\lim_{x\to -3^+} h(h(x))$ *does not exist, is not $\infty$, and is not $-\infty$.*

**Solution:** The graph is sketched in Figure 1.

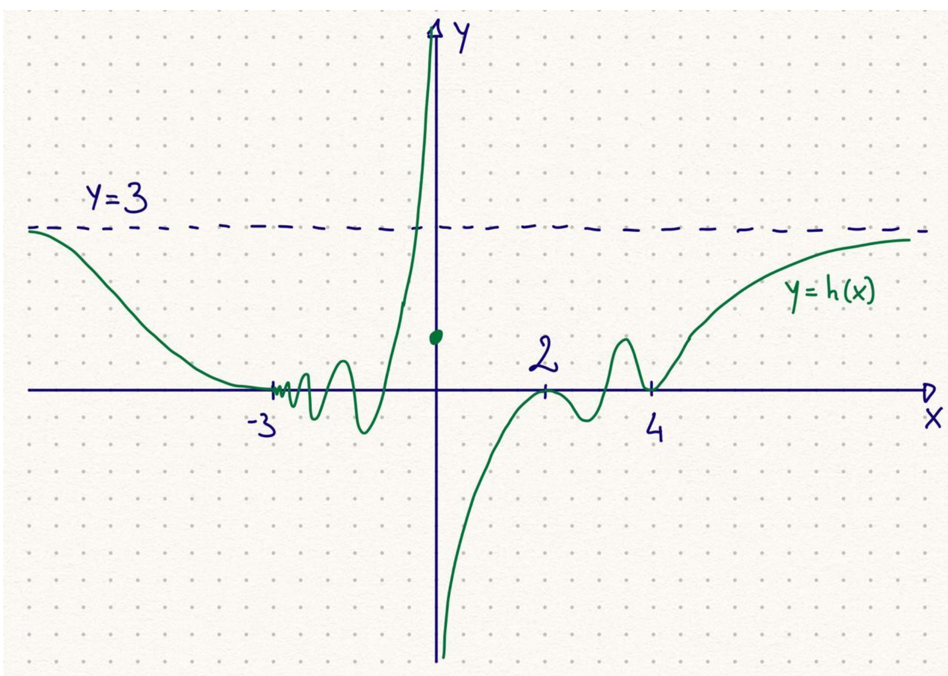


Figure 1: The function $h$ oscillates infinitely many times between positive and negative values when $x$ approaches $-3$ from the right, but the amplitude of these oscillations approaches 0 as $x \to -3^+$.

(a) The function is defined for all real numbers, so the domain is $\mathbb{R}$.

(b) When $x$ approaches 2, the value of $h(x)$ approaches 0 while remaining negative; moreover, for *negative* values of $x$ approaching 0, $h(x)$ approaches $\infty$. Therefore

$$\lim_{x\to 2} h(x) = 0\,, \qquad \lim_{x\to 2} h(h(x)) = \lim_{x\to 0^-} h(x) = \infty\,. \tag{1}$$

(c) When $x$ approaches 4, the value of $h(x)$ approaches 0 while remaining positive; moreover, for *positive* values of $x$ approaching 0, $h(x)$ approaches $-\infty$. Therefore

$$\lim_{x\to 4} h(x) = 0\,, \qquad \lim_{x\to 4} h(h(x)) = \lim_{x\to 0^+} h(x) = -\infty\,. \tag{2}$$

(d) When $x$ approaches $\infty$ or $-\infty$, $h(x)$ approaches 3. I may then write

$$\lim_{x\to 0^-} h(h(x)) = \lim_{x\to\infty} h(x) = 3 \quad \text{and} \quad \lim_{x\to 0^+} h(x) = \lim_{x\to-\infty} h(x) = 3\,. \tag{3}$$

Since the two side limits of $h(h(x))$ for $x$ approaching 0 exist and are both equal to 3, I can conclude that $\lim_{x\to 0} h(h(x)) = 3$.

(e) When $x$ approaches $-3$ (and $x > 3$), $h(x)$ has infinitely many oscillations; however the graph shows that its values are "squeezed" to 0, or in other words $\lim_{x\to -3^+} h(x) = 0$. On the other hand, since $h(x)$ oscillates between small positive and negative numbers, $h(h(x))$ swings between positive and negative numbers, arbitrarily large in absolute value. Therefore, $h(h(x))$ does not approach any real number $L$, and the limit does not exist. The limit is not $\infty$ nor $-\infty$, either, because $h(h(x))$ keeps switching between positive and negative values, rather than growing arbitrarily large in either direction.

**Question 2** *Let $a \in \mathbb{R}$. Let $f$ and $g$ be two functions that are defined, at least, on an interval centered at $a$, except maybe at $a$. Assume that $\lim_{x \to a} f(x)$ does not exist, and that $\lim_{x \to a} g(x)$ does not exist. Based only on this information, can you conclude whether $\lim_{x \to a} [f(x) + g(x)]$ exists or does not exist? Prove it.*

**Solution:**
No, we cannot conclude whether exists or does not exist based on this information.
To prove this, I will give two examples of $a$, $f$, and $g$ as above, so that the limit exists in one case but not in the other.

- Example 1: Let $a = 0$ and $f(x) = g(x) = \frac{1}{x}$.

  $\lim_{x \to 0} f(x)$ DNE, $\quad \lim_{x \to 0} g(x)$ DNE, $\quad$ and $\quad \lim_{x \to 0}(f(x) + g(x))$ DNE.

- Example 2: Let $a = 0$, $f(x) = \frac{1}{x}$, and $g(x) = -\frac{1}{x}$.

  $\lim_{x \to 0} f(x)$ DNE, $\quad \lim_{x \to 0} g(x)$ DNE, $\quad$ but $\quad \lim_{x \to 0}(f(x) + g(x)) = \lim_{x \to 0} 0 = 0.$

**Question 3** *Prove that* $\lim_{x\to 2} x^3 = 8$. *Write a proof directly from the definition of limit, without using any of the limit laws or other theorems.*

**Proof:**
I want to show that

$$\forall \varepsilon > 0,\ \exists \delta > 0, \quad 0 < |x-2| < \delta \implies |x^3 - 8| < \varepsilon$$

- Fix $\varepsilon > 0$.
- Let $\delta = \min\left\{1, \frac{\varepsilon}{20}\right\}$.
- Let $x \in \mathbb{R}$. Assume $0 < |x-2| < \delta$. I will show that $|x^3 - 8| < \varepsilon$.
- I can draw the following conclusions:
  - Since $\delta \leq \frac{\varepsilon}{20}$ and $|x-2| < \delta$, I have that $|x-2| < \frac{\varepsilon}{20}$.
  - Since $\delta \leq 1$ and $2 - \delta < x < 2 + \delta$, it follows that $1 < x < 3$.
    Combining with the triangular inequality:

    $$\left|x^2 + 2x + 4\right| \ \leq \ x^2 + 2x + 4 \ < \ 9 + 6 + 4 \ = \ 19\,.$$

- The two inequalities above imply that

  $$\left|x^3 - 8\right| = |x-2|\left|x^2 + 2x + 4\right| < \frac{\varepsilon}{20} \cdot 19 < \varepsilon\,.$$

  This is what I needed to prove.

$\square$

**Question 4** *Let $f$ and $g$ be two functions with domain $\mathbb{R}$. Let $h = f + g$. Prove that*

$$\begin{array}{lll} \textit{IF} & \lim_{x\to\infty} f(x) = \infty \quad \textit{and} \quad \lim_{x\to\infty} g(x) \textit{ exists,} \\ \textit{THEN} & \lim_{x\to\infty} h(x) = \infty. \end{array}$$

*Write a proof directly from the definition of limit, without using any of the limit laws or other theorems.*

**Proof:**

- I want to prove that $\forall M \in \mathbb{R}, \exists N \in \mathbb{R},$ such that $x > N \implies h(x) > M$
- Fix $M \in \mathbb{R}$. Call $L = \lim_{x\to\infty} g(x)$.
  - Use "$\varepsilon = 1$" in the definition of $\lim_{x\to\infty} g(x) = L$:

  $$\exists N_2 > 0 \text{ such that } x > N_2 \implies |g(x) - L| < 1$$

  - Use "$M_1 = M - L + 1$" as the cut-off in the definition of $\lim_{x\to\infty} f(x) = \infty$:

  $$\exists N_1 > 0 \text{ such that } x > N_1 \implies f(x) > M - L + 1$$

  Take $N = \max\{N_1, N_2\}$
- Let $x \in \mathbb{R}$. Assume $x > N$. I will show that $h(x) > M$.
  I can conclude that
  - $x > N \geq N_1$ so $f(x) > M - L + 1$
  - $x > N \geq N_2$ so $|g(x) - L| < 1$, and in particular $g(x) > L - 1$

  Using both inequalities:

  $$h(x) \;=\; f(x) + g(x) \;>\; (M - L + 1) + (L - 1) = M$$

  which is what I had to prove.

□