A NOTE ON MOD-p LOCAL-GLOBAL COMPATIBILITY VIA SCHOLZE'S FUNCTOR

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Abstract. Let $\mathfrak{m}$ be a Hecke system which appears in the space π of $\mathbb{Q}_p/\mathbb{Z}_p$-automorphic forms (with arbitrary level at some $\mathfrak{p} \mid p$) on certain unitary similitude group over a totally real field F^+ (which splits at $\mathfrak{p}$). Assume that $\mathfrak{m}$ corresponds to an absolutely irreducible mod-p Galois representation $\overline{\sigma}_{\mathfrak{m}}$. We prove using Scholze's functor that the $\mathrm{GL}_n(F^+_{\mathfrak{p}})$-representation $\pi[\mathfrak{m}]$ determines $\overline{\sigma}_{\mathfrak{m}}|_{\mathrm{Gal}_{F^+_{\mathfrak{p}}}}$ up to isomorphism, assuming that $\overline{\sigma}_{\mathfrak{m}}|_{\mathrm{Gal}_{F^+_{\mathfrak{p}}}}$ is multiplicity free and the dual $\pi^\vee_{\mathfrak{m}}$ is flat over the big Hecke algebra. In other words, we remove the semisimple condition on $\overline{\sigma}_{\mathfrak{m}}|_{\mathrm{Gal}_{F^+_{\mathfrak{p}}}}$ in arXiv:2106.10674.

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1. Introduction

Let L be a p-adic field and Gal_L be its absolute Galois group. In [Sch18], Scholze constructs a functor which sends an admissible $\mathbb{Z}_p$-representation π of $\mathrm{GL}_n(L)$ to certain cohomology group $H^i_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_\pi)$ (for $i \geq 0$) equipped with the action of $D^\times \times \mathrm{Gal}_L$ with D the division algebra over L with invariant $\frac{1}{n}$. Then Scholze proves a version of local-global compatibility by applying his functor to a space of $\mathbb{Q}_p/\mathbb{Z}_p$-automorphic forms on a certain inner form of GL_2 over some totally real field. In [Liu21], Liu considers a space π of $\mathbb{Q}_p/\mathbb{Z}_p$-automorphic forms (with arbitrary level at some $\mathfrak{p} \mid p$) on certain unitary similitude group over a totally real field F^+ which splits at $\mathfrak{p}$, and then relates it to the cohomology of certain Kottwitz-Harris-Taylor type Shimura varieties via Scholze's functor. Given a Hecke system $\mathfrak{m}$ which appears only in the middle degree of the cohomology of Shimura varieties and assume that it corresponds to an absolutely irreducible residual Galois representation $\overline{\sigma}_{\mathfrak{m}}$, Liu proves that $H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}})$ is $\sigma_{\mathfrak{m}}|_{\mathrm{Gal}_{F^+_{\mathfrak{p}}}}$-typic where $\sigma_{\mathfrak{m}}$ is the lift of $\overline{\sigma}_{\mathfrak{m}}$ established in [Liu21, Proposition 2.3]. Furthermore, Liu proves that $H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi[\mathfrak{m}]})$ is sufficient to determine the restriction $\overline{\sigma}_{\mathfrak{m}}|_{\mathrm{Gal}_{F^+_{\mathfrak{p}}}}$ assuming that

- the dual $\pi^\vee_{\mathfrak{m}}$ is flat as a module over the big Hecke algebra;
- $\overline{\sigma}_{\mathfrak{m}}|_{\mathrm{Gal}_{F^+_{\mathfrak{p}}}}$ is semisimple and multiplicity free.

In this short note, we completely remove the semisimple assumption in Liu's result. Namely, assuming flatness of $\pi_{\mathfrak{m}}^{\vee}$ and $\overline{\sigma}_{\mathfrak{m}}|_{\mathrm{Gal}_{F_{\mathfrak{p}}^+}}$ being multiplicity free, we prove that $H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi[\mathfrak{m}]})$ determines $\overline{\sigma}_{\mathfrak{m}}|_{\mathrm{Gal}_{F_{\mathfrak{p}}^+}}$ uniquely. We give remarks on our assumptions at the end of this note.

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2. Submodules of a typic module

Let G be a group, R be a commutative ring and ρ_0 be an $R[G]$-module of finite length. We have the following definition generalizing [Sch18, Definition 5.2].

Definition 2.1. Assume that ρ_0 is multiplicity free, namely each Jordan–Hölder factor of ρ_0 appears with multiplicity one. Then ρ_0 admits a decomposition $\rho_0 \cong \bigoplus \widetilde{\rho}$ into its non-zero indecomposable direct summands. We say that an $R[G]$-module V is *ρ_0-typic* if there exists a non-zero R-module $W_{\widetilde{\rho}}$ with trivial G-action for each $\widetilde{\rho}$, such that $V \cong \bigoplus_{\widetilde{\rho}} W_{\widetilde{\rho}} \otimes_R \widetilde{\rho}$.

We assume throughout this section that R is field and ρ_0 is multiplicity free. We fix from now a ρ_0-typic $R[G]$-module V equipped with an R-module $W_{\widetilde{\rho}}$ for each indecomposable direct summand $\widetilde{\rho}$ of ρ_0 as in Definition 2.1. The main result of this section is a criterion (see Proposition 2.8) for certain submodule of V to be ρ_0-typic.

We write Σ for the set of non-zero indecomposable $R[G]$-submodules of ρ_0, equipped with the natural partial order given by inclusion of $R[G]$-submodules. We write $\mathrm{JH}_{R[G]}(\cdot)$ for the set of Jordan–Hölder factors. As ρ_0 is multiplicity free, any $R[G]$-submodule of ρ_0 is uniquely determined by its set of Jordan–Hölder factors, and we clearly have $\#\Sigma \leq 2^{\ell}$ where ℓ is the length of ρ_0. Note that $V \cong \bigoplus_{\widetilde{\rho}} W_{\widetilde{\rho}} \otimes_R \widetilde{\rho}$ forces V to be locally finite, and so is any subquotient of V.

Lemma 2.2. *Let $\rho' \subseteq \rho$ be two elements of Σ. Then the induced map $\mathrm{Hom}_{R[G]}(\rho, V) \to \mathrm{Hom}_{R[G]}(\rho', V)$ is an isomorphism.*

Proof. We first deduce from $\rho, \rho' \in \Sigma$ and $\rho' \subseteq \rho$ that there exists a unique indecomposable direct summand $\widetilde{\rho} \in \Sigma$ of ρ_0 which contains ρ, ρ'. The canonical map $\mathrm{Hom}_{R[G]}(\rho, \widetilde{\rho}) \to \mathrm{Hom}_{R[G]}(\rho', \widetilde{\rho})$ is clearly an isomorphism of R-vector spaces of dimension one. Then the canonical map in question factors through the isomorphisms

$$\mathrm{Hom}_{R[G]}(\rho, V) \cong W_{\widetilde{\rho}} \otimes_R \mathrm{Hom}_{R[G]}(\rho, \widetilde{\rho}) \xrightarrow{\sim} W_{\widetilde{\rho}} \otimes_R \mathrm{Hom}_{R[G]}(\rho', \widetilde{\rho}) \cong \mathrm{Hom}_{R[G]}(\rho', V).$$

□

Lemma 2.3. *Let $V' \subseteq V$ be an $R[G]$-submodule with $\mathrm{cosoc}_{R[G]}(V')$ being irreducible. Then there exists $\rho \in \Sigma$ such that $V' \cong \rho$.*

Proof. We write $\tau \overset{\text{def}}{=} \mathrm{cosoc}_{R[G]}(V') \in \mathrm{JH}_{R[G]}(V) = \mathrm{JH}_{R[G]}(\rho_0)$. There exists a unique $\rho \subseteq \widetilde{\rho} \subseteq \rho_0$ such that $\widetilde{\rho}$ is an indecomposable direct summand of ρ_0 and $\mathrm{cosoc}_{R[G]}(\rho) \cong \tau$. As $V/W_{\widetilde{\rho}} \otimes_R \rho$ does not have τ as Jordan–Hölder factor, we may assume without loss of generality that $\rho = \widetilde{\rho} = \rho_0$. Then the key observation is that

$$\mathrm{Hom}_{R[G]}(\widetilde{\rho}, V) \cong W_{\widetilde{\rho}} \otimes_R \mathrm{End}_{R[G]}(\widetilde{\rho}) \xrightarrow{\sim} W_{\widetilde{\rho}} \otimes_R \mathrm{End}_{R[G]}(\tau) \cong \mathrm{Hom}_{R[G]}(\tau, W_{\widetilde{\rho}} \otimes_R \tau). \tag{2.4}$$

The $R[G]$-submodule $V' \subseteq V$ determines an embedding $\tau \hookrightarrow W_{\widetilde{\rho}} \otimes_R \tau$ and thus (by (2.4) an embedding $f : \widetilde{\rho} \hookrightarrow V$. We write $\mathrm{rad}(V')$ for the kernel of $V' \twoheadrightarrow \mathrm{cosoc}_{R[G]}(V')$. As the canonical map $V'/\mathrm{rad}(V') \to V/(\mathrm{im}(f) + \mathrm{rad}(V'))$ is zero by the choice of f, so is the map $V' \to V/\mathrm{im}(f)$, which implies that $V' \subseteq \mathrm{im}(f)$. This inclusion must be an equality as both $R[G]$-modules share the same cosocle. $\square$

Lemma 2.5. *Let $V' \subseteq V$ be an $R[G]$-submodule. If V' is multiplicity free, then there exists an embedding $V' \hookrightarrow \rho_0$.*

Proof. By writing V' as direct sum of its indecomposable direct summands, it suffices to assume that V' is indecomposable and find $\rho \in \Sigma$ such that $V' \cong \rho$. As $\mathrm{JH}_{R[G]}(V) = \mathrm{JH}_{R[G]}(\rho_0)$ is finite, we deduce that V' has finite length. By writing each $W_{\widetilde{\rho}} = \varinjlim_k W_{\widetilde{\rho},k}$ as a direct limit of its finite dimensional subspaces and then using the fact that V' has finite length, we may assume without loss of generality that $W_{\widetilde{\rho}}$ is finite dimensional for each indecomposable direct summand $\widetilde{\rho}$ of ρ_0. We write $\mathrm{soc}_{R[G]}V' \cong \bigoplus_{t=1}^{s} \tau_t$, then each $\tau_t \subseteq V' \subseteq V$ determines a unique $\widetilde{\rho}_t$ containing τ_t as well as an element $f_t \in \mathrm{Hom}_{R[G]}(\tau_t, V) \cong \mathrm{Hom}_{R[G]}(\widetilde{\rho}_t, V) \cong W_{\widetilde{\rho}_t}$. As V' is indecomposable and $W_{\widetilde{\rho}} \otimes_R \widetilde{\rho}$ do not share common Jordan–Hölder factor for different $\widetilde{\rho}$, we deduce that all $\widetilde{\rho}_t$ equal the same $\widetilde{\rho}$. As it is harmless to replace R with its algebraic closure which is an infinite field, there exists $\ell : W_{\widetilde{\rho}} \to R$ such that $\ell(f_t) \neq 0$ for each $1 \leq t \leq s$. Hence, $\ell \otimes_R \widetilde{\rho} : W_{\widetilde{\rho}} \otimes_R \widetilde{\rho} \to \widetilde{\rho}$ restricted to an injection on $\mathrm{soc}_{R[G]}(V')$, and thus an injection on V' as well. We conclude by the observation that any indecomposable $R[G]$-submodule of $\widetilde{\rho}$ is in Σ. $\square$

Lemma 2.6. *Let $V' \subseteq V$ be an $R[G]$-submodule. Assume that*

- $\mathrm{JH}_{R[G]}(V') = \mathrm{JH}_{R[G]}(\rho_0)$*; and*
- *for each indecomposable direct summand $\widetilde{\rho}$ of ρ_0 and each embedding $f : \widetilde{\rho} \hookrightarrow V$, we have either $\mathrm{im}(f) \subseteq V'$ or $\mathrm{im}(f) \cap V' = 0$.*

Then V' is ρ_0-typic.

Proof. Recall that we have $V \cong \bigoplus_{\widetilde{\rho}} W_{\widetilde{\rho}} \otimes_R \widetilde{\rho}$ and the identification $W_{\widetilde{\rho}} \cong \mathrm{Hom}_{R[G]}(\widetilde{\rho}, V)$ for each indecomposable direct summand $\widetilde{\rho}$ of ρ_0. We write $W'_{\widetilde{\rho}} \subseteq W_{\widetilde{\rho}}$ for the subspace of all morphisms $f : \widetilde{\rho} \to V$ satisfying $\mathrm{im}(f) \subseteq V'$. We claim that the natural map

$$\bigoplus_{\widetilde{\rho}} W'_{\widetilde{\rho}} \otimes_R \widetilde{\rho} \to V' \tag{2.7}$$

is an isomorphism. The compatibility with $V \cong \bigoplus_{\widetilde{\rho}} W_{\widetilde{\rho}} \otimes_R \widetilde{\rho}$ forces (2.7) to be injective. As V' is sum of its $R[G]$-submodules with irreducible cosocle, it suffices to prove that each such $R[G]$-submodule V'' of V' is contained in the image of (2.7). In fact, it follows from Lemma 2.3 that there exists $\rho \in \Sigma$ such that $V'' \cong \rho$. Hence, we deduce from Lemma 2.2 that there exists an indecomposable direct summand $\widetilde{\rho}$ of ρ_0 as well as $f \in \mathrm{Hom}_{R[G]}(\widetilde{\rho}, V)$ such that $\widetilde{\rho} \supseteq \rho$ and $V'' \subseteq \mathrm{im}(f)$. As $\mathrm{im}(f)$ is multiplicity free, it embeds into ρ_0 by Lemma 2.5, and thus embeds into $\widetilde{\rho}$ by checking Jordan–Hölder factors. This forces $\widetilde{\rho} \cong \ker(f) \oplus \mathrm{im}(f)$ and thus $\ker(f) = 0$ as $\widetilde{\rho}$ is indecomposable. In other words, f is an embedding with $0 \neq V'' \subseteq \mathrm{im}(f) \cap V'$, which together with our assumption implies that $\mathrm{im}(f) \subseteq V'$. Hence, $\mathrm{im}(f)$ is contained in the image of (2.7), and so is V''. Note that $\mathrm{JH}_{R[G]}(V') = \mathrm{JH}_{R[G]}(\rho_0)$ forces $W'_{\widetilde{\rho}} \neq 0$ for each indecomposable direct summand $\widetilde{\rho}$ of ρ_0. The proof is thus completed. $\square$

Proposition 2.8. *Let ρ_0 be a multiplicity free $R[G]$-module of finite length. Let V be a ρ_0-typic $R[G]$-module with a sequence of sub $R[G]$-modules $V_1 \subseteq V_2 \subseteq \cdots$ satisfying the following conditions*

- $V = \bigcup_{r \geq 1} V_r$*; and*
- *for each $r \geq 1$, there exists an embedding $V_{r+1}/V_r \hookrightarrow V_1^{\oplus s_r}$ for some $s_r \geq 1$.*

Then V_1 is ρ_0-typic. In particular, V_1 determines ρ_0 up to isomorphism.

Proof. Our assumption clearly implies that $\mathrm{JH}_{R[G]}(V_1) = \mathrm{JH}_{R[G]}(\rho_0)$. Let $\widetilde{\rho}$ be an indecomposable direct summand of ρ_0 and $f : \widetilde{\rho} \hookrightarrow V$ be an embedding. According to Lemma 2.6, it suffices to show that either $\mathrm{im}(f) \subseteq V_1$ or $\mathrm{im}(f) \cap V_1 = 0$. We set $V_{f,0} \stackrel{\mathrm{def}}{=} 0 \subseteq \mathrm{im}(f)$ and $V_{f,r} \stackrel{\mathrm{def}}{=} \mathrm{im}(f) \cap V_r$ for each $r \geq 1$. Our assumption on $\{V_r\}_{r\geq 1}$ implies that $\{V_{f,r}\}_{r\geq 0}$ is an increasing and exhaustive filtration on $\mathrm{im}(f)$. The inclusion $\mathrm{im}(f) \subseteq V$ induces a natural embedding

$$V_{f,r+1}/V_{f,r} \hookrightarrow V_{r+1}/V_r \hookrightarrow V_1^{\oplus s_r} \hookrightarrow V^{\oplus s_r}.$$

As $V^{\oplus s_r}$ is ρ_0-typic and $V_{f,r+1}/V_{f,r}$ is multiplicity free, we deduce from Lemma 2.5 that $V_{f,r+1}/V_{f,r}$ embeds into ρ_0, and actually embeds into $\widetilde{\rho}$ by checking Jordan–Hölder factors. As $V_{f,r+1}/V_{f,r}$ embeds into $\widetilde{\rho} \cong \mathrm{im}(f)$ for each $r \geq 0$, we deduce that

$$\widetilde{\rho} \cong \mathrm{im}(f) \cong \bigoplus_{r\geq 0} V_{r+1,f}/V_{r,f}.$$

However, $\widetilde{\rho}$ is indecomposable, and thus there exists a unique $r_f \geq 0$ such that $V_{r_f+1,f}/V_{r_f,f} \cong \widetilde{\rho}$ and $V_{r+1,f} = V_{r,f}$ for all $r \neq r_f$. In particular, we have $\mathrm{im}(f) \subseteq V_1$ if $r_f = 0$, and $\mathrm{im}(f) \cap V_1 = 0$ if $r_f \geq 1$. As the ρ_0-typic $R[G]$-module V_1 determines the isomorphism class of each indecomposable direct summand $\widetilde{\rho}$ of ρ_0 (by considering all possible indecomposable direct summands of V_1), it clearly determines ρ_0 up to isomorphism. The proof is thus finished. □

We also have the following more general result on capturing ρ_0 from an $R[G]$-submodule $V' \subseteq V$ without knowing that V' is ρ_0-typic.

Proposition 2.9. *Let $V' \subseteq V$ be an $R[G]$-submodule. Assume that $\mathrm{JH}_{R[G]}(V') = \mathrm{JH}_{R[G]}(\rho_0)$. Then V' determines ρ_0 up to isomorphism.*

Proof. As ρ_0 is multiplicity free, for each $\tau \in \mathrm{JH}_{R[G]}(\rho_0)$, there exists a unique $R[G]$-submodule $\rho_\tau \subseteq \rho_0$ such that $\mathrm{cosoc}_{R[G]}(\rho_\tau) \cong \tau$. It follows from Lemma 2.3 that, for each $\tau \in \mathrm{JH}_{R[G]}(\rho_0)$, any $R[G]$-submodule $V'' \subseteq V'$ satisfying $\mathrm{cosoc}_{R[G]}(V'') \cong \tau$ must also satisfy $V'' \cong \rho_\tau$. Consequently, we deduce from $\mathrm{JH}_{R[G]}(V') = \mathrm{JH}_{R[G]}(\rho_0)$ that V' determines the set of isomorphism classes $\{[\rho_\tau]\}_{\tau\in\mathrm{JH}_{R[G]}(\rho_0)}$. It suffices to show that $\{[\rho_\tau]\}_{\tau\in\mathrm{JH}_{R[G]}(\rho_0)}$ determines ρ_0 up to isomorphism. We prove that $\{[\rho_\tau]\}_{\tau\in\mathrm{JH}_{R[G]}(\rho_0)}$ determines ρ up to isomorphism for each $R[G]$-submodule $\rho \subseteq \rho_0$ by induction on the length of ρ. Let $\rho' \subseteq \rho$ be two $R[G]$-submodules of ρ_0 with $\rho/\rho' \cong \tau_0$ for some $\tau_0 \in \mathrm{JH}_{R[G]}(\rho_0)$. Assume first that $\{[\rho_\tau]\}_{\tau\in\mathrm{JH}_{R[G]}(\rho_0)}$ determines ρ' and $\rho' \cap \rho_{\tau_0}$ up to isomorphism. We choose two embeddings $f_1 : \rho' \cap \rho_{\tau_0} \to \rho'$ and $f_2 : \rho' \cap \rho_{\tau_0} \to \rho_{\tau_0}$ and note that the choice of the pair f_1, f_2 is unique up to automorphims of ρ', ρ_{τ_0} and $\rho' \cap \rho_{\tau_0}$. Hence, the isomorphism class of the amalgamate sum $\rho' \oplus_{\rho'\cap\rho_{\tau_0}} \rho_{\tau_0}$ does not depend on the choice of f_1, f_2. It is obvious that $\rho \cong \rho' \oplus_{\rho'\cap\rho_{\tau_0}} \rho_{\tau_0}$ and thus $\{[\rho_\tau]\}_{\tau\in\mathrm{JH}_{R[G]}(\rho_0)}$ determines ρ up to isomorphism. The proof is thus finished by an induction on length. □

3. Application to mod-p local global compatibility via Scholze's functor

We first establish the global setup for the study of the cohomology of the relevant Shimura varieties.

Fix an integer $n > 2$ and a prime p. Let F be a CM field which is a quadratic extension of its maximal totally real subfield F^+. We write c for the unique non-trivial element of $\mathrm{Gal}(F/F^+)$. Let B be the division algebra over F of dimension n^2 as chosen in Section 0.1 of [BZ99] (thus equipped with a certain involution $b \mapsto b^*$ on it) and $\widetilde{G}$ be the algebraic group over F^+ whose group of R-points for any F^+-algebra R is given by

$$\widetilde{G}(R) \overset{\text{def}}{=} \{(g, \lambda) \in (B^{\text{op}} \otimes_{F^+} R)^\times \times R^\times \,|\, gg^* = \lambda\}.$$

We assume that v splits in F for each finite place v of F^+ dividing p. We fix a finite place $\mathfrak{p}$ of F^+ (resp. $\mathfrak{q}$ of F) such that $\mathfrak{p} = \mathfrak{q}\mathfrak{q}^c$. Then our choice of the division algebra B above implies that $B_{\mathfrak{q}}$ is a division algebra over $F_{\mathfrak{q}}$ of invariant $\frac{1}{n}$.

Let $G \overset{\text{def}}{=} \mathrm{Res}_{F^+/\mathbb{Q}}(\widetilde{G})$ be the Weil restriction of scalars. Let $\mathbb{S} \overset{\text{def}}{=} \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$ be the Deligne torus and h be a morphism

$$h : \mathbb{S} \to G_{\mathbb{R}}$$

such that h defines on $W_{\mathbb{R}}$ a Hodge structure of type $(1,0), (0,1)$ and such that $\psi(w_1, h(i)w_2)$ is a symmetric positive definite bilinear form on $W_{\mathbb{R}}$. Note that h is unique up to $G(\mathbb{R})$-conjugacy and we let X denote the $G(\mathbb{R})$-conjugacy class of h. Then (G, X) defines a Shimura datum and for sufficiently small compact open subgroups $U \subseteq G(\mathbb{A}_{\mathbb{Q}}^\infty)$ we have a projective system of Shimura varieties Sh_U over its reflex field, which can be identified with F in a canonical way. We will write $\mathrm{Sh}_{KU^{\mathfrak{p}}}$ instead of $\mathrm{Sh}_{K \times (F_{\mathfrak{p}}^+)^\times \times U^{\mathfrak{p}}}$ for the Shimura variety associated with $U = K \times (F_{\mathfrak{p}}^+)^\times \times U^{\mathfrak{p}}$ for $K \subseteq (B_{\mathfrak{q}}^{\text{op}})^\times$ compact open and $U^{\mathfrak{p}} \subseteq \widetilde{G}(\mathbb{A}_{F^+}^{\infty,\mathfrak{p}})$ (sufficiently small) compact open.

We fix a tame level, i.e. a compact open subgroup $U^{\mathfrak{p}} = \prod_{v \neq \mathfrak{p}} U_v$ of $\widetilde{G}(\mathbb{A}_{F^+}^{\infty,\mathfrak{p}})$ and let $\mathcal{P}$ denote the set of finite places v of F^+ such that

- $v \nmid p$;
- v splits in F;
- $\widetilde{G}(F_v^+) \cong \mathrm{GL}_n(F_v^+) \times (F_v^+)^\times$ and U_v is a maximal compact open subgroup of $\widetilde{G}(F_v^+)$.

Consider the abstract Hecke algebra

$$\mathbb{T}_{\mathcal{P}} \overset{\text{def}}{=} \mathbb{Z}[T_w^{(j)},\ T_{w^c}^{(j)} : v = ww^c \in \mathcal{P}, 1 \leq j \leq n]$$

where $T_w^{(j)}$ is the Hecke operator corresponding to the double coset

$$\Big[\mathrm{GL}_n(\mathcal{O}_{F_w}) \begin{pmatrix} \varpi_w 1_j & 0 \\ 0 & 1_{n-j} \end{pmatrix} \mathrm{GL}_n(\mathcal{O}_{F_w})\Big].$$

Here ϖ_w is a uniformizer of the local field F_w. Then the Hecke algebra $\mathbb{T}_{\mathcal{P}}$ acts on $H^i(\mathrm{Sh}_{KU^{\mathfrak{p}},\mathbb{C}}, \mathbb{Z})$ for all compact open $K \subseteq (B_{\mathfrak{q}}^{\text{op}})^\times$.

Let $\mathbb{F}$ be a finite extension of $\mathbb{F}_p$ and $\overline{\sigma} : \mathrm{Gal}_F \to \mathrm{GL}_n(\mathbb{F})$ an n-dimensional absolutely irreducible Galois representation which is unramified at each place of F dividing some $v \in \mathcal{P}$. Hence, we can associate a maximal ideal $\mathfrak{m} \subseteq \mathbb{T}_{\mathcal{P}}$ with $\overline{\sigma}$ (cf. the paragraph before Condition 2.1 of [Liu21]).

We assume the following condition from now on:

Condition 3.1. *For each $K \subseteq (B_{\mathfrak{q}}^{\text{op}})^\times$ compact open,*

$$H^i(\mathrm{Sh}_{KU^{\mathfrak{p}},\mathbb{C}}, \mathbb{Z})_{\mathfrak{m}} \neq 0$$

only if $i = n-1$.

Let $\mathbb{T}(KU^{\mathfrak{p}})$ be the image of $\mathbb{T}_{\mathcal{P}}$ in $\mathrm{End}(H^{n-1}(\mathrm{Sh}_{KU^{\mathfrak{p}},\mathbb{C}}, \mathbb{Z}))$ and $\mathbb{T}(KU^{\mathfrak{p}})_{\mathfrak{m}}$ be its $\mathfrak{m}$-adic completion. We also consider the big Hecke algebra

$$\mathbb{T}(U^{\mathfrak{p}})_{\mathfrak{m}} \overset{\text{def}}{=} \varprojlim_{U} \mathbb{T}(KU^{\mathfrak{p}})_{\mathfrak{m}}$$

which is a complete Noetherian local ring with finite residue field. Let $\sigma : \mathrm{Gal}_F \to \mathrm{GL}_n(\mathbb{T}(U^{\mathfrak{p}})_{\mathfrak{m}})$ be the unique (up to conjugation) lift of $\overline{\sigma}$ characterized by [Liu21, Proposition 2.3].

Let G' be the inner form of G over F^+ such that $G'(F^+ \otimes_{\mathbb{Q}} \mathbb{R})$ is compact modulo center, $G'(\mathbb{A}_{F^+}^{\infty,\mathfrak{p}}) = \widetilde{G}(\mathbb{A}_{F^+}^{\infty,\mathfrak{p}})$, and $G'(F_{\mathfrak{p}}^+) \cong \mathrm{GL}_n(F_{\mathfrak{p}}^+) \times (F_{\mathfrak{p}}^+)^{\times}$. Let $\pi_{U^{\mathfrak{p}}}$ be the admissible $\mathbb{Z}_p$-representation of $\mathrm{GL}_n(F_{\mathfrak{p}}^+)$ given by the space of continuous functions

$$\pi_{U^{\mathfrak{p}}} \stackrel{\text{def}}{=} C^0(G'(F^+)\backslash G'(\mathbb{A}_{F^+}^{\infty})/((F_{\mathfrak{p}}^+)^{\times} \times U^{\mathfrak{p}}), \mathbb{Q}_p/\mathbb{Z}_p).$$

By Corollary 6.7 of [Liu21] the natural action of $\mathbb{T}_{\mathcal{P}}$ on

$$\pi_{\mathfrak{m}} \stackrel{\text{def}}{=} \pi_{U^{\mathfrak{p}},\mathfrak{m}} = C^0(G'(F^+)\backslash G'(\mathbb{A}_{F^+}^{\infty})/((F_{\mathfrak{p}}^+)^{\times} \times U^{\mathfrak{p}}), \mathbb{Q}_p/\mathbb{Z}_p)_{\mathfrak{m}}$$

extends to a continuous action of $\mathbb{T}(U^{\mathfrak{p}})_{\mathfrak{m}}$.

In [Sch18, Section 3,4], for each p-adic field L and $i \geq 0$, Scholze defines a functor which sends an admissible smooth $\mathbb{Z}_p[\mathrm{GL}_n(L)]$-module π (cf. [Sch18, Definition 4.1]) to

$$H^i_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi})$$

with a natural action by $D^{\times} \times \mathrm{Gal}_L$ (see [Sch18, Proposition 3.1] for the definition of the sheaf $\mathcal{F}_{\pi}$). Here D is the central division algebra over L of invariant $\frac{1}{n}$. Although not used in the rest of this note, we remark that the $D^{\times}$-representation $H^i_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi})$ is known to be admissible by [Sch18, Theorem 4.4].

We have the following typicity result from [Liu21, Corollary 7.1]:

Proposition 3.2. *Assume that Condition 3.1 holds for $\mathfrak{m}$. Then $H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}})$ is a $\sigma|_{\mathrm{Gal}_{F_{\mathfrak{p}}^+}}$-typic $\mathbb{T}(U^{\mathfrak{p}})_{\mathfrak{m}}[\mathrm{Gal}_{F_{\mathfrak{p}}^+}]$-module. In particular, $H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}})[\mathfrak{m}]$ is a $\overline{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}^+}}$-typic $\mathbb{F}[\mathrm{Gal}_{F_{\mathfrak{p}}^+}]$-module.*

Now we need another condition to apply various results we need from [Liu21, Section 7].

Condition 3.3. *The dual $\pi_{\mathfrak{m}}^{\vee} = \mathrm{Hom}_{\mathbb{Z}_p}(\pi_{\mathfrak{m}}, \mathbb{Q}_p/\mathbb{Z}_p)$ is flat as a module over $\mathbb{T}(U^{\mathfrak{p}})_{\mathfrak{m}}$.*

Under the Condition 3.3, there exists for each $r \geq 1$ a short exact sequence (see [Liu21, Lemma 7.5])

$$0 \to \pi_{\mathfrak{m}}[\mathfrak{m}^r] \to \pi_{\mathfrak{m}}[\mathfrak{m}^{r+1}] \to (\pi_{\mathfrak{m}}[\mathfrak{m}])^{\oplus s_r} \to 0 \tag{3.4}$$

where $s_r \geq 1$ is a positive integer. Applying the functor $H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{-})$, we obtain an exact sequence on cohomology groups (for each $r \geq 1$)

$$0 \to H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}[\mathfrak{m}^r]}) \to H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}[\mathfrak{m}^{r+1}]}) \to \bigoplus_{s_r} H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}[\mathfrak{m}]}). \tag{3.5}$$

The injectivity on the left hand side of (3.5) follows from [Liu21, Lemma 7.9]. Now we set

$$V_r \stackrel{\text{def}}{=} H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}[\mathfrak{m}^r]})[\mathfrak{m}]$$

for each $r \geq 1$ and note that $V_1 = H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}[\mathfrak{m}]})$. Taking $\mathfrak{m}$-torsion on the sequence (3.5) yields

$$0 \to V_r \to V_{r+1} \to (V_1)^{\oplus s_r}. \tag{3.6}$$

To apply our results in Section 2, we further assume that

Condition 3.7. *The $\mathbb{F}[\mathrm{Gal}_{F_{\mathfrak{p}}^+}]$-module $\overline{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}^+}}$ is multiplicity free.*

Then we take

- $R = \mathbb{F}$;
- $G = \mathrm{Gal}_{F_{\mathfrak{p}}^+}$;
- $\rho_0 = \overline{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}^+}}$; and
- $V = H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}})[\mathfrak{m}]$.

It follows from [Liu21, Lemma 7.7] that $V = \varinjlim_r V_r$, which together with (3.6) fulfills all the conditions of Proposition 2.8. Therefore we deduce:

Theorem 3.8. *Assume that Condition 3.1, Condition 3.3 and Condition 3.7 hold for* $\mathfrak{m}$. *Then*

- $H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}[\mathfrak{m}]})$ *is* $\overline{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}^+}}$*-typic; and in particular*
- $H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}[\mathfrak{m}]})$ *determines* $\overline{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}^+}}$ *up to isomorphism.*

Remark 3.9. As mentioned in Remark 7.6 of [Liu21], Condition 3.3 can be reduced to a result on the Gelfand–Kirillov dimension of $\pi_{\mathfrak{m}}[\mathfrak{m}]$ using Theorem B of [GN]. Under standard Taylor–Wiles conditions and mild genericity on $\overline{\sigma}|^{\mathrm{ss}}_{\mathrm{Gal}_{F_{\mathfrak{p}}^+}}$, the Gelfand-Kirillov dimension of $\pi_{\mathfrak{m}}[\mathfrak{m}]$ is known when $n = 2$ and $F_{\mathfrak{p}}^+$ is unramified due to [BHHMS20] and [HW20]. However, under the same assumption in [BHHMS20] and [HW20], we already know that $\pi_{\mathfrak{m}}[\mathfrak{m}]$ determines $\overline{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}^+}}$ thanks to [BD14], whose proof is significantly simpler than that of [BHHMS20] and [HW20]. One expects the determination of the Gelfand-Kirillov dimension of $\pi_{\mathfrak{m}}[\mathfrak{m}]$ for general n and $F_{\mathfrak{p}}^+$ to be a difficult problem, and so is the flatness in Condition 3.3. Concerning the alternative approach generalizing [BD14] (without assuming Condition 3.3), [LLMPQ] shows that $\pi_{\mathfrak{m}}[\mathfrak{m}]$ determines $\overline{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}^+}}$ when $F_{\mathfrak{p}}^+$ is unramified and $\overline{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}^+}}$ is Fontaine–Laffaille (assuming standard Taylor–Wiles conditions and mild genericity on $\overline{\sigma}|^{\mathrm{ss}}_{\mathrm{Gal}_{F_{\mathfrak{p}}^+}}$).

Remark 3.10. We write $V_1^\star$ for the image of

$$H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}[\mathfrak{m}]}) \to H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}})[\mathfrak{m}] \tag{3.11}$$

and consider the following condition

$$\mathrm{JH}_{\mathbb{F}[\mathrm{Gal}_{F_{\mathfrak{p}}^+}]}(V_1^\star) = \mathrm{JH}_{\mathbb{F}[\mathrm{Gal}_{F_{\mathfrak{p}}^+}]}(\overline{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}^+}}). \tag{3.12}$$

Then we have the following observations.

(i) Assuming Condition 3.1, Condition 3.7 and (3.12), we can deduce that $V_1^\star$ determines $\overline{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}^+}}$ up to isomorphism from Proposition 2.9. However, one needs to be careful that $V_1^\star$ *a priori* depends on the structure of $\pi_{\mathfrak{m}}$ rather than $\pi_{\mathfrak{m}}[\mathfrak{m}]$, and thus the result above for $V_1^\star$ is not sufficient to imply that $\pi_{\mathfrak{m}}[\mathfrak{m}]$ determines $\overline{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}^+}}$ up to isomorphism. Suppose that (3.11) is indeed an embedding, then $V_1^\star \cong H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}[\mathfrak{m}]})$ and thus Proposition 2.9 gives an alternative approach to Theorem 3.8 without showing that $H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}[\mathfrak{m}]})$ is $\overline{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}^+}}$-typic.

(ii) There are examples in [HW21] (with $n = 2$) such that
- Condition 3.1, Condition 3.7 and (3.12) hold but Condition 3.3 fails;
- the map (3.11) is an embedding; and
- $V_1^\star \cong H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}[\mathfrak{m}]})$ is not $\overline{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}^+}}$-typic.

(iii) When $n \geq 3$, we do not know how to prove (3.12) or to prove that (3.11) is an embedding without using Condition 3.3.

Remark 3.13. Let $\overline{\chi}_1, \overline{\chi}_2$ be two distinct characters $\mathrm{Gal}_{F_{\mathfrak{p}}^+} \to \mathbb{F}^\times$, $r_1, r_2 \geq 2$ be two integers and assume that $\overline{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}^+}} \cong \overline{\chi}_1^{\oplus r_1} \oplus \overline{\chi}_2^{\oplus r_2}$ which is not multiplicity free. Then for each infinite dimensional $\mathbb{F}$-space W with trivial $\mathrm{Gal}_{F_{\mathfrak{p}}^+}$-action, the isomorphism class of the $\mathbb{F}[\mathrm{Gal}_{F_{\mathfrak{p}}^+}]$-module $W \otimes_{\mathbb{F}} \overline{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}^+}}$ does not depend on the choice of r_1 and r_2. In particular, the $\mathbb{F}[\mathrm{Gal}_{F_{\mathfrak{p}}^+}]$-module $H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}})[\mathfrak{m}]$ cannot determine r_1 and r_2. In order to prove $\pi_{\mathfrak{m}}[\mathfrak{m}]$ determines $\overline{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}^+}}$ for such $\overline{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}^+}}$, we expect the $D^\times$-action on $H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}[\mathfrak{m}]})$ to be essential.

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