ON MOD p LOCAL-GLOBAL COMPATIBILITY FOR $\mathrm{GL}_n(\mathbf{Q}_p)$ IN THE ORDINARY CASE

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Abstract. Let p be a prime number, $n > 2$ an integer, and F a CM field in which p splits completely. Assume that a continuous automorphic Galois representation $\overline{r} : \mathrm{Gal}(\overline{\mathbf{Q}}/F) \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ is upper-triangular and satisfies certain genericity conditions at a place w above p, and that every subquotient of $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$ of dimension > 2 is Fontaine–Laffaille generic. In this paper, we show that the isomorphism class of $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$ is determined by $\mathrm{GL}_n(F_w)$-action on a space of mod p algebraic automorphic forms cut out by the maximal ideal of a Hecke algebra associated to $\overline{r}$. In particular, we show that the wildly ramified part of $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$ is determined by the action of Jacobi sum operators (seen as elements of $\mathbf{F}_p[\mathrm{GL}_n(\mathbf{F}_p)]$) on this space.

Contents

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1. Introduction

It is believed that one can attach a smooth $\overline{\mathbf{F}}_p$-representation of $\mathrm{GL}_n(K)$ (or a packet of such representations) to a continuous Galois representation $\mathrm{Gal}(\overline{\mathbf{Q}}_p/K) \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ in a natural way, that is called mod p Langlands program for $\mathrm{GL}_n(K)$, where K is a finite extension of $\mathbf{Q}_p$. This conjecture is well-understood for $\mathrm{GL}_2(\mathbf{Q}_p)$ ([BL94], [Ber10], [Bre03a], [Bre03b], [Col10], [Pas13], [CDP], [Eme]). Beyond the $\mathrm{GL}_2(\mathbf{Q}_p)$-case, for instance $\mathrm{GL}_n(\mathbf{Q}_p)$ for $n > 2$ or even $\mathrm{GL}_2(\mathbf{Q}_{p^f})$ for an unramified extension $\mathbf{Q}_{p^f}$ of $\mathbf{Q}_p$ of degree $f > 1$, the situation is still quite far from being understood. One of the main difficulties is that there is no classification of such smooth representations of $\mathrm{GL}_n(K)$ unless $K = \mathbf{Q}_p$ and $n = 2$: in particular, we barely understand the supercuspidal representations. Some of the difficulties in classifying the supercuspidal representations are illustrated in [BP12], [Hu10] and [Schr15].

Let F be a CM field in which p is unramified, and $\overline{r} : \mathrm{Gal}(\overline{\mathbf{Q}}/F) \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ an automorphic Galois representation. Although there is no precise statement of mod p Langlands correspondence for $\mathrm{GL}_n(K)$ unless $K = \mathbf{Q}_p$ and $n = 2$, one can define smooth representations $\Pi(\overline{r})$ of $\mathrm{GL}_n(F_w)$ in the spaces of mod p automorphic forms on a definite unitary group cut out by the maximal ideal of a Hecke algebra associated to $\overline{r}$, where w is a place of F above p. A precise definition of $\Pi(\overline{r})$ when p splits completely in F, which is our context, will be given in Section 1.4. (See also Section 5.6.) One wishes that $\Pi(\overline{r})$ is a candidate on the automorphic side corresponding to $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$ for a mod p Langlands correspondence in the spirit of Emerton [Eme]. However, we barely understand the structure of $\Pi(\overline{r})$ as a representation of $\mathrm{GL}_n(F_w)$, though the ordinary part of $\Pi(\overline{r})$ is described in [BH15] when p splits completely in F and $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$ is ordinary. In particular, it is not known whether $\Pi(\overline{r})$ and $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$ determine each other. But we have the following conjecture:

Conjecture 1.0.1. *The local Galois representation $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$ is determined by $\Pi(\overline{r})$.*

This conjecture is widely expected to be true by experts but not explicitly written down before. The case $\mathrm{GL}_2(\mathbf{Q}_{p^f})$ was treated by Breuil–Diamond [BD14]. Herzig–Le–Morra [HLM] considered the case $\mathrm{GL}_3(\mathbf{Q}_p)$ when $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$ is upper-triangular, while the case $\mathrm{GL}_3(\mathbf{Q}_p)$ when $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$ is an extension of a two dimensional irreducible representation by a character was considered by Le–Morra–Park [LMP]. We are informed that John Enns from the University of Toronto has worked on this conjecture for the group $\mathrm{GL}_3(\mathbf{Q}_{p^f})$. All of the results above are under certain generic assumptions on the tamely ramified part of $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$.

From another point of view, to a smooth admissible $\overline{\mathbf{F}}_p$-representation Π of $\mathrm{GL}_n(K)$ for a finite extension K of $\mathbf{Q}_p$, Scholze [Sch15] attaches a smooth admissible $\overline{\mathbf{F}}_p$-representation $S(\Pi)$ of $D^\times$

for a division algebra D over K with center K and invariant $\frac{1}{n}$, which also has a continuous action of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$, via the mod p cohomology of the Lubin–Tate tower. Using this construction, it was possible for Scholze to prove Conjecture 1.0.1 in full generality for $\mathrm{GL}_2(K)$ (cf. [Sch15], Theorem 1.5). On the other hand, the proof of Theorem 1.5 of [Sch15] does not tell us where the invariants that determine $S(\Pi)$ lie. We do not know if there is any relation between these two different methods.

The weight part of Serre's conjecture already gives part of the information of $\Pi(\overline{r})$: the local Serre weights of $\overline{r}$ at w determine the socle of $\Pi(\overline{r})|_{\mathrm{GL}_n(\mathcal{O}_{F_w})}$ at least up to possible multiplicities, where $\mathcal{O}_{F_w}$ is the ring of integers of F_w. If $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$ is semisimple, then it is believed that the Serre weights of $\overline{r}$ at w determine $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$ up to twisting by unramified characters, but this is no longer the case if it is not semisimple: the Serre weights are not enough to determine the wildly ramified part of $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$, so that we need to understand a deeper structure of $\Pi(\overline{r})$ than just its $\mathrm{GL}_n(\mathcal{O}_{F_w})$-socle.

In this paper, we show that Conjecture 1.0.1 is true when p splits completely in F and $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$ is upper-triangular and sufficiently generic in a precise sense. Moreover, we describe the invariants in $\Pi(\overline{r})$ that determine the wildly ramified part of $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$. The generic assumptions on $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$ ensure that very few Serre weights of $\overline{r}$ at w will occur, which we call the weight elimination conjecture, Conjecture 1.3.1. The weight elimination results are significant for our method to prove Conjecture 1.0.1. But Bao V. Le Hung pointed out that this weight elimination conjecture can be proved by constructing certain deformation rings, and the results will appear in the forthcoming paper [LHMPQ]. We follow the basic strategy in [BD14, HLM]: we define Fontaine–Laffaille parameters on the Galois side using Fontaine–Laffaille modules as well as automorphic parameters on the automorphic side using the actions of Jacobi sum operators, and then identify them via the classical local Langlands correspondence. However, there are many new difficulties that didn't occur in [BD14] or in [HLM]. For instance, the classification of semi-linear algebraic objects of rank $n>3$ on the Galois side is much more complicated. Moreover, failing of the multiplicity one property of the Jordan–Hölder factors of mod p reduction of Deligne–Lusztig representations of $\mathrm{GL}_n(\mathbf{Z}_p)$ for $n>3$ implies that new ideas are required to show crucial non-vanishing of the automorphic parameters. In the rest of the introduction, we explain our ideas and results in more detail.

1.1. **Local Galois side.** Let E be a (sufficiently large) finite extension of $\mathbf{Q}_p$ with ring of integers $\mathcal{O}_E$, a uniformizer ϖ_E, and residue field $\mathbf{F}$, and let $I_{\mathbf{Q}_p}$ be the inertia subgroup of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and ω the fundamental character of niveau 1. We also let $\overline{\rho}_0 : \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \to \mathrm{GL}_n(\mathbf{F})$ be a continuous (Fontaine-Laffaille) ordinary generic Galois representation. Namely, there exists a basis $\underline{e} := (e_{n-1}, e_{n-2}, \cdots, e_0)$ for $\overline{\rho}_0$ such that with respect to $\underline{e}$ the matrix form of $\overline{\rho}_0$ is written as follows:

$$\overline{\rho}_0|_{I_{\mathbf{Q}_p}} \cong \begin{pmatrix} \omega^{c_{n-1}+(n-1)} & *_{n-1} & * & \cdots & * & * \\ 0 & \omega^{c_{n-2}+(n-2)} & *_{n-2} & \cdots & * & * \\ 0 & 0 & \omega^{c_{n-3}+(n-3)} & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \omega^{c_1+1} & *_1 \\ 0 & 0 & 0 & \cdots & 0 & \omega^{c_0} \end{pmatrix} \tag{1.1.1}$$

for some integers c_i satisfying some genericity conditions (cf. Definition 3.0.5). We also assume that $\overline{\rho}_0$ is maximally non-split, i.e., $*_i \neq 0$ for all $i \in \{1, 2, \cdots, n-1\}$.

Our goal on the Galois side is to show that the Frobenius eigenvalues of certain potentially crystalline lifts of $\overline{\rho}_0$ determine the Fontaine–Laffaille parameters of $\overline{\rho}_0$, which parameterize the wildly ramified part of $\overline{\rho}_0$. When the unramified part and the tamely ramified part of $\overline{\rho}_0$ are fixed, we define the Fontaine–Laffaille parameters via the Fontaine–Laffaille modules corresponding to $\overline{\rho}_0$ (cf. Definition 3.2.3). These parameters vary over the space of $\frac{(n-1)(n-2)}{2}$ copies of the projective line $\mathbb{P}^1(\mathbf{F})$, and we write $\mathrm{FL}_n^{i_0,j_0}(\overline{\rho}_0) \in \mathbb{P}^1(\mathbf{F})$ for each pair of integers (i_0, j_0) with $0 \leq j_0 < j_0+1 < i_0 \leq n-1$. For each such pair (i_0, j_0), the Fontaine–Laffaille parameter $\mathrm{FL}_n^{i_0,j_0}(\overline{\rho}_0)$ is determined by the subquotient $\overline{\rho}_{i_0,j_0}$ of $\overline{\rho}_0$ which is determined by the subset $(e_{i_0}, e_{i_0-1}, \cdots, e_{j_0})$ of $\underline{e}$ (cf. (3.0.2)): in fact, we have the identity $\mathrm{FL}_n^{i_0,j_0}(\overline{\rho}_0) = \mathrm{FL}_{i_0-j_0+1}^{i_0-j_0,0}(\overline{\rho}_{i_0,j_0})$ (cf. Lemma 3.2.4).

Since potentially crystalline lifts of $\overline{\rho}_0$ are not Fontaine–Laffaille in general, we are no longer able to use Fontaine–Laffaille theory to study such lifts of $\overline{\rho}_0$; we use Breuil modules and strongly divisible modules for their lifts. It is obvious that any lift of $\overline{\rho}_0$ determines the Fontaine–Laffaille parameters, but it is not obvious how one can explicitly visualize the information that determines $\overline{\rho}_0$ in those lifts. Motivated by the automorphic side, we believe that for each pair (i_0, j_0) as above the Fontaine–Laffaille parameter $\mathrm{FL}_n^{i_0,j_0}(\overline{\rho}_0)$ is determined by a certain product of Frobenius eigenvalues of the potentially crystalline lifts of $\overline{\rho}_0$ with Hodge–Tate weights $\{-(n-1), \cdots, -1, 0\}$ and Galois type $\bigoplus_{i=0}^{n-1} \widetilde{\omega}^{k_i^{i_0,j_0}}$ where $\widetilde{\omega}$ is the Teichmüller lift of the fundamental character ω of niveau 1 and

$$k_i^{i_0,j_0} \equiv \begin{cases} c_{i_0} + i_0 - j_0 - 1 & \text{for } i = i_0; \\ c_{j_0} - (i_0 - j_0 - 1) & \text{for } i = j_0; \\ c_i & \text{otherwise} \end{cases} \tag{1.1.2}$$

modulo $(p-1)$. Here, c_i are the integers determining the tamely ramified part of $\overline{\rho}_0$ in (1.1.1) and our normalization of the Hodge–Tate weight of the cyclotomic character ε is -1.

Our main result on the Galois side is the following:

Theorem 1.1.1 (Theorem 3.7.1). *Fix $i_0, j_0 \in \mathbf{Z}$ with $0 \leq j_0 < j_0 + 1 < i_0 \leq n-1$. Assume that $\overline{\rho}_0$ is generic (cf. Definition 3.0.5) and that $\overline{\rho}_{i_0,j_0}$ is Fontaine–Laffaille generic (cf. Definition 3.2.5), and let $(\lambda_{n-1}^{i_0,j_0}, \lambda_{n-2}^{i_0,j_0}, \cdots, \lambda_0^{i_0,j_0}) \in (\mathcal{O}_E)^n$ be the Frobenius eigenvalues on the $(\widetilde{\omega}^{k_{n-1}^{i_0,j_0}}, \widetilde{\omega}^{k_{n-2}^{i_0,j_0}}, \cdots, \widetilde{\omega}^{k_0^{i_0,j_0}})$-isotypic components of $\mathrm{D}_{\mathrm{st}}^{\mathbf{Q}_p,n-1}(\rho_0)$ where ρ_0 is a potentially crystalline lift of $\overline{\rho}_0$ with Hodge–Tate weights $\{-(n-1), -(n-2), \cdots, -1, 0\}$ and Galois type $\bigoplus_{i=0}^{n-1} \widetilde{\omega}^{k_i^{i_0,j_0}}$.*

Then the Fontaine–Laffaille parameter $\mathrm{FL}_n^{i_0,j_0}$ associated to $\overline{\rho}_0$ is computed as follows:

$$\mathrm{FL}_n^{i_0,j_0}(\overline{\rho}_0) = \left[1 : \overline{\left(\frac{p^{[(n-1)-\frac{i_0+j_0}{2}](i_0-j_0-1)}}{\prod_{i=j_0+1}^{i_0-1} \lambda_i^{i_0,j_0}} \right)} \right] \in \mathbb{P}^1(\mathbf{F}).$$

Note that by $\overline{\bullet} \in \mathbf{F}$ in the theorem above we mean the image of $\bullet \in \mathcal{O}_E$ under the natural surjection $\mathcal{O}_E \twoheadrightarrow \mathbf{F}$. We also note that $\overline{\rho}_{i_0,j_0}$ being Fontaine–Laffaille generic implies $\mathrm{FL}_n^{i_0,j_0}(\overline{\rho}_0) \neq 0, \infty$ for all i_0, j_0 as in Theorem 1.1.1, but is a strictly stronger assumption if $i_0 - j_0 \geq 3$.

Let us briefly discuss our strategy for the proof of Theorem 1.1.1. Recall that the Fontaine–Laffaille parameter $\mathrm{FL}_n^{i_0,j_0}(\overline{\rho}_0)$ is defined in terms of the Fontaine–Laffaille module corresponding to $\overline{\rho}_0$. Thus we need to describe $\mathrm{FL}_n^{i_0,j_0}(\overline{\rho}_0)$ by the data of the Breuil modules of inertial type $\bigoplus_{i=0}^{n-1} \omega^{k_i^{i_0,j_0}}$ corresponding to $\overline{\rho}_0$, and we do this via étale ϕ-modules, which requires classification of such Breuil modules. If the filtration of the Breuil modules is of a certain shape, then a certain product of the Frobenius eigenvalues of the Breuil modules determines a Fontaine–Laffaille parameter (cf. Proposition 3.4.2). In order to get such a filtration, we need to assume that $\overline{\rho}_{i_0,j_0}$ is Fontaine–Laffaille generic (cf. Definition 3.2.5). Then we determine the structure of the filtration of the strongly divisible modules lifting the Breuil modules by direct computation, which immediately gives enough properties of Frobenius eigenvalues of the potentially crystalline representations we consider. But this whole process is subtle for general i_0, j_0. To resolve this issue we prove that any potentially crystalline lift of $\overline{\rho}_0$ with Hodge–Tate weights $\{-(n-1), -(n-2), \cdots, 0\}$ and Galois type $\bigoplus_{i=0}^{n-1} \widetilde{\omega}^{k_i^{i_0,j_0}}$ has a potentially crystalline subquotient ρ_{i_0,j_0} of Hodge–Tate weights $\{-i_0, \cdots, -j_0\}$ and of Galois type $\bigoplus_{i=j_0}^{i_0} \widetilde{\omega}^{k_i^{i_0,j_0}}$ lifting $\overline{\rho}_{i_0,j_0}$. More precisely,

Theorem 1.1.2 (Corollary 3.6.2). *Every potentially crystalline lift ρ_0 of $\overline{\rho}_0$ with Hodge–Tate weights $\{-(n-1),-(n-2),\cdots,0\}$ and Galois type $\bigoplus_{i=0}^{n-1}\widetilde{\omega}^{k_i^{i_0,j_0}}$ is a successive extension*

$$\rho_0 \cong \begin{pmatrix} \rho_{n-1,n-1} & \cdots & * & * & * & \cdots & * \\ & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & \rho_{i_0+1,i_0+1} & * & * & \cdots & * \\ & & & \rho_{i_0,j_0} & * & \cdots & * \\ & & & & \rho_{j_0-1,j_0-1} & \cdots & * \\ & & & & & \ddots & \vdots \\ & & & & & & \rho_{0,0} \end{pmatrix}$$

where

- *for $n-1\geq i>i_0$ and $j_0>i\geq 0$, $\rho_{i,i}$ is a 1-dimensional potentially crystalline lift of $\overline{\rho}_{i,i}$ with Hodge–Tate weight $-i$ and Galois type $\widetilde{\omega}^{k_i^{i_0,j_0}}$;*
- *ρ_{i_0,j_0} is a (i_0-j_0+1)-dimensional potentially crystalline lift of $\overline{\rho}_{i_0,j_0}$ with Hodge–Tate weights $\{-i_0,-i_0+1,\cdots,-j_0\}$ and Galois type $\bigoplus_{i=j_0}^{i_0}\widetilde{\omega}^{k_i^{i_0,j_0}}$.*

Note that we actually prove the niveau f version of Theorem 1.1.2 since it adds only little more extra work (cf. Corollary 3.6.2).

The representation $\rho_{i_0,j_0}\otimes\varepsilon^{-j_0}$ is a (i_0-j_0+1)-dimensional potentially crystalline lift of $\overline{\rho}_{i_0,j_0}$ with Hodge–Tate weights $\{-(i_0-j_0),-(i_0-j_0-1),\cdots,0\}$ and Galois type $\bigoplus_{i=j_0}^{i_0}\widetilde{\omega}^{k_i^{i_0,j_0}}$, so that, by Theorem 1.1.2, Theorem 1.1.1 reduces to the case $(i_0,j_0)=(n-1,0)$: we prove Theorem 1.1.1 when $(i_0,j_0)=(n-1,0)$, and then use the fact $\mathrm{FL}_n^{i_0,j_0}(\overline{\rho}_0)=\mathrm{FL}_{i_0-j_0+1}^{i_0-j_0,0}(\overline{\rho}_{i_0,j_0})$ to get the result for general i_0,j_0.

The Weil–Deligne representation $\mathrm{WD}(\rho_0)$ associated to ρ_0 (as in Theorem 1.1.1) contains those Frobenius eigenvalues of ρ_0. We then use the classical local Langlands correspondence for GL_n to transport the Frobenius eigenvalues of ρ_0 (and so the Fontaine–Laffaille parameters of $\overline{\rho}_0$ as well by Theorem 1.1.1) to the automorphic side (cf. Corollary 3.7.3).

1.2. **Local automorphic side.** We start by introducing the Jacobi sum operators in characteristic p. Let T (resp. B) be the maximal torus (resp. the maximal Borel subgroup) consisting of diagonal matrices (resp. of upper-triangular matrices) of GL_n. We let $X(T):=\mathrm{Hom}(T,\mathbf{G}_m)$ be the group of characters of T and Φ^+ be the set of positive roots with respect to (B,T). We define $\epsilon_i\in X(T)$ as the projection of $T\cong\mathbf{G}_m^n$ onto the i-th factor. Then the elements $\{\epsilon_i\mid 1\leq i\leq n\}$ forms a $\mathbf{Z}$-basis for the free abelian group $X(T)$. We will use the notation $(d_1,d_2,\cdots,d_n)\in\mathbf{Z}^n$ for the element $\sum_{k=1}^n d_k\epsilon_k\in X(T)$. Note that the group of characters of the finite group $T(\mathbf{F}_p)\cong(\mathbf{F}_p^\times)^n$ can be identified with $X(T)/(p-1)X(T)$, and therefore we sometimes abuse the notation $(d_1,d_2,\cdots,d_n)$ for its image in $X(T)/(p-1)X(T)$. We define $\Delta:=\{\alpha_k:=\epsilon_k-\epsilon_{k+1}\mid 1\leq k\leq n-1\}\subset\Phi^+$ as the set of simple positive roots. Note that we write s_k for the reflection of the simple root α_k. For an element w in the Weyl group W, we define $\Phi_w^+=\{\alpha\in\Phi^+\mid w(\alpha)\in-\Phi^+\}\subseteq\Phi^+$ and $U_w=\prod_{\alpha\in\Phi_w^+}U_\alpha$, where U_α is a subgroup of U whose only non-zero off-diagonal entry corresponds to α. Note in particular that $\Phi^+=\Phi_{w_0}^+$, where w_0 is the longest element in W. For $w\in W$ and for a tuple of integers $\underline{k}=(k_\alpha)_{\alpha\in\Phi_w^+}\in\{0,1,\cdots,p-1\}^{|\Phi_w^+|}$, we define the Jacobi sum operator

$$S_{\underline{k},w}:=\sum_{A\in U_w(\mathbf{F}_p)}\left(\prod_{\alpha\in\Phi_w^+}A_\alpha^{k_\alpha}\right)A\cdot w\in\mathbf{F}_p[\mathrm{GL}_n(\mathbf{F}_p)]$$

where A_α is the entry of A corresponding to $\alpha\in\Phi_w^+$. In Section 4, we establish many technical results, both conceptual and computational, around these Jacobi sum operators. The use of these

Jacobi sum operators can be traced back to at least [CL76], and are widely used for GL_2 in [BP12] and [Hu10] for instance. But systematic computation with these operators seems to be limited to GL_2 or GL_3. In this paper, we need to do some specific but technical computation on some special Jacobi sum operators for $\mathrm{GL}_n(\mathbf{F}_p)$, which is enough for our application to Theorem 1.4.1 below.

By the discussion on the local Galois side, our target on the local automorphic side is to capture the Frobenius eigenvalues coming from the local Galois side. By the classical local Langlands correspondence, the Frobenius eigenvalues of ρ_0 are transported to the unramified part of χ in the tamely ramified principal series $\mathrm{Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_n(\mathbf{Q}_p)}\chi$ corresponding to the Weil–Deligne representation $\mathrm{WD}(\rho_0)$ attached to ρ_0 in Theorem 1.1.1, and it is standard to use U_p-operators to capture the information in the unramified part of χ.

The normalizer of the Iwahori subgroup I in $\mathrm{GL}_n(\mathbf{Q}_p)$ is cyclic modulo I, and this cyclic quotient group is generated by an element $\Xi_n \in \mathrm{GL}_n(\mathbf{Q}_p)$ that is explicitly defined in (4.4.1). One of our goals is to translate the eigenvalue of U_p-operators into the action of Ξ_n on the space $(\mathrm{Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_n(\mathbf{Q}_p)}\chi)|_{\mathrm{GL}_n(\mathbf{Z}_p)}$. This is firstly done for $\mathrm{GL}_2(\mathbf{Q}_{p^f})$ in [BD14], and then the method is generalized to $\mathrm{GL}_3(\mathbf{Q}_p)$ in the ordinary case by [HLM]. Both [BD14] and [HLM] need a pair of group algebra operators: for instance, group algebra operators $\widehat{S}, \widehat{S}' \in \mathbf{Q}_p[\mathrm{GL}_3(\mathbf{Q}_p)]$ are defined in [HLM] and the authors prove an intertwining identity of the form $\widehat{S}' \cdot \Xi_3 = c\widehat{S}$ on a certain $I(1)$-fixed subspace of $\mathrm{Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_3(\mathbf{Q}_p)}\chi$ with χ assumed to be tamely ramified, where $I(1)$ is the pro-p Sylow subgroup of I. Here, the constant $c \in \mathcal{O}_E^\times$ captures the eigenvalues of U_p-operators. This is the first technical point on the local automorphic side, and we generalize the results in [BD14] and [HLM] by the following theorem.

For an n-tuple of integers $(a_{n-1}, a_{n-2}, \cdots, a_0) \in \mathbf{Z}^n$, we write $\mathcal{S}_n$ and $\mathcal{S}'_n$ for $S_{\underline{k}^1, w_0}$ with $\underline{k}^1 = (k^1_{i,j})$ and $S_{\underline{k}^{1,\prime}, w_0}$ with $\underline{k}^{1,\prime} = (k^{1,\prime}_{i,j})$ respectively, where $k^1_{i,i+1} = [a_0 - a_{n-i}]_1 + n - 2$, $k^{1,\prime}_{i,i+1} = [a_{n-i-1} - a_{n-1}]_1 + n - 2$ for $1 \leq i \leq n-1$, and $k^1_{i,j} = k^{1,\prime}_{i,j} = 0$ otherwise. Here, (i,j) is the entry corresponding to α if $\alpha = \epsilon_i - \epsilon_j \in \Phi^+$ and by $[x]_1$ for $x \in \mathbf{Z}$ we mean the integer in $[0, p-1)$ such that $x \equiv [x]_1$ modulo $(p-1)$. We define $\widehat{\mathcal{S}}_n \in \mathbf{Z}_p[\mathrm{GL}_n(\mathbf{Z}_p)]$ (resp. $\widehat{\mathcal{S}}'_n \in \mathbf{Z}_p[\mathrm{GL}_n(\mathbf{Z}_p)]$) by taking the Teichmüller lifts of the coefficients and the entries of the matrices of $\mathcal{S}_n \in \mathbf{F}_p[\mathrm{GL}_n(\mathbf{F}_p)]$ (resp. of $\mathcal{S}'_n \in \mathbf{F}_p[\mathrm{GL}_n(\mathbf{F}_p)]$).

We use the notation $\bullet$ for the composition of maps or group operators to distinguish from the notation $\circ$ for an $\mathcal{O}_E$-lattice inside a representation.

Theorem 1.2.1 (Theorem 4.4.9)**.** *Assume that the n-tuple of integers $(a_{n-1}, a_{n-2}, \cdots, a_0)$ is n-generic in the lowest alcove (cf. Definition 4.1.1), and let*

$$\Pi_n = \mathrm{Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_n(\mathbf{Q}_p)}(\chi_1 \otimes \chi_2 \otimes \chi_3 \otimes ... \otimes \chi_{n-2} \otimes \chi_{n-1} \otimes \chi_0)$$

be a tamely ramified principal series representation with the smooth characters $\chi_k : \mathbf{Q}_p^\times \to E^\times$ satisfying $\chi_k|_{\mathbf{Z}_p^\times} = \widetilde{\omega}^{a_k}$ for $0 \leq k \leq n-1$.

On the 1-dimensional subspace $\Pi_n^{I(1),(a_1,a_2,\ldots,a_{n-1},a_0)}$ we have the identity:

$$\widehat{\mathcal{S}}'_n \bullet (\Xi_n)^{n-2} = p^{n-2}\kappa_n \left(\prod_{k=1}^{n-2} \chi_k(p)\right) \widehat{\mathcal{S}}_n \tag{1.2.1}$$

for $\kappa_n \in \mathbf{Z}_p^\times$ satisfying $\kappa_n \equiv \varepsilon^* \cdot \mathcal{P}_n(a_{n-1}, \cdots, a_0)$ mod (ϖ_E) where

$$\varepsilon^* = \prod_{k=1}^{n-2} (-1)^{a_0 - a_k}$$

and

$$\mathcal{P}_n(a_{n-1},\cdots,a_0)=\prod_{k=1}^{n-2}\prod_{j=0}^{n-3}\frac{a_k-a_{n-1}+j}{a_0-a_k+j}\in\mathbf{Z}_p^\times.$$

In fact, there are many identities similar to the one in (1.2.1) for each operator U_n^i for $1\leq i\leq n-1$ (defined in (4.4.2)) which can be technically always reduced to Proposition 4.4.3, but it is clear from the proof of Theorem 1.2.1 in Section 4.4 that we need to choose U_n^{n-2} for the U_p-operator acting on $\Pi_n^{I(1),(a_1,a_2,\ldots,a_{n-1},a_0)}$, motivated from the local Galois side via Theorem 1.1.1. The crucial point here is that the constant $p^{n-2}\kappa_n\left(\prod_{k=1}^{n-2}\chi_k(p)\right)$, which is closely related to $\mathrm{FL}_n^{n-1,0}(\overline{\rho}_0)$ via Theorem 1.1.1 and classical local Langlands correspondence, should lie in $\mathcal{O}_E^\times$ for each Π_n appearing in our application of Theorem 1.2.1 to Theorem 1.4.1.

The next step is to consider the mod p reduction of the identity (1.2.1), which is effective to capture $p^{n-2}\prod_{k=1}^{n-2}\chi_k(p)$ modulo (ϖ_E) only if $\widehat{\mathcal{S}}_n\widehat{v}\not\equiv 0$ modulo (ϖ_E) for $\widehat{v}\in\Pi_n^{I(1),(a_1,a_2,\ldots,a_{n-1},a_0)}$. It turns out that this non-vanishing property is very technical to prove for general $\mathrm{GL}_n(\mathbf{Q}_p)$. Before we state our non-vanishing result, we fix a little more notation: let

$$\begin{cases}\mu^*:=(a_{n-1}-n+2,a_{n-2},\cdots,a_1,a_0+n-2);\\ \mu_0:=(a_{n-1},a_1,a_2,\cdots,a_{n-3},a_{n-2},a_0);\\ \mu_1:=(a_1,a_2,\cdots,a_{n-3},a_{n-2},a_{n-1},a_0);\\ \mu_1':=(a_{n-1},a_0,a_1,a_2,\cdots,a_{n-3},a_{n-2})\end{cases}$$

be four characters of $T(\mathbf{F}_p)$, and write π_0 (resp. $\widetilde{\pi}_0^\circ$) for the characteristic p principal series (resp. the characteristic 0 principal series) induced by the characters μ_0 (resp. by its Teichmüller lift $\widetilde{\mu}_0$). Note that we can attach an irreducible representation $F(\lambda)$ of $\mathrm{GL}_n(\mathbf{F}_p)$ to each $\lambda\in X(T)/(p-1)X(T)$ satisfying some regular conditions (cf. the beginning of Section 4). If we assume that $(a_{n-1},\cdots,a_0)\in\mathbf{Z}^n$ is n-generic in the lowest alcove, the characters μ^*, μ_0, μ_1 and μ_1' do satisfy the regular condition and thus we have four irreducible representations $F(\mu^*)$, $F(\mu_0)$, $F(\mu_1)$ and $F(\mu_1')$ of $\mathrm{GL}_n(\mathbf{F}_p)$. There is a unique (up to homothety) $\mathcal{O}_E$-lattice τ in $\widetilde{\pi}_0^\circ\otimes_{\mathcal{O}_E}E$ such that

$$\mathrm{soc}_{\mathrm{GL}_n(\mathbf{F}_p)}(\tau\otimes_{\mathcal{O}_E}\mathbf{F})=F(\mu^*).$$

We are now ready to state the non-vanishing theorem.

Theorem 1.2.2 (Corollary 4.8.3)**.** *Assume that the n-tuple of integers $(a_{n-1},a_{n-2},\cdots,a_0)$ is $2n$-generic in the lowest alcove (cf. Definition 4.1.1).*

Then we have

$$\mathcal{S}_n\left((\tau\otimes_{\mathcal{O}_E}\mathbf{F})^{U(\mathbf{F}_p),\mu_1}\right)\neq 0\quad\text{and}\quad\mathcal{S}_n'\left((\tau\otimes_{\mathcal{O}_E}\mathbf{F})^{U(\mathbf{F}_p),\mu_1'}\right)\neq 0.$$

The definition of μ_1,μ_1',μ_0 and μ^* is motivated by our application of Theorem 1.2.2 to Theorem 1.4.1 and is closely related to the Galois types we choose in Theorem 1.1.1. We emphasize that, technically speaking, it is crucial that $F(\mu^*)$ has multiplicity one in π_0. The proof of Theorem 1.2.2 is technical and makes full use of the results in Sections 4.1, 4.6, and 4.7.

1.3. **Weight elimination and automorphy of a Serre weight.** The weight part of Serre's conjecture is considered as a first step towards mod p Langlands program, since it gives a description of the socle of $\Pi(\overline{r})|_{\mathrm{GL}_n(\mathbf{Z}_p)}$ up to possible multiplicities. Substantial progress has been made for the groups $\mathrm{GL}_2(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of a finite extension K of $\mathbf{Q}_p$ ([BDJ10], [Gee11], [GK14], [GLS14], [GLS15]). For groups in higher semisimple rank, we also have a detailed description. (See [EGH15], [HLM], [LMP], [MP], [LLHLM] for GL_3; [Her09], [GG10], [BLGG], [LLL], [GHS] for general n.)

Weight elimination results are significant for the proof of our main global application, Theorem 1.4.1. For the purpose of this introduction, we quickly review some notation. Let F^+ be the maximal totally real subfield of a CM field F, and assume that p splits completely in F. Fix a place w of F above p and set $v := w|_{F^+}$. We assume that $\overline{r}$ is automorphic: this means that there exist a totally definite unitary group G_n defined over F^+ that is an outer form of GL_{n/F^+} and split at places above p, an integral model $\mathcal{G}_n$ of G_n such that $\mathcal{G}_n \times \mathcal{O}_{F^+_{v'}}$ is reductive if v' is a finite place of F^+ that splits in F, a compact open subgroup $U = \mathcal{G}_n(\mathcal{O}_{F^+_v}) \times U^v \subseteq \mathcal{G}_n(\mathcal{O}_{F^+_v}) \times G_n(\mathbf{A}^{\infty,v}_{F^+})$ that is sufficiently small and unramified above p, a Serre weight $V = \bigotimes_{v'|p} V_{v'}$ that is an irreducible smooth $\overline{\mathbf{F}}_p$-representation of $\mathcal{G}_n(\mathcal{O}_{F^+,p})$, and a maximal ideal $\mathfrak{m}_{\overline{r}}$ associated to $\overline{r}$ in the Hecke algebra acting on the space $S(U,V)$ of mod p algebraic automorphic forms such that

$$S(U,V)[\mathfrak{m}_{\overline{r}}] \neq 0. \tag{1.3.1}$$

We write $W(\overline{r})$ for the set of Serre weights V satisfying (1.3.1) for some U, and $W_w(\overline{r})$ for the set of local Serre weights V_v, that is irreducible smooth representations of $\mathcal{G}_n(\mathcal{O}_{F^+_v}) \cong \mathrm{GL}_n(\mathcal{O}_{F_w}) \cong \mathrm{GL}_n(\mathbf{Z}_p)$, such that $V_v \otimes (\bigotimes_{v'\neq v} V_{v'}) \in W(\overline{r})$ for an irreducible smooth representation $\bigotimes_{v'\neq v} V_{v'}$ of $\prod_{v'\neq v} \mathcal{G}_n(\mathcal{O}_{F^+_{v'}})$. The local Serre weights V_v have an explicit description as representations of $\mathrm{GL}_n(\mathbf{F}_p)$: there exists a p-restricted (i.e. $0 \leq a_i - a_{i-1} \leq p-1$ for all $1 \leq i \leq n-1$) weight $\underline{a} := (a_{n-1}, a_{n-2}, \cdots, a_0) \in X(T)$ such that $F(\underline{a}) \cong V_v$ where $F(\underline{a})$ is the irreducible socle of the dual Weyl module associated to $\underline{a}$ (cf. Section 5.2 as well as the beginning of Section 4).

Assume that $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)} \cong \overline{\rho}_0$, where $\overline{\rho}_0$ is defined as in (1.1.1). We define certain characters $\mu^{\square}$ and $\mu^{\square,i_1,j_1}$ of $T(\mathbf{F}_p)$ and a principal series

$$\pi_*^{i_1,j_1} = \mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_n(\mathbf{F}_p)} (\mu^{\square,i_1,j_1})^{w_0}$$

at the beginning of Section 5.3. Our main conjecture for weight elimination is

Conjecture 1.3.1 (Conjecture 5.3.1)**.** *Assume that $\overline{\rho}_{i_0,j_0}$ is Fontaine–Laffaille generic and that $\mu^{\square,i_1,j_1}$ is $2n$-generic. Then we have an inclusion*

$$W_w(\overline{r}) \cap \mathrm{JH}((\pi_*^{i_1,j_1})^\vee) \subseteq \{F(\mu^{\square})^\vee, F(\mu^{\square,i_1,j_1})^\vee\}.$$

We emphasize that the condition $\overline{\rho}_{i_0,j_0}$ is Fontaine–Laffaille generic is crucial in Conjecture 1.3.1. For example, if $n=4$ and $(i_0,j_0) = (3,0)$ and we assume merely $\mathrm{FL}_4^{3,0}(\overline{\rho}_0) \neq 0, \infty$ (which is strictly weaker than Fontaine–Laffaille generic), then we expect that an extra Serre weight can possibly appear in $W_w(\overline{r}) \cap \mathrm{JH}((\pi_*^{i_1,j_1})^\vee)$.

The Conjecture 1.3.1 is motivated by the proof of Theorem 1.1.1 and the theory of *shape* in [LLHLM]. The special case $n=3$ of Conjecture 1.3.1 was firstly proven in [HLM] and can also be deduced from the computations of Galois deformation rings in [LLHLM].

Remark 1.3.2. *In an earlier version of this paper, we prove Conjecture 1.3.1 for $n \leq 5$. But our method is rather elaborate to execute for general n. Bao V. Le Hung pointed out that one can prove Conjecture 1.3.1 completely by constructing certain potentially crystalline deformation rings. A proof of Conjecture 1.3.1 will appear in* [LHMPQ].

Finally, we also show the automorphy of the Serre weight $F(\mu^{\square})^\vee$. In other words,

$$F(\mu^{\square})^\vee \in W_w(\overline{r}) \cap \mathrm{JH}((\pi_*^{i_1,j_1})^\vee). \tag{1.3.2}$$

Showing the automorphy of a Serre weight, in general, is very subtle. But thanks to the work of [BLGG] we are able to show the automorphy of $F(\mu^{\square})^\vee$ by checking the existence of certain potentially diagonalizable crystalline lifts of $\overline{\rho}_0$ (cf. Proposition 5.3.2).

1.4. **Mod p local-global compatibility.** We now state our main results on mod p local-global compatibility. As discussed at the beginning of this introduction, we prove that $\Pi(\overline{r})$ determines the ordinary representation $\overline{\rho}_0$. Moreover, we also describe the invariants in $\Pi(\overline{r})$ that determine the wildly ramified parts of $\overline{\rho}_0$. We first recall the definition of $\Pi(\overline{r})$.

Keep the notation of the previous sections, and write $b_i = -c_{n-1-i}$ for all $0 \leq i \leq n-1$, with c_i as in (1.1.1). We fix a place w of F above p and write $v := w|_{F^+}$, and we let $\overline{r} : G_F \to \mathrm{GL}_n(\mathbf{F})$ be an irreducible automorphic representation, of a Serre weight $V \cong \bigotimes_{v'} V_{v'}$ (cf. Section 1.3), with $\overline{r}|_{G_{F_w}} \cong \overline{\rho}_0$.

Let $V' := \bigotimes_{v' \neq v} V_{v'}$ and set $S(U^v, V') := \varinjlim S(U^v \cdot U_v, V')$ where the direct limit runs over compact open subgroups $U_v \subseteq \mathcal{G}_n(\mathcal{O}_{F_v^+})$. This space $S(U^v, V')$ has a natural smooth action of $G_n(F_v^+) \cong \mathrm{GL}_n(F_w) \cong \mathrm{GL}_n(\mathbf{Q}_p)$ by right translation as well as an action of a Hecke algebra that commutes with the action of $G_n(F_v^+)$. We define

$$\Pi(\overline{r}) := S(U^v, V')[\mathfrak{m}_{\overline{r}}]$$

where $\mathfrak{m}_{\overline{r}}$ is the maximal ideal of the Hecke algebra associated to $\overline{r}$. In the spirit of [Eme], this is a candidate on the automorphic side for a mod p Langlands correspondence corresponding to $\overline{\rho}_0$. Note that the definition of $\Pi(\overline{r})$ relies on U^v and V' as well as choice of a Hecke algebra, but we suppress them in the notation.

Fix $n-1 \geq i_0 > j_0 + 1 > j_0 \geq 0$, and define i_1 and j_1 by the equation $i_1 + i_0 = j_1 + j_0 = n-1$. Note that the following Jacobi sum operators

$$\mathcal{S}^{i_1,j_1}, \quad \mathcal{S}^{i_1,j_1,\prime}, \quad \mathcal{S}_1^{i_1,j_1}, \quad \mathcal{S}_1^{i_1,j_1,\prime} \in \mathbf{F}_p[\mathrm{GL}_{j_1-i_1+1}(\mathbf{F}_p)]$$

are defined at the beginning of Section 5.5.

Now we can state the main results in this paper.

Theorem 1.4.1 (Theorem 5.6.2)**.** *Fix a pair of integers (i_0, j_0) satisfying $0 \leq j_0 < j_0 + 1 < i_0 \leq n-1$, and let $\overline{r} : G_F \to \mathrm{GL}_n(\mathbf{F})$ be an irreducible automorphic representation with $\overline{r}|_{G_{F_w}} \cong \overline{\rho}_0$. Assume that*

- *$\mu^{\square, i_1, j_1}$ is $2n$-generic;*
- *$\overline{\rho}_{i_0,j_0}$ is Fontaine–Laffaille generic.*

Assume further that

$$\{F(\mu^{\square})^\vee\} \subseteq W_w(\overline{r}) \cap \mathrm{JH}((\pi_*^{i_1,j_1})^\vee) \subseteq \{F(\mu^{\square})^\vee, F(\mu^{\square,i_1,j_1})^\vee\}. \tag{1.4.1}$$

Then there exists a primitive vector (cf. Definition 5.6.1) in $\Pi(\overline{r})^{I(1),\mu_1^{i_1,j_1}}$. Moreover, for each primitive vector $v^{i_1,j_1} \in \Pi(\overline{r})^{I(1),\mu_1^{i_1,j_1}}$, we have $\mathcal{S}^{i_1,j_1} \bullet \mathcal{S}_1^{i_1,j_1} v^{i_1,j_1} \neq 0$ and

$$\mathcal{S}^{i_1,j_1,\prime} \bullet \mathcal{S}_1^{i_1,j_1,\prime} \bullet (\Xi_n)^{j_1-i_1-1} v^{i_1,j_1} = \varepsilon^{i_1,j_1} \mathcal{P}_{i_1,j_1}(b_{n-1}, \cdots, b_0) \cdot \mathrm{FL}_n^{i_0,j_0}(\overline{r}|_{G_{F_w}}) \cdot \mathcal{S}^{i_1,j_1} \bullet \mathcal{S}_1^{i_1,j_1} v^{i_1,j_1}$$

where

$$\varepsilon^{i_1,j_1} = \prod_{k=i_1+1}^{j_1-1} (-1)^{b_{i_1} - b_k - j_1 + i_1 + 1}$$

and

$$\mathcal{P}_{i_1,j_1}(b_{n-1}, \cdots, b_0) = \prod_{k=i_1+1}^{j_1-1} \prod_{j=1}^{j_1-i_1-1} \frac{b_k - b_{j_1} - j}{b_{i_1} - b_k - j} \in \mathbf{Z}_p^\times.$$

Note that the conditions in (1.4.1) can be removed under some standard Taylor–Wiles conditions (cf. Remark 1.3.2 and (1.3.2)).

Theorem 1.4.1 relies on the choice of a principal series type (the niveau 1 Galois type $\bigoplus_{i=0}^{n-1} \widetilde{\omega}^{k_i^{i_0,j_0}}$). But this choice is somehow the unique one that could possibly make our strategy of the proof of Theorem 1.4.1 work.

Be careful that we cannot apply Theorem 1.2.1 and Theorem 1.2.2 directly to our local global-compatibility for general (i_1, j_1). Instead, we need to generalize Theorem 1.2.2 (resp. Theorem 1.2.1) to Proposition 5.5.5 (resp. Proposition 5.5.1) .

Corollary 1.4.2. *Keep the notation of Theorem 1.4.1 and assume that each assumption in Theorem 1.4.1 holds for all (i_0, j_0) such that $0 \le j_0 < j_0 + 1 < i_0 \le n-1$. Assume further that a freeness result mentioned in Remark 5.6.4 is true.*

Then the structure of $\Pi(\overline{r})$ as a smooth admissible $\mathbf{F}$-representation of $\mathrm{GL}_n(\mathbf{Q}_p)$ determines the Galois representation $\overline{\rho}_0$ up to isomorphism.

1.5. **Notation.** Much of the notation introduced in this section will also be (or have already been) introduced in the text, but we try to collect together various definitions here for ease of reading.

We let E be a (sufficiently large) extension of $\mathbf{Q}_p$ with ring of integers $\mathcal{O}_E$, a uniformizer ϖ_E, and residue field $\mathbf{F}$. We will use these rings E, $\mathcal{O}_E$, and $\mathbf{F}$ for the coefficients of our representations. We also let K be a finite extension of $\mathbf{Q}_p$ with ring of integers $\mathcal{O}_K$, a uniformizer ϖ, and residue field k. Let $W(k)$ be the ring of Witt vectors over k and write K_0 for $W(k)[\frac{1}{p}]$. (K_0 is the maximal absolutely unramified subextension of K.) In this paper, by K we always mean a tamely ramified extension of $\mathbf{Q}_p$ with $e := [K : K_0] = p^f - 1$ where $f = [k : \mathbf{F}_p]$.

For a field F, we write G_F for $\mathrm{Gal}(\overline{F}/F)$ where $\overline{F}$ is a separable closure of F. For instance, we are mainly interested in $G_{\mathbf{Q}_p}$ as well as G_{K_0} in this paper. The choice of a uniformizer $\varpi \in K$ provides us with a map:

$$\widetilde{\omega}_{\varpi} : G_{\mathbf{Q}_p} \longrightarrow W(k) \, : \, g \longmapsto \frac{g(\varpi)}{\varpi}$$

whose reduction mod (ϖ) will be denoted as ω_{ϖ}. This map factors through $\mathrm{Gal}(K/\mathbf{Q}_p)$ and $\widetilde{\omega}_{\varpi}|_{G_{K_0}}$ becomes a homomorphism. Note that the choice of the embedding $\sigma_0 : k \hookrightarrow \mathbf{F}$ provides us with a fundamental character of niveau f, namely $\omega_f := \sigma_0 \circ \omega_{\varpi}|_{\mathrm{Gal}(K/K_0)}$, and we fix the embedding in this paper.

For $a \in k$, we write $\widetilde{a}$ for its Teichmüller lift in $W(k)$. We also use the notation $\lceil a \rceil$ for $\widetilde{a}$, in particular, in Section 4.4. When the notation for an element $\bullet$ in k is quite long, we prefer $\lceil \bullet \rceil$ to $\widetilde{\bullet}$. For instance, if $a, b, c, d \in k$ then we write

$$\lceil (a-b)(a-c)(a-d)(b-c)(b-d) \rceil \;\text{ for }\; \widetilde{(a-b)(a-c)(a-d)(b-c)(b-d)}.$$

Note that $\widetilde{\omega}_{\varpi}$ is the Teichmüller lift of ω_{ϖ}.

We normalize the Hodge–Tate weight of the cyclotomic character ε to be -1. Our normalization on class field theory sends the geometric Frobenius to the uniformizers. If $a \in \mathbf{F}^{\times}$ or $a \in \mathcal{O}_E^{\times}$ then we write U_a for the unramified character sending the geometric Frobenius to a. We may regard a character of $G_{\mathbf{Q}_p}$ as a character of $\mathbf{Q}_p^{\times}$ via our normalization of class field theory.

As usual, we write S for the p-adic completion of $W(k)[u, \frac{u^{ie}}{i!}]_{i \in \mathbf{N}}$, and let $S_{\mathcal{O}_E} := S \otimes_{\mathbf{Z}_p} \mathcal{O}_E$ and $S_E := S_{\mathcal{O}_E} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. We also let $\overline{S}_{\mathbf{F}} := S_{\mathcal{O}_E}/(\varpi_E, \mathrm{Fil}^p S_{\mathcal{O}_E}) \cong (k \otimes_{\mathbf{F}_p} \mathbf{F})[u]/u^{ep}$. Choose a uniformizer ϖ of K and let $E(u) \in W(k)[u]$ be the monic minimal polynomial of ϖ. The group $\mathrm{Gal}(K/K_0)$ acts on S via the character $\widetilde{\omega}_{\varpi}$, and we write $(S_{\mathcal{O}_E})_{\widetilde{\omega}_{\varpi}^m}$ for the $\widetilde{\omega}_{\varpi}^m$-isotypical component of S for $m \in \mathbf{Z}$. We define $(\overline{S}_{\mathbf{F}})_{\omega_{\varpi}^m}$ in a similar fashion. If $\mathcal{O}_E$ or $\mathbf{F}$ are clear, we often omit them, i.e., we write $S_{\widetilde{\omega}_{\varpi}^m}$ and $\overline{S}_{\omega_{\varpi}^m}$ for $(S_{\mathcal{O}_E})_{\widetilde{\omega}_{\varpi}^m}$ and $(\overline{S}_{\mathbf{F}})_{\omega_{\varpi}^m}$ respectively. In particular,

$$\overline{S}_0 := \overline{S}_{\omega_{\varpi}^0} \cong (k \otimes_{\mathbf{F}_p} \mathbf{F})[u^e]/u^{ep}$$

and

$$S_0 := S_{\widetilde{\omega}_{\varpi}^0} = \left\{ \sum_{i=0}^{\infty} a_i \frac{E(u)^i}{i!} \mid a_i \in W(k) \otimes_{\mathbf{Z}_p} \mathcal{O}_E \text{ and } a_i \to 0 \;\, p\text{-adically} \right\}.$$

The association $a \otimes b \mapsto (\sigma(a)b)_{\sigma}$ gives rise to an isomorphism $k \otimes_{\mathbf{F}_p} \mathbf{F} \cong \coprod_{\sigma : k \hookrightarrow \mathbf{F}} \mathbf{F}$, and we write e_{σ} for the idempotent element in $k \otimes_{\mathbf{F}_p} \mathbf{F}$ that corresponds to the idempotent element in $\coprod_{\sigma : k \hookrightarrow \mathbf{F}} \mathbf{F}$ whose only non-zero entry is 1 at the position of σ.

To lighten the notation, we often write G for $\mathrm{GL}_{n/\mathbf{Z}_p}$. (By G_n, we mean an outer form of GL_n defined in Section 5.1.) We let B be the Borel subgroup of G consisting of upper-triangular matrices of G, U the unipotent subgroup of B, and T the torus of diagonal matrices of GL_n. We also write B^- and U^- for the opposite Borel of B and the unipotent subgroup of B^-, respectively. Let Φ^+ denote the set of positive roots with respect to (B,T), and $\Delta = \{\alpha_k\}_{1\leq k\leq n-1}$ the subset of simple positive roots. We also let W be the Weyl group of GL_n, which is often considered as a subgroup of GL_n, and let s_k be the simple reflection corresponding to α_k. We write w_0 for the longest Weyl element in W, and we hope that the reader is not confused with places w or w' of F.

We often write K for $\mathrm{GL}_n(\mathbf{Z}_p)$ for brevity. (Note that we use K for a tamely ramified extension of $\mathbf{Q}_p$ as well, and we hope that it does not confuse the reader.) We will often use the following three open compact subgroups of $\mathrm{GL}_n(\mathbf{Z}_p)$: if we let $\mathrm{red} : \mathrm{GL}_n(\mathbf{Z}_p) \twoheadrightarrow \mathrm{GL}_n(\mathbf{F}_p)$ be the natural mod p reduction map, then

$$K(1) := \mathrm{Ker}(\mathrm{red}) \subset I(1) := \mathrm{red}^{-1}(U(\mathbf{F}_p)) \subset I := \mathrm{red}^{-1}(B(\mathbf{F}_p)) \subset K.$$

If M is a free $\mathbf{F}$-module with a smooth action of K, then $T(\mathbf{F}_p)$ acts on the pro p Iwahori fixed subspace $M^{I(1)}$ via $I/I(1) \cong T(\mathbf{F}_p)$. We write $M^{I(1),\mu}$ for the eigenspace with respect to a character $\mu : T(\mathbf{F}_p) \to \mathbf{F}_p^\times$. $M^{I(1)}$ decomposes as

$$M^{I(1)} \cong \bigoplus M^{I(1),\mu}$$

as $T(\mathbf{F}_p)$-representations, where the direct sum runs over the characters μ of $T(\mathbf{F}_p)$. In the obvious similar fashion, we define $M^{I(1),\mu}$ when M is a free $\mathcal{O}_E$-module or a free E-module.

By $[m]_f$ for a rational number $m \in \mathbf{Z}[\frac{1}{p}] \subset \mathbf{Q}$ we mean the unique integer in $[0,e)$ congruent to m mod (e) via the natural surjection $\mathbf{Z}[\frac{1}{p}] \twoheadrightarrow \mathbf{Z}/e\mathbf{Z}$. By $\lfloor y \rfloor$ for $y \in \mathbf{R}$ we mean the floor function of y, i.e., the biggest integer less than or equal to y. For a set A, we write $|A|$ for the cardinality of A. If V is a finite-dimensional $\mathbf{F}$-representation of a group H, then we write $\mathrm{soc}_H V$ and $\mathrm{cosoc}_H V$ for the socle of V and the cosocle of V, respectively. If v is a non-zero vector in a free module over $\mathbf{F}$ (resp. over $\mathcal{O}_E$, resp. over E), then we write $\mathbf{F}[v]$ (resp. $\mathcal{O}_E[v]$, resp. $E[v]$) for the $\mathbf{F}$-line (resp. the $\mathcal{O}_E$-line, resp. the E-line) generated by v.

We write $\overline{x}$ for the image of $x \in \mathcal{O}_E$ under the natural surjection $\mathcal{O}_E \twoheadrightarrow \mathbf{F}$. We also have a natural surjection $\mathbb{P}^1(\mathcal{O}_E) \twoheadrightarrow \mathbb{P}^1(\mathbf{F})$ defined by letting $\overline{[x:y]} \in \mathbb{P}^1(\mathbf{F})$ be the image of $[x:y] \in \mathbb{P}^1(\mathcal{O}_E)$ where

$$\overline{[x:y]} = \begin{cases} [1 : \overline{(\frac{y}{x})}] & \text{if } \frac{y}{x} \in \mathcal{O}_E; \\ [\overline{(\frac{x}{y})} : 1] & \text{if } \frac{x}{y} \in \mathcal{O}_E. \end{cases}$$

We often write $\frac{y}{x}$ for $[x:y] \in \mathbb{P}^1(\mathbf{F})$ if $x \neq 0$.

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2. Integral p-adic Hodge theory

In this section, we do a quick review of some (integral) p-adic Hodge theory which will be needed later. We note that all of the results in this section are already known or easy generalization of known results. We closely follow [EGH15] as well as [HLM] in this section.

2.1. **Filtered (ϕ, N)-modules with descent data.** In this section, we review potentially semi-stable representations and their corresponding linear algebra objects, admissible filtered (ϕ, N)-modules with descent data.

Let K be a finite extension of $\mathbf{Q}_p$, and K_0 the maximal unramified subfield of K, so that $K_0 = W(k) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ where k is the residue field of K. We fix the uniformizer $p \in \mathbf{Q}_p$, so that we fix an embedding $\mathbf{B}_{\mathrm{st}} \hookrightarrow \mathbf{B}_{\mathrm{dR}}$. We also let K' be a subextension of K with K/K' Galois, and write $\phi \in \mathrm{Gal}(K_0/\mathbf{Q}_p)$ for the arithmetic Frobenius.

A p-adic Galois representation $\rho : G_{K'} \to \mathrm{GL}_n(E)$ is potentially semi-stable if there is a finite extension L of K' such that $\rho|_{G_L}$ is semi-stable, i.e., $\mathrm{rank}_{L_0 \otimes E} \mathrm{D}_{\mathrm{st}}^{K'}(V) = \dim_E V$, where V is an underlying vector space of ρ and $\mathrm{D}_{\mathrm{st}}^{K'}(V) := (\mathbf{B}_{\mathrm{st}} \otimes_{\mathbf{Q}_p} V)^{G_L}$. We often write $\mathrm{D}_{\mathrm{st}}^{K'}(\rho)$ for $\mathrm{D}_{\mathrm{st}}^{K'}(V)$. If K is the Galois closure of L over K', then $\rho|_{G_K}$ is semi-stable, provided that $\rho|_{G_L}$ is semi-stable.

Definition 2.1.1. *A filtered $(\phi, N, K/K', E)$-module of rank n is a free $K_0 \otimes E$-module D of rank n together with*
- *a $\phi \otimes 1$-automorphism ϕ on D;*
- *a nilpotent $K_0 \otimes E$-linear endomorphism N on D;*
- *a decreasing filtration $\{\mathrm{Fil}^i D_k\}_{i \in \mathbf{Z}}$ on $D_K = K \otimes_{K_0} D$ consisting of $K \otimes_{\mathbf{Q}_p} E$-submodules of D_K, which is exhaustive and separated;*
- *a K_0-semilinear, E-linear action of $\mathrm{Gal}(K/K')$ which commutes with ϕ and N and preserves the filtration on D_K.*

We say that D is *(weakly) admissible* if the underlying filtered $(\phi, N, K/K, E)$-module (with the trivial descent data) is weakly admissible in the sense of [Fon94]. The action of $\mathrm{Gal}(K/K')$ on D is often called *descent data* action. If V is potentially semi-stable, then $\mathrm{D}_{\mathrm{st}}^{K'}(V)$ is a typical example of an admissible filtered $(\phi, N, K/K', E)$-module of rank n.

Theorem 2.1.2 ([CF], Theorem 4.3)**.** *There is an equivalence of categories between the category of weakly admissible filtered $(\phi, N, K/K', E)$-modules of rank n and the category of n-dimensional potentially semi-stable E-representations of $G_{K'}$ that become semi-stable upon restriction to G_K.*

Note that Theorem 2.1.2 is proved in [CF] in the case $K = K'$, and that [Sav05] gives a generalization to the statement with non-trivial descent data.

If V is potentially semi-stable, then so is its dual $V^\vee$. We define $\mathrm{D}_{\mathrm{st}}^{*,K'}(V) := \mathrm{D}_{\mathrm{st}}^{K'}(V^\vee)$. Then $\mathrm{D}_{\mathrm{st}}^{*,K'}$ gives an anti-equivalence of categories from the category of n-dimensional potentially semi-stable E-representations of $G_{K'}$ that become semi-stable upon restriction to G_K to the category of weakly admissible filtered $(\phi, N, K/K', E)$-modules of rank n, with quasi-inverse

$$\mathrm{V}_{\mathrm{st}}^{*,K'}(D) := \mathrm{Hom}_{\phi,N}(D, \mathbf{B}_{\mathrm{st}}) \cap \mathrm{Hom}_{\mathrm{Fil}}(D_K, \mathbf{B}_{\mathrm{dR}}).$$

It will often be convenient to use covariant functors. We define an equivalence of categories: for each $r \in \mathbf{Z}$

$$\mathrm{V}_{\mathrm{st}}^{K',r}(D) := \mathrm{V}_{\mathrm{st}}^{*,K'}(D)^\vee \otimes \varepsilon^r.$$

The functor $\mathrm{D}_{\mathrm{st}}^{K',r}$ defined by $\mathrm{D}_{\mathrm{st}}^{K',r}(V) := \mathrm{D}_{\mathrm{st}}^{K'}(V \otimes \varepsilon^{-r})$ is a quasi-inverse of $\mathrm{V}_{\mathrm{st}}^{K',r}$.

For a given potentially semi-stable representation $\rho : G_{K'} \to \mathrm{GL}_n(E)$, one can attach a Weil–Deligne representation $\mathrm{WD}(\rho)$ to ρ, as in [CDT99], Appendix B.1. We refer to $\mathrm{WD}(\rho)|_{I_{\mathbf{Q}_p}}$ as to the *Galois type* associated to ρ. Note that $\mathrm{WD}(\rho)$ is defined via the filtered $(\phi, N, K/K', N)$-module $\mathrm{D}_{\mathrm{st}}^{K'}(\rho)$ and that $\mathrm{WD}(\rho)|_{I_{K'}} \cong \mathrm{WD}(\rho \otimes \varepsilon^r)|_{I_{K'}}$ for all $r \in \mathbf{Z}$.

Finally, we say that a potentially semi-stable representation ρ is *potentially crystalline* if the monodromy operator N on $\mathrm{D}_{\mathrm{st}}^{K'}(\rho)$ is trivial.

2.2. **Strongly divisible modules with descent data.** In this section, we review strongly divisible modules that correspond to Galois stable lattices in potentially semi-stable representations. We keep the notation of Section 2.1

From now on, we assume that K/K' is a tamely ramified Galois extension with ramification index $e(K/K')$. We fix a uniformizer $\varpi \in K$ with $\varpi^{e(K/K')} \in K'$. Let e be the absolute ramification index of K and $E(u) \in W(k)[u]$ the minimal polynomial of ϖ over K_0.

Let S be the p-adic completion of $W(k)[u, \frac{u^{ie}}{i!}]_{i\in\mathbf{N}}$. The ring S has additional structures:

- a continuous, ϕ-semilinear map $\phi : S \to S$ with $\phi(u) = u^p$ and $\phi(\frac{u^{ie}}{i!}) = \frac{u^{pie}}{i!}$;
- a continuous, $W(k)$-linear derivation of S with $N(u) = -u$ and $N(\frac{u^{ie}}{i!}) = -ie\frac{u^{ie}}{i!}$;
- a decreasing filtration $\{\mathrm{Fil}^i S\}_{i\in\mathbf{Z}_{\geq 0}}$ of S given by letting $\mathrm{Fil}^i S$ be the p-adic completion of the ideal $\sum_{j\geq i} \frac{E(u)^j}{j!} S$;
- a group action of $\mathrm{Gal}(K/K')$ on S defined for each $g \in \mathrm{Gal}(K/K')$ by the continuous ring isomorphism $\widehat{g} : S \to S$ with $\widehat{g}(w_i \frac{u^i}{\lfloor i/e\rfloor !}) = g(w_i)h_g^i \frac{u^i}{\lfloor i/e\rfloor !}$ for $w_i \in W(k)$, where $h_g \in W(k)$ satisfies $g(\varpi) = h_g\varpi$.

Note that ϕ and N satisfies $N\phi = p\phi N$ and that $\hat{g}(E(u)) = E(u)$ for all $g \in \mathrm{Gal}(K/K')$ since we assume $\varpi^{e(K/K')} \in K'$. We write ϕ_i for $\frac{1}{p^i}\phi$ on $\mathrm{Fil}^i S$. For $i \leq p-1$ we have $\phi(\mathrm{Fil}^i S) \subseteq p^i S$.

Let $S_{\mathcal{O}_E} := S \otimes_{\mathbf{Z}_p} \mathcal{O}_E$ and $S_E := S_{\mathcal{O}_E} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. We extend the definitions of ϕ, N, $\mathrm{Fil}^i S$, and the action of $\mathrm{Gal}(K/K')$ to $S_{\mathcal{O}_E}$ (resp. to S_E) $\mathcal{O}_E$-linearly (resp. E-linearly).

Definition 2.2.1. *Fix a positive integer $r < p-1$. A strongly divisible $\mathcal{O}_E$-module with descent data of weight r is a free $S_{\mathcal{O}_E}$-module $\widehat{\mathcal{M}}$ of finite rank together with*

- *a $S_{\mathcal{O}_E}$-submodule $\mathrm{Fil}^r \widehat{\mathcal{M}}$;*
- *additive maps $\phi, N : \widehat{\mathcal{M}} \to \widehat{\mathcal{M}}$;*
- *$S_{\mathcal{O}_E}$-semilinear bijections $\widehat{g} : \widehat{\mathcal{M}} \to \widehat{\mathcal{M}}$ for each $g \in \mathrm{Gal}(K/K')$*

such that

- $\mathrm{Fil}^r S_{\mathcal{O}_E} \cdot \widehat{\mathcal{M}} \subseteq \mathrm{Fil}^r \widehat{\mathcal{M}}$*;*
- $\mathrm{Fil}^r \widehat{\mathcal{M}} \cap I\widehat{\mathcal{M}} = I\mathrm{Fil}^r \widehat{\mathcal{M}}$ *for all ideals I in $\mathcal{O}_E$;*
- $\phi(sx) = \phi(s)\phi(x)$ *for all $s \in S_{\mathcal{O}_E}$ and for all $x \in \widehat{\mathcal{M}}$;*
- $\phi(\mathrm{Fil}^r \widehat{\mathcal{M}})$ *is contained in $p^r\widehat{\mathcal{M}}$ and generates it over $S_{\mathcal{O}_E}$;*
- $N(sx) = N(s)x + sN(x)$ *for all $s \in S_{\mathcal{O}_E}$ and for all $x \in \widehat{\mathcal{M}}$;*
- $N\phi = p\phi N$*;*
- $E(u)N(\mathrm{Fil}^r \widehat{\mathcal{M}}) \subseteq \mathrm{Fil}^r \widehat{\mathcal{M}}$*;*
- *for all $g \in \mathrm{Gal}(K/K')$ $\widehat{g}$ commutes with ϕ and N, and preserves* $\mathrm{Fil}^r \widehat{\mathcal{M}}$*;*
- $\widehat{g}_1 \circ \widehat{g}_2 = \widehat{g_1 \cdot g_2}$ *for all $g_1, g_2 \in \mathrm{Gal}(K/K')$.*

We write $\mathcal{O}_E\text{-}\mathrm{Mod}^r_{\mathrm{dd}}$ for the category of strongly divisible $\mathcal{O}_E$-modules with descent data of weight r. It is easy to see that the map $\phi_r = \frac{1}{p^r}\phi : \mathrm{Fil}^r \widehat{\mathcal{M}} \to \widehat{\mathcal{M}}$ satisfies $cN\phi_r(x) = \phi_r(E(u)N(x))$ for all $x \in \mathrm{Fil}^r \widehat{\mathcal{M}}$ where $c := \frac{\phi(E(u))}{p} \in S^\times$.

For a strongly divisible $\mathcal{O}_E$-module $\widehat{\mathcal{M}}$ with descent data of weight r, we define a $G_{K'}$-module $\mathrm{T}^{*,K'}_{\mathrm{st}}(\widehat{\mathcal{M}})$ as follows (cf. [EGH15], Section 3.1.):

$$\mathrm{T}^{*,K'}_{\mathrm{st}}(\widehat{\mathcal{M}}) := \mathrm{Hom}_{\mathrm{Fil}^r,\phi,N}(\widehat{\mathcal{M}}, \widehat{\mathbf{A}}_{\mathrm{st}}).$$

Proposition 2.2.2 ([EGH15], Proposition 3.1.4)**.** *The functor $\mathrm{T}^{*,K'}_{\mathrm{st}}$ provides an anti-equivalence of categories from the category $\mathcal{O}_E\text{-}\mathrm{Mod}^r_{\mathrm{dd}}$ to the category of $G_{K'}$-stable $\mathcal{O}_E$-lattices in finite-dimensional E-representations of $G_{K'}$ which become semi-stable over K with Hodge–Tate weights lying in $[-r, 0]$, when $0 < r < p-1$.*

Note that the case $K = K'$ and $E = \mathbf{Q}_p$ is proved by Liu [Liu08].

In this paper, we will be mainly interested in covariant functors $\mathrm{T}_{\mathrm{st}}^{K',r}$ from the category $\mathcal{O}_E$-$\mathrm{Mod}_{\mathrm{dd}}^r$ to the category $\mathrm{Rep}_{\mathcal{O}_E}^{K-\mathrm{st},[-r,0]}G_{K'}$ of $G_{K'}$-stable $\mathcal{O}_E$-lattices in finite-dimensional E-representations of $G_{K'}$ which become semi-stable over K with Hodge–Tate weights lying in $[-r,0]$ defined by

$$\mathrm{T}_{\mathrm{st}}^{K',r}(\widehat{\mathcal{M}}) := \mathrm{T}_{\mathrm{st}}^{*,K'}(\widehat{\mathcal{M}})^\vee \otimes \varepsilon^r.$$

Let $\widehat{\mathcal{M}}$ be in $\mathcal{O}_E$-$\mathrm{Mod}_{\mathrm{dd}}^r$, and define a free S_E-module $\mathcal{D} := \widehat{\mathcal{M}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. We extend ϕ and N on $\mathcal{D}$, and define a filtration on $\mathcal{D}$ as follows: $\mathrm{Fil}^r\mathcal{D} = \mathrm{Fil}^r\widehat{\mathcal{M}}[\frac{1}{p}]$ and

$$\mathrm{Fil}^i\mathcal{D} := \begin{cases} \mathcal{D} & \text{if } i \leq 0; \\ \{x \in \mathcal{D} \mid E(u)^{r-i}x \in \mathrm{Fil}^r\mathcal{D}\} & \text{if } 0 \leq i \leq r; \\ \sum_{j=0}^{i-1}(\mathrm{Fil}^{i-j}S_{\mathbf{Q}_p})(\mathrm{Fil}^j\mathcal{D}) & \text{if } i > r, \text{ inductively.} \end{cases}$$

We let $D := \mathcal{D} \otimes_{S_{\mathbf{Q}_p},s_0} K_0$ and $D_K := \mathcal{D} \otimes_{S_{\mathbf{Q}_p},s_\varpi} K$, where $s_0 : S_{\mathbf{Q}_p} \to K_0$ and $s_\varpi : S_{\mathbf{Q}_p} \to K$ are defined by $u \mapsto 0$ and $u \mapsto \varpi$ respectively, which induce ϕ and N on D and the filtration on D_K by taking $s_\varpi(\mathrm{Fil}^i\mathcal{D})$. The K_0-vector space D also inherits an E-linear action and a semi-linear action of $\mathrm{Gal}(K/K')$. Then it turns out that D is a weakly admissible filtered $(\phi, N, K/K', E)$-module with $\mathrm{Fil}^{r+1}D = 0$. Moreover, there is a compatibility (cf. [EGH15], Proof of Proposition 3.1.4.): if D corresponds to $\mathcal{D} = \widehat{\mathcal{M}}[\frac{1}{p}]$, then

$$\mathrm{T}_{\mathrm{st}}^{K',r}(\widehat{\mathcal{M}})[\frac{1}{p}] \cong \mathrm{V}_{\mathrm{st}}^{K',r}(D).$$

2.3. **Breuil modules with descent data.** In this section, we review Breuil modules with descent data. We keep the notation of Section 2.2, and assume further that $K' \subseteq K_0$.

We let $\overline{S} := S/(\varpi_E, \mathrm{Fil}^pS) \cong (k \otimes_{\mathbf{F}_p} \mathbf{F})[u]/u^{ep}$. It is easy to check that $\overline{S}$ inherits ϕ, N, the filtration of S, and the action of $\mathrm{Gal}(K/K')$.

Definition 2.3.1. *Fix a positive integer $r < p-1$. A* Breuil modules with descent data of weight *r is a free $\overline{S}$-module $\mathcal{M}$ of finite rank together with*

- *a $\overline{S}$-submodule $\mathrm{Fil}^r\mathcal{M}$ of $\mathcal{M}$;*
- *maps $\phi_r : \mathrm{Fil}^r\mathcal{M} \to \mathcal{M}$ and $N : \mathcal{M} \to \mathcal{M}$;*
- *additive bijections $\widehat{g} : \mathcal{M} \to \mathcal{M}$ for all $g \in \mathrm{Gal}(K/K')$*

such that

- *$\mathrm{Fil}^r\mathcal{M}$ contains $u^{er}\mathcal{M}$;*
- *ϕ_r is $\mathbf{F}$-linear and ϕ-semilinear (where $\phi : k[u]/u^{ep} \to k[u]/u^{ep}$ is the p-th power map) with image generating $\mathcal{M}$ as $\overline{S}$-module;*
- *N is $k \otimes_{\mathbf{F}_p} \mathbf{F}$-linear and satisfies*
  - *$N(ux) = uN(x) - ux$ for all $x \in \mathcal{M}$,*
  - *$u^eN(\mathrm{Fil}^r\mathcal{M}) \subseteq \mathrm{Fil}^r\mathcal{M}$, and*
  - *$\phi_r(u^eN(x)) = cN(\phi_r(x))$ for all $x \in \mathrm{Fil}^r\mathcal{M}$, where $c \in (k[u]/u^{ep})^\times$ is the image of $\frac{1}{p}\phi(E(u))$ under the natural map $S \to k[u]/u^{ep}$.*
- *$\widehat{g}$ preserves $\mathrm{Fil}^r\mathcal{M}$ and commutes with the ϕ_r and N, and the action satisfies $\widehat{g}_1 \circ \widehat{g}_2 = \widehat{g_1 \cdot g_2}$ for all $g_1, g_2 \in \mathrm{Gal}(K/K')$. Furthermore, if $a \in k \otimes_{\mathbf{F}_p} \mathbf{F}$ and $m \in \mathcal{M}$ then $\widehat{g}(au^im) = g(a)((\frac{g(\varpi)}{\varpi})^i \otimes 1)u^i\widehat{g}(m)$.*

We write $\mathbf{F}$-$\mathrm{BrMod}_{\mathrm{dd}}^r$ for the category of Breuil modules with descent data of weight r. For $\mathcal{M} \in \mathbf{F}$-$\mathrm{BrMod}_{\mathrm{dd}}^r$, we define a $G_{K'}$-module as follows (cf. [EGH15], Section 3.2):

$$\mathrm{T}_{\mathrm{st}}^*(\mathcal{M}) := \mathrm{Hom}_{\mathrm{BrMod}}(\mathcal{M}, \widehat{\mathbf{A}}).$$

This gives an exact faithful contravariant functor from the category $\mathbf{F}$-$\mathrm{BrMod}_{\mathrm{dd}}^r$ to the category $\mathrm{Rep}_{\mathbf{F}}G_{K'}$ of finite dimensional $\mathbf{F}$-representations of $G_{K'}$. We also define a covariant functor as follows: for each $r \in \mathbf{Z}$

$$\mathrm{T}^r_{\mathrm{st}}(\mathcal{M}) := \mathrm{T}^*_{\mathrm{st}}(\mathcal{M})^\vee \otimes \omega^r,$$

in which we will be more interested in this paper.

If $\widehat{\mathcal{M}}$ is a strongly divisible module with descent data, then

$$\mathcal{M} := \widehat{\mathcal{M}}/(\varpi_E, \mathrm{Fil}^p S)$$

is naturally an object in $\mathbf{F}$-$\mathrm{BrMod}^r_{\mathrm{dd}}$ ($\mathrm{Fil}^r\mathcal{M}$ is the image of $\mathrm{Fil}^r\widehat{\mathcal{M}}$ in $\mathcal{M}$, the map ϕ_r is induced by $\frac{1}{p^r}\phi|_{\mathrm{Fil}^r\widehat{\mathcal{M}}}$, and N and $\widehat{g}$ are those coming from $\widehat{\mathcal{M}}$). Moreover, there is a compatibility: if $\widehat{\mathcal{M}} \in \mathcal{O}_E$-$\mathrm{Mod}^r_{\mathrm{dd}}$ and we let $\mathcal{M} = \widehat{\mathcal{M}}/(\varpi_E, \mathrm{Fil}^p S)$ then

$$\mathrm{T}^{K',r}_{\mathrm{st}}(\widehat{\mathcal{M}}) \otimes_{\mathcal{O}_E} \mathbf{F} \cong \mathrm{T}^r_{\mathrm{st}}(\mathcal{M}).$$

(See [EGH15], Lemma 3.2.2 for detail.)

There is a notion of duality of Breuil modules, which will be convenient for our computation of Breuil modules as we will see later.

Definition 2.3.2. *Let $\mathcal{M} \in \mathbf{F}$-$\mathrm{BrMod}^r_{\mathrm{dd}}$. We define $\mathcal{M}^*$ as follows:*

- $\mathcal{M}^* := \mathrm{Hom}_{k[u]/u^{ep}-\mathrm{Mod}}(\mathcal{M}, k[u]/u^{ep})$*;*
- $\mathrm{Fil}^r\mathcal{M}^* := \{f \in \mathcal{M}^* \mid f(\mathrm{Fil}^r\mathcal{M}) \subseteq u^{er}k[u]/u^{ep}\}$*;*
- *$\phi_r(f)$ is defined by $\phi_r(f)(\phi_r(x)) = \phi_r(f(x))$ for all $x \in \mathrm{Fil}^r\mathcal{M}$ and $f \in \mathrm{Fil}^r\mathcal{M}^*$, where $\phi_r : u^{er}k[u]/u^{ep} \to k[u]/u^{ep}$ is the unique semilinear map sending u^{er} to c^r;*
- *$N(f) := N \circ f - f \circ N$, where $N : k[u]/u^{ep} \to k[u]/u^{ep}$ is the unique k-linear derivation such that $N(u) = -u$;*
- *$(\widehat{g}f)(x) = g(f(\widehat{g}^{-1}x))$ for all $x \in \mathcal{M}$ and $g \in \mathrm{Gal}(K/K')$, where $\mathrm{Gal}(K/K')$ acts on $k[u]/u^{ep}$ by $g(au^i) = g(a)(\frac{g(\varpi)}{\varpi})^i u^i$ for $a \in k$.*

If $\mathcal{M}$ is an object of $\mathbf{F}$-$\mathrm{BrMod}^r_{\mathrm{dd}}$ then so is $\mathcal{M}^*$. Moreover, we have $\mathcal{M} \cong \mathcal{M}^{**}$ and

$$\mathrm{T}^*_{\mathrm{st}}(\mathcal{M}^*) \cong \mathrm{T}^r_{\mathrm{st}}(\mathcal{M}).$$

(cf. [Car11]), Section 2.1.)

Finally, we review the notion of Breuil submodules developed mainly by [Car11]. See also [HLM], Section 2.3.

Definition 2.3.3. *Let $\mathcal{M}$ be an object of $\mathbf{F}$-$\mathrm{BrMod}^r_{\mathrm{dd}}$. A* Breuil submodule *of $\mathcal{M}$ is an $\overline{S}$-submodule $\mathcal{N}$ of $\mathcal{M}$ if $\mathcal{N}$ satisfies*

- *$\mathcal{N}$ is a $k[u]/u^{ep}$-direct summand of $\mathcal{M}$;*
- *$N(\mathcal{N}) \subseteq \mathcal{N}$ and $\widehat{g}(\mathcal{N}) \subseteq \mathcal{N}$ for all $g \in \mathrm{Gal}(K/K')$;*
- *$\phi_r(\mathcal{N} \cap \mathrm{Fil}^r\mathcal{M}) \subseteq \mathcal{N}$.*

If $\mathcal{N}$ is a Breuil submodule of $\mathcal{M}$, then $\mathcal{N}$ and $\mathcal{M}/\mathcal{N}$ are also objects of $\mathbf{F}$-$\mathrm{BrMod}^r_{\mathrm{dd}}$. We now state a crucial result we will use later.

Proposition 2.3.4 ([HLM], Proposition 2.3.5)**.** *Let $\mathcal{M}$ be an object in $\mathbf{F}$-$\mathrm{BrMod}^r_{\mathrm{dd}}$.*

Then there is a natural inclusion preserving bijection

$$\Theta : \{\textit{Breuil submodules in } \mathcal{M}\} \to \{G_{K'}\textit{-subrepresentations of } \mathrm{T}^r_{\mathrm{st}}(\mathcal{M})\}$$

sending $\mathcal{N} \subseteq \mathcal{M}$ to the image of $\mathrm{T}^r_{\mathrm{st}}(\mathcal{N}) \hookrightarrow \mathrm{T}^r_{\mathrm{st}}(\mathcal{M})$. Moreover, if $\mathcal{M}_2 \subseteq \mathcal{M}_1$ are Breuil submodules of $\mathcal{M}$, then $\Theta(\mathcal{M}_1)/\Theta(\mathcal{M}_2) \cong \mathrm{T}^r_{\mathrm{st}}(\mathcal{M}_1/\mathcal{M}_2)$.

We will also need classification of Breuil modules of rank 1 as follows. We denote the Breuil modules in the following lemma by $\mathcal{M}(a, s, \lambda)$.

Lemma 2.3.5 ([MP], Lemma 3.1)**.** *Let $k := \mathbf{F}_{p^f}$, $e := p^f - 1$, $\varpi := \sqrt[e]{-p}$, and $K' = \mathbf{Q}_p$. We also let $\mathcal{M}$ be a rank-one object in $\mathbf{F}$-$\mathrm{BrMod}^r_{\mathrm{dd}}$.*

Then there exists a generator $m \in \mathcal{M}$ such that:

(i) $\mathcal{M} = \overline{S}_{\mathbf{F}} \cdot m$;
(ii) $\mathrm{Fil}^r\mathcal{M} = u^{s(p-1)}\mathcal{M}$ *where* $0 \leq s \leq \frac{re}{p-1}$;
(iii) $\varphi_r(u^{s(p-1)}m) = \lambda m$ *for some* $\lambda \in (\mathbf{F}_{p^f} \otimes_{\mathbf{F}_p} \mathbf{F})^\times$;
(iv) $\widehat{g}(m) = (\omega_f(g)^a \otimes 1)m$ *for all* $g \in \mathrm{Gal}(K/K_0)$ *where* a *is an integer such that* $a + ps \equiv 0$ *mod* $(\frac{e}{p-1})$;
(v) $N(m) = 0$.

Moreover, one has

$$\mathrm{T}^r_{\mathrm{st}}(\mathcal{M})|_{I_{\mathbf{Q}_p}} = \omega_f^{a+ps}.$$

The following lemma will be used to determine if the Breuil modules violate the maximal non-splitness.

Lemma 2.3.6 ([MP], Lemma 3.2)**.** *Let* $k := \mathbf{F}_{p^f}$, $e := p^f - 1$, $\varpi := \sqrt[e]{-p}$, *and* $K' = \mathbf{Q}_p$. *We also let* $\mathcal{M}_x := \mathcal{M}(k_x, s_x, \lambda_x)$ *and* $\mathcal{M}_y := \mathcal{M}(k_y, s_y, \lambda_y)$ *be rank-one objects in* $\mathbf{F}\text{-}\mathrm{BrMod}^r_{\mathrm{dd}}$. *Assume that the integers* $k_x, k_y, s_x, s_y \in \mathbf{Z}$ *satisfy*

$$p(s_y - s_x) + [k_y - k_x]_f > 0. \tag{2.3.1}$$

Assume further that $f < p$ *and let*

$$0 \to \mathcal{M}_x \to \mathcal{M} \to \mathcal{M}_y \to 0$$

be an extension in $\mathbf{F}\text{-}\mathrm{BrMod}^r_{\mathrm{dd}}$, *with* $\mathrm{T}^*_{\mathrm{st}}(\mathcal{M})$ *being Fontaine–Laffaille.*

If the exact sequence of $\overline{S}_{\mathbf{F}}$*-modules*

$$0 \to \mathrm{Fil}^r\mathcal{M}_x \to \mathrm{Fil}^r\mathcal{M} \to \mathrm{Fil}^r\mathcal{M}_y \to 0$$

splits, then the $G_{\mathbf{Q}_p}$*-representation* $\mathrm{T}^*_{\mathrm{st}}(\mathcal{M})$ *splits as a direct sum of two characters.*

In particular, provided that $pk_y \not\equiv k_x$ *modulo* e *and that* $s_y(p-1) < re$ *if* $f > 1$, *the representation* $\mathrm{T}^*_{\mathrm{st}}(\mathcal{M})$ *splits as a direct sum of two characters if the element* $j_0 \in \mathbf{Z}$ *uniquely defined by*

$$j_0 e + [p^{-1}k_y - k_x]_f < s_x(p-1) \leq (j_0+1)e + [p^{-1}k_y - k_x]_f$$

satisfies

$$(r + j_0)e + [p^{-1}k_y - k_x]_f < (s_x + s_y)(p-1). \tag{2.3.2}$$

2.4. **Linear algebra with descent data.** In this section, we introduce the notion of framed basis for a Breuil module $\mathcal{M}$ and framed system of generators for $\mathrm{Fil}^r\mathcal{M}$. Throughout this section, we assume that $K_0 = K'$ and continue to assume that K is a tamely ramified Galois extension of K'. We also fix a positive integer $r < p - 1$.

Definition 2.4.1. *Let* $n \in \mathbf{N}$ *and let* $(k_{n-1}, k_{n-2}, \ldots, k_0) \in \mathbf{Z}^n$ *be an* n*-tuple. A rank* n *Breuil module* $\mathcal{M} \in \mathbf{F}\text{-}\mathrm{BrMod}^r_{\mathrm{dd}}$ is of (inertial) type $\omega_{\varpi}^{k_{n-1}} \oplus \cdots \oplus \omega_{\varpi}^{k_0}$ *if* $\mathcal{M}$ *has an* $\overline{S}$*-basis* $(e_{n-1}, \cdots, e_0)$ *such that* $\widehat{g}e_i = (\omega_{\varpi}^{k_i}(g) \otimes 1)e_i$ *for all* i *and all* $g \in \mathrm{Gal}(K/K_0)$. *We call such a basis a* framed basis of $\mathcal{M}$.

We also say that $\underline{f} := (f_{n-1}, f_{n-2}, \ldots, f_0)$ *is a* framed system of generators of $\mathrm{Fil}^r\mathcal{M}$ *if* $\underline{f}$ *is a system of* $\overline{S}$*-generators for* $\mathrm{Fil}^r\mathcal{M}$ *and* $\widehat{g}f_i = (\omega_{\varpi}^{p^{-1}k_i}(g) \otimes 1)f_i$ *for all* i *and all* $g \in \mathrm{Gal}(K/K_0)$.

The existence of a framed basis and a framed system of generators for a given Breuil module $\mathcal{M} \in \mathbf{F}\text{-}\mathrm{BrMod}^r_{\mathrm{dd}}$ is proved in [HLM], Section 2.2.2.

Let $\mathcal{M} \in \mathbf{F}\text{-}\mathrm{BrMod}^r_{\mathrm{dd}}$ be of inertial type $\bigoplus_{i=0}^{n-1} \omega_{\varpi}^{k_i}$, and let $\underline{e} := (e_{n-1}, \ldots, e_0)$ be a framed basis for $\mathcal{M}$ and $\underline{f} := (f_{n-1}, \ldots, f_0)$ be a framed system of generators for $\mathrm{Fil}^r\mathcal{M}$. The *matrix of the filtration*, with respect to $\underline{e}, \underline{f}$, is the matrix $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r\mathcal{M}) \in \mathrm{M}_n(\overline{S})$ such that

$$\underline{f} = \underline{e} \cdot \mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r\mathcal{M}).$$

Similarly, we define the *matrix of the Frobenius* with respect to $\underline{e}$, $\underline{f}$ as the matrix $\mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_r) \in \mathrm{GL}_n(\overline{S})$ characterized by

$$(\phi_r(f_{n-1}), \cdots, \phi_r(f_0)) = \underline{e} \cdot \mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_r).$$

As we require $\underline{e}$, $\underline{f}$ to be compatible with the framing, the entries in the matrix of the filtration satisfy the important additional properties:

$$\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r\mathcal{M})_{i,j} \in \overline{S}_{\omega_{\varpi}^{p^{f-1}k_j-k_i}}.$$

More precisely, $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r\mathcal{M})_{i,j} = u^{[p^{f-1}k_j-k_i]_f} s_{i,j}$, where $s_{i,j} \in \overline{S}_{\omega_{\varpi}^0} = k \otimes_{\mathbf{F}_p} \mathbf{F}[u^e]/(u^{ep})$.

We can therefore introduce the subspace $\mathrm{M}_n^{\square}(\overline{S})$ of matrices with framed type $\overline{\tau} = \bigoplus_{i=0}^{n-1} \omega_f^{k_i}$ as

$$\mathrm{M}_n^{\square}(\overline{S}) := \left\{ V \in \mathrm{M}_n(\overline{S}) \mid V_{i,j} \in \overline{S}_{\omega_f^{k_j-k_i}} \text{ for all } 0 \leq i,j \leq n-1 \right\}.$$

Similarly, we define

$$\mathrm{M}_n^{\square,\prime}(\overline{S}) := \left\{ V \in \mathrm{M}_n(\overline{S}) \mid V_{i,j} \in \overline{S}_{\omega_f^{p^{-1}k_j-k_i}} \text{ for all } 0 \leq i,j \leq n-1 \right\}$$

and

$$\mathrm{M}_n^{\square,\prime\prime}(\overline{S}) := \left\{ V \in \mathrm{M}_n(\overline{S}) \mid V_{i,j} \in \overline{S}_{\omega_f^{p^{-1}(k_j-k_i)}} \text{ for all } 0 \leq i,j \leq n-1 \right\}.$$

We also define

$$\mathrm{GL}_n^{\bullet}(\overline{S}) := \mathrm{GL}_n(\overline{S}) \cap \mathrm{M}_n^{\bullet}(\overline{S})$$

for $\bullet \in \{\square\} \cup \{\square,\prime\} \cup \{\square,\prime\prime\}$.

As $\varphi_r(f_i)$ is a $\omega_f^{k_i}$-eigenvector for the action of $\mathrm{Gal}(K/K_0)$ we deduce that

$$\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r\mathcal{M}) \in \mathrm{M}_n^{\square,\prime}(\overline{S}) \quad \text{and} \quad \mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_r) \in \mathrm{GL}_n^{\square}(\overline{S}).$$

Note that $\mathrm{M}_n^{\square}(\overline{S}) = \mathrm{M}_n^{\square,\prime}(\overline{S}) = \mathrm{M}_n^{\square,\prime\prime}(\overline{S})$ if the framed type $\overline{\tau}$ is of niveau 1.

We use similar terminologies for strongly divisible modules $\widehat{\mathcal{M}} \in \mathcal{O}_E\text{-}\mathrm{Mod}_{\mathrm{dd}}^r$.

Definition 2.4.2. *Let $n \in \mathbf{N}$ and let $(k_{n-1}, k_{n-2}, \ldots, k_0) \in \mathbf{Z}^n$ be an n-tuple. A rank n strongly divisible module $\widehat{\mathcal{M}} \in \mathcal{O}_E\text{-}\mathrm{Mod}_{\mathrm{dd}}^r$ is of (inertial) type $\widetilde{\omega}_{\varpi}^{k_{n-1}} \oplus \cdots \oplus \widetilde{\omega}_{\varpi}^{k_0}$ if $\widehat{\mathcal{M}}$ has an $S_{\mathcal{O}_E}$-basis $\underline{\widehat{e}} := (\widehat{e}_{n-1}, \cdots, \widehat{e}_0)$ such that $\widehat{g}\widehat{e}_i = (\widetilde{\omega}_{\varpi}^{k_i}(g) \otimes 1)\widehat{e}_i$ for all i and all $g \in \mathrm{Gal}(K/K_0)$. We call such a basis a* framed basis *for $\widehat{\mathcal{M}}$.*

We also say that $\underline{\widehat{f}} := (\widehat{f}_{n-1}, \widehat{f}_{n-2}, \ldots, \widehat{f}_0)$ is a framed system of generators for $\mathrm{Fil}^r\widehat{\mathcal{M}}$ *if $\underline{\widehat{f}}$ is a system of S-generators for $\mathrm{Fil}^r\widehat{\mathcal{M}}/\mathrm{Fil}^r S \cdot \widehat{\mathcal{M}}$ and $\widehat{g}\widehat{f}_i = (\widetilde{\omega}_{\varpi}^{p^{-1}k_i}(g) \otimes 1)\widehat{f}_i$ for all i and all $g \in \mathrm{Gal}(K/K_0)$.*

One can readily check the existence of a framed basis for $\widehat{\mathcal{M}}$ and a framed system of generators for $\mathrm{Fil}^r\widehat{\mathcal{M}}$ by Nakayama Lemma. For instance, the existence of a framed system of generators for $\mathrm{Fil}^r\widehat{\mathcal{M}}$ can be deduced as follows: if we let $\mathcal{M} := \widehat{\mathcal{M}}/(\varpi_E, \mathrm{Fil}^p S)$ is the Breuil module corresponding to the mod p reduction of the strongly divisible module $\widehat{\mathcal{M}}$ and write $\underline{f} = (f_{n-1}, f_{n-2}, \cdots, f_0)$ for a framed system of generators for $\mathrm{Fil}^r\mathcal{M}$, then it is obvious that each f_i has a lift $\widehat{f}_i \in \mathrm{Fil}^r\widehat{\mathcal{M}}$ such that $\widehat{g}\widehat{f}_i = (\widetilde{\omega}_{\varpi}^{p^{-1}k_i}(g) \otimes 1)\widehat{f}_i$ for all $g \in \mathrm{Gal}(K/K_0)$. Since $\mathrm{Fil}^r\widehat{\mathcal{M}}/\mathrm{Fil}^r S \cdot \widehat{\mathcal{M}}$ is a finitely generated $\mathcal{O}_E$-module, we conclude that the system $(\widehat{f}_{n-1}, \widehat{f}_{n-2}, \cdots, \widehat{f}_0)$ generates $\mathrm{Fil}^r\widehat{\mathcal{M}}/\mathrm{Fil}^r S \cdot \widehat{\mathcal{M}}$ by Nakayama Lemma.

We also define

$$\mathrm{Mat}_{\underline{\widehat{e}},\underline{\widehat{f}}}(\mathrm{Fil}^r\widehat{\mathcal{M}}) \quad \text{and} \quad \mathrm{Mat}_{\underline{\widehat{e}},\underline{\widehat{f}}}(\phi_r)$$

each of whose entries satisfies

$$\mathrm{Mat}_{\underline{\widehat{e}},\underline{\widehat{f}}}(\mathrm{Fil}^r\widehat{\mathcal{M}})_{i,j} \in S_{\widetilde{\omega}_{\varpi}^{p^{f-1}k_j-k_i}} \quad \text{and} \quad \mathrm{Mat}_{\underline{\widehat{e}},\underline{\widehat{f}}}(\phi_r)_{i,j} \in S_{\widetilde{\omega}_{\varpi}^{k_j-k_i}},$$

in the similar fashion to Breuil modules. In particular,

$$\mathrm{Mat}_{\underline{\widehat{e}},\underline{\widehat{f}}}(\mathrm{Fil}^r\widehat{\mathcal{M}}) \in \mathrm{M}_n^{\square,\prime}(S) \quad \text{and} \quad \mathrm{Mat}_{\underline{\widehat{e}},\underline{\widehat{f}}}(\varphi_r) \in \mathrm{GL}_n^{\square}(S)$$

where $\mathrm{M}_n^{\square,\prime}(S)$ and $\mathrm{GL}_n^{\square}(S)$ are defined in the similar way to Breuil modules. We also define $\mathrm{GL}_n^{\square,\prime\prime}(S)$ in the similar way to Breuil modules again.

The inertial types on a Breuil module $\mathcal{M}$ and on a strongly divisible modules are closely related to the Weil–Deligne representation associated to a potentially crystalline lift of $\mathrm{T}_{\mathrm{st}}^r(\mathcal{M})$.

Proposition 2.4.3 ([LMP], Proposition 2.12)**.** *Let $\widehat{\mathcal{M}}$ be an object in $\mathcal{O}_E$-$\mathrm{Mod}_{\mathrm{dd}}^r$ and let $\mathcal{M} := \widehat{\mathcal{M}} \otimes_S S/(\varpi_E, \mathrm{Fil}^p S)$ be the Breuil module corresponding to the mod p reduction of $\widehat{\mathcal{M}}$.*

If $\mathrm{T}_{\mathrm{st}}^{K_0,r}(\widehat{\mathcal{M}})[\frac{1}{p}]$ has Galois type $\bigoplus_{i=0}^{n-1} \widetilde{\omega}_f^{k_i}$ for some integers k_i, then $\widehat{\mathcal{M}}$ (resp. $\mathcal{M}$) is of inertial type $\bigoplus_{i=0}^{n-1} \widetilde{\omega}_{\varpi}^{k_i}$ (resp. $\bigoplus_{i=0}^{n-1} \omega_{\varpi}^{k_i}$).

Finally, we need a technical result on change of basis of Breuil modules with descent data.

Lemma 2.4.4 ([HLM], Lemma 2.2.8)**.** *Let $\mathcal{M} \in \mathbf{F}$-$\mathrm{BrMod}_{\mathrm{dd}}^r$ be of type $\bigoplus_{i=0}^{n-1} \omega_{\varpi}^{k_i}$, and let $\underline{e}$, $\underline{f}$ be a framed basis for $\mathcal{M}$ and a framed system of generators for $\mathrm{Fil}^r\mathcal{M}$ respectively. Write $V := \mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r\mathcal{M}) \in \mathrm{M}_n^{\square,\prime}(\overline{S})$ and $A := \mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_r) \in \mathrm{GL}_n^{\square}(\overline{S})$, and assume that there are invertible matrices $R \in \mathrm{GL}_n^{\square}(\overline{S})$ and $C \in \mathrm{GL}_n^{\square,\prime\prime}(\overline{S})$ such that*

$$R \cdot V \cdot C \equiv V' \bmod (u^{e(r+1)}),$$

for some $V' \in \mathrm{M}_n^{\square,\prime}(\overline{S})$.

Then $\underline{e}' := \underline{e} \cdot R^{-1}$ forms another framed basis for $\mathcal{M}$ and $\underline{f}' := \underline{e}' \cdot V'$ forms another framed system of generators for $\mathrm{Fil}^r\mathcal{M}$ such that

$$\mathrm{Mat}_{\underline{e}',\underline{f}'}(\mathrm{Fil}^r\mathcal{M}) = V' \in \mathrm{M}_n^{\square,\prime}(\overline{S}) \quad \textit{and} \quad \mathrm{Mat}_{\underline{e}',\underline{f}'}(\phi_r) = R \cdot A \cdot \phi(C) \in \mathrm{GL}_n^{\square}(\overline{S}).$$

In particular, if $R^{-1} = A$ then $\mathrm{Mat}_{\underline{e}',\underline{f}'}(\phi_r) = \phi(C)$.

The statement of Lemma 2.4.4 is slightly more general than [HLM], Lemma 2.2.8, but exactly the same argument works.

2.5. **Fontaine–Laffaille modules.** In this section, we briefly recall the theory of Fontaine–Laffaille modules over $\mathbf{F}$, and we continue to assume that $K_0 = K'$ and that K is a tamely ramified Galois extension of K'.

Definition 2.5.1. *A* Fontaine–Laffaille module *over $k \otimes_{\mathbf{F}_p} \mathbf{F}$ is the datum $(M, \mathrm{Fil}^\bullet M, \phi_\bullet)$ of*

- *a free $k \otimes_{\mathbf{F}_p} \mathbf{F}$-module M of finite rank;*
- *a decreasing, exhaustive and separated filtration $\{\mathrm{Fil}^j M\}_{j\in\mathbf{Z}}$ on M by $k \otimes_{\mathbf{F}_p} \mathbf{F}$-submodules;*
- *a ϕ-semilinear isomorphism $\phi_\bullet : \mathrm{gr}^\bullet M \to M$, where $\mathrm{gr}^\bullet M := \bigoplus_{j\in\mathbf{Z}} \frac{\mathrm{Fil}^j M}{\mathrm{Fil}^{j+1} M}$.*

We write $\mathbf{F}$-FLMod_k for the category of Fontaine–Laffaille modules over $k \otimes_{\mathbf{F}_p} \mathbf{F}$, which is abelian. If the field k is clear from the context, we simply write $\mathbf{F}$-FLMod to lighten the notation.

Given a Fontaine–Laffaille module M, the set of its Hodge–Tate weights in the direction of $\sigma \in \mathrm{Gal}(k/\mathbf{F}_p)$ is defined as $\mathrm{HT}_\sigma := \left\{i \in \mathbf{Z} \mid e_\sigma \mathrm{Fil}^i M \neq e_\sigma \mathrm{Fil}^{i+1} M\right\}$. In the remainder of this paper we will be focused on Fontaine–Laffaille modules with *parallel* Hodge–Tate weights, i.e. we will assume that for all $i \in \mathbf{Z}$, the submodules $\mathrm{Fil}^i M$ are free over $k \otimes_{\mathbf{F}_p} \mathbf{F}$.

Definition 2.5.2. *Let M be a Fontaine–Laffaille module with parallel Hodge–Tate weights. A $k \otimes_{\mathbf{F}_p} \mathbf{F}$ basis $\underline{f} = (f_0, f_1, \ldots, f_{n-1})$ on M is* compatible with the filtration *if for all $i \in \mathbf{Z}_{\geq 0}$ there exists $j_i \in \mathbf{Z}_{\geq 0}$ such that $\mathrm{Fil}^i M = \sum_{j=j_i}^{n} k \otimes_{\mathbf{F}_p} \mathbf{F} \cdot f_j$. In particular, the principal symbols $(\mathrm{gr}(f_0), \ldots, \mathrm{gr}(f_{n-1}))$ provide a $k \otimes_{\mathbf{F}_p} \mathbf{F}$ basis for $\mathrm{gr}^\bullet M$.*

Note that if the graded pieces of the Hodge filtration have rank at most one then any two compatible basis on M are related by a lower-triangular matrix in $\mathrm{GL}_n(k\otimes_{\mathbf{F}_p}\mathbf{F})$. Given a Fontaine–Laffaille module and a compatible basis $\underline{f}$, it is convenient to describe the Frobenius action via a matrix $\mathrm{Mat}_{\underline{f}}(\phi_\bullet) \in \mathrm{GL}_n(k \otimes_{\mathbf{F}_p} \mathbf{F})$, defined in the obvious way using the principal symbols $(\mathrm{gr}(f_0), \dots, \mathrm{gr}(f_{n-1}))$ as a basis on $\mathrm{gr}^\bullet M$.

It is customary to write $\mathbf{F}\text{-FLMod}^{[0,p-2]}$ to denote the full subcategory of $\mathbf{F}$-FLMod formed by those modules M verifying $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^{p-1}M = 0$ (it is again an abelian category). We have the following description of mod p Galois representations of G_{K_0} via Fontaine–Laffaille modules:

Proposition 2.5.3 ([FL82], Theorem 6.1)**.** *There is an exact fully faithful contravariant functor*

$$\mathrm{T}^*_{\mathrm{cris},K_0}:\ \mathbf{F}\text{-FLMod}_k^{[0,p-2]} \to \mathrm{Rep}_{\mathbf{F}}(G_{K_0})$$

which is moreover compatible with the restriction over unramified extensions: if L_0/K_0 is unramified with residue field l/k and if M is an object in $\mathbf{F}\text{-FLMod}_k^{[0,p-2]}$, then $l \otimes_k M$ is naturally regarded as an object in $\mathbf{F}\text{-FLMod}_l^{[0,p-2]}$ and

$$\mathrm{T}^*_{\mathrm{cris},L_0}(l \otimes_k M) \cong \mathrm{T}^*_{\mathrm{cris},K_0}(M)|_{G_{L_0}}.$$

We will often write $\mathrm{T}^*_{\mathrm{cris}}$ for $\mathrm{T}^*_{\mathrm{cris},K_0}$ if the base field K_0 is clear from the context.

Definition 2.5.4. *We say that $\overline{\rho} \in \mathrm{Rep}_{\mathbf{F}} G_{K_0}$ is* Fontaine–Laffaille *if $\mathrm{T}^*_{\mathrm{cris}}(M) \cong \overline{\rho}$ for some $M \in \mathbf{F}\text{-FLMod}^{[0,p-2]}$.*

2.6. **Étale ϕ-modules.** In this section, we review the theory of étale ϕ-modules, first introduced by Fontaine [Fon90], and its connection with Breuil modules and Fontaine–Laffaille modules. Throughout this section, we continue to assume that $K_0 = K'$ and that K is a tamely ramified Galois extension of K'.

Let $p_0 := -p$, and let $\underline{p}$ be identified with a sequence $(p_n)_n \in \left(\overline{\mathbf{Q}}_p\right)^{\mathbf{N}}$ verifying $p_n^p = p_{n-1}$ for all n. We also fix $\varpi := \sqrt[e]{-p} \in K$, and let $\varpi_0 = \varpi$. We fix a sequence $(\varpi_n)_n \in \left(\overline{\mathbf{Q}}_p\right)^{\mathbf{N}}$ such that $\varpi_n^e = p_n$ and $\varpi_n^p = \varpi_{n-1}$ for all $n \in \mathbf{N}$, and which is compatible with the norm maps $K(\varpi_{n+1}) \to K(\varpi_n)$ (cf. [Bre14], Appendix A). By letting $K_\infty := \cup_{n\in\mathbf{N}} K(\varpi_n)$ and $(K_0)_\infty := \cup_{n\in\mathbf{N}} K_0(p_n)$, we have a canonical isomorphism $\mathrm{Gal}(K_\infty/(K_0)_\infty) \xrightarrow{\sim} \mathrm{Gal}(K/K_0)$ and we will identify ω_ϖ as a character of $\mathrm{Gal}(K_\infty/(K_0)_\infty)$. The field of norms $k((\underline{\varpi}))$ associated to (K,ϖ) is then endowed with a residual action of $\mathrm{Gal}(K_\infty/(K_0)_\infty)$, which is completely determined by $\widehat{g}(\underline{\varpi}) = \omega_\varpi(g)\underline{\varpi}$.

We define the category $\left(\phi, \mathbf{F}\otimes_{\mathbf{F}_p} k((\underline{p}))\right)\text{-}\mathfrak{Mod}$ of étale $(\phi, \mathbf{F}\otimes_{\mathbf{F}_p} k((\underline{p})))$-modules as the category of free $\mathbf{F}\otimes_{\mathbf{F}_p} k((\underline{p}))$-modules of finite rank $\mathfrak{M}$ endowed with a semilinear map $\phi : \mathfrak{M} \to \mathfrak{M}$ with respect to the Frobenius on $k((\underline{p}))$ and inducing an isomorphism $\phi^*\mathfrak{M} \to \mathfrak{M}$ (with obvious morphisms between objects). We also define the category $(\phi, \mathbf{F}\otimes_{\mathbf{F}_p} k((\underline{\varpi})))\text{-}\mathfrak{Mod}_{\mathrm{dd}}$ of étale $(\phi, \mathbf{F}\otimes_{\mathbf{F}_p} k((\underline{\varpi})))$-modules with descent data: an object $\mathfrak{M}$ is defined as for the category $(\phi, \mathbf{F}\otimes_{\mathbf{F}_p} k((\underline{p})))\text{-}\mathfrak{Mod}$ but we moreover require that $\mathfrak{M}$ is endowed with a semilinear action of $\mathrm{Gal}(K_\infty/(K_0)_\infty)$ (semilinear with respect to the residual action on $\mathbf{F}\otimes_{\mathbf{F}_p} k((\underline{\varpi}))$ where $\mathbf{F}$ is endowed with the trivial $\mathrm{Gal}(K_\infty/(K_0)_\infty)$-action) commuting with ϕ.

By work of Fontaine [Fon90], there are anti-equivalences

$$\left(\phi, \mathbf{F}\otimes_{\mathbf{F}_p} k((\underline{p}))\right)\text{-}\mathfrak{Mod} \xrightarrow{\sim} \mathrm{Rep}_{\mathbf{F}}(G_{(K_0)_\infty})$$

and

$$\left(\phi, \mathbf{F}\otimes_{\mathbf{F}_p} k((\underline{\varpi}))\right)\text{-}\mathfrak{Mod}_{\mathrm{dd}} \xrightarrow{\sim} \mathrm{Rep}_{\mathbf{F}}(G_{(K_0)_\infty})$$

given by

$$\mathfrak{M} \longmapsto \mathrm{Hom}\left(\mathfrak{M}, k((\underline{p}))^{\mathrm{sep}}\right)$$

and

$$\mathfrak{M} \mapsto \mathrm{Hom}\left(\mathfrak{M}, k((\underline{\varpi}))^{\mathrm{sep}}\right)$$

respectively. See also [HLM], Appendix A.2.

The following proposition summarizes the relation between the various categories and functors we introduced above.

Proposition 2.6.1 ([HLM], Proposition 2.2.1)**.** *There exist faithful functors*

$$M_{k((\underline{\varpi}))} : \mathbf{F}\text{-BrMod}^r_{\mathrm{dd}} \to \left(\phi, \mathbf{F} \otimes_{\mathbf{F}_p} k((\underline{\varpi}))\right)\text{-}\mathfrak{Mod}_{\mathrm{dd}}$$

and

$$\mathcal{F} : \mathbf{F}\text{-FLMod}^{[0,p-2]} \to \left(\phi, \mathbf{F} \otimes_{\mathbf{F}_p} k((\underline{p}))\right)\text{-}\mathfrak{Mod}$$

fitting in the following commutative diagram:

$$\begin{tikzcd}
\mathbf{F}\text{-BrMod}^r_{\mathrm{dd}} \arrow[rr, "M_{k((\underline{\varpi}))}"] \arrow[d, "\mathrm{T}^*_{\mathrm{st}}"'] & & \left(\phi, \mathbf{F} \otimes_{\mathbf{F}_p} k((\underline{\varpi}))\right)\text{-}\mathfrak{Mod}_{\mathrm{dd}} \arrow[dl, "\mathrm{Hom}(-,k((\underline{\varpi}))^{\mathrm{sep}})"'] \\
\mathrm{Rep}_{\mathbf{F}}(G_{K_0}) \arrow[r, "\mathrm{Res}"] & \mathrm{Rep}_{\mathbf{F}}(G_{(K_0)_\infty}) & \\
\mathbf{F}\text{-FLMod}^{[0,p-2]} \arrow[rr, "\mathcal{F}"'] \arrow[u, "\mathrm{T}^*_{\mathrm{cris}}"] & & \left(\phi, \mathbf{F} \otimes_{\mathbf{F}_p} k((\underline{p}))\right)\text{-}\mathfrak{Mod} \arrow[ul, "\mathrm{Hom}(-,k((\underline{p}))^{\mathrm{sep}})"] \arrow[uu, "-\otimes_{k((\underline{p}))} k((\underline{\varpi}))"']
\end{tikzcd}$$

where the descent data is relative to K_0 *and the functor* $\mathrm{Res} \circ \mathrm{T}^*_{\mathrm{cris}}$ *is fully faithful.*

Note that the functors $M_{k((\underline{\varpi}))}$ and $\mathcal{F}$ are defined in [BD14]. (See also [HLM], Appendix A). The following is an immediate consequence of Proposition 2.6.1, which is also stated in [LMP], Corollary 2.14.

Corollary 2.6.2. *Let* $0 \le r \le p-2$*, and let* $\mathcal{M}$ *(resp.* M*) be an object in* $\mathbf{F}\text{-BrMod}^r_{\mathrm{dd}}$ *(resp. in* $\mathbf{F}\text{-FLMod}^{[0,p-2]}$*). Assume that* $\mathrm{T}^*_{\mathrm{st}}(\mathcal{M})$ *is Fontaine–Laffaille. If*

$$M_{k((\underline{\varpi}))}(\mathcal{M}) \cong \mathcal{F}(M) \otimes_{k((\underline{p}))} k((\underline{\varpi}))$$

then one has an isomorphism of G_{K_0}*-representations*

$$\mathrm{T}^*_{\mathrm{st}}(\mathcal{M}) \cong \mathrm{T}^*_{\mathrm{cris}}(M).$$

The following two lemmas are very crucial in this paper, as we will see later, which describe the functors $M_{k((\underline{\varpi}))}$ and $\mathcal{F}$ respectively.

Lemma 2.6.3 ([HLM], Lemma 2.2.6)**.** *Let* $\mathcal{M}$ *be a Breuil module of inertial type* $\bigoplus_{i=0}^{n-1} \omega_{\varpi}^{k_i}$ *with a framed basis* $\underline{e}$ *for* $\mathcal{M}$ *and a framed system of generators* $\underline{f}$ *for* $\mathrm{Fil}^r\mathcal{M}$*, and write* $\mathcal{M}^*$ *for its dual as defined in Definition 2.3.2. Let* $V = \mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r\mathcal{M}) \in \mathrm{M}_n^{\square,\prime}(\overline{S})$ *and* $A = \mathrm{Mat}_{\underline{e},\underline{f}}(\phi_r) \in \mathrm{GL}_n^{\square}(\overline{S})$*.*

Then there exists a basis $\mathfrak{e}$ *for* $M_{k((\underline{\varpi}))}(\mathcal{M}^*)$ *with* $\widehat{g} \cdot \mathfrak{e}_i = (\omega_{\varpi}^{-p^{-1}k_i}(g) \otimes 1)\mathfrak{e}_i$ *for all* $i \in \{0, 1, \cdots, n-1\}$ *and* $g \in \mathrm{Gal}(K/K_0)$*, such that the Frobenius* ϕ *on* $M_{k((\underline{\varpi}))}(\mathcal{M}^*)$ *is described by*

$$\mathrm{Mat}_{\underline{\mathfrak{e}}}(\phi) = \widehat{V}^t \left(\widehat{A}^{-1}\right)^t \in \mathrm{M}_n(\mathbf{F} \otimes_{\mathbf{F}_p} k[[\underline{\varpi}]])$$

where $\widehat{V}$*,* $\widehat{A}$ *are lifts of* V*,* A *in* $\mathrm{M}_n(\mathbf{F} \otimes_{\mathbf{F}_p} k[[\underline{\varpi}]])$ *via the reduction morphism* $\mathbf{F} \otimes_{\mathbf{F}_p} k[[\underline{\varpi}]] \twoheadrightarrow \overline{S}$ *induced by* $\underline{\varpi} \mapsto u$ *and* $\mathrm{Mat}_{\underline{\mathfrak{e}}}(\phi)_{i,j} \in \left(\mathbf{F} \otimes_{\mathbf{F}_p} k[[\underline{\varpi}]]\right)_{\omega_{\varpi}^{p^{-1}k_i - k_j}}$.

Lemma 2.6.4 ([HLM], Lemma 2.2.7)**.** *Let $M \in \mathbf{F}\text{-FLMod}^{[0,p-2]}$ be a rank n Fontaine–Laffaille module with parallel Hodge–Tate weights $0 \le m_0 \le \cdots \le m_{n-1} \le p-2$ (counted with multiplicity). Let $\underline{e} = (e_0, \ldots, e_{n-1})$ be a $k \otimes_{\mathbf{F}_p} \mathbf{F}$ basis for M, compatible with the Hodge filtration $\mathrm{Fil}^\bullet M$ and let $F \in \mathrm{M}_n(k \otimes_{\mathbf{F}_p} \mathbf{F})$ be the associated matrix of the Frobenius $\phi_\bullet : \mathrm{gr}^\bullet M \to M$.*

Then there exists a basis $\underline{\mathfrak{e}}$ for $\mathfrak{M} := \mathcal{F}(M)$ such that the Frobenius ϕ on $\mathfrak{M}$ is described by

$$\mathrm{Mat}_{\underline{\mathfrak{e}}}(\phi) = \mathrm{Diag}\left(\underline{p}^{m_0}, \cdots, \underline{p}^{m_{n-1}}\right) \cdot F \in \mathrm{M}_n(\mathbf{F} \otimes_{\mathbf{F}_p} k[[\underline{p}]]).$$

3. Local Galois side

In this section, we study ordinary Galois representations and their potentially crystalline lifts. In particular, we prove that the Frobenius eigenvalues of certain potentially crystalline lifts preserve the information of the wildly ramified part of ordinary representations.

Throughout this section, we let f be a positive integer, $K' = \mathbf{Q}_p$, $e = p^f - 1$, and $K = \mathbf{Q}_{p^f}(\sqrt[e]{-p})$. We also fix $\varpi := \sqrt[e]{-p}$, and let $\overline{S} = (\mathbf{F}_{p^f} \otimes_{\mathbf{F}_p} \mathbf{F})[u]/u^{ep}$ and $\overline{S}_0 := \overline{S}_{\omega_f^0} = (\mathbf{F}_{p^f} \otimes_{\mathbf{F}_p} \mathbf{F})[u^e]/u^{ep} \subseteq \overline{S}$. Recall that by $[m]_f$ for a rational number $m \in \mathbf{Z}[\frac{1}{p}]$ we mean the unique integer in $[0, e)$ congruent to m mod (e).

We say that a representation $\overline{\rho}_0 : G_{\mathbf{Q}_p} \to \mathrm{GL}_n(\mathbf{F})$ is *ordinary* if it is isomorphic to a representation whose image is contained in the Borel subgroup of upper-triangular matrices. Namely, an ordinary representation has a basis $\underline{e} := (e_{n-1}, e_{n-2}, \cdots, e_0)$ that gives rise to a matrix form as follows:

$$\overline{\rho}_0 \cong \begin{pmatrix} \mathrm{U}_{\mu_{n-1}}\omega^{c_{n-1}+(n-1)} & *_{n-1} & \cdots & * & * \\ 0 & \mathrm{U}_{\mu_{n-2}}\omega^{c_{n-2}+(n-2)} & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathrm{U}_{\mu_1}\omega^{c_1+1} & *_1 \\ 0 & 0 & \cdots & 0 & \mathrm{U}_{\mu_0}\omega^{c_0} \end{pmatrix} \tag{3.0.1}$$

Here, U_μ is the unramified character sending the geometric Frobenius to $\mu \in \mathbf{F}^\times$ and c_i are integers. By $\overline{\rho}_0$, we always mean an n-dimensional ordinary representation that is written as in (3.0.1). For $n-1 \ge i \ge j \ge 0$, we write

$$\overline{\rho}_{i,j} \tag{3.0.2}$$

for the $(i-j+1)$-dimensional subquotient of $\overline{\rho}_0$ determined by the subset $(e_i, e_{i-1}, \cdots, e_j)$ of the basis $\underline{e}$. For instance, $\overline{\rho}_{i,i} = \mathrm{U}_{\mu_i}\omega^{c_i+i}$ and $\overline{\rho}_{n-1,0} = \overline{\rho}_0$.

An ordinary representation $G_{\mathbf{Q}_p} \to \mathrm{GL}_n(\mathbf{F})$ is *maximally non-split* if its socle filtration has length n. For instance, $\overline{\rho}_0$ in (3.0.1) is maximally non-split if and only if $*_i \neq 0$ for all $i = 1, 2, \cdots, n-1$. In this paper, we are interested in ordinary maximally non-split representations satisfying a certain genericity condition.

Definition 3.0.5. *We say that $\overline{\rho}_0$ is* generic *if*

$$c_{i+1} - c_i > n-1 \text{ for all } i \in \{0, 1, \cdots, n-2\} \text{ and } c_{n-1} - c_0 < (p-1) - (n-1).$$

We say that $\overline{\rho}_0$ is strongly generic *if $\overline{\rho}_0$ is generic and*

$$c_{n-1} - c_0 < (p-1) - (3n-5).$$

Note that this strongly generic condition implies $p > n^2 + 2(n-3)$.

We describe a rough shape of the Breuil modules with descent data from K to $K' = \mathbf{Q}_p$ corresponding to $\overline{\rho}_0$. Let r be a positive integer with $p-1 > r \ge n-1$, and let $\mathcal{M} \in \mathbf{F}\text{-BrMod}^r_{\mathrm{dd}}$ be a Breuil module of inertial type $\bigoplus_{i=0}^{n-1} \omega_f^{k_i}$ such that $\mathrm{T}^r_{\mathrm{st}}(\mathcal{M}) \cong \overline{\rho}_0$, for some $k_i \in \mathbf{Z}$. By Proposition 2.3.4, we note that $\mathcal{M}$ is a successive extension of $\mathcal{M}_i$, where $\mathcal{M}_i := \mathcal{M}(k_i, r_i, \nu_i)$ (cf. Lemma 2.3.5) is a rank one Breuil module of inertial type $\omega_f^{k_i}$ such that

$$\omega_f^{k_i+pr_i} \cong \mathrm{T}^r_{\mathrm{st}}(\mathcal{M}_i)|_{I_{\mathbf{Q}_p}} \cong \omega^{c_i+i} \tag{3.0.3}$$

for each $i \in \{0, 1, \cdots, n-1\}$. More precisely, there exist a framed basis $\underline{e} = (e_{n-1}, e_{n-2}, \cdots, e_0)$ for $\mathcal{M}$ and a framed system of generators $\underline{f} = (f_{n-1}, f_{n-2}, \cdots, f_0)$ for $\mathrm{Fil}^r\mathcal{M}$ such that

$$\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r\mathcal{M}) = \begin{pmatrix} u^{r_{n-1}(p-1)} & u^{[p^{-1}k_{n-2}-k_{n-1}]_f}v_{n-1,n-2} & \cdots & u^{[p^{-1}k_0-k_{n-1}]_f}v_{n-1,0} \\ 0 & u^{r_{n-2}(p-1)} & \cdots & u^{[p^{-1}k_0-k_{n-2}]_f}v_{n-2,0} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u^{r_0(p-1)} \end{pmatrix}, \tag{3.0.4}$$

$$\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_r) = \begin{pmatrix} \nu_{n-1} & u^{[k_{n-2}-k_{n-1}]_f}w_{n-1,n-2} & \cdots & u^{[k_0-k_{n-1}]_f}w_{n-1,0} \\ 0 & \nu_{n-2} & \cdots & u^{[k_0-k_{n-2}]_f}w_{n-2,0} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \nu_0 \end{pmatrix}, \tag{3.0.5}$$

and

$$\mathrm{Mat}_{\underline{e}}(N) = \begin{pmatrix} 0 & u^{[k_{n-2}-k_{n-1}]_f}\gamma_{n-1,n-2} & \cdots & u^{[k_1-k_{n-1}]_f}\gamma_{n-1,1} & u^{[k_0-k_{n-1}]_f}\gamma_{n-1,0} \\ 0 & 0 & \cdots & u^{[k_1-k_{n-2}]_f}\gamma_{n-2,1} & u^{[k_0-k_{n-2}]_f}\gamma_{n-2,0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & u^{[k_0-k_1]_f}\gamma_{1,0} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{3.0.6}$$

for some $\nu_i \in (\mathbf{F}_{p^f} \otimes_{\mathbf{F}_p} \mathbf{F})^\times$ and for some $v_{i,j}, w_{i,j}, \gamma_{i,j} \in \overline{S}_0$.

Fix $0 \le j \le i \le n-1$. We define the Breuil submodule

$$\mathcal{M}_{i,j} \tag{3.0.7}$$

that is a subquotient of $\mathcal{M}$ determined by the basis $(e_i, e_{i-1}, \cdots, e_j)$. For instance, $\mathcal{M}_{i,i} \cong \mathcal{M}_i$ for all $0 \le i \le n-1$. We note that $\mathrm{T}^r_{\mathrm{st}}(\mathcal{M}_{i,j}) \cong \overline{\rho}_{i,j}$ by Proposition 2.3.4.

We will keep these notation and assumptions for $\mathcal{M}$ throughout this paper.

3.1. **Elimination of Galois types.** In this section, we find out the possible Galois types of niveau 1 for potentially semi-stable lifts of $\overline{\rho}_0$ with Hodge–Tate weights $\{-(n-1), -(n-2), \cdots, 0\}$.

We start this section with the following elementary lemma.

Lemma 3.1.1. *Let $\rho : G_{\mathbf{Q}_p} \to \mathrm{GL}_n(E)$ be a potentially semi-stable representation with Hodge–Tate weights $\{-(n-1), ..., -2, -1, 0\}$ and of Galois type $\bigoplus_{i=0}^{n-1} \widetilde{\omega}_f^{k_i}$.*

Then

$$\det(\rho)|_{I_{\mathbf{Q}_p}} = \varepsilon^{\frac{n(n-1)}{2}} \cdot \widetilde{\omega}_f^{\sum_{i=0}^{n-1} k_i},$$

where ε is the cyclotomic character.

Proof. $\det(\rho)$ is a potentially crystalline character of $G_{\mathbf{Q}_p}$ with Hodge–Tate weight $-(\sum_{i=0}^{n-1} i)$ and of Galois type $\widetilde{\omega}_f^{\sum_{i=0}^{n-1} k_i}$, i.e., $\det(\rho) \cdot \widetilde{\omega}_f^{-\sum_{i=0}^{n-1} k_i}$ is a crystalline character with Hodge–Tate weight $-(\sum_{i=0}^{n-1} i) = -\frac{n(n-1)}{2}$ so that $\det(\rho)|_{I_{\mathbf{Q}_p}} \cdot \widetilde{\omega}_f^{-\sum_{i=0}^{n-1} k_i} \cong \varepsilon^{\frac{n(n-1)}{2}}$. $\square$

We will only consider the Breuil modules $\mathcal{M}$ corresponding to the mod p reduction of the strongly divisible modules that correspond to the Galois stable lattices in potentially semi-stable lifts of $\overline{\rho}_0$ with Hodge–Tate weights $\{-(n-1), -(n-2), \cdots, -1, 0\}$, so that we may assume that $r = n-1$, i.e., $\mathcal{M} \in \mathbf{F}\text{-}\mathrm{BrMod}^{n-1}_{\mathrm{dd}}$.

Lemma 3.1.2. *Let $f = 1$. Assume that $\overline{\rho}_0$ is generic, and that $\mathcal{M} \in \mathbf{F}\text{-}\mathrm{BrMod}^{n-1}_{\mathrm{dd}}$ corresponds to the mod p reduction of a strongly divisible module $\widehat{\mathcal{M}}$ such that $\mathrm{T}^{n-1}_{\mathrm{st}}(\mathcal{M}) \cong \overline{\rho}_0$ and $\mathrm{T}^{\mathbf{Q}_p, n-1}_{\mathrm{st}}(\widehat{\mathcal{M}})$ is a Galois stable lattice in a potentially semi-stable lift of $\overline{\rho}_0$ with Hodge–Tate weights $\{-(n-1), -(n-2), \cdots, 0\}$ and Galois type $\bigoplus_{i=0}^{n-1} \widetilde{\omega}^{k_i}$ for some integers k_i.*

Then there exists a framed basis $\underline{e}$ for $\mathcal{M}$ and a framed system of generators $\underline{f}$ for $\mathrm{Fil}^{n-1}\mathcal{M}$ such that $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M})$, $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1})$, and $\mathrm{Mat}_{\underline{e}}(N)$ are as in (3.0.4), (3.0.5), and (3.0.6) respectively. Moreover, the (k_i, r_i) satisfy the following properties:

(i) $k_i \equiv c_i + i - r_i \mod (e)$ *for all* $i \in \{0, 1, \cdots, n-1\}$*;*
(ii) $0 \le r_i \le n-1$ *for all* $i \in \{0, 1, \cdots, n-1\}$*;*
(iii) $\sum_{i=0}^{n-1} r_i = \frac{(n-1)n}{2}$.

Proof. Note that the inertial type of $\mathcal{M}$ is $\bigoplus_{i=0}^{n-1} \omega^{k_i}$ by Proposition 2.4.3. The first part of the Lemma is obvious from the discussion at the beginning of Section 3.

We now prove the second part of the Lemma. We may assume that the rank-one Breuil modules $\mathcal{M}_i$ are of weight r_i, so that $0 \le r_i \le n-1$ for $i = \{0, 1, ..., n-1\}$ by Lemma 2.3.5. By the equation (3.0.3), we have $k_i \equiv c_i + i - r_i \mod (e)$, as $e = p-1$. By looking at the determinant of $\overline{\rho}_0$ we deduce the conditions

$$\omega^{\frac{n(n-1)}{2} + k_{n-1} + k_{n-2} + \cdots + k_0} = \det \mathrm{T}_{\mathrm{st}}^{n-1}(\mathcal{M})|_{I_{\mathbf{Q}_p}} = \det \overline{\rho}_0|_{I_{\mathbf{Q}_p}} = \omega^{c_{n-1} + c_{n-2} + \cdots + c_0 + \frac{n(n-1)}{2}}$$

from Lemma 3.1.1, and hence we have $r_{n-1} + r_{n-2} + \cdots + r_0 = \frac{n(n-1)}{2}$ (as $p > n^2 + 2(n-3)$ due to the genericity of $\overline{\rho}_0$). □

One can further eliminate Galois types of niveau 1 if $\overline{\rho}_0$ is maximally non-split.

Proposition 3.1.3. *Keep the assumptions and notation of Lemma 3.1.2. If the tuple (k_i, r_i) further satisfy one of the following conditions*

- $r_i = n-1$ *for some* $i \in \{0, 1, 2, \cdots, n-2\}$*;*
- $r_i = 0$ *for some* $i \in \{1, 2, 3, \cdots, n-1\}$*,*

then $\overline{\rho}_0$ is not maximally non-split.

Proof. The main ingredient is Lemma 2.3.6. Following the notation in Lemma 2.3.6, we fix $i \in \{0, 1, 2, \cdots, n-2\}$ and identify $x = i+1$ and $y = i$ so that $r_x = s_x$ and $r_y = s_y$. From the results in Lemma 3.1.2, it is easy to compute that $[k_i - k_{i+1}]_1 = e - (c_{i+1} - c_i + 1) + (r_{i+1} - r_i)$. By the genericity conditions in Definition 3.0.5 and by part (ii) of Lemma 3.1.2, we see that $0 < [k_i - k_{i+1}]_1 < e$ so that if $r_i \ge r_{i+1}$ then the equation (2.3.1) in Lemma 2.3.6 holds.

If $r_{i+1}e \le [k_i - k_{i+1}]_1$ and $r_i \ge r_{i+1}$, then $*_{i+1} = 0$ by Lemma 2.3.6. Since $0 < [k_i - k_{i+1}]_1 < e$, we have $r_{i+1}e \le [k_i - k_{i+1}]_1$ if and only if $r_{i+1} = 0$, in which case $\overline{\rho}_0$ is not maximally non-split.

We now apply the second part of Lemma 2.3.6. It is easy to check that $j_0 = r_{i+1} - 1$. One can again readily check that the equation (2.3.2) is equivalent to $r_i = n-1$, in which case $*_{i+1} = 0$ so that $\overline{\rho}_0$ is not maximally non-split. □

Note that all of the Galois types that will appear later in this section will satisfy the conditions in Lemma 3.1.2, and Proposition 3.1.3 as well if we further assume that $\overline{\rho}_0$ is maximally non-split.

3.2. Fontaine–Laffaille parameters.

In this section, we parameterize the wildly ramified part of generic and maximally non-split ordinary representations using Fontaine–Laffaille theory.

We start this section by recalling that if $\overline{\rho}_0$ is generic then $\overline{\rho}_0 \otimes \omega^{-c_0}$ is Fontaine–Laffaille (cf. [GG10], Lemma 3.1.5), so that there is a Fontaine–Laffaille module M with Hodge–Tate weights $\{0, c_1 - c_0 + 1, \cdots, c_{n-1} - c_0 + (n-1)\}$ such that $\mathrm{T}_{\mathrm{cris}}^*(M) \cong \overline{\rho}_0 \otimes \omega^{-c_0}$ (if we assume that $\overline{\rho}_0$ is generic).

Lemma 3.2.1. *Assume that $\overline{\rho}_0$ is generic, and let $M \in \mathbf{F}\text{-}\mathrm{FLMod}_{\mathbf{F}_p}^{[0,p-2]}$ be a Fontaine–Laffaille module such that $\mathrm{T}_{\mathrm{cris}}^*(M) \cong \overline{\rho}_0 \otimes \omega^{-c_0}$.*

Then there exists a basis $\underline{e} = (e_0, e_1, \cdots, e_{n-1})$ for M such that

$$\mathrm{Fil}^j M = \begin{cases} M & \text{if } j \le 0; \\ \mathbf{F}(e_i, \cdots, e_{n-1}) & \text{if } c_{i-1} - c_0 + i - 1 < j \le c_i - c_0 + i; \\ 0 & \text{if } c_{n-1} - c_0 + n - 1 < j. \end{cases}$$

and

$$\mathrm{Mat}_{\underline{e}}(\phi_\bullet) = \begin{pmatrix} \mu_0^{-1} & \alpha_{0,1} & \alpha_{0,2} & \cdots & \alpha_{0,n-2} & \alpha_{0,n-1} \\ 0 & \mu_1^{-1} & \alpha_{1,2} & \cdots & \alpha_{1,n-2} & \alpha_{1,n-1} \\ 0 & 0 & \mu_2^{-1} & \cdots & \alpha_{2,n-2} & \alpha_{2,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_{n-2}^{-1} & \alpha_{n-2,n-1} \\ 0 & 0 & 0 & \cdots & 0 & \mu_{n-1}^{-1} \end{pmatrix} \tag{3.2.1}$$

where $\alpha_{i,j} \in \mathbf{F}$.

Note that the basis $\underline{e}$ on M in Lemma 3.2.1 is compatible with the filtration.

Proof. This is an immediate generalization of [HLM], Lemma 2.1.7. □

For $i \geq j$, the subset $(e_j, \cdots, e_i)$ of $\underline{e}$ determines a subquotient $M_{i,j}$ of the Fontaine–Laffaille module M, which is also a Fontaine–Laffaille module with the filtration induced from $\mathrm{Fil}^s M$ in the obvious way and with Frobenius described as follows:

$$A_{i,j} := \begin{pmatrix} \mu_j^{-1} & \alpha_{j,j+1} & \cdots & \alpha_{j,i-1} & \alpha_{j,i} \\ 0 & \mu_{j+1}^{-1} & \cdots & \alpha_{j+1,i-1} & \alpha_{j+1,i} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mu_{i-1}^{-1} & \alpha_{i-1,i} \\ 0 & 0 & \cdots & 0 & \mu_i^{-1} \end{pmatrix}.$$

Note that $\mathrm{T}^*_{\mathrm{cris}}(M_{i,j}) \otimes \omega^{c_0} \cong \overline{\rho}_{i,j}$. We let $A'_{i,j}$ be the $(i-j) \times (i-j)$-submatrix of $A_{i,j}$ obtained by deleting the left-most column and the lowest row of $A_{i,j}$.

Lemma 3.2.2. *Keep the assumptions and notation of Lemma 3.2.1, and let* $0 \leq j < j+1 < i \leq n-1$. *Assume further that* $\overline{\rho}_0$ *is maximally non-split.*

If $\det A'_{i,j} \neq (-1)^{i-j+1}\mu_{j+1}^{-1} \cdots \mu_{i-1}^{-1}\alpha_{j,i}$, *then* $[\alpha_{j,i} : \det A'_{i,j}] \in \mathbb{P}^1(\mathbf{F})$ *does not depend on the choice of basis* $\underline{e}$ *compatible with the filtration.*

Proof. This is an immediate generalization of [HLM], Lemma 2.1.9. □

Definition 3.2.3. *Keep the assumptions and notation of Lemma 3.2.2, and assume further that* $\overline{\rho}_0$ *satisfies*

$$\det A'_{i,j} \neq (-1)^{i-j+1}\mu_{j+1}^{-1} \cdots \mu_{i-1}^{-1}\alpha_{j,i} \tag{3.2.2}$$

for all $i, j \in \mathbf{Z}$ *with* $0 \leq j < j+1 < i \leq n-1$.

The Fontaine–Laffaille parameter associated to $\overline{\rho}_0$ *is defined as*

$$\mathrm{FL}_n(\overline{\rho}_0) := \left(\mathrm{FL}_n^{i,j}(\overline{\rho}_0)\right)_{i,j} \in [\mathbb{P}^1(\mathbf{F})]^{\frac{(n-2)(n-1)}{2}}$$

where

$$\mathrm{FL}_n^{i,j}(\overline{\rho}_0) := \left[\alpha_{j,i} : (-1)^{i-j+1} \cdot \det A'_{i,j}\right] \in \mathbb{P}^1(\mathbf{F})$$

for all $i, j \in \mathbf{Z}$ *such that* $0 \leq j < j+1 < i \leq n-1$.

We often write $\frac{y}{x}$ for $[x : y] \in \mathbb{P}^1(\mathbf{F})$ if $x \neq 0$. The conditions in (3.2.2) for i, j guarantee the well-definedness of $\mathrm{FL}_n^{i,j}(\overline{\rho}_0)$ in $\mathbb{P}^1(\mathbf{F})$. We also point out that $\mathrm{FL}_n^{i,j}(\overline{\rho}_0) \neq (-1)^{i-j}\mu_{j+1}^{-1} \cdots \mu_{i-1}^{-1}$ in $\mathbb{P}^1(\mathbf{F})$.

One can define the inverses of the elements in $\mathbb{P}^1(\mathbf{F})$ in a natural way: for $[x_1 : x_2] \in \mathbb{P}^1(\mathbf{F})$, $[x_1 : x_2]^{-1} := [x_2 : x_1] \in \mathbb{P}^1(\mathbf{F})$.

Lemma 3.2.4. *Assume that* $\overline{\rho}_0$ *is generic. Then*

(i) $\overline{\rho}_0^\vee$ *is generic;*

(ii) *if* $\overline{\rho}_0$ *is strongly generic, then so is* $\overline{\rho}_0^\vee$*;*
(iii) *if* $\overline{\rho}_0$ *is maximally non-split, then so is* $\overline{\rho}_0^\vee$*;*
(iv) *if* $\overline{\rho}_0$ *is maximally non-split, then the conditions in (3.2.2) are stable under* $\overline{\rho}_0 \mapsto \overline{\rho}_0^\vee$*.*

Assume further that $\overline{\rho}_0$ *is maximally non-split and satisfies the conditions in (3.2.2).*

(v) *for all* $i, j \in \mathbf{Z}$ *with* $0 \leq j < j+1 < i \leq n-1$, $\mathrm{FL}_n^{i,j}(\overline{\rho}_0) = \mathrm{FL}_n^{i,j}(\overline{\rho}_0 \otimes \omega^b)$ *for any* $b \in \mathbf{Z}$*;*
(vi) *for all* $i, j \in \mathbf{Z}$ *with* $0 \leq j < j+1 < i \leq n-1$, $\mathrm{FL}_n^{i,j}(\overline{\rho}_0) = \mathrm{FL}_{i-j+1}^{i-j,0}(\overline{\rho}_{i,j})$*;*
(vii) *for all* $i, j \in \mathbf{Z}$ *with* $0 \leq j < j+1 < i \leq n-1$, $\mathrm{FL}_n^{i,j}(\overline{\rho}_0)^{-1} = \mathrm{FL}_n^{n-1-j,n-1-i}(\overline{\rho}_0^\vee)$*.*

Proof. (i), (ii) and (iii) are easy to check. We leave them for the reader.

The only effect on Fontaine–Laffaille module by twisting ω^b is shifting the jumps of the filtration. Thus (v) and (vi) are obvious.

For (iv) and (vii), one can check that the Frobenius of the Fontaine–Laffaille module associated to $\overline{\rho}_0^\vee$ is described by

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \cdot [\mathrm{Mat}_{\underline{e}}(\phi_\bullet)^t]^{-1} \cdot \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

where $\mathrm{Mat}_{\underline{e}}(\phi_\bullet)$ is as in (3.2.1). Now one can check them by direct computation. □

We end this section by defining certain numerical conditions on Fontaine–Laffaille parameters. We consider the matrix $(1,n)w_0 \mathrm{Mat}_{\underline{e}}(\phi_\bullet)^t$, where $\mathrm{Mat}_{\underline{e}}(\phi_\bullet)$ is the upper-triangular matrix in (3.2.1). Here, w_0 is the longest element of the Weyl group W associated to T and $(1,n)$ is a permutation in W. Note that the anti-diagonal matrix displayed in the proof of Lemma 3.2.4 is w_0 seen as an element in $\mathrm{GL}_n(\mathbf{F})$. For $1 \leq i \leq n-1$ we let B_i be the square matrix of size i that is the left-bottom corner of $(1,n)w_0 \mathrm{Mat}_{\underline{e}}(\phi_\bullet)^t$.

Definition 3.2.5. *Keep the notation and assumptions of Definition 3.2.3. We say that* $\overline{\rho}_0$ *is* Fontaine–Laffaille generic *if moreover* $\det B_i \neq 0$ *for all* $1 \leq i \leq n-1$ *and* $\overline{\rho}_0$ *is strongly generic.*

We emphasize that by an ordinary representation $\overline{\rho}_0$ being Fontaine–Laffaille generic, we *always mean* that $\overline{\rho}_0$ satisfies the maximally non-splitness and the conditions in (3.2.2) as well as $\det B_i \neq 0$ for all $1 \leq i \leq n-1$ and the strongly generic assumption (cf. Definition 3.0.5).

Although the Frobenius matrix of a Fontaine–Laffaille module depends on the choice of basis, it is easy to see that the non-vanishing of the determinants above is independent of the choice of basis compatible with the filtration. Note that the conditions in Definition 3.2.5 are necessary and sufficient conditions for

$$(1,n)w_0 \mathrm{Mat}_{\underline{e}}(\phi_\bullet)^t \in B(\mathbf{F})w_0B(\mathbf{F})$$

in the Bruhat decomposition, which will significantly reduce the size of the paper (cf. Remark 3.2.6). We also note that

- ◦ $\det B_1 \neq 0$ if and only if $\mathrm{FL}_n^{n-1,0}(\overline{\rho}_0) \neq \infty$;
- ◦ $\det B_{n-1} \neq 0$ if and only if $\mathrm{FL}_n^{n-1,0}(\overline{\rho}_0) \neq 0$.

Finally, we point out that the locus of Fontaine–Laffaille generic ordinary Galois representations $\overline{\rho}_0$ forms a (Zariski) open subset in $[\mathbb{P}^1(\mathbf{F})]^{\frac{(n-1)(n-2)}{2}}$.

Remark 3.2.6. *Definition 3.2.5 comes from the fact that the list of Serre weights of* $\overline{\rho}_0$ *is then minimal in the sense of Conjecture 5.3.1. It is very crucial in the proof of Theorem 5.6.2 as it is more difficult to track the Fontaine–Laffaille parameters on the automorphic side if we have too many Serre weights. Moreover, these conditions simplify our proof for Theorem 3.7.1.*

3.3. **Breuil modules of certain inertial types of niveau 1.** In this section, we classify the Breuil modules with certain inertial types, corresponding to the ordinary Galois representations $\overline{\rho}_0$ as in (3.0.1), and we also study their corresponding Fontaine–Laffaille parameters.

Throughout this section, we always assume that $\overline{\rho}_0$ is strongly generic. Since we are only interested in inertial types of niveau 1, we let $f = 1$, $e = p-1$, and $\varpi = \sqrt[e]{-p}$. We define the following integers for $0 \le i \le n-1$:

$$r_i^{(0)} := \begin{cases} 1 & \text{if } i = n-1; \\ i & \text{if } 0 < i < n-1; \\ n-2 & \text{if } i = 0. \end{cases} \tag{3.3.1}$$

We also set

$$k_i^{(0)} := c_i + i - r_i^{(0)}$$

for all $i \in \{0, 1, \cdots, n-1\}$.

We first classify the Breuil modules of inertial types described as above.

Lemma 3.3.1. *Assume that $\overline{\rho}_0$ is strongly generic and that $\mathcal{M} \in \mathbf{F}\text{-BrMod}_{\mathrm{dd}}^{n-1}$ corresponds to the mod p reduction of a strongly divisible modules $\widehat{\mathcal{M}}$ such that $\mathrm{T}_{\mathrm{st}}^{\mathbf{Q}_p, n-1}(\widehat{\mathcal{M}})$ is a Galois stable lattice in a potentially semi-stable lift of $\overline{\rho}_0$ with Hodge–Tate weights $\{-(n-1), -(n-2), \cdots, 0\}$ and Galois type $\bigoplus_{i=0}^{n-1} \widetilde{\omega}^{k_i^{(0)}}$.*

Then $\mathcal{M} \in \mathbf{F}\text{-BrMod}_{\mathrm{dd}}^{n-1}$ can be described as follows: there exist a framed basis $\underline{e}$ for $\mathcal{M}$ and a framed system of generators $\underline{f}$ for $\mathrm{Fil}^{n-1}\mathcal{M}$ such that

$$\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M}) = \begin{pmatrix} u^{r_{n-1}^{(0)}e} & \beta_{n-1,n-2}u^{r_{n-1}^{(0)}e - k_{n-1,n-2}^{(0)}} & \cdots & \beta_{n-1,0}u^{r_{n-1}^{(0)}e - k_{n-1,0}^{(0)}} \\ 0 & u^{r_{n-2}^{(0)}e} & \cdots & \beta_{n-2,0}u^{r_{n-2}^{(0)}e - k_{n-2,0}^{(0)}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u^{r_0^{(0)}e} \end{pmatrix}$$

and

$$\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1}) = \mathrm{Diag}\,(\nu_{n-1}, \nu_{n-2}, \cdots, \nu_0)$$

where $k_{i,j}^{(0)} := k_i^{(0)} - k_j^{(0)}$, $\nu_i \in \mathbf{F}^\times$ and $\beta_{i,j} \in \mathbf{F}$. Moreover,

$$\mathrm{Mat}_{\underline{e}}(N) = \left(\gamma_{i,j} \cdot u^{[k_j^{(0)} - k_i^{(0)}]_1}\right)$$

where $\gamma_{i,j} = 0$ if $i \le j$ and $\gamma_{i,j} \in u^{e[k_j^{(0)} - k_i^{(0)}]_1}\overline{S}_0$ if $i > j$.

Note that $\underline{e}$ and $\underline{f}$ in Lemma 3.3.1 are not necessarily the same as the ones in Lemma 3.1.2.

Proof. We keep the notation in (3.0.4), (3.0.5), and (3.0.6). That is, there exist a framed basis $\underline{e}$ for $\mathcal{M}$ and a framed system of generators $\underline{f}$ for $\mathrm{Fil}^{n-1}\mathcal{M}$ such that $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M})$, $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1})$, $\mathrm{Mat}_{\underline{e}}(N)$ are given as in (3.0.4), (3.0.5), and (3.0.6) respectively. Since $k_i \equiv k_i^{(0)} \mod (p-1)$, we have $r_i = r_i^{(0)}$ for all $i \in \{0, 1, \cdots, n-1\}$ by Lemma 3.1.2, following the notation of Lemma 3.1.2.

We start to prove the following claim: if $n-1 \ge i > j \ge 0$ then

$$e - (k_i^{(0)} - k_j^{(0)}) \ge n. \tag{3.3.2}$$

Indeed, by the strongly generic assumption, Definition 3.0.5

$$\begin{aligned} e - (k_i^{(0)} - k_j^{(0)}) &= (p-1) - (c_i + i - r_i^{(0)}) + (c_j + j - r_j^{(0)}) \\ &= (p-1) - (c_i - c_j) - (i-j) + (r_i^{(0)} - r_j^{(0)}) \\ &\ge (p-1) - (c_{n-1} - c_0) - (n-1-0) + (1 - (n-2)) \\ &\ge 3n - 4 - 2n + 4 = n. \end{aligned}$$

Note that this claim will be often used during the proof later.

We now diagonalize $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1})$ with some restriction on the powers of the entries of the matrix $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M})$. Let $V_0 = \mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M}) \in \mathrm{M}_n^{\square}(\overline{S})$ and $A_0 = \mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1}) \in \mathrm{GL}_n^{\square}(\overline{S})$. We also let $V_1 \in \mathrm{M}_n^{\square}(\overline{S})$ be the matrix obtained from V_0 by replacing $v_{i,j}$ by $v'_{i,j} \in \overline{S}_0$, and $B_1 \in \mathrm{GL}_n^{\square}(\overline{S})$ the matrix obtained from A_0 by replacing $w_{i,j}$ by $w'_{i,j} \in \overline{S}_0$. It is straightforward to check that $A_0 \cdot V_1 = V_0 \cdot B_1$ if and only if for all $i > j$

$$\begin{aligned}(3.3.3)\quad \nu_i v'_{i,j} u^{[k_j^{(0)}-k_i^{(0)}]_1} &+ \sum_{s=j+1}^{i-1} w_{is} v'_{s,j} u^{[k_s^{(0)}-k_i^{(0)}]_1+[k_j^{(0)}-k_s^{(0)}]_1} + w_{i,j} u^{r_j^{(0)}e+[k_j^{(0)}-k_i^{(0)}]_1} \\ &= w'_{i,j} u^{r_i^{(0)}e+[k_j^{(0)}-k_i^{(0)}]_1} + \sum_{s=j+1}^{i-1} v_{i,s} w'_{s,j} u^{[k_s^{(0)}-k_i^{(0)}]_1+[k_j^{(0)}-k_s^{(0)}]_1} + \nu_j v_{i,j} u^{[k_j^{(0)}-k_i^{(0)}]_1}.\end{aligned}$$

Note that the power of u in each term of (3.3.3) is congruent to $[k_j^{(0)} - k_i^{(0)}]_1$ modulo (e). It is immediate that for all $i > j$ there exist $v'_{i,j} \in \overline{S}_0$ and $w'_{i,j} \in \overline{S}_0$ satisfying the equation (3.3.3) with the following additional properties: for all $i > j$

$$(3.3.4)\qquad \deg v'_{i,j} < r_i^{(0)} e.$$

Letting $\underline{e}' := \underline{e}A_0$, we have

$$\mathrm{Mat}_{\underline{e}',\underline{f}'}(\mathrm{Fil}^{n-1}\mathcal{M}) = V_1 \text{ and } \mathrm{Mat}_{\underline{e}',\underline{f}'}(\phi_{n-1}) = \phi(B_1)$$

where $\underline{f}' = \underline{e}'V_1$, by Lemma 2.4.4. Note that $\phi(B_1)$ is congruent to a diagonal matrix modulo (u^{ne}) by (3.3.2). We repeat this process one more time. We may assume that $w_{i,j} \in u^{ne}\overline{S}_0$, i.e., that $A_0 \equiv B_1$ modulo (u^{ne}) where B_1 is assumed to be a diagonal matrix. It is obvious that there exists an upper-triangular matrix $V_1 = (v'_{i,j} u^{[p^{-1}k_j^{(0)}-k_i^{(0)}]_1})$ whose entries have bounded degrees as in (3.3.4), satisfying the equation $A_0V_1 \equiv V_0B_1$ modulo (u^{ne}). By Lemma 2.4.4, we get $\mathrm{Mat}_{\underline{e}',\underline{f}'}(\phi_{n-1})$ is diagonal. Hence, we may assume that $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1})$ is diagonal and that $\deg v_{i,j}$ in $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M})$ is bounded as in (3.3.4), and we do so. Moreover, this change of basis do not change the shape of $\mathrm{Mat}_{\underline{e}}(N)$, so that we also assume that $\mathrm{Mat}_{\underline{e}}(N)$ is still as in (3.0.6).

We now prove that for all $n-1 \geq i > j \geq 0$

$$(3.3.5)\qquad v_{i,j} u^{[k_j^{(0)}-k_i^{(0)}]_1} = \beta_{i,j} u^{r_i^{(0)}e-(k_i^{(0)}-k_j^{(0)})}$$

for some $\beta_{i,j} \in \mathbf{F}$. Note that this is immediate for $i = n-1$ and $i = 1$, since $r_i^{(0)} = 1$ if $i = n-1$ or $i = 1$. To prove (3.3.5), we induct on i. The case $i = 1$ is done as above. Fix $p_0 \in \{2, 3, \cdots, n-2\}$, and assume that (3.3.5) holds for all $i \in \{1, 2, \cdots, p_0 - 1\}$ and for all $j < i$. We consider the subquotient $\mathcal{M}_{p_0,0}$ of $\mathcal{M}$ defined in (3.0.7). By abuse of notation, we write $\underline{e} = (e_{p_0}, \cdots, e_0)$ for the induced framed basis for $\mathcal{M}_{p_0,0}$ and $\underline{f} = (f_{p_0}, \cdots, f_0)$ for the induced framed system of generators for $\mathrm{Fil}^{n-1}\mathcal{M}_{p_0,0}$.

We claim that for $p_0 \geq j \geq 0$

$$u^e N(f_j) \in \overline{S}_0 u^e f_j + \sum_{t=j+1}^{p_0} \overline{S}_0 u^{[k_j^{(0)}-k_t^{(0)}]_1} f_t.$$

It is clear that it is true when $j = p_0$. For $j < p_0$, consider $N(f_j) = N(f_j - u^{r_j^{(0)}e} e_j) + N(u^{r_j^{(0)}e} e_j)$. It is easy to check that $N(f_j - u^{r_j^{(0)}e} e_j)$ and $N(u^{r_j^{(0)}e} e_j) + r_j^{(0)} e f_j$ are $\overline{S}$-linear combinations of $e_{p_0}, \cdots, e_{j+1}$, and they are, in fact, $\overline{S}_0$-linear combinations of $u^{[k_j^{(0)}-k_{p_0}^{(0)}]_1} e_{p_0}, \cdots, u^{[k_j^{(0)}-k_{j+1}^{(0)}]_1} e_{j+1}$ since they are $\omega^{k_j^{(0)}}$-invariant. Since

$$u^e N(f_j) \in \mathrm{Fil}^{n-1}\mathcal{M}_{p_0,0} \supset u^{(n-1)e}\mathcal{M}_{p_0,0}$$

and

$$u^eN(f_j)+r_j^{(0)}eu^ef_j=u^e[N(f_j-u^{r_j^{(0)}e}e_j)]+u^e[N(u^{r_j^{(0)}e}e_j)+r_j^{(0)}ef_j],$$

we conclude that

$$u^eN(f_j)+r_j^{(0)}eu^ef_j\in\sum_{t=j+1}^{p_0}\overline{S}_0u^{[k_j^{(0)}-k_t^{(0)}]_1}f_t,$$

which completes the claim.

Let

$$\mathrm{Mat}_{\underline{e},\underline{f}}(N|_{\mathcal{M}_{p_0,0}})=\left(\gamma_{i,j}\cdot u^{[k_j^{(0)}-k_i^{(0)}]_1}\right)$$

where $\gamma_{i,j}=0$ if $i\le j$ and $\gamma_{i,j}\in\overline{S}_0$ if $i>j$. We also claim that

$$\gamma_{i,j}\in u^{e[k_j^{(0)}-k_i^{(0)}]_1}\overline{S}_0$$

for $p_0\ge i>j\ge 0$, which can be readily checked from the equation $cN\phi_{n-1}(f_j)=\phi_{n-1}(u^eN(f_j))$. (Note that $c=1\in\overline{S}$ as $E(u)=u^e+p$.) Indeed, we have

$$cN\phi_{n-1}(f_j)=N(\nu_je_j)=\nu_j\sum_{i=j+1}^{p_0}\gamma_{i,j}u^{[k_j^{(0)}-k_i^{(0)}]_1}e_i.$$

On the other hand, since $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1}|_{\mathcal{M}_{p_0,0}})$ is diagonal, the previous claim immediately implies that

$$\phi_{n-1}(u^eN(f_j))\in\sum_{t=j+1}^{p_0}\overline{S}_0u^{p[k_j^{(0)}-k_t^{(0)}]_1}e_t.$$

Hence, we conclude the claim.

We now finish the proof of (3.3.5) by inducting on p_0-j as well. Write $v_{i,j}=\sum_{t=0}^{r_i^{(0)}-1}x_{i,j}^{(t)}u^{te}$ for $x_{i,j}^{(t)}\in\mathbf{F}$. We need to prove $x_{p_0,j}^{(t)}=0$ for $t\in\{0,1,\cdots,r_{p_0}^{(0)}-2\}$. Assume first $j=p_0-1$, and we compute $N(f_j)$ as follows:

$$\begin{aligned}N(f_{p_0-1})=-&\sum_{t=0}^{r_{p_0}^{(0)}-1}x_{p_0,p_0-1}^{(t)}[e(t+1)-(k_{p_0}^{(0)}-k_{p_0-1}^{(0)})]u^{e(t+1)-(k_{p_0}^{(0)}-k_{p_0-1}^{(0)})}e_{p_0}\\&+\gamma_{p_0,p_0-1}u^{(r_{p_0-1}^{(0)}+1)e-(k_{p_0}^{(0)}-k_{p_0-1}^{(0)})}e_{p_0}-r_{p_0-1}^{(0)}eu^{r_{p_0-1}^{(0)}e}e_{p_0-1}.\end{aligned}$$

Since $f_{p_0-1}=u^{r_{p_0-1}^{(0)}e}e_{p_0-1}+\sum_{t=0}^{r_{p_0}^{(0)}-1}x_{p_0,p_0-1}^{(t)}u^{te+[k_{p_0-1}-k_{p_0}]_1}e_{p_0}$, we get

$$\begin{aligned}N(f_{p_0-1})\equiv&\sum_{t=0}^{r_{p_0}^{(0)}-1}x_{p_0,p_0-1}^{(t)}[er_{p_0-1}^{(0)}-e(t+1)+(k_{p_0}^{(0)}-k_{p_0-1}^{(0)})]u^{e(t+1)-(k_{p_0}^{(0)}-k_{p_0-1}^{(0)})}e_{p_0}\\&+\gamma_{p_0,p_0-1}u^{(r_{p_0-1}^{(0)}+1)e-(k_{p_0}^{(0)}-k_{p_0-1}^{(0)})}e_{p_0}\end{aligned}\tag{3.3.6}$$

modulo $\mathrm{Fil}^{n-1}\mathcal{M}_{p_0,0}$. Since $\gamma_{p_0,p_0-1}\in u^{e[e-(k_{p_0}^{(0)}-k_{p_0-1}^{(0)})]}\overline{S}_0$ and $e-(k_{p_0}^{(0)}-k_{p_0-1}^{(0)})\ge n$ by (3.3.2), we get

$$N(f_{p_0-1})\equiv\sum_{t=0}^{r_{p_0}^{(0)}-1}x_{p_0,p_0-1}^{(t)}[er_{p_0-1}^{(0)}-e(t+1)+(k_{p_0}^{(0)}-k_{p_0-1}^{(0)})]u^{e(t+1)-(k_{p_0}^{(0)}-k_{p_0-1}^{(0)})}e_{p_0}$$

modulo $\mathrm{Fil}^{n-1}\mathcal{M}_{p_0,0}$, so that

$$u^eN(f_{p_0-1})\equiv\sum_{t=0}^{r_{p_0}^{(0)}-1}x_{p_0,p_0-1}^{(t)}[er_{p_0-1}^{(0)}-e(t+1)+(k_{p_0}^{(0)}-k_{p_0-1}^{(0)})]u^{e(t+2)-(k_{p_0}^{(0)}-k_{p_0-1}^{(0)})}e_{p_0}$$

modulo $\mathrm{Fil}^{n-1}\mathcal{M}_{p_0,0}$. But if $t = r_{p_0}^{(0)} - 1$ then $e(t+2) - (k_{p_0}^{(0)} - k_{p_0-1}^{(0)}) \geq r_{p_0}^{(0)}$, so that we have

$$u^e N(f_{p_0-1}) \equiv \sum_{t=0}^{r_{p_0}^{(0)}-2} x_{p_0,p_0-1}^{(t)} [er_{p_0-1}^{(0)} - e(t+1) + (k_{p_0}^{(0)} - k_{p_0-1}^{(0)})] u^{e(t+2)-(k_{p_0}^{(0)} - k_{p_0-1}^{(0)})} e_{p_0} \tag{3.3.7}$$

modulo $\mathrm{Fil}^{n-1}\mathcal{M}_{p_0,0}$.

It is easy to check that

$$er_{p_0-1}^{(0)} - e(t+1) + (k_{p_0}^{(0)} - k_{p_0-1}^{(0)}) \not\equiv 0 \tag{3.3.8}$$

modulo (p) for all $0 \leq t \leq r_{p_0}^{(0)} - 2$. Indeed, since $k_i^{(0)} = c_i$ for $0 < i < n-1$ by (3.3.1), we have

$$er_{p_0-1}^{(0)} - e(t+1) + (k_{p_0}^{(0)} - k_{p_0-1}^{(0)}) \equiv -r_{p_0-1}^{(0)} + (t+1) + (c_{p_0} - c_{p_0-1})$$

modulo (p), and so

$$er_{p_0-1}^{(0)} - e(t+1) + (k_{p_0}^{(0)} - k_{p_0-1}^{(0)}) \equiv (t+1) + (c_{p_0} - c_{p_0-1} + 1) - r_{p_0}^{(0)}$$

modulo (p) since $r_i^{(0)} = i$ for $0 < i < n-1$ by (3.3.1).

Since $0 \leq t \leq r_{p_0}^{(0)} - 2$,

$$0 < (c_{p_0} - c_{p_0-1} + 2) - r_{p_0}^{(0)} \leq (t+1) + (c_{p_0} - c_{p_0-1} + 1) - r_{p_0}^{(0)} \leq (c_{p_0} - c_{p_0-1} - 1) < p$$

by the strongly generic conditions, Definition 3.0.5. Hence, we conclude that $x_{p_0,p_0-1}^{(t)} = 0$ for all $0 \leq t \leq r_{p_0}^{(0)} - 2$ since $u^e N(f_{p_0-1}) \in \mathrm{Fil}^{n-1}\mathcal{M}_{p_0,0}$. This completes the proof of (3.3.5) for $j = p_0 - 1$.

Assume that (3.3.5) holds for $i = p_0$ and $j \in \{p_0-1, p_0-2, \cdots, s+1\}$. We compute $N(f_s)$ for $p_0 - 1 > s \geq 0$ as follows: using the induction hypothesis on $i \in \{1, 2, \cdots, p_0-1\}$

$$\begin{aligned} N(f_s) = &- \sum_{t=0}^{r_{p_0}^{(0)}-1} x_{p_0,s}^{(t)} [e(t+1) - (k_{p_0}^{(0)} - k_s^{(0)})] u^{e(t+1)-(k_{p_0}^{(0)} - k_s^{(0)})} e_{p_0} \\ &+ \sum_{i=s+1}^{p_0-1} \beta_{i,s} u^{r_i^{(0)}e - (k_i^{(0)} - k_s^{(0)})} \left(\sum_{s=i+1}^{p_0} \gamma_{s,i} u^{e-(k_s^{(0)} - k_i^{(0)})} e_s - [r_i^{(0)} e - (k_i^{(0)} - k_s^{(0)})] e_i \right) \\ &+ u^{r_s^{(0)}e} \sum_{i=s+1}^{p_0} \gamma_{i,s} u^{e-(k_i^{(0)} - k_s^{(0)})} e_i - r_s^{(0)} e u^{r_s^{(0)}e} e_s. \end{aligned}$$

Since $\gamma_{i,j} \in u^{e[e-(k_i^{(0)} - k_j^{(0)})]}\overline{S}_0$, we have

$$\begin{aligned} N(f_s) \equiv &- \sum_{t=0}^{r_{p_0}^{(0)}-1} x_{p_0,s}^{(t)} [e(t+1) - (k_{p_0}^{(0)} - k_s^{(0)})] u^{e(t+1)-(k_{p_0}^{(0)} - k_s^{(0)})} e_{p_0} \\ &- \sum_{i=s+1}^{p_0-1} \beta_{i,s} [r_i^{(0)} e - (k_i^{(0)} - k_s^{(0)})] u^{r_i^{(0)}e - (k_i^{(0)} - k_s^{(0)})} e_i - r_s^{(0)} e u^{r_s^{(0)}e} e_s \end{aligned}$$

modulo $\mathrm{Fil}^{n-1}\mathcal{M}_{p_0,0}$. By the same argument as in (3.3.6), we have

$$\begin{aligned} N(f_s) \equiv &\sum_{t=0}^{r_{p_0}^{(0)}-1} x_{p_0,s}^{(t)} [r_s^{(0)} e - e(t+1) + (k_{p_0}^{(0)} - k_s^{(0)})] u^{e(t+1)-(k_{p_0}^{(0)} - k_s^{(0)})} e_{p_0} \\ &+ \sum_{i=s+1}^{p_0-1} \beta_{i,s} [r_s^{(0)} e - r_i^{(0)} e + (k_i^{(0)} - k_s^{(0)})] u^{r_i^{(0)}e - (k_i^{(0)} - k_s^{(0)})} e_i \end{aligned}$$

modulo $\mathrm{Fil}^{n-1}\mathcal{M}_{p_0,0}$. Now, from the induction hypothesis on $j \in \{p_0-1, p_0-2, \cdots, s+1\}$,

$$u^e \sum_{i=s+1}^{p_0-1} \beta_{i,s}[r_s^{(0)}e - r_i^{(0)}e + (k_i^{(0)} - k_s^{(0)})]u^{r_i^{(0)}e-(k_i^{(0)}-k_s^{(0)})}e_i \in \mathrm{Fil}^{n-1}\mathcal{M}_{p_0,0}$$

and so we have

$$u^e N(f_s) \equiv \sum_{t=0}^{r_{p_0}^{(0)}-1} x_{p_0,s}^{(t)}[r_s^{(0)}e - e(t+1) + (k_{p_0}^{(0)} - k_s^{(0)})]u^{e(t+2)-(k_{p_0}^{(0)}-k_s^{(0)})}e_{p_0}$$

modulo $\mathrm{Fil}^{n-1}\mathcal{M}_{p_0,0}$. By the same argument as in (3.3.7), we have

$$u^e N(f_s) \equiv \sum_{t=0}^{r_{p_0}^{(0)}-2} x_{p_0,s}^{(t)}[r_s^{(0)}e - e(t+1) + (k_{p_0}^{(0)} - k_s^{(0)})]u^{e(t+2)-(k_{p_0}^{(0)}-k_s^{(0)})}e_{p_0}$$

modulo $\mathrm{Fil}^{n-1}\mathcal{M}_{p_0,0}$. By the same argument as in (3.3.8), one can readily check that $r_s^{(0)}e - e(t+1) + (k_{p_0}^{(0)} - k_s^{(0)}) \not\equiv 0$ modulo (p) for all $0 \le t \le r_{p_0}^{(0)} - 2$. Hence, we conclude that $x_{p_0,s}^{(t)} = 0$ for all $0 \le t \le r_{p_0}^{(0)} - 2$ as $u^e N(f_s) \in \mathrm{Fil}^{n-1}\mathcal{M}_{p_0,0}$, which completes the proof. $\square$

Proposition 3.3.2. *Keep the assumptions and notation of Lemma 3.3.1. Assume further that $\overline{\rho}_0$ is maximally non-split and satisfies the conditions in (3.2.2).*

Then $\beta_{i,i-1} \in \mathbf{F}^\times$ for $i \in \{1, 2, \cdots, n-1\}$ and we have the following identities: for $0 \le j < j+1 < i \le n-1$

$$\mathrm{FL}_n^{i,j}(\overline{\rho}_0) = \left[\beta_{i,j}\nu_{j+1}\cdots\nu_{i-1} : (-1)^{i-j+1}\det A'_{i,j}\right] \in \mathbb{P}^1(\mathbf{F})$$

where

$$A'_{i,j} = \begin{pmatrix} \beta_{j+1,j} & \beta_{j+2,j} & \beta_{j+3,j} & \cdots & \beta_{i-1,j} & \beta_{i,j} \\ 1 & \beta_{j+2,j+1} & \beta_{j+3,j+1} & \cdots & \beta_{i-1,j+1} & \beta_{i,j+1} \\ 0 & 1 & \beta_{j+3,j+2} & \cdots & \beta_{i-1,j+2} & \beta_{i,j+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_{i-1,i-2} & \beta_{i,i-2} \\ 0 & 0 & 0 & \cdots & 1 & \beta_{i,i-1} \end{pmatrix}.$$

Proof. We may assume $c_0 = 0$ by Lemma 3.2.4. We let $V := \mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M})$ and $A := \mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1})$ be as in the statement of Lemma 3.3.1. By Lemma 2.6.3, the ϕ-module over $\mathbf{F} \otimes_{\mathbf{F}_p} \mathbf{F}_p((\underline{\varpi}))$ defined by $\mathfrak{M} := M_{\mathbf{F}_p((\underline{\varpi}))}(\mathcal{M}^*)$ is described as follows:

$$\mathrm{Mat}_{\underline{\mathfrak{e}}}(\phi) = (U_{i,j})$$

where

$$U_{i,j} = \begin{cases} \nu_j^{-1} \cdot \underline{\varpi}^{r_j^{(0)}e} & \text{if } i = j; \\ 0 & \text{if } i > j; \\ \nu_j^{-1} \cdot \beta_{j,i} \cdot \underline{\varpi}^{r_j^{(0)}e-(k_j^{(0)}-k_i^{(0)})} & \text{if } i < j \end{cases}$$

in a framed basis $\underline{\mathfrak{e}} = (\mathfrak{e}_{n-1}, \mathfrak{e}_{n-2}, \cdots, \mathfrak{e}_0)$ with dual type $\omega^{-k_{n-1}^{(0)}} \oplus \omega^{-k_{n-2}^{(0)}} \cdots \oplus \omega^{-k_0^{(0)}}$.

By considering the change of basis $\underline{\mathfrak{e}}' = (\underline{\varpi}^{k_{n-1}^{(0)}}\mathfrak{e}_{n-1}, \underline{\varpi}^{k_{n-2}^{(0)}}\mathfrak{e}_{n-2}, \cdots, \underline{\varpi}^{k_0^{(0)}}\mathfrak{e}_0)$, $\mathrm{Mat}_{\underline{\mathfrak{e}}'}(\phi)$ is described as follows:

$$\mathrm{Mat}_{\underline{\mathfrak{e}}'}(\phi) = (V_{i,j})$$

where

$$V_{i,j} = \begin{cases} \nu_j^{-1} \cdot \underline{\varpi}^{e(k_j^{(0)}+r_j^{(0)})} & \text{if } i = j; \\ 0 & \text{if } i > j; \\ \nu_j^{-1} \cdot \beta_{j,i} \cdot \underline{\varpi}^{e(k_j^{(0)}+r_j^{(0)})} & \text{if } i < j. \end{cases}$$

Since $k_i^{(0)} = c_i + i - r_i^{(0)}$ for each $n-1 \geq i \geq 0$, we easily see that the ϕ-module $\mathfrak{M}_0$ is the base change via $\mathbf{F}\otimes_{\mathbf{F}_p}\mathbf{F}_p((\underline{p})) \to \mathbf{F}\otimes_{\mathbf{F}_p}\mathbf{F}_p((\underline{\varpi}))$ of the ϕ-module $\mathfrak{M}_0$ over $\mathbf{F}\otimes_{\mathbf{F}_p}\mathbf{F}_p((\underline{p}))$ described by

$$\mathrm{Mat}_{\underline{\mathfrak{e}}''}(\phi) = \begin{pmatrix} \nu_{n-1}^{-1}\underline{p}^{c_{n-1}+(n-1)} & 0 & \cdots & 0 \\ \nu_{n-1}^{-1}\beta_{n-1,n-2}\underline{p}^{c_{n-1}+(n-1)} & \nu_{n-2}^{-1}\underline{p}^{c_{n-2}+(n-2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \nu_{n-1}^{-1}\beta_{n-1,0}\underline{p}^{c_{n-1}+(n-1)} & \nu_{n-2}^{-1}\beta_{n-2,0}\underline{p}^{c_{n-2}+(n-2)} & \cdots & \nu_0^{-1}\underline{p}^{c_0} \end{pmatrix}$$

in an appropriate basis $\underline{\mathfrak{e}}'' = (\mathfrak{e}''_{n-1}, \mathfrak{e}''_{n-2}, \cdots, \mathfrak{e}''_0)$, which can be rewritten as

$$\mathrm{Mat}_{\underline{\mathfrak{e}}''}(\phi) = \underbrace{\begin{pmatrix} \nu_{n-1}^{-1} & 0 & \cdots & 0 \\ \nu_{n-1}^{-1}\beta_{n-1,n-2} & \nu_{n-2}^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \nu_{n-1}^{-1}\beta_{n-1,0} & \nu_{n-2}^{-1}\beta_{n-2,0} & \cdots & \nu_0^{-1} \end{pmatrix}}_{=:H'} \cdot \mathrm{Diag}\left(\underline{p}^{c_{n-1}+n-1}, \cdots, \underline{p}^{c_1+1}, \underline{p}^{c_0}\right).$$

By considering the change of basis $\underline{\mathfrak{e}}''' = \underline{\mathfrak{e}}'' \cdot H'$ and then reversing the order of the basis $\underline{\mathfrak{e}}'''$, the Frobenius ϕ of $\mathfrak{M}_0$ with respect to this new basis is described as follows:

$$\mathrm{Mat}(\phi) = \mathrm{Diag}\left(\underline{p}^{c_0}, \underline{p}^{c_1+1}, \cdots, \underline{p}^{c_{n-1}+(n-1)}\right) \underbrace{\begin{pmatrix} \nu_0^{-1} & \nu_1^{-1}\beta_{1,0} & \cdots & \nu_{n-1}^{-1}\beta_{n-1,0} \\ 0 & \nu_1^{-1} & \cdots & \nu_{n-1}^{-1}\beta_{n-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \nu_{n-1}^{-1} \end{pmatrix}}_{=:H} \tag{3.3.9}$$

with respect to the new basis described as above.

The last displayed upper-triangular matrix H is the Frobenius of the Fontaine–Laffaille module M such that $\mathrm{T}^*_{\mathrm{cris}}(M) \cong \overline{\rho}_0 \cong \mathrm{T}^r_{\mathrm{st}}(\mathcal{M})$, by Lemma 2.6.4. Hence, we get the desired results (cf. Definition 3.2.3). □

Remark 3.3.3. *We emphasize that the matrix H is the Frobenius of the Fontaine–Laffaille module M, with respect to a basis $(e_0, e_1, \cdots, e_{n-1})$ compatible with the filtration, such that $\mathrm{T}^*_{\mathrm{cris}}(M) \cong \overline{\rho}_0 \cong \mathrm{T}^r_{\mathrm{st}}(\mathcal{M})$, so that we can now apply the conditions in (3.2.2) as well as Definition 3.2.5 to the Breuil modules in Lemma 3.3.1. Moreover, H can be written as*

$$H = \underbrace{\begin{pmatrix} 1 & \beta_{1,0} & \cdots & \beta_{n-1,0} \\ 0 & 1 & \cdots & \beta_{n-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{=:H''} \cdot \mathrm{Diag}\left(\nu_0^{-1}, \nu_1^{-1}, \cdots, \nu_{n-1}^{-1}\right),$$

so that we have $(1,n)w_0 H^t \in B(\mathbf{F})w_0B(\mathbf{F})$ if and only if $(1,n)w_0(H'')^t \in B(\mathbf{F})w_0B(\mathbf{F})$. Hence, $\overline{\rho}_0$ being Fontaine–Laffaille generic is a matter only of the entries of the filtration of the Breuil modules if the Breuil modules are written as in Lemma 3.3.1.

3.4. **Fontaine–Laffaille parameters vs Frobenius eigenvalues.** In this section, we study further the Breuil modules of Lemma 3.3.1. We show that if the filtration is of a certain shape then a certain product of Frobenius eigenvalues (of the Breuil modules) corresponds to the newest Fontaine–Laffaille parameter, $\mathrm{FL}_n^{n-1,0}(\overline{\rho}_0)$. To get such a shape of the filtration, we assume further that $\overline{\rho}_0$ is Fontaine–Laffaille generic.

Lemma 3.4.1. *Keep the assumptions and notation of Lemma 3.3.1. Assume further that $\overline{\rho}_0$ is Fontaine–Laffaille generic (cf. Definition 3.2.5).*

Then $\mathcal{M} \in \mathbf{F}\text{-}\mathrm{BrMod}_{\mathrm{dd}}^{n-1}$ can be described as follows: there exist a framed basis $\underline{e}$ for $\mathcal{M}$ and a framed system of generators $\underline{f}$ for $\mathrm{Fil}^{n-1}\mathcal{M}$ such that

$$\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1}) = \mathrm{Diag}\,(\mu_{n-1},\, \mu_{n-2},\, \cdots,\, \mu_0)$$

and

$$\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M}) = (U_{i,j})$$

where

$$U_{i,j} = \begin{cases} u^{r_{n-1}^{(0)}e-(k_{n-1}^{(0)}-k_0^{(0)})} & \text{if } i = n-1 \text{ and } j = 0; \\ u^{r_i^{(0)}e} & \text{if } 0 < i = j < n-1; \\ x_{i,j}\cdot u^{r_i^{(0)}e-(k_i^{(0)}-k_j^{(0)})} & \text{if } n-1 > i > j; \\ u^{r_0^{(0)}e+(k_{n-1}^{(0)}-k_0^{(0)})} & \text{if } i = 0 \text{ and } j = n-1; \\ x_{0,j}\cdot u^{r_0^{(0)}e+(k_j^{(0)}-k_0^{(0)})} & \text{if } i = 0 \leq j < n-1; \\ 0 & \text{otherwise.} \end{cases} \tag{3.4.1}$$

Here, $\mu_i \in \mathbf{F}^\times$ and $x_{i,j} \in \mathbf{F}$.

Moreover, we have the following identity:

$$\mathrm{FL}_n^{n-1,0}(\overline{\rho}_0) = \prod_{i=1}^{n-2} \mu_i^{-1}.$$

Due to the size of the matrix, we decide to describe the matrix $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M})$ as (3.4.1). But for the reader we visualize the matrix $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M})$ below, although it is less accurate:

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & u^{r_{n-1}^{(0)}e-k_{n-1,0}^{(0)}} \\ 0 & u^{r_{n-2}^{(0)}} & \cdots & x_{n-2,1}u^{r_{n-2}^{(0)}e-k_{n-2,1}^{(0)}} & x_{n-2,0}u^{r_{n-2}^{(0)}e-k_{n-2,0}^{(0)}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & u^{r_1^{(0)}} & x_{1,0}u^{r_1^{(0)}e-k_{1,0}^{(0)}} \\ u^{r_0^{(0)}e+k_{n-1,0}^{(0)}} & x_{0,n-2}u^{r_0^{(0)}e+k_{n-2,0}^{(0)}} & \cdots & x_{0,1}u^{r_0^{(0)}e+k_{1,0}^{(0)}} & x_{0,0}u^{r_0^{(0)}e} \end{pmatrix}$$

where $k_{i,j}^{(0)} := k_i^{(0)} - k_j^{(0)}$.

Proof. Let $\underline{e}_0$ be a framed basis for $\mathcal{M}$ and $\underline{f}_0$ a framed system of generators for $\mathrm{Fil}^{n-1}\mathcal{M}$ such that $V_0 := \mathrm{Mat}_{\underline{e}_0,\underline{f}_0}(\mathrm{Fil}^{n-1}\mathcal{M})$ and $A_0 := \mathrm{Mat}_{\underline{e}_0,\underline{f}_0}(\phi_{n-1})$ are given as in Lemma 3.3.1. So, in particular, V_0 is upper-triangular and A_0 is diagonal.

By Proposition 3.3.2, the upper-triangular matrix H in (3.3.9) is the Frobenius of the Fontaine–Laffaille module corresponding to $\overline{\rho}_0$, as in Definition 3.2.3. Since we assume that $\overline{\rho}_0$ is Fontaine–Laffaille generic, we have $(1,n)w_0H^t \in B(\mathbf{F})w_0B(\mathbf{F})$ as discussed right after Definition 3.2.3, so that we have $w_0H^tw_0 \in (1,n)B(\mathbf{F})w_0B(\mathbf{F})w_0$. Equivalently, $w_0(H')^tw_0 \in (1,n)B(\mathbf{F})w_0B(\mathbf{F})w_0$ by Remark 3.3.3, where H' is defined in Remark 3.3.3. Hence, comparing V_0 with $w_0(H')^tw_0$, there exists a lower-triangular matrix $C \in \mathrm{GL}_n^\square(\overline{S})$ such that

$$V_0 \cdot C = V_1 := (U_{i,j})_{0\leq i,j\leq n-1}$$

where $U_{i,j}$ is described as in (3.4.1), since any matrix in $w_0B(\mathbf{F})w_0$ is lower-triangular. From the identity $V_0 \cdot C = V_1$, we have $V_1 = \mathrm{Mat}_{\underline{e}_1,\underline{f}_1}(\mathrm{Fil}^{n-1}\mathcal{M})$ and $A_1 := \mathrm{Mat}_{\underline{e}_1,\underline{f}_1}(\phi_{n-1}) = A_0 \cdot \phi(C)$ by Lemma 2.4.4, where $\underline{e}_1 := \underline{e}_0$ and $\underline{f}_1 := \underline{e}_1V_1$. If $i < j$, then $[k_j^{(0)} - k_i^{(0)}]_1 = k_j^{(0)} - k_i^{(0)} \geq n$ as $\overline{\rho}_0$ is strongly generic, so that A_1 is congruent to a diagonal matrix $B_2' \in \mathrm{GL}_n(\mathbf{F})$ modulo (u^{ne}) as $C = (c_{i,j}\cdot u^{[k_j^{(0)}-k_i^{(0)}]_1})$ is a lower-triangular and A_0 is diagonal.

Let V_2 be the matrix obtained from V_1 by replacing $x_{i,j}$ in (3.4.1) by $y_{i,j}$, and $B_2 = (b_{i,j})$ is the diagonal matrix defined by taking $b_{i,i} = b'_{i,i}$ if $1 \le i \le n-2$ and $b_{i,i} = b'_{n-1-i,n-1-i}$ otherwise, where $B'_2 = (b'_{i,j})$. Then it is obvious that there exist $y_{i,j} \in \mathbf{F}$ such that

$$A_1 \cdot V_2 \equiv V_1 \cdot B_2$$

modulo (u^{ne}). Letting $\underline{e}_2 := \underline{e}_1 \cdot A_1$, we have $V_2 = \mathrm{Mat}_{\underline{e}_2,\underline{f}_2}(\mathrm{Fil}^{n-1}\mathcal{M})$ and $\mathrm{Mat}_{\underline{e}_2,\underline{f}_2}(\phi_{n-1}) = \phi(B_2)$ by Lemma 2.4.4. Note that $A_2 := \mathrm{Mat}_{\underline{e}_2,\underline{f}_2}(\phi_{n-1})$ is diagonal. Hence, there exist a framed basis for $\mathcal{M}$ and a framed system of generators for $\mathrm{Fil}^{n-1}\mathcal{M}$ such that $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1})$ and $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M})$ are described as in the statement.

We now prove the second part of the lemma. It is harmless to assume $c_0 = 0$ by Lemma 3.2.4. Let $V := \mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M})$ and $A := \mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1})$ be as in the first part of the lemma. By Lemma 2.6.3, the ϕ-module over $\mathbf{F} \otimes_{\mathbf{F}_p} \mathbf{F}_p((\underline{\varpi}))$ defined by $\mathfrak{M} := M_{\mathbf{F}_p((\underline{\varpi}))}(\mathcal{M}^*)$ is described as follows: there exists a basis $\underline{\mathfrak{e}} = (\mathfrak{e}_{n-1}, \mathfrak{e}_{n-2}, \cdots, \mathfrak{e}_0)$, compatible with decent data, such that $\mathrm{Mat}_{\underline{\mathfrak{e}}}(\phi) = (\widehat{A}^{-1}\widehat{V})^t$ where $\widehat{V}^t$ and $(\widehat{A}^{-1})^t$ are computed as follows:

$$\widehat{V}^t = \begin{pmatrix} 0 & 0 & \cdots & 0 & \underline{\varpi}^{r_0^{(0)}e+k_{n-1,0}^{(0)}} \\ 0 & \underline{\varpi}^{r_{n-2}^{(0)}} & \cdots & 0 & x_{0,n-2}\underline{\varpi}^{r_0^{(0)}e+k_{n-2,0}^{(0)}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & x_{n-2,1}\underline{\varpi}^{r_{n-2}^{(0)}e-k_{n-2,1}^{(0)}} & \cdots & \underline{\varpi}^{r_1^{(0)}} & x_{0,1}\underline{\varpi}^{r_0^{(0)}e+k_{1,0}^{(0)}} \\ \underline{\varpi}^{r_{n-1}^{(0)}e-k_{n-1,0}^{(0)}} & x_{n-2,0}\underline{\varpi}^{r_{n-2}^{(0)}e-k_{n-2,0}^{(0)}} & \cdots & x_{1,0}\underline{\varpi}^{r_1^{(0)}e-k_{1,0}^{(0)}} & x_{0,0}\underline{\varpi}^{r_0^{(0)}e} \end{pmatrix}$$

and

$$\widehat{A}^{-1} = \mathrm{Diag}\left(\mu_{n-1}^{-1},\, \mu_{n-2}^{-1},\, \cdots,\, \mu_0^{-1}\right).$$

By considering the change of basis $\underline{\mathfrak{e}}' = (\underline{\varpi}^{k_{n-1}^{(0)}}\mathfrak{e}_{n-1}, \underline{\varpi}^{k_{n-2}^{(0)}}\mathfrak{e}_{n-2}, \cdots, \underline{\varpi}^{k_1^{(0)}}\mathfrak{e}_1, \underline{\varpi}^{k_0^{(0)}}\mathfrak{e}_0)$, we have

$$\mathrm{Mat}_{\underline{\mathfrak{e}}'}(\phi) = (\widehat{V}^t)' \cdot \mathrm{Diag}\left(\mu_{n-1}^{-1},\, \mu_{n-2}^{-1},\, \cdots,\, \mu_0^{-1}\right)$$

where

$$(\widehat{V}^t)' = \begin{pmatrix} 0 & 0 & \cdots & 0 & \underline{\varpi}^{e(k_0^{(0)}+r_0^{(0)})} \\ 0 & \underline{\varpi}^{e(k_{n-2}^{(0)}+r_{n-2}^{(0)})} & \cdots & 0 & x_{0,n-2}\underline{\varpi}^{e(k_0^{(0)}+r_0^{(0)})} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & x_{n-2,1}\underline{\varpi}^{e(k_{n-2}^{(0)}+r_{n-2}^{(0)})} & \cdots & \underline{\varpi}^{e(k_1^{(0)}+r_1^{(0)})} & x_{0,1}\underline{\varpi}^{e(k_0^{(0)}+r_0^{(0)})} \\ \underline{\varpi}^{e(k_{n-1}^{(0)}+r_{n-1}^{(0)})} & x_{n-2,0}\underline{\varpi}^{e(k_{n-2}^{(0)}+r_{n-2}^{(0)})} & \cdots & x_{1,0}\underline{\varpi}^{e(k_1^{(0)}+r_1^{(0)})} & x_{0,0}\underline{\varpi}^{e(k_0^{(0)}+r_0^{(0)})} \end{pmatrix}.$$

Since $k_j^{(0)} + r_j^{(0)} = c_j + j$ for all j, it is immediate that the ϕ-module $\mathfrak{M}$ over $\mathbf{F} \otimes_{\mathbf{F}_p} \mathbf{F}_p((\underline{\varpi}))$ is the base change via $\mathbf{F} \otimes_{\mathbf{F}_p} \mathbf{F}_p((\underline{p})) \to \mathbf{F} \otimes_{\mathbf{F}_p} \mathbf{F}_p((\underline{\varpi}))$ of the ϕ-module $\mathfrak{M}_0$ over $\mathbf{F} \otimes_{\mathbf{F}_p} \mathbf{F}_p((\underline{p}))$ described by

$$\mathrm{Mat}_{\underline{\mathfrak{e}}''}(\phi) = F'' \cdot \mathrm{Diag}\left(\underline{p}^{c_{n-1}+n-1}, \underline{p}^{c_{n-2}+n-2}, \cdots, \underline{p}^{c_0}\right)$$

where

$$F'' = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \mu_0^{-1} \\ 0 & \mu_{n-2}^{-1} & 0 & \cdots & 0 & \mu_0^{-1}x_{0,n-2} \\ 0 & \mu_{n-2}^{-1}x_{n-2,n-3} & \mu_{n-3}^{-1} & \cdots & 0 & \mu_0^{-1}x_{0,n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \mu_{n-2}^{-1}x_{n-2,1} & \mu_{n-3}^{-1}x_{n-3,1} & \cdots & \mu_1^{-1} & \mu_0^{-1}x_{0,1} \\ \mu_{n-1}^{-1} & \mu_{n-2}^{-1}x_{n-2,0} & \mu_{n-3}^{-1}x_{n-3,0} & \cdots & \mu_1^{-1}x_{1,0} & \mu_0^{-1}x_{0,0} \end{pmatrix},$$

in an appropriate basis $\underline{\mathfrak{e}}''$.

Now, consider the change of basis $\underline{\mathfrak{e}}''' = \underline{\mathfrak{e}}'' \cdot F''$ and then reverse the order of the basis $\mathfrak{e}'''$. Then the matrix of the Frobenius ϕ for $\mathfrak{M}_0$ with respect to this new basis is given by

$$\mathrm{Diag}\left(\underline{p}^{c_0}, \underline{p}^{c_1+1}, \cdots, \underline{p}^{c_{n-1}+n-1}\right) \cdot F$$

where

$$F = \begin{pmatrix} \mu_0^{-1}x_{0,0} & \mu_1^{-1}x_{1,0} & \mu_2^{-1}x_{2,0} & \cdots & \mu_{n-2}^{-1}x_{n-2,0} & \mu_{n-1}^{-1} \\ \mu_0^{-1}x_{0,1} & \mu_1^{-1} & \mu_2^{-1}x_{2,1} & \cdots & \mu_{n-2}^{-1}x_{n-2,1} & 0 \\ \mu_0^{-1}x_{0,2} & 0 & \mu_2^{-1} & \cdots & \mu_{n-2}^{-1}x_{n-2,2} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_0^{-1}x_{0,n-2} & 0 & 0 & \cdots & \mu_{n-2}^{-1} & 0 \\ \mu_0^{-1} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

By Lemma 2.6.4, there exists a Fontaine–Laffaille module M such that $\mathcal{F}(M) = \mathfrak{M}_0$ with Hodge–Tate weights $(c_0, c_1+1, \cdots, c_{n-1}+n-1)$ and $\mathrm{Mat}_{\underline{e}}(\phi_\bullet) = F$ for some basis $\underline{e}$ of M compatible with the Hodge filtration on M. On the other hand, since $\mathrm{T}^*_{\mathrm{cris}}(M) \cong \overline{\rho}_0$, there exists a basis $\underline{e}'$ of M compatible with the Hodge filtration on M such that

$$\mathrm{Mat}_{\underline{e}'}(\phi_\bullet) = \underbrace{\begin{pmatrix} w_0 & w_{0,1} & \cdots & w_{0,n-2} & w_{0,n-1} \\ 0 & w_1 & \cdots & w_{1,n-2} & w_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & w_{n-2} & w_{n-2,n-1} \\ 0 & 0 & \cdots & 0 & w_{n-1} \end{pmatrix}}_{=:G}$$

where $w_{i,j} \in \mathbf{F}$ and $w_i \in \mathbf{F}^\times$ by Lemma 3.2.1. Since both $\underline{e}$ and $\underline{e}'$ are compatible with the Hodge filtration on M, there exists a unipotent lower-triangular $n \times n$-matrix U such that

$$U \cdot F = G.$$

Note that we have $w_{0,n-1} = \mu_{n-1}^{-1}$ by direct computation.

Let U' be the $(n-1) \times (n-1)$-matrix obtained from U by deleting the right-most column and the lowest row, and F' (resp. G') the $(n-1) \times (n-1)$-matrix obtained from F (resp. G) by deleting the left-most column and the lowest row. Then they still satisfy $G' = U' \cdot F'$ as U is a lower-triangular unipotent matrix, so that

$$\mathrm{FL}_n^{n-1,0}(\overline{\rho}_0) = [w_{0,n-1} : (-1)^n \det G'] = \left[\mu_{n-1}^{-1} : (-1)^n \det F'\right] = \left[1 : \prod_{i=1}^{n-2} \mu_i^{-1}\right],$$

which completes the proof. □

Proposition 3.4.2. *Keep the assumptions and notation of Lemma 3.4.1.*

Then $\mathcal{M} \in \mathbf{F}\text{-}\mathrm{BrMod}_{\mathrm{dd}}^{n-1}$ can be described as follows: there exist a framed basis $\underline{e}$ for $\mathcal{M}$ and a framed system of generators $\underline{f}$ for $\mathrm{Fil}^{n-1}\mathcal{M}$ such that

$$\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M}) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & u^{e-(k_{n-1}^{(0)}-k_0^{(0)})} \\ 0 & u^{(n-2)e} & 0 & \cdots & 0 & 0 \\ 0 & 0 & u^{(n-3)e} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & u^e & 0 \\ u^{(n-2)e+(k_{n-1}^{(0)}-k_0^{(0)})} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Moreover, if we let

$$\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1}) = \left(\alpha_{i,j} u^{[k_j^{(0)}-k_i^{(0)}]_1}\right)$$

for $\alpha_{i,i}\in\overline{S}_0^{\times}$ and $\alpha_{i,j}\in\overline{S}_0$ if $i\neq j$ then we have the following identity:

$$\mathrm{FL}_n^{n-1,0}(\overline{\rho}_0)=\prod_{i=1}^{n-2}(\alpha_{i,i}^{(0)})^{-1}=\prod_{i=1}^{n-2}\mu_i^{-1}$$

where $\alpha_{i,j}^{(0)}\in\mathbf{F}$ is determined by $\alpha_{i,j}^{(0)}\equiv\alpha_{i,j}$ modulo (u^e).

Note that $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1})$ always belong to $\mathrm{GL}_n^{\square}(\overline{S})$ as $\underline{e}$ and $\underline{f}$ are framed.

Proof. We let $\underline{e}_0$ (resp. $\underline{e}_1$) be a framed basis for $\mathcal{M}$ and $\underline{f}_0$ (resp. $\underline{f}_1$) be a framed system of generators for $\mathrm{Fil}^{n-1}\mathcal{M}$ such that $\mathrm{Mat}_{\underline{e}_0,\underline{f}_0}(\mathrm{Fil}^{n-1}\mathcal{M})$ and $\mathrm{Mat}_{\underline{e}_0,\underline{f}_0}(\phi_{n-1})$ (resp. $\mathrm{Mat}_{\underline{e}_1,\underline{f}_1}(\mathrm{Fil}^{n-1}\mathcal{M})$ and $\mathrm{Mat}_{\underline{e}_1,\underline{f}_1}(\phi_{n-1})$) are given as in the statement of Lemma 3.4.1 (resp. in the statement of Proposition 3.4.2). We also let $V_0=\mathrm{Mat}_{\underline{e}_0,\underline{f}_0}(\mathrm{Fil}^{n-1}\mathcal{M})$ and $A_0=\mathrm{Mat}_{\underline{e}_0,\underline{f}_0}(\phi_{n-1})$ as well as $V_1=\mathrm{Mat}_{\underline{e}_1,\underline{f}_1}(\mathrm{Fil}^{n-1})\mathcal{M}$ and $A_1=\mathrm{Mat}_{\underline{e}_1,\underline{f}_1}(\phi_{n-1})$.

It is obvious that there exist $R=(r_{i,j}u^{[k_j^{(0)}-k_i^{(0)}]_1})$ and $C=(c_{i,j}u^{[k_j^{(0)}-k_i^{(0)}]_1})$ in $\mathrm{GL}_n^{\square}(\overline{S})$ such that

$$R\cdot V_0\cdot C=V_1\text{ and }\underline{e}_1=\underline{e}_0R^{-1}$$

for $r_{i,j}$ and $c_{i,j}$ in $\overline{S}_0$. From the first equation above, we immediately get the identities:

$$r_{n-1,n-1}^{(0)}\cdot c_{0,0}^{(0)}=1=r_{0,0}^{(0)}\cdot c_{n-1,n-1}^{(0)}\text{ and }r_{i,i}^{(0)}\cdot c_{i,i}^{(0)}=1$$

for $0<i<n-1$, where $r_{i,j}^{(0)}\in\mathbf{F}$ (resp. $c_{i,j}^{(0)}\in\mathbf{F}$) is determined by $r_{i,j}^{(0)}\equiv r_{i,j}$ modulo (u^e) (resp. $c_{i,j}^{(0)}\equiv c_{i,j}$ modulo (u^e)). By Lemma 2.4.4, we see that $A_1=R\cdot A_0\cdot\phi(C)$.

Hence, if we let $A_1=\left(\alpha_{i,j}u^{[k_j^{(0)}-k_i^{(0)}]_1}\right)$ then

$$r_{i,i}^{(0)}\cdot\mu_i\cdot c_{i,i}^{(0)}=\alpha_{i,i}^{(0)}$$

for each $0<i<n-1$ since R and C are diagonal modulo (u), so that we have

$$\prod_{i=1}^{n-2}\mu_i=\prod_{i=1}^{n-2}\alpha_{i,i}^{(0)}$$

which completes its proof. □

Note that the matrix in the statement of Proposition 3.4.2 gives rise to the elementary divisors of $\mathcal{M}/\mathrm{Fil}^{n-1}\mathcal{M}$.

3.5. **Filtration of strongly divisible modules.** In this section, we describe the filtration of the strongly divisible modules lifting the Breuil modules described in Proposition 3.4.2. Throughout this section, we keep the notation $r_i^{(0)}$ as in (3.3.1) as well as $k_i^{(0)}$.

We start to recall the following lemma, which is easy to prove but very useful.

Lemma 3.5.1. *Let $0<f\leq n$ be an integer, and let $\widehat{\mathcal{M}}\in\mathcal{O}_E\text{-}\mathrm{Mod}_{\mathrm{dd}}^{n-1}$ be a strongly divisible module corresponding to a lattice in a potentially semi-stable representation $\rho:G_{\mathbf{Q}_p}\to\mathrm{GL}_n(E)$ with Hodge–Tate weights $\{-(n-1),-(n-2),\cdots,0\}$ and Galois type of niveau f such that $\mathrm{T}_{\mathrm{st}}^{\mathbf{Q}_p,n-1}(\widehat{\mathcal{M}})\otimes_{\mathcal{O}_E}\mathbf{F}\cong\overline{\rho}_0$.*

If we let

$$X^{(i)}:=\left(\frac{\mathrm{Fil}^{n-1}\widehat{\mathcal{M}}\cap\mathrm{Fil}^iS\cdot\widehat{\mathcal{M}}}{\mathrm{Fil}^{n-1}S\cdot\widehat{\mathcal{M}}}\right)\otimes_{\mathcal{O}_E}E$$

for $i \in \{0, 1, \cdots, n-1\}$, then for any character $\xi : \mathrm{Gal}(K/K_0) \to K^\times$ we have that the ξ-isotypical component $X_\xi^{(i)}$ of $X^{(i)}$ is a free $K_0 \otimes E$-module of finite rank

$$\mathrm{rank}_{K_0\otimes_{\mathbf{Q}_p}E} X_\xi^{(i)} = \frac{n(n-1)}{2} - \frac{i(i+1)}{2}.$$

Moreover, multiplication by $u \in S$ induces an isomorphism $X_\xi^{(0)} \xrightarrow{\sim} X_{\xi\widetilde{\omega}}^{(0)}$.

Proof. We follow the strategy of the proof of [HLM], Lemma 2.4.9. Since ρ has Hodge–Tate weights $\{-(n-1), -(n-2), \cdots, 0\}$, by the analogue with E-coefficients of [Bre97], Proposition A.4, we deduce that

$$\mathrm{Fil}^{n-1}\mathcal{D} = \mathrm{Fil}^{n-1}S_E\widehat{f}_{n-1} \oplus \mathrm{Fil}^{n-2}S_E\widehat{f}_{n-2} \oplus \cdots \oplus \mathrm{Fil}^1 S_E\widehat{f}_1 \oplus S_E\widehat{f}_0$$

for some S_E-basis $\widehat{f}_0, \cdots, \widehat{f}_{n-1}$ of $\mathcal{D}$, where $\mathcal{D} := \widehat{\mathcal{M}}[\frac{1}{p}] \cong S_E \otimes_E \mathrm{D}_{\mathrm{st}}^{\mathbf{Q}_p, n-1}(V)$, so that we also have

$$\mathrm{Fil}^{n-1}\mathcal{D} \cap \mathrm{Fil}^i S_E\mathcal{D} = \mathrm{Fil}^{n-1}S_E\widehat{f}_{n-1} \oplus \mathrm{Fil}^{n-2}S_E\widehat{f}_{n-2} \oplus \cdots \oplus \mathrm{Fil}^i S_E\widehat{f}_i \oplus \cdots \oplus \mathrm{Fil}^i S_E\widehat{f}_0.$$

Since $\rho \cong \mathrm{T}_{\mathrm{st}}^{\mathbf{Q}_p, n-1}(\widehat{\mathcal{M}}) \otimes_{\mathcal{O}_E} E$ is a $G_{\mathbf{Q}_p}$-representation, $\mathrm{Fil}^i(K \otimes_{K_0} \mathrm{D}_{\mathrm{st}}^{\mathbf{Q}_p, n-1}(\rho)) \cong K \otimes_{\mathbf{Q}_p} \mathrm{Fil}^i \mathrm{D}_{\mathrm{dR}}(\rho \otimes \varepsilon^{1-n})$, so that $X^{(i)} \cong \frac{\mathrm{Fil}^{n-1}\mathcal{D} \cap \mathrm{Fil}^i S_E\mathcal{D}}{\mathrm{Fil}^{n-1}S_E\mathcal{D}}$ is a free $K_0 \otimes_{\mathbf{Q}_p} E$-module. Since $\frac{S_E}{\mathrm{Fil}^{n-1}S_E} \cong \bigoplus_{i=0}^{n-2}\bigoplus_{j=0}^{e-1}(K_0 \otimes_{\mathbf{Q}_p} E)u^j E(u)^i$, we have $\mathrm{rank}_{K_0\otimes_{\mathbf{Q}_p}E} X^{(i)} = \left[\frac{n(n-1)}{2} - \frac{i(i+1)}{2}\right] e$. We note that $\mathrm{Gal}(K/K_0)$ acts semisimply and that multiplication by u gives rise to a $K_0 \otimes_{\mathbf{Q}_p} E$-linear isomorphism on $S_E/\mathrm{Fil}^p S_E$ which cyclically permutes the isotypical components, which completes the proof. $\square$

Note that Lemma 3.5.1 immediately implies that

$$\mathrm{rank}_{K_0\otimes_{\mathbf{Q}_p}E} X_\xi^{(i)} - \mathrm{rank}_{K_0\otimes_{\mathbf{Q}_p}E} X_\xi^{(i+1)} = i+1. \tag{3.5.1}$$

We will use this fact frequently to prove the main result, Proposition 3.5.3, in this subsection.

To describe the filtration of strongly divisible modules, we need to analyze the $\mathrm{Fil}^{n-1}\mathcal{M}$ of the Breuil modules $\mathcal{M}$ we consider.

Lemma 3.5.2. *Keep the notation and assumptions of Lemma 3.3.1.*

(i) *If u^a is an elementary divisor of $\mathcal{M}/\mathrm{Fil}^{n-1}\mathcal{M}$ then*

$$e - (k_{n-1}^{(0)} - k_0^{(0)}) \le a \le (n-2)e + (k_{n-1}^{(0)} - k_0^{(0)}).$$

Moreover, $\mathrm{FL}_n^{n-1,0}(\overline{\rho}_0) \neq \infty$ (resp. $\mathrm{FL}_n^{n-1,0}(\overline{\rho}_0) \neq 0$) if and only if $u^{e-(k_{n-1}^{(0)} - k_0^{(0)})}$ (resp. $u^{(n-2)e+(k_{n-1}^{(0)} - k_0^{(0)})}$) is an elementary divisor of $\mathcal{M}/\mathrm{Fil}^{n-1}\mathcal{M}$.

(ii) *If we further assume that $\overline{\rho}_0$ is Fontaine–Laffaille generic, then*

$$\{u^{(n-2)e+(k_{n-1}^{(0)} - k_0^{(0)})},\, u^{(n-2)e},\, u^{(n-3)e},\, \cdots,\, u^e,\, u^{e-(k_{n-1}^{(0)} - k_0^{(0)})}\}$$

are the elementary divisors of $\mathcal{M}/\mathrm{Fil}^{n-1}\mathcal{M}$.

Proof. The first part of (i) is obvious since one can obtain the Smith normal form of $\mathrm{Mat}_{\underline{e},\underline{f}}\,\mathrm{Fil}^{n-1}\mathcal{M}$ by elementary row and column operations. By Proposition 3.3.2, we know that $\mathrm{FL}_n^{n-1,0}(\overline{\rho}_0) \neq \infty$ if and only if $\beta_{n-1,0} \neq 0$. Since $u^{e-(k_{n-1}^{(0)} - k_0^{(0)})}$ has the minimal degree among the entries of $\mathrm{Mat}_{\underline{e},\underline{f}}\,\mathrm{Fil}^{n-1}\mathcal{M}$, we conclude the equivalence statement for $\mathrm{FL}_n^{n-1,0}(\overline{\rho}_0) \neq \infty$ holds. The equivalence statement for $\mathrm{FL}_n^{n-1,0}(\overline{\rho}_0) \neq 0$ is immediate from the equivalence statement for $\mathrm{FL}_n^{n-1,0}(\overline{\rho}_0) \neq \infty$ by considering $\mathcal{M}^*$ and using Lemma 3.2.4, (vi).

Part (ii) is obvious from Proposition 3.4.2. $\square$

Proposition 3.5.3. *Assume that $\overline{\rho}_0$ is Fontaine–Laffaille generic and keep the notation $r_i^{(0)}$ as in (3.3.1) as well as $k_i^{(0)}$. Let $\widehat{\mathcal{M}} \in \mathcal{O}_E\text{-}\mathrm{Mod}_{\mathrm{dd}}^{n-1}$ be a strongly divisible module corresponding to a lattice in a potentially semi-stable representation $\rho : G_{\mathbf{Q}_p} \to \mathrm{GL}_n(E)$ with Galois type $\bigoplus_{i=0}^{n-1} \widetilde{\omega}^{k_i^{(0)}}$ and Hodge–Tate weights $\{-(n-1), -(n-2), \cdots, 0\}$ such that $\mathrm{T}_{\mathrm{st}}^{\mathbf{Q}_p, n-1}(\widehat{\mathcal{M}}) \otimes_{\mathcal{O}_E} \mathbf{F} \cong \overline{\rho}_0$.*

Then there exists a framed basis $(\widehat{e}_{n-1}, \widehat{e}_{n-2}, \cdots, \widehat{e}_0)$ for $\widehat{\mathcal{M}}$ and a framed system of generators $(\widehat{f}_{n-1}, \widehat{f}_{n-2}, \cdots, \widehat{f}_0)$ for $\mathrm{Fil}^{n-1}\widehat{\mathcal{M}}$ modulo $\mathrm{Fil}^{n-1}S \cdot \widehat{\mathcal{M}}$ such that $\mathrm{Mat}_{\underline{\widehat{e}}, \underline{\widehat{f}}} \mathrm{Fil}^{n-1}\widehat{\mathcal{M}}$ is described as follows:

$$\begin{pmatrix} -\frac{p^{n-1}}{\alpha} & 0 & 0 & \cdots & 0 & u^{e-(k_{n-1}^{(0)} - k_0^{(0)})} \\ 0 & E(u)^{n-2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & E(u)^{n-3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & E(u) & 0 \\ u^{k_{n-1}^{(0)} - k_0^{(0)}} \sum_{i=0}^{n-2} p^{n-2-i} E(u)^i & 0 & 0 & \cdots & 0 & \alpha \end{pmatrix}$$

where $\alpha \in \mathcal{O}_E$ with $0 < v_p(\alpha) < n-1$.

Proof. Note that we write the elements of $\widehat{\mathcal{M}}$ in terms of coordinates with respect to a framed basis $\underline{\widehat{e}} := (\widehat{e}_{n-1}, \widehat{e}_{n-2}, \cdots, \widehat{e}_0)$. We let $\mathcal{M} := \widehat{\mathcal{M}} \otimes_S \overline{S}$, which is a Breuil module of weight $n-1$ and of type $\bigoplus_{i=0}^{n-1} \omega^{k_i^{(0)}}$ by Proposition 2.4.3. Note also that $\mathcal{M}$ can be described as in Proposition 3.4.2, and we assume that $\mathcal{M}$ has such a framed basis for $\mathcal{M}$ and such a framed system of generators for $\mathrm{Fil}^{n-1}\mathcal{M}$. During the proof, we write $(\mathrm{Fil}^{n-1}\widehat{\mathcal{M}})_\xi$ for the ξ-isotypical component of $\mathrm{Fil}^{n-1}\widehat{\mathcal{M}}$ for any character $\xi : \mathrm{Gal}(K/K_0) \to K^\times$, and by abuse of notation we often write $\widehat{f}_i$ for the image of $\widehat{f}_i$ in $\mathrm{Fil}^{n-1}\widehat{\mathcal{M}}/\mathrm{Fil}^{n-1}S \cdot \widehat{\mathcal{M}}$ without mentioning.

Since $\mathrm{Fil}^{n-1}S \cdot \widehat{\mathcal{M}} \subset \mathrm{Fil}^{n-1}\widehat{\mathcal{M}}$, we may let

$$\widehat{f}_0 = \begin{pmatrix} u^{e-(k_{n-1}^{(0)} - k_0^{(0)})} \sum_{k=0}^{n-2} x_{n-1,k} E(u)^k \\ u^{e-(k_{n-2}^{(0)} - k_0^{(0)})} \sum_{k=0}^{n-2} x_{n-2,k} E(u)^k \\ \vdots \\ u^{e-(k_1^{(0)} - k_0^{(0)})} \sum_{k=0}^{n-2} x_{1,k} E(u)^k \\ \sum_{k=0}^{n-2} x_{0,k} E(u)^k \end{pmatrix} \in \left(\mathrm{Fil}^{n-1}\widehat{\mathcal{M}}\right)_{\widetilde{\omega}^{k_0^{(0)}}},$$

where $x_{i,j} \in \mathcal{O}_E$. The vector $\widehat{f}_0$ can be written as follows:

$$\widehat{f}_0 = u^{e-(k_{n-1}^{(0)} - k_0^{(0)})} \underbrace{\begin{pmatrix} \sum_{k=0}^{n-2} x_{n-1,k} E(u)^k \\ u^{(k_{n-1}^{(0)} - k_{n-2}^{(0)})} \sum_{k=0}^{n-2} x_{n-2,k} E(u)^k \\ \vdots \\ u^{(k_{n-1}^{(0)} - k_1^{(0)})} \sum_{k=0}^{n-2} x_{1,k} E(u)^k \\ u^{(k_{n-1}^{(0)} - k_0^{(0)})} \sum_{k=1}^{n-2} x_{0,k} [E(u)^k - p^k]/u^e \end{pmatrix}}_{=:\widehat{e}'_{n-1}} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ x_{0,0} + \sum_{k=1}^{n-2} x_{0,k} p^k \end{pmatrix}.$$

By (ii) of Lemma 3.5.2, we know that $u^{e-(k_{n-1}^{(0)} - k_0^{(0)})}$ is an elementary divisor of $\mathcal{M}/\mathrm{Fil}^{n-1}\mathcal{M}$ and all other elementary divisors have bigger powers, so that we may assume $v_p(x_{n-1,0}) = 0$. Since $\mathrm{Fil}^{n-1}\mathcal{M} \subseteq u^{e-(k_{n-1}^{(0)} - k_0^{(0)})}\mathcal{M}$, we must have $v_p(x_{0,0}) > 0$. So $\underline{\widehat{e}}_1 := (\widehat{e}'_{n-1}, \widehat{e}_{n-2}, \cdots, \widehat{e}_0)$ is a framed basis for $\widehat{\mathcal{M}}$ by Nakayama lemma and we have the following coordinates of $\widehat{f}_0$ with respect to $\underline{\widehat{e}}_1$:

$$\widehat{f}_0 = \begin{pmatrix} u^{e-(k^{(0)}_{n-1}-k^{(0)}_0)} \\ 0 \\ \vdots \\ 0 \\ 0 \\ \alpha \end{pmatrix} \in \left(\mathrm{Fil}^{n-1}\widehat{\mathcal{M}}\right)_{\widetilde{\omega}^{k^{(0)}_0}}$$

for $\alpha \in \mathcal{O}_E$ with $v_p(\alpha) > 0$.

Since $u^{k^{(0)}_1-k^{(0)}_0}\widehat{f}_0 \in \left(\mathrm{Fil}^{n-1}\widehat{\mathcal{M}}\right)_{\widetilde{\omega}^{k^{(0)}_1}}$ and $\mathrm{Fil}^{n-1}S\cdot\widehat{\mathcal{M}} \subset \mathrm{Fil}^{n-1}\widehat{\mathcal{M}}$, $\widehat{f}_1$ can be written as

$$\widehat{f}_1 = \begin{pmatrix} 0 \\ u^{e-(k^{(0)}_{n-2}-k^{(0)}_1)}\sum_{k=0}^{n-2} y_{n-2,k}E(u)^k \\ \vdots \\ \sum_{k=0}^{n-2} y_{1,k}E(u)^k \\ u^{k^{(0)}_1-k^{(0)}_0}\sum_{k=0}^{n-2} y_{0,k}E(u)^k \end{pmatrix} \in \left(\mathrm{Fil}^{n-1}\widehat{\mathcal{M}}\right)_{\widetilde{\omega}^{k^{(0)}_1}},$$

where $y_{i,j} \in \mathcal{O}_E$. By Lemma 3.5.1, we have $y_{i,0} = 0$ for all i: otherwise, both $u^{k^{(0)}_1-k^{(0)}_0}\widehat{f}_0$ and $\widehat{f}_1$ belong to $X^{(0)}_{\widetilde{\omega}^{k^{(0)}_1}} - X^{(1)}_{\widetilde{\omega}^{k^{(0)}_1}}$ which violates (3.5.1). Since u^e is an elementary divisor of $\mathcal{M}/\mathrm{Fil}^{n-1}\mathcal{M}$ by (ii) of Lemma 3.5.2, we may also assume $y_{1,1} = 1$. Hence, by the obvious change of basis we get $\widehat{f}_1$ as follows:

$$\widehat{f}_1 = E(u)\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{pmatrix} \in \left(\mathrm{Fil}^{n-1}\widehat{\mathcal{M}}\right)_{\widetilde{\omega}^{k^{(0)}_1}}.$$

By the same arguments, we get $\widehat{f}_i \in \left(\mathrm{Fil}^{n-1}\widehat{\mathcal{M}}\right)_{\widetilde{\omega}^{k^{(0)}_i}}$ for $i = 1, 2, \cdots, n-2$ as in the statement.

Note that the elements in the set

$$\begin{aligned}\{u^{k^{(0)}_{n-1}-k^{(0)}_0}\widehat{f}_0, E(u)u^{k^{(0)}_{n-1}-k^{(0)}_0}\widehat{f}_0, \cdots, E(u)^{n-2}u^{k^{(0)}_{n-1}-k^{(0)}_0}\widehat{f}_0\}\\ \cup\{u^{k^{(0)}_{n-1}-k^{(0)}_1}\widehat{f}_1, E(u)u^{k^{(0)}_{n-1}-k^{(0)}_1}\widehat{f}_1, \cdots, E(u)^{n-3}u^{k^{(0)}_{n-1}-k^{(0)}_1}\widehat{f}_1\}\\ \cup\cdots\cup\{u^{k^{(0)}_{n-1}-k^{(0)}_{n-2}}\widehat{f}_{n-2}\}\end{aligned}$$

are linearly independent in $X^{(0)}_{\widetilde{\omega}^{k^{(0)}_{n-1}}}$ over E, so that the set forms a basis for $X^{(0)}_{\widetilde{\omega}^{k^{(0)}_{n-1}}}$ by Lemma 3.5.1. Hence, $\widehat{f}_{n-1}$ is a linear combination of those elements over E. We have

$$u^{k^{(0)}_{n-1}-k^{(0)}_0}\left(\sum_{i=0}^{n-2}p^{n-2-i}E(u)^i\right)\widehat{f}_0 = \begin{pmatrix} -p^{n-1} \\ 0 \\ \vdots \\ 0 \\ \alpha u^{k^{(0)}_{n-1}-k^{(0)}_0}\sum_{i=0}^{n-2}p^{n-2-i}E(u)^i \end{pmatrix}.$$

Hence, we may let

$$\widehat{f}_{n-1} := \frac{1}{\alpha}u^{k^{(0)}_{n-1}-k^{(0)}_0}\left(\sum_{i=0}^{n-2}p^{n-2-i}E(u)^i\right)\widehat{f}_0 \in \left(\mathrm{Fil}^{n-1}\widehat{\mathcal{M}}\right)_{\widetilde{\omega}^{k^{(0)}_{n-1}}}$$

since $u^{(n-2)e+(k^{(0)}_{n-1}-k^{(0)}_0)}$ is an elementary divisor for $\mathcal{M}/\mathrm{Fil}^{n-1}\mathcal{M}$ by (ii) of Lemma 3.5.2. Moreover, $v_p\left(\frac{p^{n-1}}{\alpha}\right) > 0$ since $\mathrm{Fil}^{n-1}\mathcal{M} \subseteq u^{e-(k^{(0)}_{n-1}-k^{(0)}_0)}\mathcal{M} \subseteq u\mathcal{M}$ by Proposition 3.4.2.

It is obvious that the $\widehat{f}_i$ mod $(\varpi_E, \mathrm{Fil}^p S)$ generate $\mathrm{Fil}^{n-1}\mathcal{M}$ for $\mathcal{M}$ written as in Proposition 3.3.2, so that they generate $\mathrm{Fil}^{n-1}\mathcal{M}/u^{e(n-1)}\mathcal{M}$. By Nakayama Lemma, we conclude that the $\widehat{f}_i$ generate $\mathrm{Fil}^{n-1}\widehat{\mathcal{M}}/\mathrm{Fil}^{n-1}S\cdot\widehat{\mathcal{M}}$, which completes the proof. □

Corollary 3.5.4. *Keep the notation and assumptions of Proposition 3.5.3, and let*

$$(\lambda_{n-1}, \lambda_{n-2}, \cdots, \lambda_0) \in (\mathcal{O}_E)^n$$

be the Frobenius eigenvalues on the $(\widetilde{\omega}^{k^{(0)}_{n-1}}, \widetilde{\omega}^{k^{(0)}_{n-2}}, \cdots, \widetilde{\omega}^{k^{(0)}_0})$*-isotypic component of* $\mathrm{D}^{\mathbf{Q}_p,n-1}_{\mathrm{st}}(\rho)$. *Then*

$$v_p(\lambda_i) = \begin{cases} v_p(\alpha) & \text{if } i = n-1 \\ (n-1)-i & \text{if } n-1 > i > 0 \\ (n-1)-v_p(\alpha) & \text{if } i = 0. \end{cases}$$

Proof. The proof goes parallel to the proof of [HLM], Corollary 2.4.11. □

3.6. **Reducibility of certain lifts.** In this section, we let $1 \leq f \leq n$ and $e = p^f - 1$, and we prove that every potentially semi-stable lift of $\overline{\rho}_0$ with Hodge–Tate weights $\{-(n-1), -(n-2), \cdots, 0\}$ and certain prescribed Galois types $\bigoplus_{i=0}^{n-1} \widetilde{\omega}_f^{k_i}$ is reducible. We emphasize that we only assume that $\overline{\rho}_0$ is generic (cf. Definition 3.0.5) for the results in this section.

Proposition 3.6.1. *Assume that* $\overline{\rho}_0$ *is generic, and let* $(k_{n-1}, k_{n-2}, \cdots, k_0)$ *be an n-tuple of integers. Assume further that* $k_0 \equiv (p^{f-1} + p^{f-2} + \cdots + p + 1)c_0$ *modulo* (e) *and that k_i are pairwise distinct modulo* (e).

Then every potentially semi-stable lift of $\overline{\rho}_0$ *with Hodge–Tate weights* $\{-(n-1), -(n-2), \cdots, 0\}$ *and Galois types* $\bigoplus_{i=0}^{n-1} \widetilde{\omega}_f^{k_i}$ *is an extension of a 1-dimensional potentially semi-stable lift of* $\overline{\rho}_{0,0}$ *with Hodge–Tate weight 0 and Galois type* $\widetilde{\omega}_f^{k_0}$ *by an $(n-1)$-dimensional potentially semi-stable lift of* $\overline{\rho}_{n-1,1}$ *with Hodge–Tate weights* $\{-(n-1), -(n-2), \cdots, 1\}$ *and Galois types* $\bigoplus_{i=1}^{n-1} \widetilde{\omega}_f^{k_i}$.

Note that if $f = 1$ then the assumption that $\overline{\rho}_0$ is generic implies that k_i are pairwise distinct modulo (e) by Lemma 3.1.2. In fact, we believe that this is true for any $1 \leq f \leq n$, but this requires extra works as we did in Lemma 3.1.2. Since we will need the results in this section only when $f = 1$, we will add the assumption that k_i are pairwise distinct modulo (e) in the proposition.

Proof. Let $\widehat{\mathcal{M}} \in \mathcal{O}_E\text{-}\mathrm{Mod}^{n-1}_{\mathrm{dd}}$ be a strongly divisible module corresponding to a Galois stable lattice in a potentially semi-stable representation $\rho : G_{\mathbf{Q}_p} \to \mathrm{GL}_n(E)$ with Galois type $\bigoplus_{i=0}^{n-1} \widetilde{\omega}_f^{k_i}$ and Hodge–Tate weights $\{-(n-1), -(n-2), \cdots, 0\}$ such that $\mathrm{T}^{\mathbf{Q}_p,n-1}_{\mathrm{st}}(\widehat{\mathcal{M}}) \otimes_{\mathcal{O}_E} \mathbf{F} \cong \overline{\rho}_0$. We also let $\mathcal{M}$ be the Breuil module corresponding to the mod p reduction of $\widehat{\mathcal{M}}$. $\widehat{\mathcal{M}}$ (resp. $\mathcal{M}$) is of inertial type $\bigoplus_{i=0}^{n-1} \widetilde{\omega}_f^{k_i}$ (resp. $\bigoplus_{i=0}^{n-1} \omega_f^{k_i}$) by Proposition 2.4.3.

We let $\underline{f} = (f_{n-1}, f_{n-2}, \cdots, f_0)$ (resp. $\underline{\widehat{f}} = (\widehat{f}_{n-1}, \widehat{f}_{n-2}, \cdots, \widehat{f}_0)$) be a framed system of generators for $\mathrm{Fil}^{n-1}\mathcal{M}$ (resp. for $\mathrm{Fil}^{n-1}\widehat{\mathcal{M}}$). We also let $\underline{e} = (e_{n-1}, e_{n-2}, \cdots, e_0)$ (resp. $\underline{\widehat{e}} = (\widehat{e}_{n-1}, \widehat{e}_{n-2}, \cdots, \widehat{e}_0)$) be a framed basis for $\mathcal{M}$ (resp. for $\widehat{\mathcal{M}}$). If $x = a_{n-1}e_{n-1} + \cdots + a_0 e_0 \in \mathcal{M}$, we will write $[x]_{e_i}$ for a_i for $i \in \{0, 1, \cdots, n-1\}$. We define $[x]_{\widehat{e}_i}$ for $x \in \widehat{\mathcal{M}}$ in the obvious similar way. We may assume that $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M})$, $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1})$, and $\mathrm{Mat}_{\underline{e}}(N)$ are written as in (3.0.4), (3.0.5), and (3.0.6) respectively, and we do so.

By the equation (3.0.3), we deduce $r_0 \equiv 0$ modulo (e) from our assumption on k_0. Recall that $p > n^2 + 2(n-3)$ by the generic condition. Since $0 \leq r_0 \leq (n-1)(p^f - 1)/(p-1)$ by (ii) of

Lemma 2.3.5, we conclude that $r_0 = 0$. Thus, we may let f_0 satisfy that $[f_0]_{e_i} = 0$ if $0 < i \leq n-1$ and $[f_0]_{e_0} = 1$, so that we can also let

$$\widehat{f}_0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Hence, we can also assume that $[\widehat{f}_j]_{\widehat{e}_0} = 0$ for $0 < j \leq n-1$. We let $V_0 = \mathrm{Mat}_{\underline{\widehat{e}},\underline{\widehat{f}}}(\mathrm{Fil}^{n-1}\widehat{\mathcal{M}}) \in \mathrm{M}_n^{\square,\prime}(S_{\mathcal{O}_E})$ and $A_0 = \mathrm{Mat}_{\underline{\widehat{e}},\underline{\widehat{f}}}(\phi_{n-1}) \in \mathrm{GL}_n^{\square}(S_{\mathcal{O}_E})$.

We construct a sequence of framed bases $\{\underline{\widehat{e}}^{(m)}\}$ for $\widehat{\mathcal{M}}$ by change of basis, satisfying that

$$\mathrm{Mat}_{\underline{\widehat{e}}^{(m)},\underline{\widehat{f}}^{(m)}}(\mathrm{Fil}^{n-1}\widehat{\mathcal{M}}) \in \mathrm{M}_n^{\square,\prime}(S_{\mathcal{O}_E}) \quad \text{and} \quad \mathrm{Mat}_{\underline{\widehat{e}}^{(m)},\underline{\widehat{f}}^{(m)}}(\phi_{n-1}) \in \mathrm{GL}_n^{\square}(S_{\mathcal{O}_E})$$

converge to certain desired forms as m goes to ∞. We let $V^{(m)} \in \mathrm{M}_n^{\square,\prime}(S_{\mathcal{O}_E})$ and $A^{(m)} \in \mathrm{GL}_n^{\square}(S_{\mathcal{O}_E})$ for a non-negative integer m. We may write

$$(x_{n-1}^{(m+1)}u^{[k_{n-1}-k_0]_f}, x_{n-2}^{(m+1)}u^{[k_{n-2}-k_0]_f}, \cdots, x_{m+1}^{(1)}u^{[k_{m+1}-k_0]_f}, x_0^{(m+1)})$$

for the last row of $(A^{(m)})^{-1}$, where $x_0^{(m+1)} \in (S_{\mathcal{O}_E}^{\times})_0$ and $x_j^{(m+1)} \in (S_{\mathcal{O}_E})_0$ for $0 < j \leq n-1$. We define an $n \times n$-matrix $R^{(m+1)}$ as follows:

$$R^{(m+1)} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ \frac{x_{n-1}^{(m+1)}}{x_0^{(m+1)}}u^{[k_{n-1}-k_0]_f} & \frac{x_{n-2}^{(m+1)}}{x_0^{(m+1)}}u^{[k_{n-2}-k_0]_f} & \cdots & \frac{x_1^{(m+1)}}{x_0^{(m+1)}}u^{[k_1-k_0]_f} & 1 \end{pmatrix}.$$

We also define

$$C^{(m+1)} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ y_{n-1}^{(m+1)}u^{[p^{-1}(k_{n-1}-k_0)]_f} & y_{n-2}^{(m+1)}u^{[p^{-1}(k_{n-2}-k_0)]_f} & \cdots & y_1^{(m+1)}u^{[p^{-1}(k_1-k_0)]_f} & 1 \end{pmatrix}$$

by the equation

$$R^{(m+1)} \cdot V^{(m)} \cdot C^{(m+1)} = V^{(m)}$$

where $y_j^{(m+1)} \in (S_{\mathcal{O}_E})_0$ for $0 < j \leq n-1$. Note that the existence of such a matrix $C^{(m+1)}$ is obvious, since $p^{-1}k_0 \equiv k_0$ modulo (e) by our assumption on k_0 immediately implies $[p^{-1}(k_j - k_0)]_f \leq [k_s - k_0]_f + [p^{-1}k_j - k_s]_f$. We also note that $R^{(m+1)} \in \mathrm{GL}_n^{\square}(S_{\mathcal{O}_E})$ and $C^{(m+1)} \in \mathrm{GL}_n^{\square,\prime\prime}(S_{\mathcal{O}_E})$.

Let $V^{(m+1)} = V^{(m)}$ for all $m \geq 0$. Assume that $V^{(m)} = \mathrm{Mat}_{\underline{\widehat{e}}^{(m)},\underline{\widehat{f}}^{(m)}}(\mathrm{Fil}^{n-1}\widehat{\mathcal{M}})$ and $A^{(m)} = \mathrm{Mat}_{\underline{\widehat{e}}^{(m)},\underline{\widehat{f}}^{(m)}}(\phi_{n-1})$, with respect to a framed basis $\underline{\widehat{e}}^{(m)}$ and a framed system of generators $\underline{\widehat{f}}^{(m)}$. If we let $\underline{\widehat{e}}^{(m+1)} = \underline{\widehat{e}}^{(m)} \cdot (R^{(m+1)})^{-1}$, then

$$\begin{aligned} \phi_{n-1}(\underline{\widehat{e}}^{(m+1)}V^{(m+1)}) &= \phi_{n-1}(\underline{\widehat{e}}^{(m)}(R^{(m+1)})^{-1}V^{(m+1)}) \\ &= \phi_{n-1}(\underline{\widehat{e}}^{(m)}V^{(m)}C^{(m+1)}) \\ &= \underline{\widehat{e}}^{(m)}A^{(m)}\phi(C^{(m+1)}) \\ &= \underline{\widehat{e}}^{(m+1)}R^{(m+1)} \cdot A^{(m)} \cdot \phi(C^{(m+1)}). \end{aligned}$$

Hence, we get

$$V^{(m+1)} = \mathrm{Mat}_{\underline{\widehat{e}}^{(m+1)},\underline{\widehat{f}}^{(m+1)}}(\mathrm{Fil}^{n-1}\widehat{\mathcal{M}})$$

and

$$R^{(m+1)} \cdot A^{(m)} \cdot \phi(C^{(m+1)}) = \mathrm{Mat}_{\underline{\widehat{e}}^{(m+1)},\underline{\widehat{f}}^{(m+1)}}(\phi_{n-1}),$$

where $\underline{\widehat{f}}^{(m+1)} := \underline{\widehat{e}}^{(m+1)}V^{(m+1)}$.

We compute the matrix product $A^{(m+1)} := R^{(m+1)} \cdot A^{(m)} \cdot \phi(C^{(m+1)})$ as it follows. If we let $A^{(m)} = \left(\alpha_{i,j}^{(m)} u^{[k_j-k_i]_f}\right)_{0\le i,j\le n-1}$ for $\alpha_{i,j}^{(m)} \in (S_{\mathcal{O}_E})_0$ if $i\neq j$ and $\alpha_{i,i}^{(m)} \in (S_{\mathcal{O}_E}^{\times})_0$, then

$$A^{(m+1)} = \left(\alpha_{i,j}^{(m+1)} u^{[k_j-k_i]_f}\right)_{0\le i,j\le n-1} \in \mathrm{GL}_n^{\square}(S_{\mathcal{O}_E}) \tag{3.6.1}$$

where $\alpha_{i,j}^{(m+1)} u^{[k_j-k_i]_f}$ is described as follows:

$$\begin{cases} \alpha_{i,j}^{(m)} u^{[k_j-k_i]_f} + \alpha_{i,0}^{(m)} u^{[k_0-k_i]_f}\phi(y_j^{(m+1)})u^{p[p^{-1}(k_j-k_0)]_f} & \text{if } i>0 \text{ and } j>0;\\ \alpha_{i,0}^{(m)} u^{[k_0-k_i]_f} & \text{if } i>0 \text{ and } j=0;\\ \frac{1}{x_0^{(m+1)}}\phi(y_j^{(m+1)})u^{p[p^{-1}(k_j-k_0)]_f} & \text{if } i=0 \text{ and } j>0;\\ \frac{1}{x_0^{(m+1)}} & \text{if } i=0 \text{ and } j=0. \end{cases}$$

Let $V^{(0)} = V_0$ and $A^{(0)} = A_0$. We apply the algorithm above to $V^{(0)}$ and $A^{(0)}$. By the algorithm above, we have two matrices $V^{(m)}$ and $A^{(m)}$ for each $m\ge 0$. We claim that

$$\begin{cases} \alpha_{i,j}^{(m+1)} - \alpha_{i,j}^{(m)} \in u^{(1+p+\cdots+p^m)e}S_{\mathcal{O}_E} & \text{if } i>0 \text{ and } j>0;\\ \alpha_{i,j}^{(m+1)} = \alpha_{i,j}^{(m)} & \text{if } i>0 \text{ and } j=0;\\ \alpha_{i,j}^{(m+1)} \in u^{(1+p+\cdots+p^m)e}S_{\mathcal{O}_E} & \text{if } i=0 \text{ and } j>0;\\ \alpha_{i,j}^{(m+1)} - \alpha_{i,j}^{(m)} \in u^{(1+p+\cdots+p^{m-1})e}S_{\mathcal{O}_E} & \text{if } i=0 \text{ and } j=0. \end{cases}$$

It is obvious that the case $i>0$ and $j=0$ from the computation (3.6.1). For the case $i=0$ and $j>0$ we induct on m. Note that $p[p^{-1}(k_j-k_0)]_f - [k_j-k_0]_f = p([p^{-1}k_j]_f - k_0) - (k_j-k_0) \ge e$ if $j>0$. From the computation (3.6.1) again, it is obvious that it is true for $m=0$. Assume that it holds for m. This implies that $x_j^{(m+1)} \in u^{(1+p+\cdots+p^{m-1})e}S_{\mathcal{O}_E}$ for $0<j\le n-1$ and so $y_j^{(m+1)} \in u^{(1+p+\cdots+p^{m-1})e}S_{\mathcal{O}_E}$. Since $\phi(y_j^{(m+1)})u^{p[p^{-1}(k_j-k_0)]_f-[k_j-k_0]_f}) \in u^{(1+p+\cdots+p^m)e}S_{\mathcal{O}_E}$, by the computation (3.6.1) we conclude that the case $i=0$ and $j>0$ holds. The case $i>0$ and $j>0$ follows easily from the case $i=0$ and $j>0$, since $[p^{-1}(k_j-k_0)]_f + [k_0-k_i]_f - [k_j-k_i]_f = p([p^{-1}k_j]_f - k_0) + e + k_0 - k_i - [k_j-k_i]_f \ge p[p^{-1}k_j]_f - k_j - (p-1)k_0 \ge e$. Finally, we check the case $i=0$ and $j=0$. We also induct on m for this case. It is obvious that it holds for $m=0$. Note that $R^{(m+1)} \equiv I_n$ modulo $u^{(1+p+\cdots+p^{m-1})e}S_{\mathcal{O}_E}$. Since $A^{(m+1)} = R^{(m+1)}\cdot A^{(m)}\cdot\phi(C^{(m+1)})$, we conclude that the case $i=0$ and $j=0$ holds.

The previous claim says the limit of $A^{(m)}$ exists (entrywise), say $A^{(\infty)}$. By definition, we have $V^{(\infty)} = V^{(m)}$ for all $m\ge 0$. In other words, there exist a framed basis $\underline{\widehat{e}}^{(\infty)}$ for $\widehat{\mathcal{M}}$ and a framed system of generators $\underline{\widehat{f}}^{(\infty)}$ for $\mathrm{Fil}^{n-1}\widehat{\mathcal{M}}$ such that

$$\mathrm{Mat}_{\underline{\widehat{e}}^{(\infty)},\underline{\widehat{f}}^{(\infty)}}(\mathrm{Fil}^{n-1}\widehat{\mathcal{M}}) = V^{(\infty)} \in \mathrm{M}_n^{\square,\prime}(S_{\mathcal{O}_E})$$

and

$$\mathrm{Mat}_{\underline{\widehat{e}}^{(\infty)},\underline{\widehat{f}}^{(\infty)}}(\phi_{n-1}) = A^{(\infty)} \in \mathrm{GL}_n^{\square}(S_{\mathcal{O}_E}).$$

Note that $(V^{(\infty)})_{i,j} = 0$ if either $i=0$ and $j>0$ or $i>0$ and $j=0$, and that $(A^{(\infty)})_{i,j} = 0$ if $i=0$ and $j>0$.

Since $\underline{\widehat{e}}^{(\infty)}$ is a framed basis for $\widehat{\mathcal{M}}$, we may write

$$\mathrm{Mat}_{\underline{\widehat{e}}^{(\infty)}}(N) = \left(\gamma_{i,j} u^{[k_j-k_i]_f}\right)_{0\leq i,j\leq n-1} \in \mathrm{M}_n^{\square}(S_{\mathcal{O}_E})$$

for the matrix of the monodromy operator of $\widehat{\mathcal{M}}$ where $\gamma_{i,j} \in (S_{\mathcal{O}_E})_0$, and let

$$A^{(\infty)} = \left(\alpha_{i,j}^{(\infty)} u^{[k_j-k_i]_f}\right)_{0\leq i,j\leq n-1} \in \mathrm{GL}_n^{\square}(S_{\mathcal{O}_E}).$$

We claim that $\gamma_{0,j} = 0$ for $n-1 \geq j > 0$. Recall that $\alpha_{0,j}^{(\infty)} = 0$ for $j > 0$, and write $\underline{\widehat{f}}^{(\infty)} = (\widehat{f}_{n-1}^{(\infty)}, \widehat{f}_{n-2}^{(\infty)}, \cdots, \widehat{f}_0^{(\infty)})$ and $\underline{\widehat{e}}^{(\infty)} = (\widehat{e}_{n-1}^{(\infty)}, \widehat{e}_{n-2}^{(\infty)}, \cdots, \widehat{e}_0^{(\infty)})$. We also write

$$\widehat{f}_j^{(\infty)} = \sum_{i=1}^{n-1} \beta_{i,j}^{(\infty)} u^{[p^{-1}k_j-k_i]} \widehat{e}_i^{(\infty)}$$

where $\beta_{i,j}^{(\infty)} \in (S_{\mathcal{O}_E})_0$, for each $0 < j \leq n-1$. From the equation

$$[cN\phi_{n-1}(\widehat{f}_j^{(\infty)})]_{\widehat{e}_0^{(\infty)}} = [\phi_{n-1}(E(u)N(\widehat{f}_j^{(\infty)}))]_{\widehat{e}_0^{(\infty)}}$$

for $n-1 \geq j > 0$, we have the identity

$$\sum_{i=1}^{n-1} \alpha_{i,j}^{(\infty)} u^{[k_j-k_i]_f+[k_i-k_0]_f} \gamma_{0,i} = p \sum_{i=1}^{n-1} \beta_{i,j}^{(\infty)} u^{p[p^{-1}k_j-k_i]_f+p[k_i-k_0]_f} \phi(\gamma_{0,i}) \alpha_{0,0}^{(\infty)} \tag{3.6.2}$$

for each $n-1 \geq j > 0$. Choose an integer s such that $\mathrm{ord}_u(\gamma_{0,s} u^{[k_s-k_0]_f}) \leq \mathrm{ord}_u(\gamma_{0,i} u^{[k_i-k_0]_f})$ for all $n-1 \geq i > 0$, and consider the identity (3.6.2) for $j = s$. Then the identity (3.6.2) induces

$$\alpha_{s,s}^{(\infty)} u^{[k_s-k_0]_f} \gamma_{0,s} \equiv 0$$

modulo $(u^{\mathrm{ord}_u(\gamma_{0,s})+[k_s-k_0]_f+1})$. Note that $\alpha_{s,s}^{(\infty)} \in S_{\mathcal{O}_E}^{\times}$, so that we get $\gamma_{0,s} = 0$. Recursively, we conclude that $\gamma_{0,j} = 0$ for all $0 < j \leq n-1$.

Finally, it is now easy to check that $(\widehat{e}_{n-1}^{(\infty)}, \widehat{e}_{n-2}^{(\infty)}, \cdots, \widehat{e}_1^{(\infty)})$ determines a strongly divisible modules of rank $n-1$, that is a submodule of $\widehat{\mathcal{M}}$. This completes the proof. $\square$

Corollary 3.6.2. *Fix a pair of integers (i_0, j_0) with $0 \leq j_0 \leq i_0 \leq n-1$. Assume that $\overline{\rho}_0$ is generic, and let $(k_{n-1}, k_{n-2}, \cdots, k_0)$ be an n-tuple of integers. Assume further that*

$$k_i = (p^{f-1} + p^{f-2} + \cdots + p + 1)c_i$$

for $i > i_0$ and for $i < j_0$ and that the k_i are pairwise distinct modulo (e).

Then every potentially semi-stable lift ρ of $\overline{\rho}_0$ with Hodge–Tate weights $\{-(n-1), -(n-2), \cdots, 0\}$ and Galois types $\bigoplus_{i=0}^{n-1} \widetilde{\omega}_f^{k_i}$ is a successive extension

$$\rho \cong \begin{pmatrix} \rho_{n-1,n-1} & \cdots & * & * & * & \cdots & * \\ & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & \rho_{i_0+1,i_0+1} & * & * & \cdots & * \\ & & & \rho_{i_0,j_0} & * & \cdots & * \\ & & & & \rho_{j_0-1,j_0-1} & \cdots & * \\ & & & & & \ddots & \vdots \\ & & & & & & \rho_{0,0} \end{pmatrix}$$

where

- ○ *$\rho_{i,i}$ is a 1-dimensional potentially semi-stable lift of $\overline{\rho}_{i,i}$ with Hodge–Tate weights $-i$ and Galois type $\widetilde{\omega}_f^{k_i}$ for $n-1 \geq i > i_0$ and for $j_0 > i \geq 0$;*

- ρ_{i_0,j_0} *is a* (i_0-j_0+1)*-dimensional potentially semi-stable lift of* $\overline{\rho}_{i_0,j_0}$ *with Hodge–Tate weights* $\{-i_0,-i_0+1,\cdots,-j_0\}$ *and Galois types* $\bigoplus_{i=j_0}^{i_0}\widetilde{\omega}_f^{k_i}$.

Proof. Proposition 3.6.1 implies this corollary recursively. Let $\mathcal{M}\in\mathbf{F}\text{-BrMod}_{\mathrm{dd}}^{n-1}$ be a Breuil module corresponding to the mod p reduction of a strongly divisible module $\widehat{\mathcal{M}}\in\mathcal{O}_E\text{-Mod}_{\mathrm{dd}}^{n-1}$ corresponding to a Galois stable lattice in a potentially semi-stable representation $\rho:G_{\mathbf{Q}_p}\to\mathrm{GL}_n(E)$ with Galois type $\bigoplus_{i=0}^{n-1}\widetilde{\omega}_f^{k_i}$ and Hodge–Tate weights $\{-(n-1),-(n-2),\cdots,0\}$ such that $\mathrm{T}_{\mathrm{st}}^{\mathbf{Q}_p,n-1}(\widehat{\mathcal{M}})\otimes_{\mathcal{O}_E}\mathbf{F}\cong\overline{\rho}_0$. $\widehat{\mathcal{M}}$ (resp. $\mathcal{M}$) is of inertial type $\bigoplus_{i=0}^{n-1}\widetilde{\omega}_f^{k_i}$ (resp. $\bigoplus_{i=0}^{n-1}\omega_f^{k_i}$) by Proposition 2.4.3. We may assume that $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M})$, $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1})$, and $\mathrm{Mat}_{\underline{e}}(N)$ are written as in (3.0.4), (3.0.5), and (3.0.6) respectively, and we do so.

By the equation (3.0.3), it is easy to see that $r_i=(p^{f-1}+p^{f-2}+\cdots+p+1)i$ for $i>i_0$ and for $i<j_0$, by our assumption on k_i. By Proposition 3.6.1, there exists an $(n-1)$-dimensional subrepresentation $\rho'_{n-1,1}$ of ρ whose quotient is $\rho_{0,0}$ which is a potentially semi-stable lift of $\overline{\rho}_{0,0}$ with Hodge–Tate weight 0 and Galois type $\widetilde{\omega}_f^{k_0}$. Now consider $\rho'_{n-1,1}\otimes\varepsilon^{-1}$. Apply Proposition 3.6.1 to $\rho'_{n-1,1}\otimes\varepsilon^{-1}$. Recursively, one can readily check that ρ has subquotients $\rho_{i,i}$ for $0\leq i\leq j_0-1$. Considering $\rho^\vee\otimes\varepsilon^{n-1}$, one can also readily check that ρ has subquotients $\rho_{i,i}$ lifting $\overline{\rho}_{i,i}$ for $n-1\geq i\geq i_0+1$. □

The results in Corollary 3.6.2 reduce many of our computations for the main results on the Galois side.

3.7. **Main results on the Galois side.** In this section, we state and prove the main local results on the Galois side, that connects the Fontaine–Laffaille parameters of $\overline{\rho}_0$ with the Frobenius eigenvalues of certain potentially semi-stable lifts of $\overline{\rho}_0$. Throughout this section, we assume that $\overline{\rho}_0$ is Fontaine–Laffaille generic. We also fix $f=1$ and $e=p-1$.

Fix $i_0,j_0\in\mathbf{Z}$ with $0\leq j_0<j_0+1<i_0\leq n-1$, and define the n-tuple of integers

$$(r_{n-1}^{i_0,j_0},\ r_{n-2}^{i_0,j_0},\ \cdots,\ r_0^{i_0,j_0})$$

as follows:

$$r_i^{i_0,j_0}:=\begin{cases} i & \text{if } i_0\neq i\neq j_0;\\ j_0+1 & \text{if } i=i_0;\\ i_0-1 & \text{if } i=j_0.\end{cases} \tag{3.7.1}$$

We note that if we replace n by i_0-j_0+1 in the definition of $r_i^{(0)}$ in (3.3.1) then we have the identities:

$$r_{j_0+i}^{i_0,j_0}=j_0+r_i^{(0)} \tag{3.7.2}$$

for all $0\leq i\leq i_0-j_0$. In particular, $r_i^{n-1,0}=r_i^{(0)}$ for all $0\leq i\leq n-1$.

From the equation $k_i^{i_0,j_0}\equiv c_i+i-r_i^{i_0,j_0}$ mod (e) (cf. Lemma 3.1.2, (i)), this tuple immediately determines an n-tuple $(k_{n-1}^{i_0,j_0},\ k_{n-2}^{i_0,j_0},\ \cdots,\ k_0^{i_0,j_0})$ of integers mod (e), which will determine the Galois types of our representations. We set

$$k_i^{i_0,j_0}:=c_i+i-r_i^{i_0,j_0}$$

for all $i\in\{0,1,\cdots,n-1\}$.

The following is the main result on the Galois side.

Theorem 3.7.1. *Let* i_0,j_0 *be integers with* $0\leq j_0<j_0+1<i_0\leq n-1$. *Assume that* $\overline{\rho}_0$ *is generic and that* $\overline{\rho}_{i_0,j_0}$ *is Fontaine–Laffaille generic. Let* $(\lambda_{n-1}^{i_0,j_0},\lambda_{n-2}^{i_0,j_0},\cdots,\lambda_0^{i_0,j_0})\in(\mathcal{O}_E)^n$ *be the Frobenius eigenvalues on the* $(\widetilde{\omega}^{k_{n-1}^{i_0,j_0}},\widetilde{\omega}^{k_{n-2}^{i_0,j_0}},\cdots,\widetilde{\omega}^{k_0^{i_0,j_0}})$*-isotypic components of* $\mathrm{D}_{\mathrm{st}}^{\mathbf{Q}_p,n-1}(\rho_0)$ *where* ρ_0 *is a potentially semi-stable lift of* $\overline{\rho}_0$ *with Hodge–Tate weights* $\{-(n-1),-(n-2),\cdots,-1,0\}$ *and Galois types* $\bigoplus_{i=0}^{n-1}\widetilde{\omega}^{k_i^{i_0,j_0}}$.

Then the Fontaine–Laffaille parameter $\mathrm{FL}_n^{i_0,j_0}$ *associated to* $\overline{\rho}_0$ *is computed as follows:*

$$\mathrm{FL}_n^{i_0,j_0}(\overline{\rho}_0) = \overline{\left(\frac{p^{[(n-1)-\frac{i_0+j_0}{2}](i_0-j_0-1)}}{\prod_{i=j_0+1}^{i_0-1}\lambda_i^{i_0,j_0}}\right)} \in \mathbb{P}^1(\mathbf{F}).$$

We first prove Theorem 3.7.1 for the case $(i_0, j_0) = (n-1, 0)$ in the following proposition, which is the key step to prove Theorem 3.7.1.

Proposition 3.7.2. *Keep the assumptions and notation of Theorem 3.7.1, and assume further* $(i_0, j_0) = (n-1, 0)$*. Then Theorem 3.7.1 holds.*

Recall that $(k_{n-1}^{n-1,0}, \cdots, k_0^{n-1,0})$ in Proposition 3.7.2 is the same as $(k_{n-1}^{(0)}, \cdots, k_0^{(0)})$ in (3.3.1). To lighten the notation, we let $k_i = k_i^{n-1,0}$ and $\lambda_i = \lambda_i^{n-1,0}$ during the proof of Proposition 3.7.2. We heavily use the results in Sections 3.3, 3.4 and 3.5 to prove this proposition.

Proof. Let $\widehat{\mathcal{M}} \in \mathcal{O}_E\text{-}\mathrm{Mod}_{\mathrm{dd}}^{n-1}$ be a strongly divisible module corresponding to a Galois stable lattice in a potentially semi-stable representation $\rho_0 : G_{\mathbf{Q}_p} \to \mathrm{GL}_n(E)$ with Galois type $\bigoplus_{i=0}^{n-1}\widetilde{\omega}^{k_i}$ and Hodge–Tate weights $\{-(n-1), -(n-2), \cdots, 0\}$ such that $\mathrm{T}_{\mathrm{st}}^{\mathbf{Q}_p,n-1}(\widehat{\mathcal{M}}) \otimes_{\mathcal{O}_E} \mathbf{F} \cong \overline{\rho}_0$. We also let $\mathcal{M}$ be the Breuil module corresponding to the mod p reduction of $\widehat{\mathcal{M}}$. $\widehat{\mathcal{M}}$ (resp. $\mathcal{M}$) is of inertial type $\bigoplus_{i=0}^{n-1}\widetilde{\omega}^{k_i}$ (resp. $\bigoplus_{i=0}^{n-1}\omega^{k_i}$) by Proposition 2.4.3.

We let $\underline{\widehat{f}} = (\widehat{f}_{n-1}, \widehat{f}_{n-2}, \cdots, \widehat{f}_1, \widehat{f}_0)$ be a framed system of generators for $\mathrm{Fil}^{n-1}\widehat{\mathcal{M}}$, and $\underline{\widehat{e}} = (\widehat{e}_{n-1}, \widehat{e}_{n-2}, \cdots, \widehat{e}_1, \widehat{e}_0)$ be a framed basis for $\widehat{\mathcal{M}}$. We may assume that $\mathrm{Mat}_{\underline{\widehat{e}},\underline{\widehat{f}}}(\mathrm{Fil}^{n-1}\widehat{\mathcal{M}})$ is described as in Proposition 3.5.3, and we do so.

Define $\alpha_i \in \mathbf{F}^\times$ by the condition $\phi_{n-1}(\widehat{f}_i) \equiv \widetilde{\alpha}_i\widehat{e}_i$ modulo (ϖ_E, u) for all $i \in \{0, 1, \cdots, n-1\}$. There exists a framed basis $\underline{e} = (e_{n-1}, e_{n-2}, \cdots, e_0)$ for $\mathcal{M}$ and a framed system of generators $\underline{f} = (f_{n-1}, f_{n-2}, \cdots, f_0)$ for $\mathrm{Fil}^{n-1}\mathcal{M}$ such that $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M})$ is given as in Proposition 3.4.2 and

$$\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1}) = \left(\alpha_{i,j}u^{[k_j-k_i]_1}\right) \in \mathrm{GL}_n^{\square}(\overline{S})$$

for some $\alpha_{i,j} \in \overline{S}_0$ with $\alpha_{i,i} \equiv \alpha_i \bmod (u^e)$.

Recall that $\widehat{f}_i = E(u)^i\widehat{e}_i$ for $i \in \{1, 2, \cdots, n-2\}$ by Proposition 3.5.3. Write $\phi_{n-1}(\widehat{f}_j) = \sum_{i=0}^{n-1}\widehat{\alpha}_{i,j}u^{[k_j-k_i]_1}\widehat{e}_i$ for some $\widehat{\alpha}_{i,j} \in S_0$. Then we get

$$s_0(\widehat{\alpha}_{i,i}) \equiv \frac{p^i\lambda_i}{p^{n-1}} \pmod{\varpi_E}$$

for $i \in \{1, 2, \cdots, n-2\}$ since $\phi_{n-1} = \frac{1}{p^{n-1}}\phi$ for the Frobenius ϕ on $\mathrm{D}_{\mathrm{st}}^{\mathbf{Q}_p,n-1}(\rho_0)$, so that we have

$$\prod_{i=1}^{n-2}\widetilde{\alpha}_i \equiv \prod_{i=1}^{n-2}\frac{\lambda_i}{p^{n-1-i}} \pmod{\varpi_E}.$$

(Note that $\frac{\lambda_i}{p^{n-1-i}} \in \mathcal{O}_E^\times$ by Corollary 3.5.4.) This completes the proof, by applying the results in Proposition 3.4.2. $\square$

We now prove Theorem 3.7.1, which is the main result on the Galois side.

Proof of Theorem 3.7.1. Recall from the identities in (3.7.2) that

$$(r_{i_0}^{i_0,j_0}, r_{i_0-1}^{i_0,j_0}, \cdots, r_{j_0}^{i_0,j_0}) = j_0 + (1, n'-2, n'-3, \cdots, 1, n'-2)$$

where $n' := i_0 - j_0 + 1$. Recall also that ρ_0 has a subquotient ρ_{i_0,j_0} that is a potentially semi-stable lift of $\overline{\rho}_{i_0,j_0}$ with Hodge–Tate weights $\{-i_0, -(i_0-1), ..., -j_0\}$ and of Galois type $\bigoplus_{i=j_0}^{i_0} k_i^{i_0,j_0}$, by Corollary 3.6.2.

It is immediate that $\rho'_{i_0,j_0} := \rho_{i_0,j_0}\varepsilon^{-j_0}\widetilde{\omega}^{j_0}$ is another potentially semi-stable lift of $\overline{\rho}_{i_0,j_0}$ with Hodge–Tate weights $\{-(i_0-j_0), -(i_0-j_0-1), ..., 0\}$ and of Galois type $\bigoplus_{i=j_0}^{i_0}\widetilde{\omega}^{k_i^{i_0,j_0}+j_0}$. We let $(\eta_{i_0}, \eta_{i_0-1}, \cdots, \eta_{j_0}) \in (\mathcal{O}_E)^{i_0-j_0+1}$ (resp. $(\delta_{i_0}, \delta_{i_0-1}, \cdots, \delta_{j_0}) \in (\mathcal{O}_E)^{i_0-j_0+1}$) be the Frobenius eigenvalues on the $(\widetilde{\omega}^{k_{i_0}^{i_0,j_0}}, \widetilde{\omega}^{k_{i_0-1}^{i_0,j_0}}, \cdots, \widetilde{\omega}^{k_{j_0}^{i_0,j_0}})$-isotypic component of $\mathrm{D}_{\mathrm{st}}^{\mathbf{Q}_p, i_0-j_0}(\rho_{i_0,j_0})$ (resp. on the $(\widetilde{\omega}^{k_{i_0}^{i_0,j_0}+j_0}, \widetilde{\omega}^{k_{i_0-1}^{i_0,j_0}+j_0}, \cdots, \widetilde{\omega}^{k_{j_0}^{i_0,j_0}+j_0})$-isotypic component of $\mathrm{D}_{\mathrm{st}}^{\mathbf{Q}_p, i_0-j_0}(\rho'_{i_0,j_0})$). Then we have

$$p^{-j_0}\delta_i = \eta_i$$

for all $i \in \{j_0, j_0+1, \cdots, i_0\}$ and, by Proposition 3.7.2,

$$\mathrm{FL}_{i_0-j_0+1}^{i_0-j_0,0}(\overline{\rho}_{i_0,j_0}) = \overline{\left[\left(\prod_{i=j_0+1}^{i_0-1}\delta_i\right) : p^{\frac{(i_0-j_0)(i_0-j_0-1)}{2}}\right]} \in \mathbb{P}^1(\mathbf{F}).$$

But we also have that

$$p^{n-1-(i_0-j_0)}\eta_i = \lambda_i^{i_0,j_0}$$

for all $i \in \{j_0, j_0+1, \cdots, i_0\}$ by Corollary 3.6.2. Hence, we have $\delta_i = p^{-(n-1-i_0)}\lambda_i^{i_0,j_0}$ for all $i \in \{j_0, j_0+1, \cdots, i_0\}$ and we conclude that

$$\mathrm{FL}_n^{i_0,j_0}(\overline{\rho}_0) = \mathrm{FL}_{i_0-j_0+1}^{i_0-j_0,0}(\overline{\rho}_{i_0,j_0}) = \overline{\left[\left(\prod_{i=j_0+1}^{i_0-1}\lambda_i^{i_0,j_0}\right) : p^{[(n-1)-\frac{i_0+j_0}{2}](i_0-j_0-1)}\right]} \in \mathbb{P}^1(\mathbf{F}).$$

(Note that $\mathrm{FL}_n^{i_0,j_0}(\overline{\rho}_0) = \mathrm{FL}_{i_0-j_0+1}^{i_0-j_0,0}(\overline{\rho}_{i_0,j_0})$ by Lemma 3.2.4.) □

In the following corollary, we prove that the Weil–Deligne representation $\mathrm{WD}(\rho_0)$ associated to ρ_0 still contains Fontaine–Laffaille parameters. As we will see later, we will transport this information to the automorphic side via local Langlands correspondence.

Corollary 3.7.3. *Keep the assumptions and notation of Theorem 3.7.1.*
Then ρ_0 is, in fact, potentially crystalline and

$$\mathrm{WD}(\rho_0)^{\mathrm{F-ss}} = \mathrm{WD}(\rho_0) \cong \bigoplus_{i=0}^{n-1}\Omega_i$$

where $\Omega_i : \mathbf{Q}_p^\times \to E^\times$ is defined by $\Omega_i := \mathrm{U}_{\lambda_i^{i_0,j_0}/p^{n-1}} \cdot \widetilde{\omega}^{k_i^{i_0,j_0}}$ for all $i \in \{0, 1, \cdots, n-1\}$. Moreover,

$$\mathrm{FL}_n^{i_0,j_0}(\overline{\rho}_0) = \overline{\left(\frac{\prod_{i=j_0+1}^{i_0-1}\Omega_i^{-1}(p)}{p^{\frac{(i_0+j_0)(i_0-j_0-1)}{2}}}\right)} \in \mathbb{P}^1(\mathbf{F}).$$

Proof. Notice that ϕ is diagonal on $D := \mathrm{D}_{\mathrm{st}}^{\mathbf{Q}_p}(\rho_0)$ with respect to a framed basis $\underline{e} := (e_{n-1}, \cdots, e_0)$ (which satisfies $ge_i = \widetilde{\omega}^{k_i^{i_0,j_0}}(g)e_i$ for all i and for all $g \in \mathrm{Gal}(K/\mathbf{Q}_p)$) since $\widetilde{\omega}^{k_i^{i_0,j_0}}$ are all distinct. Hence, we have $\mathrm{WD}(\rho_0) = \mathrm{WD}(\rho_0)^{\mathrm{F-ss}}$. Similarly, since the descent data action on D commutes with the monodromy operator N, it is immediate that $N = 0$.

From the definition of $\mathrm{WD}(\rho_0)$ (cf. [CDT99]), the action of $W_{\mathbf{Q}_p}$ on D can be described as follows: let $\alpha(g) \in \mathbf{Z}$ be determined by $\bar{g} = \phi^{\alpha(g)}$, where ϕ is the arithmetic Frobenius in $G_{\mathbf{F}_p}$ and $\bar{g}$ is the image under the surjection $W_{\mathbf{Q}_p} \twoheadrightarrow \mathrm{Gal}(K/\mathbf{Q}_p)$. Then

$$\mathrm{WD}(\rho_0)(g)\cdot e_i = \left(\frac{\lambda_i^{i_0,j_0}}{p^{n-1}}\right)^{-\alpha(g)} \cdot \widetilde{\omega}^{k_i^{i_0,j_0}}(g)\cdot e_i$$

for all $i \in \{0, 1, \cdots, n-1\}$. (Recall that $\mathrm{D}_{\mathrm{st}}^{\mathbf{Q}_p, n-1}(\rho_0) = \mathrm{D}_{\mathrm{st}}^{\mathbf{Q}_p}(\rho_0 \otimes \varepsilon^{-(n-1)})$, so that the $\frac{\lambda_i^{i_0, j_0}}{p^{n-1}}$ are the Frobenius eigenvalues of the Frobenius on D.) Write Ω_i for the eigen-character with respect to e_i.

Recall that we identify the geometric Frobenius with the uniformizers in $\mathbf{Q}_p^\times$ (by our normalization of class field theory), so that $\Omega_i(p) = \frac{p^{n-1}}{\lambda_i^{i_0, j_0}}$ which completes the proof by applying Theorem 3.7.1. □

4. Local automorphic side

In this section, we establish several results concerning representation theory of GL_n, that will be applied to the proof of our main results on mod p local-global compatibility, Theorem 5.6.2. The main results in this section are the non-vanishing result, Corollary 4.8.3, as well as the intertwining identity in characteristic 0, Theorem 4.4.9.

We start this section by fixing some notation. Let $G := \mathrm{GL}_{n/\mathbf{Z}_p}$ and T be the maximal split torus consisting of diagonal matrices. We fix a Borel subgroup $B \subseteq G$ consisting of upper-triangular matrices, and let $U \subseteq B$ be the maximal unipotent subgroup. Let Φ^+ denote the set of positive roots with respect to (B, T), and $\Delta = \{\alpha_k\}_{1 \le k \le n-1}$ the subset of simple positive roots. Let $X(T)$ and $X^\vee(T)$ denote the abelian group of characters and cocharacters respectively. We often say a weight for an element in $X(T)$, and write $X(T)_+$ for the set of dominant weights. The set Φ^+ induces a partial order on $X(T)$: for $\lambda, \mu \in X(T)$ we say that $\lambda \le \mu$ if $\mu - \lambda \in \sum_{\alpha \in \Phi^+} \mathbf{Z}_{\ge 0}\alpha$. We will also write $\lambda < \mu$ if $\lambda \le \mu$ and $\lambda \ne \mu$.

We use the n-tuple of integers $\lambda = (d_1, d_2, \cdots, d_n)$ to denote the character associated to the weight $\sum_{k=1}^n d_k \epsilon_k$ of T where for each $1 \le i \le n$ ϵ_i is a weight of T defined by

$$\mathrm{diag}(x_1, x_2, \cdots, x_n) \overset{\epsilon_i}{\mapsto} x_i.$$

We will often use the following weight

$$\eta := (n-1, n-2, \cdots, 1, 0).$$

We let $\overline{G}$, $\overline{B}$, $\cdots$ be the base change to $\mathbf{F}_p$ of G, B, $\cdots$ respectively. The Weyl group of G, denoted by W, has a standard lifting inside G as the group of permutation matrix, likewise with $\overline{G}$. We denote the longest Weyl element by w_0 which is lifted to the antidiagonal permutation matrix in G or $\overline{G}$. We use the notation s_i for the simple reflection corresponding to $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \le i \le n-1$. We define the length $\ell(w)$ of $w \in W$ to be its minimal length of decomposition into product of s_i for $1 \le i \le n-1$. Given $A \in U(\mathbf{F}_p)$, we use A_α or equivalently $A_{i,j}$ to denote the (i,j)-entry of A, where $\alpha = \epsilon_i - \epsilon_j$ is the positive root corresponding to the pair (i,j) with $1 \le i < j \le n$.

Given a representation π of $G(\mathbf{F}_p)$, we use the notation π^μ for the $T(\mathbf{F}_p)$-eigenspace with the eigencharacter μ. Given an algebraic representation V of G or $\overline{G}$, we use the notation V_λ for the weight space of V associated to the weight λ. For any representation V of $\overline{G}$ or $G(\mathbf{F}_p)$ with coefficient in $\mathbf{F}_p$, we define

$$V_{\mathbf{F}} := V \otimes_{\mathbf{F}_p} \mathbf{F}$$

to be the extension of coefficient of V from $\mathbf{F}_p$ to $\mathbf{F}$. Similarly, we write $V_{\overline{\mathbf{F}}_p}$ for $V \otimes_{\mathbf{F}_p} \overline{\mathbf{F}}_p$.

It is easy to observe that we can identify the character group of $T(\mathbf{F}_p)$ with $X(T)/(p-1)X(T)$. The natural action of the Weyl group W on T and thus on $T(\mathbf{F}_p)$ induces an action of W on the character group $X(T)$ and $X(T)/(p-1)X(T)$. We carefully distinguish the notation between them. We use the notation $w\lambda$ (resp. μ^w) for the action of W on $X(T)$ (resp. $X(T)/(p-1)X(T)$) satisfying

$$w\lambda(x) = \lambda(w^{-1}xw) \text{ for all } x \in T$$

and

$$\mu^w(x) = \mu(w^{-1}xw) \text{ for all } x \in T(\mathbf{F}_p).$$

As a result, without further comments, the notation $w\lambda$ is a weight but μ^w is just a character of $T(\mathbf{F}_p)$. There is another dot action of W on $X(T)$ defined by

$$w \cdot \lambda = w(\lambda + \eta) - \eta \text{ for all } \lambda \in X(T) \text{ and } w \in W.$$

The affine Weyl group $\widetilde{W}$ of G is defined as the semi-direct product of W and $X(T)$ with respect to the natural action of W on $X(T)$. We denote the image of $\lambda \in X(T)$ in $\widetilde{W}$ by t_λ. Then the dot action of W on $X(T)$ extends to the dot action of $\widetilde{W}$ on $X(T)$ through the following formula

$$\widetilde{w} \cdot \lambda = w \cdot (\lambda + p\mu)$$

if $\widetilde{w} = wt_\mu$. We use the notation $\lambda \uparrow \mu$ for $\lambda, \mu \in X(T)$ if $\lambda \leq \mu$ and $\lambda \in \widetilde{W} \cdot \mu$. We define a specific element of $\widetilde{W}$ by

$$\widetilde{w}_h := w_0 t_{-\eta}$$

following Section 4 of [LLL].

We usually write K for $\mathrm{GL}_n(\mathbf{Z}_p)$ for brevity. We will also often use the following three open compact subgroups of $\mathrm{GL}_n(\mathbf{Z}_p)$: if we let $\mathrm{red} : \mathrm{GL}_n(\mathbf{Z}_p) \twoheadrightarrow \mathrm{GL}_n(\mathbf{F}_p)$ be the natural mod p reduction map, then

$$K(1) := \mathrm{Ker}(\mathrm{red}) \subset I(1) := \mathrm{red}^{-1}(U(\mathbf{F}_p)) \subset I := \mathrm{red}^{-1}(B(\mathbf{F}_p)) \subset K.$$

For each $\alpha \in \Phi^+$, there exists a subgroup U_α of G such that $xu_\alpha(t)x^{-1} = u_\alpha(\alpha(x)t)$ where $x \in T$ and $u_\alpha : \mathbb{G}_a \to U_\alpha$ is an isomorphism sending 1 to 1 in the entry corresponding to α. For each $\alpha \in \Phi^+$, we have the following equalities by [Jan03] II 1.19 (5) and (6):

$$u_\alpha(t) = \sum_{m \geq 0} t^m (X_{\alpha,m}^{\mathrm{alg}}). \tag{4.0.3}$$

where $X_{\alpha,m}^{\mathrm{alg}}$ is an element in the algebra of distributions on G supported at the origin $1 \in G$. The equation (4.0.3) is actually just the Taylor expansion with respect to t of the operation $u_\alpha(t)$ at the origin 1. We use the same notation $X_{\alpha,m}^{\mathrm{alg}}$ if G is replaced by $\overline{G}$.

We define the set of p-restricted weights as

$$X_1(T) := \{\lambda \in X(T) \mid 0 \leq \langle \lambda, \alpha^\vee \rangle \leq p - 1 \text{ for all } \alpha \in \Delta\}$$

and the set of central weights as

$$X_0(T) := \{\lambda \in X(T) \mid \langle \lambda, \alpha^\vee \rangle = 0 \text{ for all } \alpha \in \Delta\}.$$

We also define the set of p-regular weights as

$$X_1^{\mathrm{reg}}(T) := \{\lambda \in X(T) \mid 1 \leq \langle \lambda, \alpha^\vee \rangle \leq p - 2 \text{ for all } \alpha \in \Delta\},$$

and in particular we have $X_1^{\mathrm{reg}}(T) \subsetneq X_1(T)$. We say that $\lambda \in X(T)$ *lies in the lowest p-restricted alcove* if

$$0 < \langle \lambda + \eta, \alpha^\vee \rangle < p \text{ for all } \alpha \in \Phi^+. \tag{4.0.4}$$

We define a subset $\widetilde{W}^+$ of $\widetilde{W}$ as

$$\widetilde{W}^+ := \{\widetilde{w} \in \widetilde{W} \mid \widetilde{w} \cdot \lambda \in X(T)_+ \text{ for each } \lambda \text{ in the lowest } p\text{-restricted alcove}\}.$$

We define another subset $\widetilde{W}^{\mathrm{res}}$ of $\widetilde{W}$ as

$$\widetilde{W}^{\mathrm{res}} := \{\widetilde{w} \in \widetilde{W} \mid \widetilde{w} \cdot \lambda \in X_1(T) \text{ for each } \lambda \text{ in the lowest } p\text{-restricted alcove}\}. \tag{4.0.5}$$

In particular, we have the inclusion

$$\widetilde{W}^{\mathrm{res}} \subseteq \widetilde{W}^+.$$

For any weight $\lambda \in X(T)$, we let

$$H^0(\lambda) := \left(\mathrm{Ind}_{\overline{B}}^{\overline{G}} w_0\lambda\right)_{/\mathbf{F}_p}^{\mathrm{alg}}$$

be the associated dual Weyl module. Note by [Jan03], Proposition II 2.6 that $H^0(\lambda) \neq 0$ if and only if $\lambda \in X(T)_+$. Assume that $\lambda \in X(T)_+$, we write $F(\lambda) := \mathrm{soc}_{\overline{G}}(H^0(\lambda))$ for its irreducible socle as an algebraic representation (cf. [Jan03] part II, section 2). When λ is running through $X_1(T)$, the $F(\lambda)$ exhaust all the irreducible representations of $G(\mathbf{F}_p)$. On the other hand, two weights $\lambda, \lambda' \in X_1(T)$ satisfies

$$F(\lambda) \cong F(\lambda')$$

as $G(\mathbf{F}_p)$-representation if and only if

$$\lambda - \lambda' \in (p-1)X_0(T).$$

If $\lambda \in X_1^{\mathrm{reg}}(T)$, then the structure of $F(\lambda)$ as a $G(\mathbf{F}_p)$-representation depends only on the image of λ in $X(T)/(p-1)X(T)$, namely as a character of $T(\mathbf{F}_p)$. Conversely, given a character μ of $T(\mathbf{F}_p)$ which lies in the image of

$$X_1^{\mathrm{reg}}(T) \to X(T)/(p-1)X(T),$$

its inverse image in $X_1^{\mathrm{reg}}(T)$ is uniquely determined up to a translation of $(p-1)X_0(T)$. In this case, we say that μ is p-regular. Whenever it is necessary for us to lift an element in $X(T)/(p-1)X(T)$ back into $X_1(T)$ (or maybe $X_1^{\mathrm{reg}}(T)$), we will clarify the choice of the lift.

Consider the standard Bruhat decomposition

$$G = \bigsqcup_{w\in W} BwB = \bigsqcup_{w\in W} U_w wB = \bigsqcup_{w\in W} BwU_{w^{-1}}.$$

where U_w is the unique subgroup of U satisfying $BwB = U_w wB$ as schemes over $\mathbf{Z}_p$. The group U_w has a unique form of $\prod_{\alpha\in\Phi_w^+} U_\alpha$ for the subset Φ_w^+ of Φ^+ defined by $\Phi_w^+ = \{\alpha \in \Phi^+, w(\alpha) \in -\Phi^+\}$. (If $w = 1$, we understand $\prod_{\alpha\in\Phi_w^+} U_\alpha$ to be the trivial group 1.) We also have the following Bruhat decomposition:

$$G(\mathbf{F}_p) = \bigsqcup_{w\in W} B(\mathbf{F}_p)wB(\mathbf{F}_p) = \bigsqcup_{w\in W} U_w(\mathbf{F}_p)wB(\mathbf{F}_p) = \bigsqcup_{w\in W} B(\mathbf{F}_p)wU_{w^{-1}}(\mathbf{F}_p). \tag{4.0.6}$$

Given any integer x, recall that we use the notation $[x]_1$ to denote the only integer satisfying $0 \leq [x]_1 \leq p-2$ and $[x]_1 \equiv x \bmod (p-1)$. Given two non-negative integers m and k with $m \geq k$, we use the notation $c_{m,k}$ for the binomial number $\frac{m!}{(m-k)!k!}$. We use the notation $\bullet$ for composition of maps and, in particular, composition of elements in the group algebra $\mathbf{F}_p[G(\mathbf{F}_p)]$.

4.1. **Jacobi sums in characteristic p.** In this section we establish several fundamental properties of Jacobi sum operators on mod p principal series representations.

Definition 4.1.1. *A weight $\lambda \in X(T)$ is called* k-generic *for $k \in \mathbf{Z}_{>0}$ if for each $\alpha \in \Phi^+$ there exists $m_\alpha \in \mathbf{Z}$ such that*

$$m_\alpha p + k < \langle \lambda, \alpha^\vee \rangle < (m_\alpha + 1)p - k.$$

In particular, the n-tuple of integers $(a_{n-1}, \cdots, a_1, a_0)$ is called k-generic in the lowest alcove *if*

$$a_i - a_{i-1} > k \quad \forall\, 1 \leq i \leq n-1 \text{ and } a_{n-1} - a_0 < p-k.$$

Note that $(a_{n-1}, \cdots, a_0)-\eta$ lies the lowest p-restricted alcove in the sense of (4.0.4) if $(a_{n-1}, \cdots, a_0)$ is k-generic in the lowest alcove for some $k > 0$. Note also that the existence of an n-tuple of integers satisfying k-generic in the lowest alcove implies $p > n(k+1) - 1$.

We use the notation π for a principal series representation:

$$\pi := \mathrm{Ind}_{B(\mathbf{F}_p)}^{G(\mathbf{F}_p)} \mu_\pi = \{f : G(\mathbf{F}_p) \to \mathbf{F}_p \mid f(bg) = \mu_\pi(b)f(g) \quad \forall (b,g) \in B(\mathbf{F}_p) \times G(\mathbf{F}_p)\}$$

where μ_π is a mod p character of $T(\mathbf{F}_p)$. The action of $\mathrm{GL}_n(\mathbf{F}_p)$ on π is given by $(g\cdot f)(g') = f(g'g)$. We will assume throughout this article that μ_π is p-regular. By definition we have

$$\mathrm{cosoc}_{G(\mathbf{F}_p)}(\pi) = F(\mu_\pi) \text{ and } \mathrm{soc}_{G(\mathbf{F}_p)}(\pi) = F(\mu_\pi^{w_0}).$$

By Bruhat decomposition we can deduce that

$$\dim_{\mathbf{F}_p}\pi^{U(\mathbf{F}_p),\mu_\pi^w}=1$$

for each $w\in W$. We denote by v_π a non-zero fixed vector in $\pi^{U(\mathbf{F}_p),\mu_\pi}$. We also consider the natural lift $\widetilde{\pi}^\circ$ of π defined as

$$\widetilde{\pi}^\circ:=\mathrm{Ind}_{B(\mathbf{F}_p)}^{G(\mathbf{F}_p)}\widetilde{\mu}_\pi=\{f:G(\mathbf{F}_p)\to\mathbf{Z}_p\mid f(bg)=\widetilde{\mu}_\pi(b)f(g)\quad\forall(b,g)\in B(\mathbf{F}_p)\times G(\mathbf{F}_p)\}\tag{4.1.1}$$

where $\widetilde{\mu}_\pi$ is the Teichmüller lift of μ_π.

Given $w\in W$ with $w\neq 1$ and $\underline{k}=(k_\alpha)_{\alpha\in\Phi_w^+}\in\{0,1,\cdots,p-1\}^{|\Phi_w^+|}$, we define the Jacobi sum operators

$$S_{\underline{k},w}:=\sum_{A\in U_w(\mathbf{F}_p)}\left(\prod_{\alpha\in\Phi_w^+}A_\alpha^{k_\alpha}\right)A\cdot w\in\mathbf{F}_p[G(\mathbf{F}_p)].$$

These Jacobi sum operators play a main role on the local automorphic side as a crucial computation tool. These operators already appeared in [CL76] for example.

For each $\alpha\in\Phi^+$ and each integer m satisfying $0\leq m\leq p-2$, we define the operator

$$X_{\alpha,m}:=\sum_{t\in\mathbf{F}_p}t^{p-1-m}u_\alpha(t)\in\mathbf{F}_p[U(\mathbf{F}_p)]\subseteq\mathbf{F}_p[G(\mathbf{F}_p)].\tag{4.1.2}$$

The transition matrix between $\{u_\alpha(t)\mid t\in\mathbf{F}_p^\times\}$ and $\{X_{\alpha,m}\mid 0\leq m\leq p-2\}$ is a Vandermonde matrix

$$\left(t^k\right)_{t\in\mathbf{F}_p^\times,1\leq k\leq p-1}$$

which has a non-zero determinant. Hence, we also have a converse formula

$$u_\alpha(t)=-\sum_{m=0}^{p-2}t^mX_{\alpha,m}\text{ for all }t\in\mathbf{F}_p.\tag{4.1.3}$$

By the equation (4.0.3), we note that we have the equality

$$X_{\alpha,m}=-\sum_{k\geq 0}X^{\mathrm{alg}}_{\alpha,m+(p-1)k}.\tag{4.1.4}$$

Lemma 4.1.2. *Fix $w\in W$ and $\alpha_0=(i_0,j_0)\in\Phi_w^+$. Given a tuple of integers $\underline{k}=(k_{i,j})\in\{0,1,\cdots,p-1\}^{|\Phi_w^+|}$ satisfying*

$$k_{i_0,j}=0\ \textit{for all}\ (i_0,j)\in\Phi_w^+\ \textit{with}\ j\geq j_0+1,\tag{4.1.5}$$

we have

$$X_{\alpha_0,m}\bullet S_{\underline{k},w}=\begin{cases}(-1)^{m+1}c_{k_{\alpha_0},m}S_{\underline{k}',w} & \textit{if}\quad m\leq k_{\alpha_0}\\ 0 & \textit{if}\quad m>k_{\alpha_0}\end{cases}$$

where $\underline{k}'=(k'_\alpha)_{\alpha\in\Phi_w}$ satisfies

$$k'_\alpha=\begin{cases}k_{\alpha_0}-m & \textit{if }\alpha=\alpha_0;\\ k_\alpha & \textit{otherwise.}\end{cases}$$

Proof. We prove this lemma by direct computation.

$$X_{\alpha,m} \bullet S_{\underline{k},w} = \sum_{t \in \mathbf{F}_p} t^{p-1-m} \left(\sum_{A \in U_w(\mathbf{F}_p)} \left(\prod_{\alpha \in \Phi_w^+} A_\alpha^{k_\alpha} \right) u_{\alpha_0}(t) A w \right) \tag{4.1.6}$$
$$= \sum_{t \in \mathbf{F}_p} t^{p-1-m} \left(\sum_{A \in U_w(\mathbf{F}_p)} \left(\prod_{\alpha \in \Phi_w^+, \alpha \neq \alpha_0} A_\alpha^{k_\alpha} \right) (A_{\alpha_0} - t)^{k_{\alpha_0}} A w \right)$$
$$= \sum_{A \in U_w(\mathbf{F}_p)} \left(\prod_{\alpha \in \Phi_w^+, \alpha \neq \alpha_0} A_\alpha^{k_\alpha} \right) \left(\sum_{t \in \mathbf{F}_p} t^{p-1-m} (A_{\alpha_0} - t)^{k_{\alpha_0}} \right) A w$$

where the second equality follows from the change of variable $A \leftrightarrow u_{\alpha_0}(t)A$ and the assumption (4.1.5).

Notice that

$$\sum_{t \in \mathbf{F}_p} t^{p-1-m} (A_{\alpha_0} - t)^{k_{\alpha_0}} = \sum_{t \in \mathbf{F}_p} t^{p-1-m} \left(\sum_{j=0}^{k_{\alpha_0}} (-1)^j c_{k_{\alpha_0},j} A_{k_{\alpha_0}}^{k_{\alpha_0}-j} t^j \right)$$
$$= \sum_{j=0}^{k_{\alpha_0}} (-1)^j c_{k_{\alpha_0},j} A_{k_{\alpha_0}}^{k_{\alpha_0}-j} \left(\sum_{t \in \mathbf{F}_p} t^{p-1-m+j} \right),$$

which can be easily seen to be

$$\begin{cases} (-1)^{m+1} c_{k_{\alpha_0},m} A_{k_{\alpha_0}}^{k_{\alpha_0}-m} & \text{if} \quad m \leq k_{\alpha_0} \\ 0 & \text{if} \quad m > k_{\alpha_0}. \end{cases} \tag{4.1.7}$$

The last computation (4.1.7) follows from the fact that

$$\sum_{t \in \mathbf{F}_p} t^\ell = \begin{cases} 0 & \text{if} \quad p-1 \nmid \ell; \\ -1 & \text{if} \quad p-1 \mid \ell \text{ and } \ell \neq 0. \end{cases}$$

Applying (4.1.7) back to (4.1.6) gives us the result. □

Lemma 4.1.3. *Fix $w \in W$ and $\alpha_0 = (i_0, j_0) \in \Phi_w^+$. Given a tuple of integers $\underline{k} = (k_{i,j}) \in \{0, 1, \cdots, p-1\}^{|\Phi_w^+|}$ satisfying*

$$k_{i_0,j} = 0 \textit{ for all } (i_0, j) \in \Phi_w^+ \textit{ with } j \geq j_0,$$

we have

$$u_{\alpha_0}(t) \bullet S_{\underline{k},w} = S_{\underline{k},w}.$$

Proof. By Lemma 4.1.2 we deduce that

$$X_{\alpha_0,m} \bullet S_{\underline{k},w} = \begin{cases} -S_{\underline{k},w} & \text{if} \quad m = 0 \\ 0 & \text{if} \quad 1 \leq m \leq p-2 \end{cases}$$

Therefore we conclude this lemma from (4.1.3). □

By the definition of principal series representations, we have the decomposition

$$\pi = \oplus_{w \in W} \pi_w \tag{4.1.8}$$

where $\pi_w \subset \pi|_{B(\mathbf{F}_p)}$ consists of the functions supported on a non-empty subset of the Bruhat cell

$$B(\mathbf{F}_p) w^{-1} B(\mathbf{F}_p) = B(\mathbf{F}_p) w^{-1} U_w(\mathbf{F}_p).$$

Proposition 4.1.4. *Fix a non-zero vector $v_\pi \in \pi^{U(\mathbf{F}_p),\mu_\pi}$. For each $w \in W$ with $w \neq 1$, the set*

$$\left\{S_{\underline{k},w}v_\pi \mid \underline{k} = (k_\alpha)_{\alpha\in\Phi_w^+} \in \{0,1,\cdots,p-1\}^{|\Phi_w^+|}\right\} \tag{4.1.9}$$

forms a $T(\mathbf{F}_p)$-eigenbasis of π_w.

Proof. We have a decomposition $\pi_w = \oplus_{A\in U_w(\mathbf{F}_p)}\pi_{w,A}$ where $\pi_{w,A}$ is the subspace of π_w consisting of functions supported on $B(\mathbf{F}_p)w^{-1}A^{-1}$. It is easy to observe by the definition of parabolic induction that $\dim_{\mathbf{F}_p}\pi_{w,A} = 1$ and $\pi_{w,A}$ is generated by Awv_π.

We claim that, for a fixed $w \in W$, the set of vectors (4.1.9) can be linearly represented by the set of vectors $\{Awv_\pi, A \in U_w(\mathbf{F}_p)\}$ through the matrix $\left(m_{\underline{k},A}\right)$ where

$$\underline{k} = (k_\alpha)_{\alpha\in\Phi_w^+} \in \{0,1,\cdots,p-1\}^{|\Phi_w^+|}, \qquad A \in U_w(\mathbf{F}_p)$$

and $m_{\underline{k},A} := \prod_{\alpha\in\Phi_w^+} A_\alpha^{k_\alpha}$. Note that this matrix is the $|\Phi_w^+|$-times tensor of the Vandermonde matrix

$$\left(\lambda^k\right)_{\lambda\in\mathbf{F}_p, 0\leq k\leq p-1},$$

and therefore has a non-zero determinant. As a result, the matrix $\left(m_{\underline{k},A}\right)$ is invertible and $\{S_{\underline{k},w}v_\pi \mid 0 \leq k_\alpha \leq p-1 \quad \forall\alpha \in \Phi_w^+\}$ forms a basis of π_w.

The fact that this is a $T(\mathbf{F}_p)$-eigenbasis is immediate by the following calculation: if we let $x = \mathrm{diag}(x_1, x_2, \cdots, x_n)$

$$\begin{aligned}
x \bullet S_{\underline{k},w}v_\pi &= x \bullet \left(\sum_{A\in U_w(\mathbf{F}_p)}\left(\prod_{\alpha\in\Phi_w^+} A_\alpha^{k_\alpha}\right) A\ w\right) v_\pi \\
&= \left(\sum_{A\in U_w(\mathbf{F}_p)}\left(\prod_{(i,j)\in\Phi_w^+} A_{i,j}^{k_{i,j}}\right) xAx^{-1}\ w\right)\ \left(w^{-1}xw\right)\ v_\pi \\
&= \left(\sum_{B=xAx^{-1}\in U_w(\mathbf{F}_p)}\left(\prod_{(i,j)\in\Phi_w^+} (B_{i,j}x_jx_i^{-1})^{k_{i,j}}\right) B\ w\right)\ \left(w^{-1}xw\right)\ v_\pi \\
&= \mu_\pi(w^{-1}xw)\left(\prod_{(i,j)\in\Phi_w^+}(x_jx_i^{-1})^{k_{i,j}}\right)\left(\sum_{A\in U_w(\mathbf{F}_p)}\prod_{\alpha\in\Phi_w^+} A_\alpha^{k_\alpha} A\ w\right) v_\pi \\
&= (\mu_\pi^w\lambda)(x)S_{\underline{k},w}v_\pi,
\end{aligned}$$

where $\lambda(x) = \prod_{1\leq i<j\leq n}(x_jx_i^{-1})^{k_{i,j}}$ and $B_{i,j} = A_{i,j}x_ix_j^{-1}$ for $1 \leq i < j \leq n$. $\square$

We can further describe the action of $T(\mathbf{F}_p)$ on $S_{\underline{k},w}v_\pi$. By $\lfloor y\rfloor$ for $y \in \mathbf{R}$ we mean the floor function of y, i.e., the biggest integer less than or equal to y.

Lemma 4.1.5. *Let $\mu_\pi = (d_1, d_2, \cdots, d_{n-1}, d_n)$. If we write $(\ell_1, \ell_2 \cdots, \ell_{n-1}, \ell_n)$ for the $T(\mathbf{F}_p)$-eigencharacter of $S_{\underline{k},w}v_\pi$, then we have*

$$\ell_r \equiv d_{w^{-1}(r)} + \sum_{1\leq i<r} k_{i,r} - \sum_{r<j\leq n} k_{r,j} \pmod{p-1}$$

for all $1 \leq r \leq n$, where $k_{i,j} = k_\alpha$ if $\alpha \in \Phi_w^+$ and (i,j) corresponds to α, and $k_{i,j} = 0$ otherwise.

In particular,

(i) *if $k_\alpha = 0$ for any $\alpha \in \Phi_w^+ \setminus \Delta$, then for all $1 \leq r \leq n$*

$$\ell_r \equiv d_{w^{-1}(r)} + (1 - \lfloor 1/r\rfloor)k_{r-1,r} - (1 - \lfloor 1/(n+1-r)\rfloor)k_{r,r+1} \pmod{p-1};$$

(ii) *if* $w = w_0$ *and* $k_{i,j} = 0$ *for any* $2 \leq i < j \leq n$*, then*

$$\ell_r \equiv \begin{cases} d_n - \sum_{j=2}^{n} k_{1,j} \pmod{p-1} & \text{if } r = 1; \\ d_{n+1-r} + k_{1,r} \pmod{p-1} & \text{if } 2 \leq r \leq n. \end{cases}$$

Proof. The first part of the Lemma is a direct calculation as shown at the end of the proof of Proposition 4.1.4. The second part follows trivially from the first part. □

Given any $w \in W$, we write $S_{\underline{0},w}$ for $S_{\underline{k},w}$ with $k_\alpha = 0$ for all $\alpha \in \Phi_w^+$.

Lemma 4.1.6. *We have*

$$\mathbf{F}_p[S_{\underline{0},w} v_\pi] = \pi^{U(\mathbf{F}_p),\mu_\pi^w}.$$

Proof. Pick an arbitrary positive root α. If $\alpha \in \Phi_w^+$, then we have (since $u_\alpha(t) \in U_w(\mathbf{F}_p)$)

$$u_\alpha(t) \left(\sum_{A \in U_w(\mathbf{F}_p)} A \right) = \left(\sum_{A \in U_w(\mathbf{F}_p)} A \right)$$

and therefore $u_\alpha(t) S_{\underline{0},w} v_\pi = S_{\underline{0},w} v_\pi$ for any $t \in \mathbf{F}_p$. On the other hand, if $\alpha \notin \Phi_w^+$, then

$$u_\alpha(t) \left(\sum_{A \in U_w(\mathbf{F}_p)} A \right) = \left(\sum_{A \in U_w(\mathbf{F}_p)} A \right) u'_\alpha(t)$$

and

$$u'_\alpha(t) w v_\pi = w u''_\alpha(t) v_\pi = w v_\pi$$

where $u'_\alpha(t) \in \prod_{\alpha \notin \Phi_w^+} \overline{U}_\alpha(\mathbf{F}_p)$ and $u''_\alpha(t) \in U(\mathbf{F}_p)$ are elements depending on x, w and α. Hence, $u_\alpha(t) S_{\underline{0},w} v_\pi = S_{\underline{0},w} v_\pi$ for any $t \in \mathbf{F}_p$ and any $\alpha \in \Phi^+$. So we conclude that $S_{\underline{0},w} v_\pi$ is $U(\mathbf{F}_p)$-invariant as $\{u_\alpha(t)\}_{\alpha \in \Phi^+, t \in \mathbf{F}_p}$ generate $U(\mathbf{F}_p)$.

Finally, we check that $x \cdot S_{\underline{0},w} v_\pi = \mu_\pi^w(x) S_{\underline{0},w} v_\pi$ for $x \in T(\mathbf{F}_p)$. But this is immediate from the following two easy computations:

$$x \bullet \left(\sum_{A \in U_w(\mathbf{F}_p)} A \right) = \left(\sum_{A \in U_w(\mathbf{F}_p)} A \right) \bullet x \in \mathbf{F}_p[G(\mathbf{F}_p)]$$

and

$$x w v_\pi = w \left(w^{-1} x w \right) v_\pi = w \mu_\pi(w^{-1} x w) v_\pi = \mu_\pi^w(x) w v_\pi.$$

This completes the proof. □

Note that Proposition 4.1.4, Lemma 4.1.5, and Lemma 4.1.6 are very elementary and have essentially appeared in [CL76]. In this article, we formulate them and give quick proofs of them for the convenience.

Definition 4.1.7. *Given* $\alpha, \alpha' \in \Phi^+$*, we say that* α *is* strongly smaller than α' *with the notation*

$$\alpha \,\tilde{\preceq}\, \alpha'$$

if there exist $1 \leq i \leq j \leq k \leq n-1$ *such that*

$$\alpha = \sum_{r=i}^{j} \alpha_r \text{ and } \alpha' = \sum_{r=i}^{k} \alpha_r.$$

A subset Φ' *of* Φ^+ *is said to be* good *if it satisfies the following:*

(i) *if* $\alpha, \alpha' \in \Phi'$ *and* $\alpha + \alpha' \in \Phi^+$*, then* $\alpha + \alpha' \in \Phi'$*;*
(ii) *if* $\alpha \in \Phi'$ *and* $\alpha \,\tilde{\preceq}\, \alpha'$*, then* $\alpha' \in \Phi'$*.*

We associate a subgroup of U to Φ' by

$$U_{\Phi'} := \langle U_\alpha \mid \alpha \in \Phi' \rangle \tag{4.1.10}$$

and denote its reduction mod p by $\overline{U}_{\Phi'}$. We define U_1 to be the subgroup scheme of U generated by U_{α_r} for $2 \leq r \leq n-1$, and denote its reduction mod p by $\overline{U}_1$.

Example 4.1.8. *The following are two examples of good subsets of Φ^+, that will be important for us:*

$$\left\{\sum_{r=i}^{j} \alpha_r \mid 1 \leq i < j \leq n-1\right\} \text{ and } \left\{\sum_{r=i}^{j} \alpha_r \mid 2 \leq i \leq j \leq n-1\right\}.$$

The subgroups of U associated with the two good subsets via (4.1.10) are $[U,U]$ and U_1 respectively.

We recall that we have defined $\pi_w \subsetneq \pi$ in (4.1.8) for each $w \in W$.

Proposition 4.1.9. *Let $\Phi' \subseteq \Phi^+$ be good. Pick an element $w \in W$ with $w \neq 1$. The following set of vectors*

$$\left\{S_{\underline{k},w} v_\pi \mid \underline{k} = (k_\alpha)_{\alpha \in \Phi_w^+} \in \{0, 1, \cdots, p-1\}^{|\Phi_w^+|} \text{ with } k_\alpha = 0 \ \forall \alpha \in \Phi' \cap \Phi_w^+\right\} \tag{4.1.11}$$

forms a basis of the subspace $\pi_w^{U_{\Phi'}(\mathbf{F}_p)}$ of π_w.

Proof. By Proposition 4.1.4, the set of vectors (4.1.9) forms a $T(\mathbf{F}_p)$-eigenbasis of π_w. Hence we fix a $U_{\Phi'}(\mathbf{F}_p)$-invariant vector v in π_w and can write it as a unique linear combination of vectors of the form $S_{\underline{k},w} v_\pi$, namely

$$v = \sum_{\underline{k} \in \{0, \cdots, p-1\}^{|\Phi_w^+|}} C_{\underline{k},w} S_{\underline{k},w} v_\pi \text{ for some } C_{\underline{k},w} \in \mathbf{F}_p.$$

We define

$$\mathrm{Supp}(v)_\alpha := \{\underline{k} = (k_\alpha)_{\alpha \in \Phi_w^+} \mid C_{\underline{k},w} \neq 0 \text{ and } k_\alpha > 0\}$$

for each $\alpha \in \Phi_w^+$, and then consider

$$\Phi'_{w,v,>0} := \{\alpha \in \Phi' \cap \Phi_w^+ \mid \mathrm{Supp}(v)_\alpha \neq \varnothing\}.$$

We have a natural partial order on $\Phi'_{w,v,>0}$ induced from the partial order $\widetilde{\preceq}$ on Φ^+. Assume that

$$\Phi'_{w,v,>0} \neq \varnothing \tag{4.1.12}$$

which means that $\mathrm{Supp}(v)_\alpha \neq \varnothing$ for some $\alpha \in \Phi' \cap \Phi_w^+$, and thus we can choose one maximal element $\alpha_0 \in \Phi'_{w,v,>0}$ with respect to the order $\widetilde{\preceq}$. We may write v as

$$v = \sum_{\substack{\underline{k} \in \{0, \cdots, p-1\}^{|\Phi_w^+|} \\ k_{\alpha_0} = 0}} C_{\underline{k},w} S_{\underline{k},w} v_\pi + \sum_{\substack{\underline{k} \in \{0, \cdots, p-1\}^{|\Phi_w^+|} \\ k_{\alpha_0} > 0}} C_{\underline{k},w} S_{\underline{k},w} v_\pi. \tag{4.1.13}$$

By the maximality assumption on α_0 we know that if $C_{\underline{k},w} \neq 0$ and $\alpha_0 \widetilde{\preceq} \alpha$, then $k_\alpha = 0$. As a result, we deduce from Lemma 4.1.3 that

$$u_{\alpha_0}(t) \sum_{\substack{\underline{k} \in \{0, \cdots, p-1\}^{|\Phi_w^+|} \\ k_{\alpha_0} = 0}} C_{\underline{k},w} S_{\underline{k},w} v_\pi = \sum_{\substack{\underline{k} \in \{0, \cdots, p-1\}^{|\Phi_w^+|} \\ k_{\alpha_0} = 0}} C_{\underline{k},w} S_{\underline{k},w} v_\pi \tag{4.1.14}$$

for all $t \in \mathbf{F}_p$.

We define

$$\Phi_w^{\alpha_0,+} := \{\alpha \in \Phi_w^+ \mid \alpha_0 \widetilde{\preceq} \alpha\} \text{ and } \Phi_w^{\alpha_0,-} := \Phi_w^+ \setminus \Phi_w^{\alpha_0,+},$$

and we use the notation

$$\underline{\ell} := (\ell_\alpha)_{\alpha \in \Phi_w^{\alpha_0,-}} \in \{0, \cdots, p-1\}^{|\Phi_w^{\alpha_0,-}|}$$

for a tuple of integers indexed by $\Phi_w^{\alpha_0,-}$. Given a tuple $\underline{\ell}$, we can define

$$\Lambda(\underline{\ell},\alpha_0) := \left\{ \underline{k} = (k_\alpha)_{\alpha\in\Phi_w^+} \in \{0,\cdots,p-1\}^{|\Phi_w^+|} \;\middle|\; \begin{array}{ll} \cdot\ k_\alpha = 0 & \text{if } \alpha\in\Phi_w^{\alpha_0,+}\setminus\{\alpha_0\}; \\ \cdot\ k_\alpha > 0 & \text{if } \alpha = \alpha_0; \\ \cdot\ k_\alpha = \ell_\alpha & \text{if } \alpha\in\Phi_w^{\alpha_0,-} \end{array} \right\}.$$

Now we can define a polynomial

$$f_{(\underline{\ell},\alpha_0)}(x) = \sum_{\underline{k}\in\Lambda(\underline{\ell},\alpha_0)} C_{\underline{k},w} x^{k_{\alpha_0}} \in \mathbf{F}_p[x]$$

for each tuple of integers $\underline{\ell}$. By the maximality assumption on α_0 and the notation introduced above, we have

$$\sum_{\substack{\underline{k}\in\{0,\cdots,p-1\}^{|\Phi_w^+|}\\ k_{\alpha_0}>0}} C_{\underline{k},w} S_{\underline{k},w} v_\pi = \sum_{\underline{\ell}\in\{0,\cdots,p-1\}^{|\Phi_w^{\alpha_0,-}|}} \left(\sum_{A\in U_w(\mathbf{F}_p)} \left(\prod_{\alpha\in\Phi_w^{\alpha_0,-}} A_\alpha^{\ell_\alpha} \right) f_{(\underline{\ell},\alpha_0)}(A_{\alpha_0}) A \right) w v_\pi.$$

By the assumption on v we know that $u_{\alpha_0}(t)v = v$ for all $t\in\mathbf{F}_p$. Using (4.1.14) and (4.1.13) we have

$$u_{\alpha_0}(t) \sum_{\substack{\underline{k}\in\{0,\cdots,p-1\}^{|\Phi_w^+|}\\ k_{\alpha_0}>0}} C_{\underline{k},w} S_{\underline{k},w} v_\pi = \sum_{\substack{\underline{k}\in\{0,\cdots,p-1\}^{|\Phi_w^+|}\\ k_{\alpha_0}>0}} C_{\underline{k},w} S_{\underline{k},w} v_\pi$$

and so

$$\begin{aligned} &\sum_{\underline{\ell}\in\{0,\cdots,p-1\}^{|\Phi_w^{\alpha_0,-}|}} \left(\sum_{A\in U_w(\mathbf{F}_p)} \left(\prod_{\alpha\in\Phi_w^{\alpha_0,-}} A_\alpha^{\ell_\alpha} \right) f_{(\underline{\ell},\alpha_0)}(A_{\alpha_0}) A \right) w v_\pi \\ &\quad = u_{\alpha_0}(t) \sum_{\underline{\ell}\in\{0,\cdots,p-1\}^{|\Phi_w^{\alpha_0,-}|}} \left(\sum_{A\in U_w(\mathbf{F}_p)} \left(\prod_{\alpha\in\Phi_w^{\alpha_0,-}} A_\alpha^{\ell_\alpha} \right) f_{(\underline{\ell},\alpha_0)}(A_{\alpha_0}) A \right) w v_\pi \\ &\quad = \sum_{\underline{\ell}\in\{0,\cdots,p-1\}^{|\Phi_w^{\alpha_0,-}|}} \left(\sum_{A\in U_w(\mathbf{F}_p)} \left(\prod_{\alpha\in\Phi_w^{\alpha_0,-}} A_\alpha^{\ell_\alpha} \right) f_{(\underline{\ell},\alpha_0)}(A_{\alpha_0}-t) A \right) w v_\pi \end{aligned}$$

where the last equality follows from a change of variable $A \leftrightarrow u_{\alpha_0}(t)A$.

By the linear independence of Jacobi sums from Proposition 4.1.4, we deduce an equality

$$\begin{aligned} &\left(\sum_{A\in U_w(\mathbf{F}_p)} \left(\prod_{\alpha\in\Phi_w^{\alpha_0,-}} A_\alpha^{\ell_\alpha} \right) f_{(\underline{\ell},\alpha_0)}(A_{\alpha_0}) A \right) w v_\pi \\ &\qquad = \left(\sum_{A\in U_w(\mathbf{F}_p)} \left(\prod_{\alpha\in\Phi_w^{\alpha_0,-}} A_\alpha^{\ell_\alpha} \right) f_{(\underline{\ell},\alpha_0)}(A_{\alpha_0}-t) A \right) w v_\pi \end{aligned}$$

for each fixed tuple $\underline{\ell}$.

Therefore, again by the linear independence of Jacobi sum operators in Proposition 4.1.4 we deduce that

$$f_{(\underline{\ell},\alpha_0)}(A_{\alpha_0}-t) = f_{(\underline{\ell},\alpha_0)}(A_{\alpha_0})$$

for each $t\in\mathbf{F}_p$ and each $(\underline{\ell},\alpha_0)$. We know that if $f\in\mathbf{F}_p[x]$ satisfies $\deg f\le p-1$, $f(0)=0$ and $f(x-t)=f(x)$ for each $t\in\mathbf{F}_p$ then $f=0$. Thus we deduce that

$$f_{(\underline{\ell},\alpha_0)} = 0$$

for each tuple of integers $\underline{\ell}$, which is a contradiction to (4.1.12) and so we have $k_\alpha = 0$ for any $\alpha \in \Phi' \cap \Phi_w^+$ for each tuple of integers $\underline{k}$ such that $C_{\underline{k},w} \neq 0$.

As a result, we have shown that each vector in $\pi_w^{U_{\Phi'}(\mathbf{F}_p)}$ can be written as certain linear combination of vectors in (4.1.11). On the other hand, by Proposition 4.1.4 we know that vectors in (4.1.11) are linear independent, and therefore they actually form a basis of $\pi_w^{U_{\Phi'}(\mathbf{F}_p)}$. $\square$

Corollary 4.1.10. *Let $\mu_\pi = (d_1, \cdots, d_n)$ and fix a non-zero vector $v_\pi \in \pi^{U(\mathbf{F}_p),\mu_\pi}$. Given a weight $\mu = (\ell_1, \cdots, \ell_n) \in X_1(T)$ the space*

$$\pi_{w_0}^{[U(\mathbf{F}_p),U(\mathbf{F}_p)],\mu}$$

has a basis whose elements are of the form

$$S_{\underline{k},w_0} v_\pi$$

where $\underline{k} = (k_\alpha)$ satisfies

$$\ell_r \equiv d_{n+1-r} + (1 - \lfloor 1/r \rfloor) k_{r-1,r} - (1 - \lfloor 1/(n+1-r) \rfloor) k_{r,r+1} \ mod\ (p-1)$$

for all $1 \leq r \leq n$ and $k_\alpha = 0$ if $\alpha \in \Phi^+ \setminus \Delta$.

Proof. By a special case of Proposition 4.1.9 when $\Phi' = \{\sum_{r=i}^{j} \alpha_r \mid 1 \leq i < j \leq n-1\}$, we know that

$$\{S_{\underline{k},w_0} v_\pi \mid k_\alpha = 0 \text{ if } \alpha \in \Phi^+ \setminus \Delta\}$$

forms a basis of $\pi_{w_0}^{[U(\mathbf{F}_p),U(\mathbf{F}_p)]}$. On the other hand, we know from Proposition 4.1.4 that the above basis is actually an $T(\mathbf{F}_p)$-eigenbasis. Therefore the vectors in this basis with a fixed eigencharacter μ form a basis of the eigensubspace $\pi_{w_0}^{[U(\mathbf{F}_p),U(\mathbf{F}_p)],\mu}$. Finally, using (i) of the second part of Lemma 4.1.5 we conclude this lemma. $\square$

Corollary 4.1.11. *Let $\mu_\pi = (d_1, d_2, \cdots, d_n)$ and fix a non-zero vector $v_\pi \in \pi^{U(\mathbf{F}_p),\mu_\pi}$. Given a weight $\mu = (\ell_1, \cdots, \ell_n) \in X_1(T)$, the space*

$$\pi_{w_0}^{U_1(\mathbf{F}_p),\mu}$$

has a basis whose elements are of the form

$$S_{\underline{k},w_0} v_\pi$$

where $\underline{k} = (k_{i,j})_{i,j}$ satisfies

$$k_{1,j} \equiv \ell_j - d_{n+1-j} \ mod\ (p-1)$$

for $2 \leq j \leq n$ and $k_{i,j} = 0$ for all $2 \leq i < j \leq n$.

Proof. By a special case of Proposition 4.1.9 when $\Phi' = \{\sum_{r=i}^{j} \alpha_r \mid 2 \leq i \leq j \leq n-1\}$, we know that

$$\{S_{\underline{k},w_0} v_\pi \mid k_{i,j} = 0 \text{ if } 2 \leq i < j \leq n\}$$

forms a basis of $\pi_{w_0}^{U_1(\mathbf{F}_p)}$. On the other hand, we know from Proposition 4.1.4 that the above basis is actually an $T(\mathbf{F}_p)$-eigenbasis. Therefore the vectors in this basis with a fixed eigencharacter μ form a basis of the eigensubspace $\pi_{w_0}^{U_1(\mathbf{F}_p),\mu}$. Finally, using (ii) of the second part of Lemma 4.1.5 we conclude this lemma. $\square$

4.2. **Summary of results on Deligne–Lusztig representations.** In this section, we recall some standard facts on Deligne–Lusztig representations and fix the notation that will be used throughout this paper. We closely follow [Her09]. Throughout this article we will only focus the group $G(\mathbf{F}_p) = \mathrm{GL}_n(\mathbf{F}_p)$, which is the fixed point set of the standard (p-power) Frobenius F inside $\mathrm{GL}_n(\overline{\mathbf{F}}_p)$. We will identify a variety over $\overline{\mathbf{F}}_p$ with the set of its $\overline{\mathbf{F}}_p$-rational points for simplicity. Then our fixed maximal torus $\mathbb{T}$ is F-stable and split.

To each pair $(\mathbb{T}, \theta)$ consisting of an F-stable maximal torus $\mathbb{T}$ and a homomorphism $\theta : \mathbb{T}^F \to \overline{\mathbf{Q}}_p^\times$, Deligne–Lusztig [DL76] associate a virtual representation $R_{\mathbb{T}}^\theta$ of $\mathrm{GL}_n(\mathbf{F}_p)$. (We restrict ourself to $\mathrm{GL}_n(\mathbf{F}_p)$ although the result in [DL76] is much more general.) On the other hand, given a pair $(w, \mu) \in W \times X(T)$, one can construct a pair $(\mathbb{T}_w, \theta_{w,\mu})$ by the method in the third paragraph of [Her09], Section 4.1. Then we denote by $R_w(\mu)$ the representation corresponding to $R_{\mathbb{T}_w}^{\theta_{w,\mu}}$ after multiplying a sign. This is the so-called Jantzen parametrization in [Jan81] 3.1.

The representations $R_{\mathbb{T}}^\theta$ (resp. $R_w(\mu)$) can be isomorphic for different pairs $(\mathbb{T}, \theta)$ (resp. (w, μ)), and the explicit relation between is summarized in [Her09], Lemma 4.2. As each p-regular character $\mu \in X(T)/(p-1)X(T)$ of $T(\mathbf{F}_p)$ can be lift to an element in $X_1^{\mathrm{reg}}(T)$ which is unique up to $(p-1)X_0(T)$, the representation $R_w(\mu)$ is well defined for each $w \in W$ and such a μ.

We recall the notation $\Theta(\theta)$ for a cuspidal representation for $\mathrm{GL}_n(\mathbf{F}_p)$ from [Her06], Section 2.1 where θ is a *primitive* character of $\mathbf{F}_{p^n}^\times$ defined in [Her09], Section 4.2. We refer further discussion about the basic properties and references of $\Theta(\theta)$ to [Her06], Section 2.1. The relation between the notation $R_w(\mu)$ and the notation $\Theta(\theta)$ is summarized in [Her09], Lemma 4.7. In this paper, we will use the notation $\Theta_m(\theta_m)$ for a cuspidal representation for $\mathrm{GL}_m(\mathbf{F}_p)$ where θ_m is a primitive character of $\mathbf{F}_{p^m}^\times$.

We emphasize that, as a special case of [Her09], Lemma 4.7, we have the natural isomorphism

$$R_1(\mu) \cong \mathrm{Ind}_{B(\mathbf{F}_p)}^{G(\mathbf{F}_p)} \widetilde{\mu}$$

for a p-regular character μ of $T(\mathbf{F}_p)$, where $\widetilde{\mu}$ is the Teichmüller lift of μ.

4.3. **A multiplicity one theorem.** The main target of this section is to prove Corollary 4.3.7, which immediately implies our main multiplicity one theorem, Theorem 4.8.2. In fact, Theorem 4.8.2 is a special case of Corollary 4.3.7.

We recall some notation from [Jan03]. We use the notation $\overline{G}_r$ for the r-th Frobenius kernel defined in [Jan03] Chapter I 9 as kernel of r-th iteration of Frobenius morphism on the group scheme $\overline{G}$ over $\mathbf{F}_p$. We will consider the subgroup scheme $\overline{G}_r\overline{T}$, $\overline{G}_r\overline{B}$, $\overline{G}_r\overline{B}^-$ of $\overline{G}$ in the following. Note that our $\overline{B}$ (resp. $\overline{B}^-$) is denoted by B^+ (resp. B) in [Jan03] Chapter II 9. We define

$$\begin{array}{rcl} \widehat{Z}_r'(\lambda) & := & \mathrm{ind}_{\overline{B}^-}^{\overline{G}_r\overline{B}^-}\lambda; \\ \widehat{Z}_r(\lambda) & := & \mathrm{coind}_{\overline{B}}^{\overline{G}_r\overline{B}}\lambda \end{array}$$

where ind and coind are defined in I 3.3 and I 8.20 of [Jan03] respectively. By [Jan03] Proposition II 9.6 we know that there exists a simple $\overline{G}_r\overline{T}$-module $\widehat{L}_r(\lambda)$ satisfying

$$\mathrm{soc}_{\overline{G}_r}\left(\widehat{Z}_r'(\lambda)\right) \cong \widehat{L}_r(\lambda) \cong \mathrm{cosoc}_{\overline{G}_r}\left(\widehat{Z}_r(\lambda)\right).$$

The properties of $\widehat{Z}_r'(\lambda)$ and $\widehat{Z}_r(\lambda)$ are systematically summarized in [Jan03] II 9, and therefore we will frequently refer to results over there.

From now on we assume $r = 1$ in this section.

Now we recall several well-known results from [Jan81], [Jan84] and [Jan03]. We recall the definition of $\widetilde{W}^{res}$ from (4.0.5).

Theorem 4.3.1 ([Jan81], Satz 4.3)**.** *Assume that $\mu+\eta$ is in the lowest p-restricted alcove and $2n$-generic (Definition 4.1.1). Then we have*

$$\overline{R_w(\mu+\eta)} = \sum_{\substack{\widetilde{w}'\in\widetilde{W}^{\mathrm{res}}\\ \nu\in X(T)}} [\widehat{Z}_1(\mu-p\nu+p\eta):\widehat{L}_1(\widetilde{w}'\cdot\mu)]F(\widetilde{w}'\cdot(\mu+w\nu)).$$

Proposition 4.3.2 ([Jan03], Corollary II 6.24)**.** *Let $\lambda\in X(T)_+$. Suppose $\mu\in X(T)$ is maximal for $\mu\uparrow\lambda$ and $\mu\neq\lambda$. If $\mu\in X(T)_+$ and if $\mu\neq\lambda-p\alpha$ for all $\alpha\in\Phi^+$, then*

$$[H^0(\lambda):F(\mu)]=1.$$

If M is an arbitrary $\overline{G}$-module, we use the notation $M^{[1]}$ for the Frobenius twist of M as defined in [Jan03], I 9.10.

Proposition 4.3.3 ([Jan03], Proposition II 9.14)**.** *Let $\lambda\in X(T)_+$. Suppose each composition factor of $\widehat{Z}_1'(\lambda)$ has the form $\widehat{L}_1(\mu_0+p\mu_1)$ with $\mu_0\in X_1(T)$ and $\mu_1\in X(T)$ such that*

$$\langle\mu_1+\eta,\beta^\vee\rangle\geq 0$$

for all $\beta\in\Delta$. Then $H^0(\lambda)$ has a filtration with factors of the form $F(\mu_0)\otimes H^0(\mu_1)^{[1]}$. Each such module occurs as often as $\widehat{L}_1(\mu_0+p\mu_1)$ occurs in a composition series of $\widehat{Z}_1'(\lambda)$.

Remark 4.3.4. *Note that if μ_1 is in the lowest p-restricted alcove, then $F(\mu_0)\otimes H^0(\mu_1)^{[1]}=F(\mu)$ by Steinberg tensor product theorem.*

Lemma 4.3.5 ([Jan03], Lemma II 9.18 (a))**.** *Let $\widehat{L}_1(\mu)$ be a composition factor of $\widehat{Z}_1'(\lambda)$, and write*

$$\lambda+\eta=p\lambda_1+\lambda_0 \text{ and } \mu=p\mu_1+\mu_0$$

with $\lambda_0,\mu_0\in X_1(T)$ and $\lambda_1,\mu_1\in X(T)$.

If

$$\langle\lambda,\alpha^\vee\rangle\geq n-2 \tag{4.3.1}$$

for all $\alpha\in\Phi^+$, then

$$\langle\mu_1+\eta,\beta^\vee\rangle\geq 0$$

for all $\beta\in\Phi^+$.

Proof. We only need to mention that $h_\alpha=n$ for all $\alpha\in\Phi^+$ and for our group $\overline{G}=\mathrm{GL}_{n/\mathbf{F}_p}$, where h_α is defined in [Jan03], Lemma II 9.18. □

We define an element $s_{\alpha,m}\in\widetilde{W}$ by

$$s_{\alpha,m}\cdot\lambda=s_\alpha\cdot\lambda+mp\alpha$$

for each $\alpha\in\Phi^+$ and $m\in\mathbf{Z}$.

Theorem 4.3.6. *Let $\lambda,\mu\in X(T)$ such that*

$$\mu=s_{\alpha,m}\cdot\lambda \quad\text{and}\quad mp<\langle\lambda+\eta,\alpha^\vee\rangle<(m+1)p. \tag{4.3.2}$$

Assume further that there exists $\nu\in X(T)$ such that $\lambda+p\nu$ satisfies the condition (4.3.1) and that ν and $\mu_1+\nu$ are in the lowest p-restricted alcove.

Then we have

$$[\widehat{Z}_1(\lambda):\widehat{L}_1(\mu)]=1.$$

Proof. The condition (4.3.2) ensures that for any fixed $\nu \in X(T)$, $\mu + p\nu$ is maximal for $\mu + p\nu \uparrow \lambda + p\nu$ and $\mu + p\nu \neq \lambda + p\nu$. Notice that we have

$$[\widehat{Z}_1(\lambda) : \widehat{L}_1(\mu)] = [\widehat{Z}'_1(\lambda) : \widehat{L}_1(\mu)]$$

by II 9.2(3) in [Jan03], as the character of a $\overline{G}_r\overline{T}$-module determine its Jordan–Hölder factors with multiplicities (or equivalently, determine the semisimplification of the $\overline{G}_r\overline{T}$-module).

By II 9.2(5) and II 9.6(6) in [Jan03] we have

$$[\widehat{Z}'_1(\lambda) : \widehat{L}_1(\mu)] = [\widehat{Z}'_1(\lambda) \otimes p\nu : \widehat{L}_1(\mu) \otimes p\nu] = [\widehat{Z}'_1(\lambda + p\nu) : \widehat{L}_1(\mu + p\nu)],$$

and thus we may assume that

$$\langle \lambda, \alpha^\vee \rangle \geq n - 2$$

for all $\alpha \in \Phi^+$ by choosing appropriate ν (which exists by our assumption) and replacing λ by $\lambda + p\nu$ and μ by $\mu + p\nu$. Then by Lemma 4.3.5 we know that

$$\langle \mu'_1 + \eta, \beta^\vee \rangle \geq 0$$

for any $\mu' = p\mu'_1 + \mu'_0$ such that $\widehat{L}_1(\mu')$ is a factor of $\widehat{Z}'_1(\lambda)$.

Thus by Proposition 4.3.3, Proposition 4.3.2 and Remark 4.3.4 we know that

$$[\widehat{Z}'_1(\lambda) : \widehat{L}_1(\mu)] = [H^0(\lambda) : F(\mu_0) \otimes H^0(\mu_1)^{[1]}] = [H^0(\lambda) : F(\mu)] = 1$$

which finishes the proof. □

We pick an arbitrary principal series π and write

$$\mu_\pi = (d_1, \cdots, d_n)$$

For each pair of integers (i_1, j_1) satisfying $0 \leq i_1 < i_1 + 1 < j_1 \leq n - 1$, we define

$$\mu_\pi^{i_1,j_1} := (d_1^{i_1,j_1}, \cdots, d_n^{i_1,j_1})$$

where

$$d_k^{i_1,j_1} = \begin{cases} d_k & \text{if } k \neq n - j_1 \text{ and } k \neq n - i_1; \\ d_{n-i_1} + j_1 - i_1 - 1 & \text{if } k = n - i_1; \\ d_{n-j_1} - j_1 + i_1 + 1 & \text{if } k = n - j_1. \end{cases}$$

Corollary 4.3.7. *Assume that μ_π is $2n$-generic in the lowest alcove (cf. Definition 4.1.1). Then $F(\mu_\pi^{i_1,j_1})$ has multiplicity one in π, or equivalently in $\mathrm{Ind}_{B(\mathbf{F}_p)}^{G(\mathbf{F}_p)}\mu_\pi^w$ for any $w \in W$.*

Proof. We notice at first that each $\mathrm{Ind}_{B(\mathbf{F}_p)}^{G(\mathbf{F}_p)}\mu_\pi^w$ has the same Jordan–Hölder factor as π with the same multiplicity as each of them is a mod p reduction of certain lattice of the same characteristic 0 representation of $G(\mathbf{F}_p)$. We are going to apply Theorem 4.3.6 and Theorem 4.3.1 to determine the multiplicity of $F(\mu_\pi^{i_1,j_1})$ in π. We use the shortened notation

$$\alpha'_{i_1,j_1} := \sum_{r=n-j_1}^{n-1-i_1} \alpha_r.$$

We choose $w = 1$ in Theorem 4.3.1 and take

$$\mu + \eta := \mu_\pi = \mu_\pi^{i_1,j_1} + (j_1 - i_1 - 1)\alpha'_{i_1,j_1}.$$

We would like to consider the multiplicity of $F(\mu_\pi^{i_1,j_1})$ in $\pi = R_1(\mu+\eta)$. We will follow the notation of Theorem 4.3.1 except that we will replace the notation ν in Theorem 4.3.1 with the notation ν_0. We take $\widetilde{w}' := 1 \in \widetilde{W}^{\mathrm{res}}$ as well as

$$\nu_0 := \eta - (j_1 - i_1 - 1)\alpha'_{i_1,j_1}$$

and then note that

$$\mu_\pi^{i_1,j_1} = \mu + \nu_0.$$

We deduce from II 9.16 (5) in [Jan03] the following equality

$$[\widehat{Z}_1((\mu+\eta-p\nu_0)+(p-1)\eta):\widehat{L}_1(\mu)]=[\widehat{Z}_1((n-j_1,n-i_1)(\mu+\eta-p\nu_0)+(p-1)\eta):\widehat{L}_1(\mu)]. \tag{4.3.3}$$

We set

$$\lambda:=(n-j_1,n-i_1)(\mu+\eta-p\nu_0)+(p-1)\eta$$

and observe that

$$\begin{aligned}\lambda&=(n-j_1,n-i_1)\cdot(\mu-p\nu_0)+p\eta\\&=(n-j_1,n-i_1)\cdot\mu+p\left(\eta-(n-j_1,n-i_1)\eta-(j_1-i_1-1)\alpha'_{i_1,j_1}\right)\\&=(n-j_1,n-i_1)\cdot\mu+p\alpha'_{i_1,j_1}.\end{aligned} \tag{4.3.4}$$

Therefore we have

$$p<\langle\lambda,\alpha'_{i_1,j_1}\rangle<2p$$

and that

$$\mu=s_{\alpha'_{i_1,j_1},p}\cdot\lambda.$$

Moreover, it is easy to see that

$$\lambda+p\eta=(n-j_1,n-i_1)\cdot\mu+p\alpha'_{i_1,j_1}+p\eta$$

satisfies (4.3.1).

We take $\nu:=\lambda$ and then apply Theorem 4.3.6, (4.3.3) as well as the obvious equality

$$(\mu-p\nu_0)+p\eta=(\mu+\eta-p\nu_0)+(p-1)\eta$$

and conclude that

$$[\widehat{Z}_1((\mu-p\nu_0)+p\eta):\widehat{L}_1(\mu)]=[\widehat{Z}_1(\lambda):\widehat{L}_1(\mu)]=1$$

which implies that $F(\mu_\pi^{i_1,j_1})=F(\mu+\nu_0)$ has multiplicity one in $\overline{R_1(\mu+\eta)}=\mathrm{Ind}_{B(\mathbf{F}_p)}^{G(\mathbf{F}_p)}\mu_\pi$ by Theorem 4.3.1. □

4.4. **Jacobi sums in characteristic 0.** In this section, we establish an intertwining identity for lifts of Jacobi sums in characteristic 0 in Theorem 4.4.9, which is one of the main ingredients of the proof of Theorem 5.6.2. All of our calculations here are in the setting of $G(\mathbf{Q}_p)=\mathrm{GL}_n(\mathbf{Q}_p)$. We first fix some notation.

Let $A\in G(\mathbf{F}_p)$. By $\lceil A\rceil$ we mean the matrix in $G(\mathbf{Q}_p)$ whose entries are the classical Teichmüller lifts of the entries of A. The map $A\mapsto\lceil A\rceil$ is obviously not a group homomorphism but only a map between sets. On the other hand, we use the notation $\widetilde{\mu}$ for the Teichmüller lift of a character μ of $T(\mathbf{F}_p)$.

We denote the standard lifts of simple reflections in $G(\mathbf{Q}_p)$ by

$$s_i=\begin{pmatrix}\mathrm{Id}_{i-1}&&&\\&&1&\\&1&&\\&&&\mathrm{Id}_{n-i-1}\end{pmatrix}$$

for $1\leq i\leq n-1$. We also use the following notation

$$t_i=\begin{pmatrix}p\mathrm{Id}_i&\\&\mathrm{Id}_{n-i}\end{pmatrix}$$

for $1\leq i\leq n$. Let

$$\Xi_n:=w^*t_1, \tag{4.4.1}$$

where $w^* := s_{n-1} \bullet \ldots \bullet s_1$. We recall the Iwahori subgroup I and the pro-p Iwahori subgroup $I(1)$ from the beginning of Section 4. Note that the operator Ξ_n and the group I actually generate the normalizer of I inside $G(\mathbf{Q}_p)$. One easily sees that Ξ_n is nothing else than the following matrix:

$$\Xi_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ p & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \in G(\mathbf{Q}_p).$$

For each $1 \leq i \leq n-1$, we consider the maximal parabolic subgroup P_i^- of G containing lower-triangular Borel subgroup B^- such that its Levi subgroup can be chosen to be $\mathrm{GL}_i \times \mathrm{GL}_{n-i}$ which embeds into G in the standard way. We denote the unipotent radical of P_i^- by N_i^-. Then we introduce

$$U_n^i = \sum_{A \in N_i^-(\mathbf{F}_p)} t_i^{-1} \lceil A \rceil \text{ for each } 1 \leq i \leq n-1. \tag{4.4.2}$$

Note that each $A \in N_i^-$ has the form

$$\begin{pmatrix} \mathrm{Id}_i & 0_{(n-i)\times i} \\ *_{i\times(n-i)} & \mathrm{Id}_{n-i} \end{pmatrix}$$

for each $1 \leq i \leq n-1$.

For each $w \in W$ and each tuple $\underline{k} = (k_\alpha)_{\alpha \in \Phi_w^+} \in \{0, \ldots, p-1\}^{|\Phi_w^+|}$, we consider the following Jacobi sum

$$\widehat{\mathcal{S}}_{\underline{k},w} := \left(\sum_{A \in U_w(\mathbf{F}_p)} \left(\prod_{\alpha \in \Phi_w^+} \lceil A_\alpha \rceil^{k_\alpha} \right) \lceil A \rceil \right) w \in \mathbf{Z}_p[G(\mathbf{Z}_p)].$$

In particular, we consider

$$\widehat{\mathcal{S}}_w := \left(\sum_{A \in U_w(\mathbf{F}_p)} \lceil A \rceil \right) w \in \mathbf{Z}_p[G(\mathbf{Z}_p)]$$

which is a characteristic 0 lift of $S_{\underline{0},w}$.

Recall the notation $\widetilde{\pi}^\circ$ from (4.1.1).

Lemma 4.4.1. *Assume that μ_π is n-generic (Definition 4.1.1). We have the equality*

$$\widehat{\mathcal{S}}_w \bullet \widehat{\mathcal{S}}_{w'} = p^{\frac{\ell(w)+\ell(w')-\ell(ww')}{2}} \widehat{\mathcal{S}}_{ww'}$$

on $(\widetilde{\pi}^\circ)^{I(1)}$ for all $w, w' \in W$.

Proof. One can quickly reduce the general case to the following two elementary equalities on $(\widetilde{\pi}^\circ)^{I(1)}$:

$$\widehat{\mathcal{S}}_w \bullet \widehat{\mathcal{S}}_{w'} = \widehat{\mathcal{S}}_{ww'} \text{ if } \ell(ww') = \ell(w) + \ell(w') \tag{4.4.3}$$

and

$$\widehat{\mathcal{S}}_{s_r} \bullet \widehat{\mathcal{S}}_{s_r} = p \text{ for all } 1 \leq r \leq n-1. \tag{4.4.4}$$

The equality (4.4.3) follows directly from the definition of the Jacobi sum operators. The equality (4.4.4) follows from a simple Bruhat decomposition. In fact, we have for each $t \neq 0$

$$s_r u_{\alpha_r}(t) s_r = u_{\alpha_r}(t^{-1}) s_r \mathrm{diag}(1, \cdots, 1, t, -t^{-1}, 1, \cdots, 1) u_{\alpha_r}(t^{-1})$$

where the diagonal matrix has t at (r,r)-entry and $-t^{-1}$ at $(r+1,r+1)$-entry. Therefore for each $\widehat{v}\in(\widetilde{\pi}^\circ)^{I(1)}$ there exists an integer $n\leq\ell\leq p-n$ such that

$$\mathrm{diag}(1,\cdots,1,t,-t^{-1},1,\cdots,1)\widehat{v}=\pm\lceil t\rceil^\ell\widehat{v}$$

and thus

$$\begin{aligned}\widehat{\mathcal{S}}_{s_r}\bullet\widehat{\mathcal{S}}_{s_r}\widehat{v}&=\left(\sum_{t'\in\mathbf{F}_p}u_{\alpha_r}(t')\right)\left(\widehat{v}\pm\sum_{t\in\mathbf{F}_p'}\lceil t\rceil^\ell\lceil u_{\alpha_r}(t^{-1})\rceil s_r\widehat{v}\right)\\&=p\widehat{v}\pm\left(\sum_{t'\in\mathbf{F}_p,t\in\mathbf{F}_p}\lceil t\rceil^\ell\lceil u_{\alpha_r}(t'+t^{-1})\rceil s_r\right)\widehat{v}\\&=p\widehat{v}.\end{aligned}$$

This finishes the proof. □

Lemma 4.4.2. *We have the equality*

$$(\Xi_n)^k\bullet U_n^k=\widehat{\mathcal{S}}_{(w^*)^k}.$$

Proof. This is immediate by definition. □

We quickly recall some standard facts about Jacobi sums and Gauss sums. We fix a primitive p-th root of unity $\xi\in E$ and set $\epsilon:=\xi-1$. For each pair of integers (a,b) with $0\leq a,b\leq p-1$, we set

$$J(a,b):=\sum_{\lambda\in\mathbf{F}_p}\lceil\lambda\rceil^a\lceil 1-\lambda\rceil^b.$$

We also set

$$G(a):=\sum_{\lambda\in\mathbf{F}_p}\lceil\lambda\rceil^a\xi^\lambda$$

for each integers a with $0\leq a\leq p-1$. For example, we have $G(p-1)=-1$.

It is known by section 1.1, $GS3$ of [Lang] that if $a+b\not\equiv 0\mod(p-1)$, we have

$$J(a,b)=\frac{G(a)G(b)}{G(a+b)}.$$

It is also obvious from the definition that if $a,b,a+b\not\equiv 0\mod(p-1)$ then

$$J(b,a)=J(a,b)=(-1)^bJ(b,[-a-b]_1)=(-1)^aJ(a,[-a-b]_1).$$

By Stickelberger's theorem ([Lang], Section 1.2, Theorem 2.1), we know that

$$\mathrm{ord}_p(G(a))=1-\frac{a}{p-1}\quad\text{and}\quad\frac{G(a)}{\epsilon^{p-1-a}}\equiv a!\pmod{p}.\tag{4.4.5}$$

Let $r\in\mathbf{Z}$ with $1\leq r\leq n-1$ and $w\in W$. Given the data $\mu_\pi=(d_1,d_2,\cdots,d_n)$ and tuple $\underline{k}\in\{0,\dots,p-1\}^{|\Phi_w^+|}$, we define a tuple

$$\underline{k}'\in\begin{cases}\{0,\dots,p-1\}^{|\Phi_w^+|}&\text{if }\ell(ws_r)<\ell(w);\\\{0,\dots,p-1\}^{|\Phi_{ws_r}^+|}&\text{if }\ell(ws_r)>\ell(w)\end{cases}$$

by

$$k'_\alpha=\begin{cases}k_\alpha&\text{if }\alpha\in\Phi_w^+;\\0&\text{if }\alpha=w\alpha_r\end{cases}$$

in the first case and

$$k'_\alpha=\begin{cases}[k_{w\alpha_r}-d_r+d_{r+1}]_1&\text{if }\alpha=w\alpha_r;\\k_\alpha&\text{if }\alpha\in\Phi_w^+\text{ and }\alpha\neq w\alpha_r\end{cases}$$

in the second case.

Proposition 4.4.3. *Assume that $\mu_\pi = (d_1, d_2, \cdots, d_n)$ is n-generic and that*

$$k_\alpha = 0 \text{ for all } \alpha \in \Phi_w^+ \text{ with } w\alpha_r < \alpha.$$

Assume further that if $\ell(ws_r) < \ell(w)$ then $k_{w\alpha_r} \notin \{0,\, p-1,\, [d_r - d_{r+1}]_1\}$.

Then for each $1 \le r \le n-1$ we have

$$\widehat{\mathcal{S}}_{\underline{k},w} \bullet \widehat{\mathcal{S}}_{s_r} = \begin{cases} \widehat{\mathcal{S}}_{\underline{k}',ws_r} & \text{if } \ell(ws_r) > \ell(w); \\ (-1)^{d_{r+1}} J(k_{w\alpha_r}, [d_{r+1} - d_r]_1)\widehat{\mathcal{S}}_{\underline{k}',w} & \text{if } \ell(ws_r) < \ell(w) \end{cases}$$

on $(\widetilde{\pi}^\circ)^{I(1),\widetilde{\mu}_\pi}$.

Proof. By definition we have

$$\widehat{\mathcal{S}}_{\underline{k},w} \bullet \widehat{\mathcal{S}}_{s_r} = \sum_{A \in U(\mathbf{F}_p), t \in \mathbf{F}_p} \left(\prod_{\alpha \in \Phi_w^+} \lceil A_\alpha \rceil^{k_\alpha} \right) \lceil A \rceil w \lceil u_{\alpha_r}(t) \rceil s_r.$$

We divide it into two cases:

(i) $\ell(ws_r) > \ell(w)$;
(ii) $\ell(ws_r) < \ell(w)$.

In case (i), we have the Bruhat decomposition

$$Awu_{\alpha_r}(t)s_r = Au_{w\alpha_r}(t)ws_r$$

and thus

$$\widehat{\mathcal{S}}_{\underline{k},w} \bullet \widehat{\mathcal{S}}_{s_r} = \widehat{\mathcal{S}}_{\underline{k}',ws_r}.$$

In case (ii), we have the Bruhat decompositions: if $t = 0$

$$Awu_{\alpha_r}(0)s_r = A(ws_r) = A'' ws_r u_{\alpha_r}(A_{w\alpha_r})$$

where A'' is the unipotent matrix that has the same entries as A except a zero at $w\alpha_r$-entry; if $t \neq 0$

$$Awu_{\alpha_r}(t)s_r = Au_{w\alpha_r}(t^{-1})w\mathrm{diag}(1, \cdots, t, -t^{-1}, \cdots, 1)u_{\alpha_r}(t^{-1}).$$

We fix a vector $\widehat{v}_\pi \in (\widetilde{\pi}^\circ)^{I(1),\widetilde{\mu}_\pi}$ whose mod p reduction is non-zero. Therefore, we have

$$\begin{aligned} \widehat{\mathcal{S}}_{\underline{k},w} \bullet \widehat{\mathcal{S}}_{s_r}\widehat{v}_\pi = (-1)^{d_{r+1}} &\sum_{A \in U_w(\mathbf{F}_p), t \in \mathbf{F}_p^\times} \left(\prod_{\alpha \in \Phi_w^+} \lceil A_\alpha \rceil^{k_\alpha} \right) \lceil t \rceil^{d_r - d_{r+1}} \lceil A \rceil u_{w\alpha_r}(t^{-1}) w\widehat{v}_\pi \\ &+ \sum_{A \in U_w(\mathbf{F}_p)} \left(\prod_{\alpha \in \Phi_w^+} \lceil A_\alpha \rceil^{k_\alpha} \right) \lceil A \rceil ws_r\widehat{v}_\pi. \end{aligned}$$

The summation $\sum_{A \in U_w(\mathbf{F}_p)} \left(\prod_{\alpha \in \Phi_w^+} \lceil A_\alpha \rceil^{k_\alpha} \right) Aws_r\widehat{v}_\pi$ can be rewritten as

$$\sum_{A'' \in U_{ws_r}(\mathbf{F}_p)} \left(\prod_{\alpha \in \Phi_{ws_r}^+} \lceil A_\alpha \rceil^{k_\alpha} \right) \left(\sum_{A_{w\alpha_r} \in \mathbf{F}_p} \lceil A_{w\alpha_r} \rceil^{k_{w\alpha_r}} \right) A'' ws_r \widehat{v}_\pi$$

which is 0 as we assume $0 < k_{w\alpha_r} < p-1$. Hence, we have

$$\widehat{\mathcal{S}}_{\underline{k},w_0} \bullet \widehat{\mathcal{S}}_{s_r}\widehat{v}_\pi = (-1)^{d_{r+1}} \sum_{A \in U_w(\mathbf{F}_p), t \in \mathbf{F}_p^\times} \left(\prod_{\alpha \in \Phi_w^+} \lceil A_\alpha \rceil^{k_\alpha} \right) \lceil t \rceil^{d_r - d_{r+1}} \lceil Au_{w\alpha_r}(t^{-1}) \rceil w\widehat{v}_\pi.$$

On the other hand, after setting $A' = Au_{w\alpha_r}(t^{-1})$ we have

$$\begin{aligned}(4.4.6)\qquad &\sum_{A\in U_w(\mathbf{F}_p),t\in\mathbf{F}_p^\times}\left(\prod_{\alpha\in\Phi_w^+}\lceil A_\alpha\rceil^{k_\alpha}\right)\lceil t\rceil^{d_r-d_{r+1}}\lceil Au_{w\alpha_r}(t^{-1})\rceil w\widehat{v}_\pi\\ &= \sum_{A'\in U_w(\mathbf{F}_p),t\in\mathbf{F}_p^\times}\left(\prod_{\alpha\in\Phi_{ws_r}^+}\lceil A_\alpha\rceil^{k_\alpha}\right)\lceil (A'_{w\alpha_r}-t^{-1})\rceil^{k_{w\alpha_r}}\lceil t\rceil^{d_r-d_{r+1}}\lceil A'\rceil w\widehat{v}_\pi\end{aligned}$$

since $k_\alpha = 0$ for all $w\alpha_r < \alpha$.

One can easily check that if $A'_{w\alpha_r} = 0$ then

$$\sum_{t\in\mathbf{F}_p^\times}\lceil (A'_{w\alpha_r}-t^{-1})\rceil^{k_{w\alpha_r}}\lceil t\rceil^{d_r-d_{r+1}} = (-1)^{k_{w\alpha_r}}\sum_{t\in\mathbf{F}_p}\lceil t\rceil^{d_r-d_{r+1}-k_{w\alpha_r}} = 0,$$

and if $A'_{w\alpha_r}\neq 0$ then

$$\begin{aligned}&\sum_{t\in\mathbf{F}_p^\times}\lceil (A'_{w\alpha_r}-t^{-1})\rceil^{k_{w\alpha_r}}\lceil t\rceil^{d_r-d_{r+1}}\\ &\quad= \lceil A'_{w\alpha_r}\rceil^{k_{w\alpha_r}-d_r+d_{r+1}}\left(\sum_{t\in\mathbf{F}_p}\lceil (1-(A'_{w\alpha_r}t)^{-1})\rceil^{k_{w\alpha_r}}\lceil (A'_{w\alpha_r}t)^{-1})^{d_{r+1}-d_r}\rceil\right)\\ &\quad= J(k_{w\alpha_r},[d_{r+1}-d_r]_1)\lceil A'_{w\alpha_r}\rceil^{[k_{w\alpha_r}-d_{r+1}+d_r]_1}.\end{aligned}$$

Combining these computations with (4.4.6) finishes the proof. □

Remark 4.4.4. *Proposition 4.4.3 is the technical heart of this section. It roughly says that $[U(\mathbf{F}_p),U(\mathbf{F}_p)]$-invariant vectors behave well under intertwining of principal series, which is essentially why the identities in Theorem 4.4.9 and Proposition 5.5.1 exist. On the other hand, it is crucial that the vector $\widehat{v}_\pi$ is invariant under $\lceil u_{\alpha_r}(t)\rceil$ for $t\in\mathbf{F}_p$.*

From now on we fix an n-tuple of integers $(a_{n-1},\cdots,a_0)$ which is assumed to be n-generic in the lowest alcove (cf. Definition 4.1.1). We let

$$\begin{cases}\mu^* := (a_{n-1}-n+2,a_{n-2},a_{n-3},\cdots,a_2,a_1,a_0+n-2);\\ \mu_1 := (a_1,a_2,\cdots,a_{n-3},a_{n-2},a_{n-1},a_0);\\ \mu'_1 := (a_{n-1},a_0,a_1,a_2,\cdots,a_{n-3},a_{n-2});\\ \mu_0 := (a_{n-1},a_1,a_2\cdots,a_{n-3},a_{n-2},a_0)\end{cases}$$

and

$$(4.4.7)\qquad \begin{cases}\pi_0 := \mathrm{Ind}_{B(\mathbf{F}_p)}^{G(\mathbf{F}_p)}\mu_0;\\ \widetilde{\pi}_0^\circ := \mathrm{Ind}_{B(\mathbf{F}_p)}^{G(\mathbf{F}_p)}\widetilde{\mu}_0\end{cases}$$

where $\widetilde{\mu}_0$ is the Teichmüller lift of μ_0. Then we recursively define sequences of elements in the Weyl group W by

$$\begin{cases}w_1 = 1,\ w_m = s_{n-m}w_{m-1};\\ w'_1 = 1,\ w'_m = s_m w'_{m-1}\end{cases}$$

for all $2\le m\le n-1$, where s_m are the reflection of the simple roots α_m. We define the sequences of characters of $T(\mathbf{F}_p)$

$$\mu_m := \mu_1^{w_m}\ \text{ and }\ \mu'_m := (\mu'_1)^{w'_m}$$

for all $1\le m\le n-1$. In particular, we have $\mu_{n-1} = \mu_0 = \mu'_{n-1}$.

We let $\underline{k}^1 = (k^1_{i,j})$, $\underline{k}^{1,\prime} = (k^{1,\prime}_{i,j})$ and $\underline{k}^0 = (k^0_{i,j})$, where

$$\begin{cases} k^1_{i,i+1} &= [a_0 - a_{n-i}]_1 + n - 2; \\ k^{1,\prime}_{i,i+1} &= [a_{n-i-1} - a_{n-1}]_1 + n - 2; \\ k^0_{i,i+1} &= [a_0 - a_{n-1}]_1 + n - 2 \end{cases} \tag{4.4.8}$$

for $1 \leq i \leq n-1$ and $k^1_{i,j} = k^{1,\prime}_{i,j} = k^0_{i,j} = 0$ otherwise.

We also define several families of Jacobi sums:

$$\widehat{\mathcal{S}}_{\underline{k}^m, w_0} \quad \text{and} \quad \widehat{\mathcal{S}}_{\underline{k}^{m,\prime}, w_0}$$

for all integers m with $1 \leq m \leq n-1$, where $\underline{k}^m = (k^m_{i,j})$ satisfies

$$k^m_{i,j} = \begin{cases} n - 2 + [a_0 - a_{n-1}]_1 & \text{if } 1 \leq i = j - 1 \leq m; \\ n - 2 + [a_0 - a_{n-i}]_1 & \text{if } m + 1 \leq i = j - 1 \leq n - 1; \\ 0 & \text{otherwise} \end{cases}$$

and $\underline{k}^{m,\prime} = (k^{m,\prime}_{i,j})$ satisfies

$$k^{m,\prime}_{i,j} = \begin{cases} n - 2 + [a_{n-i-1} - a_{n-1}]_1 & \text{if } 1 \leq i = j - 1 \leq n - m - 1; \\ n - 2 + [a_0 - a_{n-1}]_1 & \text{if } n - m \leq i = j - 1 \leq n - 1; \\ 0 & \text{otherwise.} \end{cases}$$

We keep the notation in (4.4.7) and recall that $\underline{k}^0$ is defined in (4.4.8) and satisfies

$$\underline{k}^0 = \underline{k}^{n-1} = \underline{k}^{n-1,\prime}. \tag{4.4.9}$$

We also define

$$\begin{cases} \kappa^{(1)}_n := (-1)^{\sum_{m=1}^{n-2} a_m} \prod_{m=1}^{n-2} J(n - 2 + [a_0 - a_{n-m-1}]_1, [a_{n-m-1} - a_{n-1}]_1); \\ \kappa^{(2)}_n := (-1)^{(n-2)a_0} \prod_{m=1}^{n-2} J(n - 2 + [a_m - a_{n-1}]_1, [a_0 - a_m]_1). \end{cases} \tag{4.4.10}$$

Proposition 4.4.5. *Assume that $(a_{n-1}, \cdots, a_0)$ is n-generic.*

Then we have

$$\widehat{\mathcal{S}}_{\underline{k}^1, w_0} \bullet \widehat{\mathcal{S}}_{w_{n-1}^{-1}} = \kappa^{(1)}_n \widehat{\mathcal{S}}_{\underline{k}^0, w_0} \quad \text{and} \quad \widehat{\mathcal{S}}_{\underline{k}^{1,\prime}, w_0} \bullet \widehat{\mathcal{S}}_{(w'_{n-1})^{-1}} = \kappa^{(2)}_n \widehat{\mathcal{S}}_{\underline{k}^0, w_0}$$

on the 1-dimensional space $(\widetilde{\pi}^\circ_0)^{I(1), \widetilde{\mu}_0}$.

Proof. By the case $w = w_0$ of Proposition 4.4.3 and the fact that

$$k^m_{m+1,m+2} = n - 2 + [a_0 - a_{n-m-1}]_1 \text{ and } k^{m,\prime}_{n-m-1,n-m} = n - 2 + [a_m - a_{n-1}]_1$$

we have

$$\widehat{\mathcal{S}}_{\underline{k}^m, w_0} \bullet \widehat{\mathcal{S}}_{s_{n-m-1}} = (-1)^{a_{n-m-1}} J(n - 2 + [a_0 - a_{n-m-1}]_1, [a_{n-m-1} - a_{n-1}]_1) \widehat{\mathcal{S}}_{\underline{k}^{m+1}, w_0}$$

and

$$\widehat{\mathcal{S}}_{\underline{k}^{m,\prime}, w_0} \bullet \widehat{\mathcal{S}}_{s_{m+1}} = (-1)^{a_0} J(n - 2 + [a_m - a_{n-1}]_1, [a_0 - a_m]_1) \widehat{\mathcal{S}}_{\underline{k}^{m+1,\prime}, w_0}$$

on the 1-dimensional space $(\pi^\circ_0)^{I(1), \widetilde{\mu}_0}$ for all $1 \leq m \leq n-2$. Using the equality (4.4.9) together with Lemma 4.4.1 one can write

$$\widehat{\mathcal{S}}_{w_{n-1}^{-1}} = \widehat{\mathcal{S}}_{s_{n-2}} \bullet \cdots \bullet \widehat{\mathcal{S}}_{s_1}, \quad \text{and } \widehat{\mathcal{S}}_{(w'_{n-1})^{-1}} = \widehat{\mathcal{S}}_{s_2} \bullet \cdots \bullet \widehat{\mathcal{S}}_{s_{n-1}}.$$

Hence, we finish the proof by induction on m. $\square$

Lemma 4.4.6. *We have*

$$\begin{cases} \kappa^{(1)}_n \equiv (-1)^{\sum_{m=1}^{n-2} a_m} \left(\prod_{m=1}^{n-2} \frac{(n-2+[a_0-a_{n-m-1}]_1)!([a_{n-m-1}-a_{n-1}]_1)!}{(n-2+[a_0-a_{n-1}]_1)!} \right) \pmod p; \\ \kappa^{(2)}_n \equiv (-1)^{(n-2)a_0} \left(\prod_{m=1}^{n-2} \frac{(n-2+[a_m-a_{n-1}]_1)!([a_0-a_m]_1)!}{(n-2+[a_0-a_{n-1}]_1)!} \right) \pmod p. \end{cases}$$

In particular,

$$\mathrm{ord}_p(\kappa^{(1)}_n) = \mathrm{ord}_p(\kappa^{(2)}_n) = 0.$$

Proof. This follows directly from (4.4.5), the definition of $\kappa_n^{(1)}$, $\kappa_n^{(2)}$, and the fact that $(a_{n-1},\cdots,a_0)$ is n-generic. $\square$

Corollary 4.4.7. *Assume that $(a_{n-1},\cdots,a_0)$ is n-generic.*

Then we have

$$\widehat{\mathcal{S}}_{\underline{k}^1,w_0}=p^{n-2}\kappa_n^{(1)}\widehat{\mathcal{S}}_{\underline{k}^0,w_0}\bullet\widehat{\mathcal{S}}_{w_{n-1}}\quad and\quad \widehat{\mathcal{S}}_{\underline{k}^{1,\prime},w_0}=p^{n-2}\kappa_n^{(2)}\widehat{\mathcal{S}}_{\underline{k}^0,w_0}\bullet\widehat{\mathcal{S}}_{w'_{n-1}}$$

on the 1*-dimensional space* $(\widetilde{\pi}_0^\circ)^{I(1),\widetilde{\mu}_0}$.

Proof. It follows from Lemma 4.4.1 that

$$\widehat{\mathcal{S}}_{w_{n-1}^{-1}}\bullet\widehat{\mathcal{S}}_{w_{n-1}}=p^{n-2}=\widehat{\mathcal{S}}_{(w'_{n-1})^{-1}}\bullet\widehat{\mathcal{S}}_{w'_{n-1}},$$

so that this follows from Proposition 4.4.5 and Lemma 4.4.1. $\square$

We define two important Jacobi sum operators (in characteristic p) $\mathcal{S}_n$ and $\mathcal{S}'_n$ to be

$$\mathcal{S}_n:=S_{\underline{k}^1,w_0}\qquad\text{and}\qquad\mathcal{S}'_n:=S_{\underline{k}^{1,\prime},w_0}.\tag{4.4.11}$$

Corollary 4.4.8. *We have the equality*

$$\mathcal{S}_n\left(\pi_0^{U(\mathbf{F}_p),\mu_1}\right)=\mathcal{S}'_n\left(\pi_0^{U(\mathbf{F}_p),\mu'_1}\right)=S_{\underline{k}^0,w_0}\left(\pi_0^{U(\mathbf{F}_p),\mu_0}\right).$$

Proof. It follows from Lemma 4.1.6 that

$$S_{\underline{0},w_{n-1}^{-1}}\left(\pi_0^{U(\mathbf{F}_p),\mu_0}\right)=\pi_0^{U(\mathbf{F}_p),\mu_1}\quad\text{and}\quad S_{\underline{0},(w'_{n-1})^{-1}}\left(\pi_0^{U(\mathbf{F}_p),\mu_0}\right)=\pi_0^{U(\mathbf{F}_p),\mu'_1}.$$

Hence we finish the proof by the reduction modulo p of identities in Proposition 4.4.5 and the fact that the reduction modulo p of $\widehat{\mathcal{S}}_w$ is $S_{\underline{0},w}$ for each $w\in W$. $\square$

As in (4.4.11), we use the shortened notation

$$\widehat{\mathcal{S}}_n:=\widehat{\mathcal{S}}_{\underline{k}^1,w_0}\quad\text{and}\quad\widehat{\mathcal{S}}'_n:=\widehat{\mathcal{S}}_{\underline{k}^{1,\prime},w_0}$$

and note that $\mathcal{S}_n$ (resp. $\mathcal{S}'_n$) is the reduction modulo p of $\widehat{\mathcal{S}}_n$ (resp. $\widehat{\mathcal{S}}'_n$).

To state the main result in this section, we also define

$$\mathcal{P}_n:=\prod_{k=1}^{n-2}\prod_{j=1}^{n-2}\frac{[a_k-a_{n-1}]_1+j}{[a_0-a_k]_1+j}=\prod_{k=1}^{n-2}\prod_{j=0}^{n-3}\frac{a_k-a_{n-1}+j}{a_0-a_k+j}\in\mathbf{Z}_p^\times,\tag{4.4.12}$$

$$\varepsilon^*:=\prod_{m=1}^{n-2}(-1)^{a_0-a_m},\tag{4.4.13}$$

and

$$\kappa_n:=\kappa_n^{(2)}(\kappa_n^{(1)})^{-1}.\tag{4.4.14}$$

The main result of this section is the following theorem, which is a generalization of the case $n=3$ in [HLM], (3.2.1).

Theorem 4.4.9. *Let*

$$\Pi_n:=\mathrm{Ind}_{B(\mathbf{Q}_p)}^{G(\mathbf{Q}_p)}\chi$$

be a tamely ramified principal series representation where the $\chi=\chi_1\otimes\cdots\otimes\chi_n:T(\mathbf{Q}_p)\to E^\times$ is a smooth character satisfying $\chi|_{T(\mathbf{Z}_p)}=\widetilde{\mu}_1$.

On the 1*-dimensional subspace $\Pi_n^{I(1),\widetilde{\mu}_1}$ we have the identity:*

$$\widehat{\mathcal{S}}'_n\bullet(\Xi_n)^{n-2}=p^{n-2}\kappa_n\left(\prod_{k=1}^{n-2}\chi_k(p)\right)\widehat{\mathcal{S}}_n$$

for $\kappa_n \in \mathcal{O}_E^\times$ *(defined in (4.4.14)) such that*

$$\kappa_n \equiv \varepsilon^* \mathcal{P}_n(a_{n-1}, \cdots, a_0) \pmod{\varpi_E}$$

where $\varepsilon^* = \pm 1$ *is the sign function defined in (4.4.13) and* $\mathcal{P}_n$ *is the rational function defined in (4.4.12).*

The following is a direct generalization of Lemma 3.2.5 in [HLM].

Lemma 4.4.10. *We have the equality*

$$U_n^r = \left(\prod_{k=1}^{r} \chi_k(p)\right)^{-1}$$

on the 1*-dimensional space* $\Pi_n^{I(1),\widetilde{\mu}_1}$ *for each* $1 \leq r \leq n-1$.

Proof. The proof of this lemma is an immediate calculation which is parallel to that of [HLM], Lemma 3.2.5. □

Proof of Theorem 4.4.9. Notice that

$$w'_{n-1}(w^*)^{n-2} = w_{n-1} \quad \text{and} \quad \ell(w'_{n-1}) + \ell((w^*)^{n-2}) = 3(n-2) = \ell(w_{n-1}) + 2(n-2),$$

so that by Lemma 4.4.1 we have

$$\widehat{\mathcal{S}}_{w'_{n-1}} \bullet \widehat{\mathcal{S}}_{(w^*)^{n-2}} = p^{n-2} \widehat{\mathcal{S}}_{w_{n-1}}. \tag{4.4.15}$$

By composing $\widehat{\mathcal{S}}_{\underline{k}^0, w_0}$ on both sides of (4.4.15), we deduce from Proposition 4.4.5 that

$$(\kappa_n^{(2)})^{-1} \widehat{\mathcal{S}}'_n \bullet \widehat{\mathcal{S}}_{(w^*)^{n-2}} = p^{n-2} (\kappa_n^{(1)})^{-1} \widehat{\mathcal{S}}_n$$

and thus

$$\widehat{\mathcal{S}}'_n \bullet \widehat{\mathcal{S}}_{(w^*)^{n-2}} = p^{n-2} \kappa_n \widehat{\mathcal{S}}_n$$

on the 1-dimensional subspace $\Pi_n^{I(1),\widetilde{\mu}_1}$. Now Lemma 4.4.2 together with Lemma 4.4.10 gives rise to the identity in the statement of this theorem.

Finally, one can readily check from Lemma 4.4.6 that

$$\begin{aligned}
\kappa_n &= \kappa_n^{(2)} (\kappa_n^{(1)})^{-1} \\
&\equiv (-1)^{\sum_{m=1}^{n-2} a_0 - a_m} \prod_{m=1}^{n-2} \frac{(n-2+[a_0 - a_{n-m-1}]_1)!([a_{n-m-1} - a_{n-1}]_1)!}{(n-2+[a_m - a_{n-1}]_1)!([a_0 - a_m]_1)!} \\
&\equiv (-1)^{\sum_{m=1}^{n-2} a_0 - a_m} \prod_{m=1}^{n-2} \prod_{\ell=1}^{n-2} \frac{\ell + [a_0 - a_m]_1}{\ell + [a_m - a_{n-1}]_1} \\
&\equiv \varepsilon^* \mathcal{P}_n \pmod{\varpi_E}.
\end{aligned}$$

Note that $\mathrm{ord}_p(\kappa_n) = 0$. This completes the proof. □

4.5. **Special vectors in a dual Weyl module.** We fix a tuple of integers $\underline{h} := (h_1, \cdots, h_s)$ for some $1 \leq s \leq n-1$ such that

$$1 \leq h_r \leq n-1 \text{ for all } 1 \leq r \leq s$$

and

$$\sum_{r=1}^{s} h_r = n-1.$$

Then we can define $n-1$ positive roots $\beta_{\underline{h},i}$ for $1 \leq i \leq n-1$ as follows. Given an integer $1 \leq i \leq n-1$, there exists a unique integer $0 \leq r_0 \leq s-1$ such that

$$\sum_{r=1}^{r_0} h_r < i \leq \sum_{r=1}^{r_0+1} h_r,$$

and we use the notation

$$[i]_{\underline{h}} := \sum_{r=1}^{r_0} h_r.$$

Then we define

$$\beta_{\underline{h},i} := \sum_{k=1+[i]_{\underline{h}}}^{i} \alpha_k.$$

Note in particular that we always have

$$\beta_{\underline{h},1} = \alpha_1.$$

Then we define

$$\Phi^+_{\underline{h}} := \{\alpha \in \Phi^+ \mid \alpha \neq \beta_{\underline{h},i} \text{ for all } 1 \leq i \leq n-1\}$$

and notice that this set gives an unipotent group $U_{\underline{h}} \subsetneq U$ by setting

$$U_{\underline{h}} := \prod_{\alpha \in \Phi^+_{\underline{h}}} U_\alpha.$$

We emphasize that all $U_{\underline{h}}$ constructed here are *good* in the sense of Definition 4.1.7. In particular, if $s = n-1$ and $h_r = 1$ for $1 \leq r \leq n-1$ we recover $[U, U]$, and if $s = 1$ and $h_1 = n-1$ we recover U_1 (cf. Example 4.1.8). We define $\overline{U}_{\underline{h}}$ as the reduction of $U_{\underline{h}}$ mod p. If we mark the positive roots $\beta_{\underline{h},i}$ by a $\bullet$ on their corresponding upper-triangular entry, we get the following matrix looking like a ladder with s steps

$$\begin{pmatrix}
1 & \bullet & \cdots & \bullet & \bullet & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 \\
 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 \\
 & & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
 & & & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 \\
 & & & & 1 & \bullet & \bullet & \cdots & \bullet & 0 & 0 & \cdots & \cdots & 0 \\
 & & & & & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 \\
 & & & & & & 1 & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 \\
 & & & & & & & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
 & & & & & & & & 1 & \bullet & \bullet & \cdots & \cdots & 0 \\
 & & & & & & & & & 1 & 0 & \cdots & \cdots & 0 \\
 & & & & & & & & & & 1 & \cdots & \cdots & 0 \\
 & & & & & & & & & & & \ddots & \ddots & \vdots \\
 & & & & & & & & & & & & \ddots & \vdots \\
 & & & & & & & & & & & & & 1
\end{pmatrix}$$

Let R be a $\mathbf{F}_p$-algebra, and $A \in \overline{G}(R)$ a matrix. For $J_1, J_2 \subseteq \{1, 2, \cdots, n-1, n\}$, we write A_{J_1,J_2} for the submatrix of A consisting of the entries of A at the (i,j)-position for $i \in J_1$, $j \in J_2$. We define

$$J^i_0 := \{1, 2, \cdots, i\} \subseteq \{1, \cdots, n\}$$

for each $1 \leq i \leq n$. Given a tuple $\underline{h}$ as above, we define the subsets $J^i_{\underline{h}} \subseteq \{1, \cdots, n\}$ for $1 \leq i \leq n-1$ as

$$J^i_{\underline{h}} := \{1, 2, \cdots, i+1\} \setminus \{[i+1]_{\underline{h}} + 1\}.$$

It is easy to see that $|J^i_{\underline{h}}| = i$ for $1 \le i \le n-1$. We define

$$D_{\underline{h},i} := \det\left((w_0A)_{J^i_0, J^i_{\underline{h}}}\right)$$

for all $1 \le i \le n-1$. We also set $D_i := \det(w_0A)_{J^i_0, J^i_0}$ for $1 \le i \le n$. Hence, $D_{\underline{h},i}$ $(1 \le i \le n-1)$ and D_i $(1 \le i \le n)$ are polynomials over the entries of A.

Given a weight $\lambda \in X_+(T)$, we now introduce an explicit model for the representation $H^0(\lambda)$, and then start some explicit calculation. Consider the space of polynomials on $\overline{G}_{/\mathbf{F}_p}$, which is denoted by $\mathcal{O}(\overline{G})$. The space $\mathcal{O}(\overline{G})$ has both a left action and a right action of $\overline{B}$ induced by right translation and left translation by $\overline{B}$ on $\overline{G}$ respectively. The fraction field of $\mathcal{O}(\overline{G})$ is denoted by $\mathcal{M}(\overline{G})$.

Consider the subspace

$$\mathcal{O}(\lambda) := \{f \in \mathcal{O}(\overline{G}) \mid f \cdot b = w_0\lambda(b)f \quad \forall b \in \overline{B}\},$$

which has a natural left $\overline{G}$-action by right translation. As the right action of $\overline{T}$ on $\mathcal{O}(\overline{G})$ is semisimple (and normalizes $\overline{U}$), we have a decomposition of algebraic representations of $\overline{G}$:

$$\mathcal{O}(\overline{G})^{\overline{U}} := \{f \in \mathcal{O}(\overline{G}) \mid f \cdot u = f \quad \forall u \in \overline{U}\} = \oplus_{\lambda \in X(\overline{T})}\mathcal{O}(\lambda).$$

It follows from the definition of the dual Weyl module as an algebraic induction that we have a natural isomorphism

$$H^0(\lambda) \cong \mathcal{O}(\lambda). \tag{4.5.1}$$

Note by [Jan03], Proposition II 2.6 that $H^0(\lambda) \neq 0$ if and only if $\lambda \in X(T)_+$.

We often write the weight λ explicitly as $(d_1, d_2, \cdots, d_n)$ where $d_i \in \mathbf{Z}$ for $1 \le i \le n$. We will restrict our attention to a p-restricted and dominant λ, i.e., $d_1 \ge d_2 \ge ... \ge d_n$ and $d_{i-1} - d_i < p$ for $2 \le i \le n$. We recall from the beginning of Section 4 the notation $(\cdot)_{\lambda'}$ for a weight space with respect to the weight λ'. We define Σ to be the set of $(n-1)$-tuple of integers $\underline{m} = (m_1, ..., m_{n-1})$ satisfying

$$0 \le m_i \le d_i - d_{i+1} \text{ for } 1 \le i \le n-1.$$

For each tuple $\underline{m}$, we can define a vector

$$v^{\mathrm{alg}}_{\underline{h},\underline{m}} := D_n^{d_n} \prod_{i=1}^{n-1} D_i^{d_i - d_{i+1} - m_i}(D_{\underline{h},i})^{m_i}.$$

Proposition 4.5.1. *Let $\lambda = (d_1, d_2, \cdots, d_n) \in X_1(T)$. The set*

$$\{v^{\mathrm{alg}}_{\underline{h},\underline{m}} \mid \underline{m} \in \Sigma\} \tag{4.5.2}$$

forms a basis of $H^0(\lambda)^{\overline{U}_{\underline{h}}}$. Moreover, the weight of $v^{\mathrm{alg}}_{\underline{h},\underline{m}}$ is

$$\lambda - \left(\sum_{i=1}^{n-1} m_i \beta_{\underline{h},i}\right)$$

and thus each element in (4.5.2) has distinct weight.

Proof. We define

$${}^{\overline{U}_{\underline{h}}}\mathcal{O}(\overline{G})^{\overline{U}} := \{f \in \mathcal{O}(\overline{G}) \mid u_1 \cdot f = f \cdot u = f \quad \forall u \in \overline{U} \ \&\ \forall u_1 \in \overline{U}_{\underline{h}}\}$$

and

$${}^{\overline{U}_{\underline{h}}}\mathcal{M}(\overline{G})^{\overline{U}} := \{f \in \mathcal{M}(\overline{G}) \mid u_1 \cdot f = f \cdot u = f \quad \forall u \in \overline{U} \ \&\ \forall u_1 \in \overline{U}_{\underline{h}}\}.$$

We consider a matrix A such that its entries $A_{i,j}$ are indefinite variables. Then we can formally do Bruhat decomposition

$$A = U_A w_0 T_{A,\underline{h}} U_{A,\underline{h}}$$

such that the entries of U_A, $T_{A,\underline{h}}$, $U_{A,\underline{h}}$ are rational functions of $A_{i,j}$ satisfying

$$(U_A)_{i,j} = \begin{cases} 1 & \text{if } i=j; \\ 0 & \text{if } i>j, \end{cases}$$

$$(T_{A,\underline{h}})_{i,j} = \begin{cases} D_i(A) & \text{if } i=j; \\ D_{\underline{h},k}(A) & \text{if } (i,j)=\beta_{\underline{h},k}; \\ 0 & \text{otherwise }, \end{cases}$$

$$(U_{A,\underline{h}})_{i,j} = \begin{cases} 1 & \text{if } i=j; \\ 0 & \text{if } i>j \text{ or } (i,j)=\beta_{\underline{h},k} \text{ for some } 1\leq k\leq n-1. \end{cases}$$

For each rational function $f \in {}^{\overline{U}_{\underline{h}}}\mathcal{M}(\overline{G})^{\overline{U}}$, we notice that f only depends on $T_{A,\underline{h}}$, which means that f is rational function of D_i for $1\leq i\leq n$ and $D_{\underline{h},i}$ for $1\leq i\leq n-1$. In other word, we have

$${}^{\overline{U}_{\underline{h}}}\mathcal{M}(\overline{G})^{\overline{U}} = \mathbf{F}_p\left(D_1,\cdots,D_n,D_{\underline{h},1},\cdots,D_{\underline{h},n-1}\right) \subseteq \mathcal{M}(\overline{G}).$$

Then we define

$${}^{\overline{U}_{\underline{h}},\lambda'}\mathcal{O}(\overline{G})^{\overline{U},\lambda} := \{f\in {}^{\overline{U}_{\underline{h}}}\mathcal{O}(\overline{G})^{\overline{U}} \mid x\cdot f=\lambda'(x)f, \text{ and } f\cdot x=\lambda(x)f \quad \forall x\in\overline{T}\}$$

and

$${}^{\overline{U}_{\underline{h}},\lambda'}\mathcal{M}(\overline{G})^{\overline{U},\lambda} := \{f\in {}^{\overline{U}_{\underline{h}}}\mathcal{M}(\overline{G})^{\overline{U}} \mid x\cdot f=\lambda'(x)f, \text{ and } f\cdot x=\lambda(x)f \quad \forall x\in\overline{T}\}.$$

Note that we have and an obvious inclusion

$${}^{\overline{U}_{\underline{h}},\lambda'}\mathcal{O}(\overline{G})^{\overline{U},\lambda} \subseteq {}^{\overline{U}_{\underline{h}},\lambda'}\mathcal{M}(\overline{G})^{\overline{U},\lambda}.$$

We can also identify ${}^{\overline{U}_{\underline{h}},\lambda'}\mathcal{O}(\overline{G})^{\overline{U},\lambda}$ with $H^0(\lambda)^{\overline{U}_{\underline{h}}}_{\lambda'}$ via the isomorphism (4.5.1). By definition of D_i (resp. $D_{\underline{h},i}$) we know that they are $\overline{T}$-eigenvector with eigencharacter $\sum_{k=1}^{i}\epsilon_k$ (resp. $(\sum_{k=1}^{i+1}\epsilon_k)-\epsilon_{[i]'_{\underline{h}}}$) for $1\leq i\leq n$ (resp. for $1\leq i\leq n-1$). Therefore we observe that ${}^{\overline{U}_{\underline{h}},\lambda'}\mathcal{M}(\overline{G})^{\overline{U},\lambda}$ is one dimensional for any $\lambda,\lambda'\in X(T)$ and is spanned by

$$D_n^{d_n}\prod_{i=1}^{n-1}D_i^{d_i-d_{i+1}-m_i}(D_{\underline{h},i})^{m_i}$$

where $\lambda=(d_1,\cdots,d_n)$ and

$$\lambda'=\lambda-\left(\sum_{i=1}^{n-1}m_i\beta_{\underline{h},i}\right).$$

As $\mathcal{O}(\overline{G})$ is a UFD and $D_i, D_{\underline{h},i}$ are irreducible, we deduce that

$$D_n^{d_n}\prod_{i=1}^{n-1}D_i^{d_i-d_{i+1}-m_i}(D_{\underline{h},i})^{m_i}\in\mathcal{O}(\overline{G})$$

if and only if

$$0\leq m_i\leq d_i-d_{i+1} \text{ for all } 1\leq i\leq n-1$$

if and only if

$$H^0(\lambda)^{\overline{U}_{\underline{h}}}_{\lambda'}\neq 0$$

which finishes the proof. □

Remark 4.5.2. *The groups $\overline{U}_{\underline{h}}$ we defined have the advantage that the $\overline{U}_{\underline{h}}$-invariant subspace $H^0(\lambda)^{\overline{U}_{\underline{h}}}\subseteq H^0(\lambda)$ is a direct sum of its one dimensional weight spaces. In other word, one can easily distinguish vectors in $H^0(\lambda)^{\overline{U}_{\underline{h}}}$ using the $\overline{T}$-action. Note that the weight spaces of $H^0(\lambda)$ have very large dimensions in general.*

We consider the special case of Proposition 4.5.1 when $s=1$, $h_1=n-1$ and hence $\underline{h}=\{n-1\}$.

Corollary 4.5.3. *Let $\lambda = (d_1, d_2, \cdots, d_n) \in X_1(T)$. For $\lambda' \in X(T)$, we have*

$$\dim_{\mathbf{F}_p} H^0(\lambda)^{\overline{U}_1}_{\lambda'} \leq 1.$$

Moreover, the set of λ' such that the space above is nontrivial is described explicitly as follows: consider the set $\Sigma_{\{n-1\}}$ of $(n-1)$-tuple of integers $\underline{m} = (m_1, ..., m_{n-1})$ satisfying $m_i \leq d_i - d_{i+1}$ for $1 \leq i \leq n-1$, and

$$v^{\mathrm{alg}}_{\{n-1\},\underline{m}} = D_n^{d_n} \prod_{i=1}^{n-1} D_i^{d_i - d_{i+1} - m_i} (D_{\{n-1\},i})^{m_i}.$$

Then the set

$$\{v^{\mathrm{alg}}_{\{n-1\},\underline{m}} \mid \underline{m} \in \Sigma_{\{n-1\}}\}$$

forms a basis of the space $H^0(\lambda)^{\overline{U}_1}$, and the weight of the vector $v^{\mathrm{alg}}_{\{n-1\},\underline{m}}$ is

$$(d_1 - \sum_{i=1}^{n-1} m_i, d_2 + m_1, ..., d_{n-1} + m_{n-2}, d_n + m_{n-1}).$$

Remark 4.5.4. *Corollary 4.5.3 essentially describes the decomposition of an irreducible algebraic representation of GL_n after restricting to a maximal Levi subgroup which is isomorphic to $\mathrm{GL}_1 \times \mathrm{GL}_{n-1}$. This classical result is crucial in the proof of Theorem 4.7.17.*

4.6. **Some technical formula.** In this section, we prove a technical formula that will be used in Section 4.7. The main result of this section is Proposition 4.6.5.

Throughout this section, we assume that $(a_{n-1}, \cdots, a_0)$ is n-generic in the lowest alcove (cf. Definition 4.1.1). We need to do some elementary calculation of Jacobi sums. For this purpose we need to define the following group operators for $2 \leq r \leq n-1$:

$$X_r^+ := \sum_{t \in \mathbf{F}_p} t^{p-2} u_{\sum_{i=r}^{n-1} \alpha_i}(t) \in \mathbf{F}_p[G(\mathbf{F}_p)],$$

and similarly

$$X_r^- := \sum_{t \in \mathbf{F}_p} t^{p-2} w_0 u_{\sum_{i=r}^{n-1} \alpha_i}(t) w_0 \in \mathbf{F}_p[G(\mathbf{F}_p)].$$

We notice that by definition we have the identification $X_r^+ = X_{\sum_{i=r}^{n-1} \alpha_i, 1}$, where $X_{\sum_{i=r}^{n-1} \alpha_i, 1}$ is defined in (4.1.2).

Lemma 4.6.1. *For a tuple of integers $\underline{k} = (k_{i,j}) \in \{0, 1, \cdots, p-1\}^{|\Phi^+_{w_0}|}$, we have*

$$X_r^+ \bullet S_{\underline{k}, w_0} = k_{r,n} S_{\underline{k}^{r,-}, w_0}$$

where $\underline{k}^{r,-} = (k^{r,-}_{i,j})$ satisfies $k^{r,-}_{r,n} = k_{r,n} - 1$, and $k^{r,-}_{i,j} = k_{i,j}$ if $(i,j) \neq (r,n)$.

Proof. This is just a special case of Lemma 4.1.2 when $\alpha_0 = \sum_{i=r}^{n-1} \alpha_i$ and $m = 1$. □

For the following lemma, we set

$$\mathbf{I} := \{(i_1, i_2, \cdots, i_s) \mid 1 \leq i_1 < i_2 < \cdots < i_s = n \text{ for some } 1 \leq s \leq n\}$$

to lighten the notation.

Lemma 4.6.2. *Let $X = (X_{i,j})_{1 \leq i,j \leq n}$ be a matrix satisfying*

$$X_{i,j} = 0 \text{ if } 1 \leq j < i \leq n-1.$$

Then the determinant of X is

$$\det(X) = \sum_{(i_1, \cdots, i_s) \in \mathbf{I}} (-1)^{s-1} X_{n,i_1} \left(\prod_{j \neq i_k,\, 1 \leq k \leq s} X_{j,j} \right) \left(\prod_{k=1}^{s-1} X_{i_k, i_{k+1}} \right). \tag{4.6.1}$$

Proof. By definition of the determinant we know that

$$\det(X) = \sum_{w\in W} (-1)^{\ell(w)} \prod_{k=1}^{n} X_{k,w(k)}.$$

From the assumption on X, we know that each w that appears in the sum satisfies

$$w(k) < k \tag{4.6.2}$$

for all $2 \leq k \leq n-1$.

Assume that w has the decomposition into disjoint cycles

$$w = (i_1^1, i_2^1, \cdots, i_{n_1}^1) \cdots (i_1^m, i_2^m, \cdots, i_{n_m}^m)$$

where m is the number of disjoint cycles and $n_k \geq 2$ is the length for the k-th cycle appearing in the decomposition.

We observe that the largest integer in $\{i_j^k \mid 1 \leq j \leq n_k\}$ must be n for each $1 \leq k \leq m$ by condition (4.6.2). Therefore we must have $m = 1$ and we can assume without loss of generality that $i_{n_1}^1 = n$. It follows from the condition (4.6.2) that

$$i_j^1 < i_{j+1}^1$$

for all $1 \leq j \leq n_1 - 1$. Hence we can set

$$s := n_1, \quad i_1 := i_1^1, \cdots, i_s := i_{n_1}^1.$$

We observe that $\ell(w) = s-1$ and the formula (4.6.1) follows. □

Recall from the beginning of Section 4.6 that we use the notation A_{J_1,J_2} for the submatrix of A consisting of the entries at the (i,j)-position with $i \in J_1, j \in J_2$, where J_1, J_2 are two subsets of $\{1, 2, \cdots, n\}$ with the same cardinality. For a pair of integers (m, r) with $1 \leq m \leq r-1 \leq n-2$, we let

$$J_0^{m,r} := \{1, 2, \cdots, r, n-m+1\}.$$

For a matrix $A \in U(\mathbf{F}_p)$, an element $t \in \mathbf{F}_p$, and a triple of integers (m, r, ℓ) satisfying $1 \leq m \leq r-1 \leq n-2$ and $1 \leq \ell \leq n-1$, we define some polynomials as follows:

$$\begin{cases} D_{m,r}(A,t) := \det\left(u_{\sum_{i=r}^{n-1}\alpha_i}(t) w_0 A w_0\right)_{J_0^{m,r}, J_0^{n-r+1}} & \text{when } 1 \leq m \leq r-1; \\ D_r^{(\ell)}(A,t) := \det\left(u_{\sum_{i=r}^{n-1}\alpha_i}(t) w_0 A w_0\right)_{J_0^{\ell}, J_0^{\ell}} & \text{when } 1 \leq \ell \leq n-r. \end{cases} \tag{4.6.3}$$

We define the following subsets of $\mathbf{I}$: for each $1 \leq \ell \leq n-1$

$$\mathbf{I}_\ell := \{(i_1, i_2, \cdots, i_s) \in \mathbf{I} \mid n-\ell+1 \leq i_1 < i_2 < \cdots < i_s = n \text{ for some } 1 \leq s \leq \ell\}.$$

Note that we have natural inclusions

$$\mathbf{I}_\ell \subseteq \mathbf{I}_{\ell'} \subseteq \mathbf{I}$$

if $1 \leq \ell \leq \ell' \leq n-1$. In particular, $\mathbf{I}_1$ has a unique element (n). Similarly, for each $1 \leq \ell' \leq n-1$ we define

$$\mathbf{I}^{\ell'} := \{(i_1, i_2, \cdots, i_s) \mid 1 \leq i_1 < i_2 < \cdots < i_{s-1} \leq n-\ell' < i_s = n \text{ for some } 1 \leq s \leq \ell'\},$$

and we set

$$\mathbf{I}_\ell^{\ell'} := \mathbf{I}_\ell \cap \mathbf{I}^{\ell'}$$

for all $1 \leq \ell' \leq \ell - 1 \leq n-2$. We often write $\underline{i} = (i_1, \cdots, i_s)$ for an arbitrary element of $\mathbf{I}$, and define the sign of $\underline{i}$ by

$$\varepsilon(\underline{i}) := (-1)^s.$$

We emphasize that all the matrices $\left(w_0 u_{\sum_{i=r}^{n-1}\alpha_i}(t) w_0 A w_0\right)_{J_1^{m,r}, J_2^{n-r+1}}$ for $1 \leq m \leq r-1$, and all the matrices $\left(w_0 u_{\sum_{i=r}^{n-1}\alpha_i}(t) w_0 A w_0\right)_{J_1^{\ell}, J_2^{\ell}}$ for $1 \leq \ell \leq n-r$, after multiplying a permutation matrix, satisfy the conditions on the matrix X in Lemma 4.6.2. Hence, by Lemma 4.6.2 we notice that

$$\left\{\begin{array}{ll} D_{m,r}(A,t) = A_{m,r} + tf_{m,r}(A) & \text{when } 1 \leq m \leq r-1; \\ D_r^{(\ell)}(A,t) = 1 - tf_{r,n-\ell+1}(A) & \text{wehn } 1 \leq \ell \leq n-r \end{array}\right. \tag{4.6.4}$$

where for all $1 \leq m \leq r-1$

$$f_{m,r}(A) := \sum_{\underline{i} \in \mathbf{I}_{n-r+1}} \left(\varepsilon(\underline{i}) A_{m,i_1} \prod_{j=2}^{s} A_{i_{j-1},i_j} \right). \tag{4.6.5}$$

Let (m,r) be a tuple of integers with $1 \leq m \leq r-1 \leq n-2$. Given a tuple of integers $\underline{k} \in \{0,1,\cdots,p-1\}^{|\Phi^+_{w_0}|}$, $\underline{i} = (i_1, i_2, \cdots, i_s) \in \mathbf{I}_{n-r+1}$, and an integer r' satisfying $1 \leq r' \leq r$, we define four tuples of integers in $\{0,1,\cdots,p-1\}^{|\Phi^+_{w_0}|}$

$$\underline{k}^{\underline{i},m,r,+} = (k^{\underline{i},m,r,+}_{i,j}),\ \ \underline{k}^{\underline{i},m,r} = (k^{\underline{i},m,r}_{i,j}),\ \ \underline{k}^{\underline{i},m,r,r',+} = (k^{\underline{i},m,r,r',+}_{i,j}),\ \ \underline{k}^{\underline{i},m,r,r'} = (k^{\underline{i},m,r,r'}_{i,j})$$

as follows:

$$k^{\underline{i},m,r,+}_{i,j} := \left\{\begin{array}{ll} k_{m,i_1}+1 & \text{if } (i,j) = (m,i_1) \text{ and } i_1 > r; \\ k_{m,r} & \text{if } (i,j) = (m,r); \\ k_{i,j}+1 & \text{if } (i,j) = (i_h, i_{h+1}) \text{ for } 1 \leq h \leq s-1; \\ k_{i,j} & \text{otherwise,} \end{array}\right.$$

$$k^{\underline{i},m,r}_{i,j} := \left\{\begin{array}{ll} k^{\underline{i},m,r,+}_{i,j} - 1 & \text{if } (i,j) = (m,r) \text{ and } i_1 > r; \\ k^{\underline{i},m,r,+}_{i,j} & \text{otherwise,} \end{array}\right.$$

and

$$k^{\underline{i},m,r,r',*}_{i,j} := \left\{\begin{array}{ll} k^{\underline{i},m,r,*}_{r',n} - 1 & \text{if } (i,j) = (r',n); \\ k^{\underline{i},m,r,*}_{i,j} & \text{otherwise} \end{array}\right.$$

where $* \in \{+, \varnothing\}$. Finally, we define one more tuple of integers $\underline{k}^{r,+} = (k^{r,+}_{i,j}) \in \{0,1,\cdots,p-1\}^{|\Phi^+_{w_0}|}$ by

$$k^{r,+}_{i,j} := \left\{\begin{array}{ll} k_{r,n}+1 & \text{if } (i,j) = (r,n); \\ k_{i,j} & \text{otherwise.} \end{array}\right.$$

Remark 4.6.3. *If we use the shortened notation $\alpha_{i,j} = \sum_{k=i}^{j-1} \alpha_k$, then we clearly have the equality*

$$\alpha_{m,n} = \alpha_{m,i_1} + \sum_{1 \leq h \leq s-1} \alpha_{i_h,i_{h+1}} = \alpha_{m,r} + \alpha_{r,n} \tag{4.6.6}$$

as we always have $i_s = n$ by definition of the tuple $\underline{i}$. The equality (4.6.6) would imply by Lemma 4.1.5 that $S_{\underline{k}^{\underline{i},m,r},w_0} v_0$ and $S_{\underline{k}^{r,+},w_0} v_0$ have the same $T(\mathbf{F}_p)$-eigencharacter, which differs from the one for $S_{\underline{k},w_0} v_0$ by $\alpha_{r,n} = \epsilon_r - \epsilon_n$. Very roughly speaking, $S_{\underline{k}^{\underline{i},m,r},w_0} v_0$ and $S_{\underline{k}^{r,+},w_0} v_0$ exhaust minimal modifications of $S_{\underline{k},w_0} v_0$ that modify the corresponding $T(\mathbf{F}_p)$-eigencharacter by $\alpha_{r,n}$, if we vary m and $\underline{i}$.

Lemma 4.6.4. *Fix two integers r and m such that $1 \leq m \leq r-1 \leq n-2$, and let $\underline{k} = (k_{i,j}) \in \{0,1,\cdots,p-1\}^{|\Phi^+_{w_0}|}$. Assume that $k_{i,j} = 0$ for $r+1 \leq j \leq n-1$ and that $k_{i,r} = 0$ for $i \neq m$, and assume further that*

$$a_{n-r} - a_1 + [a_1 - a_{n-1} - \sum_{i=1}^{n-1} k_{i,n}]_1 + k_{m,r} < p.$$

Then we have

$$\begin{aligned}X_r^- \bullet S_{\underline{k},w_0}v_0 = k_{m,r}\sum_{\underline{i}\in\mathbf{I}_{n-r}}\varepsilon(\underline{i})S_{\underline{k}^{\underline{i},m,r},w_0}v_0 \\ + ([a_{n-r}-a_{n-1}-\sum_{i=1}^{n-1}k_{i,n}]_1+k_{m,r})S_{\underline{k}^{r,+},w_0}v_0 \\ -\sum_{\ell=2}^{n-r}(a_{n-r}-a_{\ell-1}+k_{m,r})\left(\sum_{\underline{i}\in\mathbf{I}_\ell\setminus\mathbf{I}_{\ell-1}}\varepsilon(\underline{i})S_{\underline{k}^{\underline{i},r,n-\ell+1,+},w_0}v_0\right).\end{aligned}$$

Proof. By the definition of X_r^-, we have

$$X_r^- \bullet S_{\underline{k},w_0}v_0 = \sum_{A\in U(\mathbf{F}_p),t\in\mathbf{F}_p}\left(t^{p-2}\left(\prod_{1\leq i<j\leq n}A_{i,j}^{k_{i,j}}\right)w_0u_{\sum_{h=r}^{n-1}\alpha_h}(t)w_0Aw_0\right)v_0.$$

For an element $w\in W$, we use E_w to denote the subset of $U(\mathbf{F}_p)\times\mathbf{F}_p$ consisting of all (A,t) such that

$$w_0u_{\sum_{h=r}^{n-1}\alpha_h}(t)w_0Aw_0\in B(\mathbf{F}_p)wB(\mathbf{F}_p).$$

We consider the standard parabolic subgroup $P\supseteq U$ of G with standard Levi subgroup isomorphic to $\mathbf{G}_m^{r-1}\times\mathrm{GL}_{n-r+1}$ which induces an embedding $\mathrm{GL}_{n-r+1}\hookrightarrow G$. We consider the longest element in the Weyl group of GL_{n-r+1} and denote its image under the embedding $\mathrm{GL}_{n-r+1}\hookrightarrow G$ by w_P. We notice that

$$w_0u_{\sum_{h=r}^{n-1}\alpha_h}(t)w_0Aw_0\in\mathrm{GL}_{n-r+1}(\mathbf{F}_p)\cdot U(\mathbf{F}_p)w_0=P(\mathbf{F}_p)w_0\subseteq\bigsqcup_{w_1\leq w_P}B(\mathbf{F}_p)w_1w_0B(\mathbf{F}_p),$$

and deduce that if $E_w\neq\varnothing$ then $ww_0\leq w_P$ and in particular $ww_0(i)=i$ for all $1\leq i\leq r-1$.

We define M_w to be

$$M_w:=\sum_{(A,t)\in E_w}\left(t^{p-2}\left(\prod_{1\leq i<j\leq n}A_{i,j}^{k_{i,j}}\right)w_0u_{\sum_{h=r}^{n-1}\alpha_h}(t)w_0Aw_0\right)v_0.$$

By the definition of E_w, we deduce that there exist $A'\in U_w(\mathbf{F}_p)$, $A''\in U(\mathbf{F}_p)$, and $T\in T(\mathbf{F}_p)$ for each given $(A,t)\in E_w$ such that

$$w_0u_{\sum_{h=r}^{n-1}\alpha_h}(t)w_0Aw_0=A'wTA''. \tag{4.6.7}$$

Here, the entries of A', T and A'' are rational functions of t and the entries of A. We can rewrite the identity (4.6.7) as

$$w_0u_{\sum_{h=r}^{n-1}\alpha_h}(-t)w_0A'w=Aw_0T^{-1}(T(A'')^{-1}T^{-1}). \tag{4.6.8}$$

Note that the right side of (4.6.8) can also be viewed as the Bruhat decomposition of the left side. In fact, if we define E'_w as the set of elements $(A',t)\in U_w(\mathbf{F}_p)\times\mathbf{F}_p$ satisfying

$$w_0u_{\sum_{h=r}^{n-1}\alpha_h}(-t)w_0A'w\in B(\mathbf{F}_p)w_0B(\mathbf{F}_p), \tag{4.6.9}$$

then (4.6.7) and (4.6.8) imply that we have a natural bijection

$$E_w\xrightarrow{\sim}E'_w,\ (A,t)\mapsto(A',t)$$

induced from isomorphism of schemes by considering $\mathbf{F}_p$-points. Therefore the entries of A, T, A'' can also be expressed as rational functions of the entries of A'.

For each $A' \in U_w(\mathbf{F}_p)$ and $w \in W$, we define

$$\begin{cases} D^w_{m,r}(A',t) := \det\left(\left(u_{\sum_{i=r}^{n-1}\alpha_i}(t)w_0A'w\right)_{J_0^{m,r},J_0^{n-r+1}}\right) & \text{when } 1 \le m \le r-1;\\ D^{w,(\ell)}_r(A',t) := \det\left(\left(u_{\sum_{i=r}^{n-1}\alpha_i}(t)w_0A'w\right)_{J_0^{\ell},J_0^{\ell}}\right) & \text{when } 1 \le \ell \le n-r. \end{cases} \tag{4.6.10}$$

Note that if $w = w_0$, then the definition in (4.6.10) specializes to (4.6.3). We notice that for a given matrix $A' \in U_w(\mathbf{F}_p)$, the inclusion (4.6.9) holds if and only if

$$D^{w,(\ell)}_r(A',-t) \neq 0 \text{ for all } 1 \le \ell \le n-r. \tag{4.6.11}$$

On the other hand, using the bijection $E_w \xrightarrow{\sim} E'_w$, we deduce that (4.6.11) holds for $(A',t) \in U_w(\mathbf{F}_p)\times\mathbf{F}_p$ if and only if there exists a unique determined pair $(A,t) \in E_w$ such that (4.6.7) (or equivalently (4.6.8)) holds for some $T \in T(\mathbf{F}_p)$, $A'' \in U(\mathbf{F}_p)$ uniquely determined by (A',t).

By the Bruhat decomposition in (4.6.8), we have

$$T^{-1} = \mathrm{diag}\left(D^{w,(1)}_r, \frac{D^{w,(2)}_r}{D^{w,(1)}_r}, \cdots, \frac{D^{w,(n-r)}_r}{D^{w,(n-1-r)}_r}, \frac{1}{D^{w,(n-r)}_r}, 1, \cdots, 1\right) \tag{4.6.12}$$

in which we write $D^{w,(i)}_r$ for $D^{w,(i)}_r(A',-t)$ for brevity. We also have

$$A_{i,j} = \begin{cases} A'_{i,j} & \text{if } 1 \le i < j \le n \text{ and } j \le r-1;\\ D^w_{m,r}(A',-t) & \text{if } (i,j) = (m,r);\\ \frac{A'_{i,n}}{D^{w,(1)}_r(A',-t)} & \text{if } 1 \le i \le n-1 \text{ and } j = n. \end{cases} \tag{4.6.13}$$

We apply (4.6.7), (4.6.13) and (4.6.12) to M_w and get

$$M_w = \sum_{(A,t)\in E_w}\left(F(A',w,t)\left(\prod_{\substack{1\le i<j\le n\\ j\le r-1 \text{ or } j=n}}(A'_{i,j})^{k_{i,j}}\right)A'w_0\right)v_0$$

where

$$F(A',w,t) := t^{p-2}\left((D^w_{m,r})^{k_{m,r}}(D^{w,(1)}_r)^{a_1-a_{n-1}-\sum_{i=1}^{n-1}k_{i,n}}\prod_{s=2}^{n-r}(D^{w,(s)}_r)^{a_s-a_{s-1}}\right)$$

in which we let $D^w_{m,r} := D^w_{m,r}(A',-t)$ and $D^{w,(s)}_r := D^{w,(s)}_r(A',-t)$ for brevity. We have discussed in (4.6.11) that $(A,t) \in E_w$ is equivalent to $(A',t) \in U_w(\mathbf{F}_p)\times\mathbf{F}_p$ satisfying the conditions in (4.6.11). As each $D^{w,(s)}_r(A',-t)$ appears in $F(A',w,t)$ with a positive power, we can automatically drop the condition (4.6.11) and get

$$M_w = \sum_{(A,t)\in U_w(\mathbf{F}_p)\times\mathbf{F}_p}\left(F(A',w,t)\left(\prod_{\substack{1\le i<j\le n\\ j\le r-1 \text{ or } j=n}}(A'_{i,j})^{k_{i,j}}\right)A'w_0\right)v_0. \tag{4.6.14}$$

If $w \neq w_0$, then there exist a unique integer i_0 satisfying $r \le i_0 \le n$ such that $ww_0(i_0) < i_0$ but $ww_0(i) = i$ for all $i_0+1 \le i \le n$.

By applying Lemma 4.6.2 to $D^{w,(n+1-i_0)}_r(A',-t)$ (as $(u_{\sum_{i=r}^{n-1}\alpha_i}(t)w_0A'w)_{J_0^\ell,J_0^\ell}$ satisfy the condition of Lemma 4.6.2 after multiplying a permutation matrix), we deduce that

$$D^{w,(n+1-i_0)}_r(A',-t) = tf(A')$$

where $f(A')$ is certain polynomial of entries of A'.

Now we consider $F(A',w,t)$ as a polynomial of t. The minimal degree of monomials of t appearing in $F(A',w,t)$ is at least

$$p-2+a_{n+1-i_0}-a_{n-i_0}>p-1,$$

and the maximal degree of monomials of t appearing in $F(A',w,t)$ is

$$\begin{aligned}p-2+k_{m,r}+[a_1-a_{n-1}-\sum_{i=1}^{n-1}k_{i,n}]_1+\sum_{s=2}^{n-r}a_s-a_{s-1}\\ &=p-2+k_{m,r}+[a_1-a_{n-1}-\sum_{i=1}^{n-1}k_{i,n}]_1+a_{n-r}-a_1\\ &<2(p-1).\end{aligned}$$

As a result, the degree of each monomials of t in $F(A',w,t)$ is not divisible by $p-1$. Hence, we conclude that

$$M_w=0 \text{ for all } w\neq w_0$$

as we know that $\sum_{t\in\mathbf{F}_p}t^k\neq 0$ if and only if $p-1\mid k$ and $k\neq 0$.

Finally, we compute M_{w_0} explicitly using (4.6.14). Indeed, by applying (4.6.4), the monomials of t appearing in $F(A',w_0,t)$ is nothing else than

$$\begin{aligned}t^{p-1}(A'_{m,r})^{k_{m,r}}\Bigg(-k_{m,r}f_{m,r}(A')(A'_{m,r})^{-1}+[a_1-a_{n-1}-\sum_{i=1}^{n-1}k_{i,n}]_1f_{r,n}(A')\\ +\sum_{s=2}^{n-r}(a_s-a_{s-1})f_{r,n+1-s}(A')\Bigg).\end{aligned}$$

As $\sum_{t\in\mathbf{F}_p}t^{p-1}=-1$, we conclude that

$$X_r^-\bullet S_{\underline{k},w_0}v_0=M_{w_0}=\sum_{A'\in U(\mathbf{F}_p)}\left(F_0(A')\left(\prod_{\substack{1\leq i<j\leq n\\ j\leq r\text{ or }j=n}}(A'_{i,j})^{k_{i,j}}\right)A'w_0\right)v_0 \tag{4.6.15}$$

where

$$\begin{aligned}F_0(A'):=(A'_{m,r})^{k_{m,r}}\Bigg(k_{m,r}f_{m,r}(A')(A'_{m,r})^{-1}-[a_1-a_{n-1}-\sum_{i=1}^{n-1}k_{i,n}]_1f_{r,n}(A')\\ -\sum_{s=2}^{n-r}(a_s-a_{s-1})f_{r,n+1-s}(A')\Bigg).\end{aligned}$$

Recalling the explicit formula of $f_{m,r}$ and $f_{r,n+1-s}$ for $1\leq s\leq n-r$ from (4.6.5) and then rewriting (4.6.15) as a sum of distinct monomials of entries of A' finishes the proof. $\square$

Proposition 4.6.5. *Keep the assumptions and the notation of Lemma 4.6.4.*

Then we have

$$\begin{aligned}X_r^+ \bullet X_r^- \bullet S_{\underline{k},w_0}v_0 &= k_{m,r}k_{r,n}\sum_{\underline{i}\in\mathbf{I}_{n-r}}\varepsilon(\underline{i})S_{\underline{k}^{\underline{i},m,r,r},w_0}v_0\\ &\quad+(k_{r,n}+1)\left([a_{n-r}-a_{n-1}-\sum_{i=1}^{n-1}k_{i,n}]_1+k_{m,r}\right)S_{\underline{k},w_0}v_0\\ &\quad-k_{r,n}\sum_{\ell=2}^{n-r}(a_{n-r}-a_{\ell-1}+k_{m,r})\left(\sum_{\underline{i}\in\mathbf{I}_\ell\setminus\mathbf{I}_{\ell-1}}\varepsilon(\underline{i})S_{\underline{k}^{\underline{i},r,n-\ell+1,r,+},w_0}v_0\right).\end{aligned}$$

Proof. This is just a direct combination of Lemma 4.6.4 and Lemma 4.6.1. $\square$

Remark 4.6.6. *The effect of X_r^+ (resp. X_r^-) on $T(\mathbf{F}_p)$-eigencharacter is essentially $\chi\mapsto\chi+\alpha_{r,n}$ (resp. $\chi\mapsto\chi-\alpha_{r,n}$) where χ is the $T(\mathbf{F}_p)$-eigencharacter of $S_{\underline{k},w_0}v_0$. The conditions assumed in Lemma 4.6.4 are crucial for the formula in Proposition 4.6.5. In fact, the formula in Proposition 4.6.5 is relatively simple in the sense that all the coefficients are totally explicit when we write $X_r^+\bullet X_r^-\bullet S_{\underline{k},w_0}v_0$ as a linear combination of $S_{\underline{k}',w_0}v_0$ for various $\underline{k}'$.*

4.7. **A non-vanishing theorem.** The main target of this section is to prove Theorem 4.7.17. This theorem together with Corollary 4.4.8 immediately implies Theorem 4.8.1. We start this section by introducing some notation.

We first define a subset Λ_{w_0} of $\{0,\cdots,p-1\}^{|\Phi_{w_0}^+|}$ consisting of the tuples $\underline{k}=(k_{i,j})_{i,j}$ such that for each $1\leq r\leq n-1$

$$\sum_{1\leq i\leq r<j\leq n}k_{i,j}=[a_0-a_{n-1}]_1+n-2.$$

Note that the set Λ_{w_0} embeds into π_0 by sending $\underline{k}$ to $S_{\underline{k},w_0}v_0$. Moreover, this family of vectors $\{S_{\underline{k},w_0}v_0\mid\underline{k}\in\Lambda_{w_0}\}$ shares the same eigencharacter by Lemma 4.1.5.

We define $\underline{k}^\sharp\in\Lambda_{w_0}$ where $\underline{k}^\sharp=(k^\sharp_{i,j})$ is defined by $k^\sharp_{1,n}=[a_0-a_{n-1}]_1+n-2$ and $k^\sharp_{i,j}=0$ otherwise. We set

$$V^\sharp:=\langle G(\mathbf{F}_p)\cdot S_{\underline{k}^\sharp,w_0}v_0\rangle\subseteq\pi_0.$$

We also need to define some other useful elements of Λ_{w_0}. For each $1\leq r\leq n-1$, we define $\underline{k}^{\sharp,r}=(k^{\sharp,r}_{i,j})\in\Lambda_{w_0}$ by

$$k^{\sharp,r}_{i,j}:=\begin{cases}n-2+[a_0-a_{n-1}]_1 & \text{if } 2\leq j=i+1\leq r;\\ n-2+[a_0-a_{n-1}]_1 & \text{if } (i,j)=(r,n);\\ 0 & \text{otherwise.}\end{cases}$$

In particular, we have

$$\underline{k}^{\sharp,1}=\underline{k}^\sharp \text{ and } \underline{k}^{\sharp,n-1}=\underline{k}^0 \qquad (4.7.1)$$

where $\underline{k}^0$ is defined in (4.4.8). If we represent $\underline{k}$ by a matrix in $U(\mathbf{Z})$ with (i,j)-entry given by $k_{i,j}$, then $\underline{k}^{\sharp,r}$ has the following form

$$\begin{pmatrix} 1 & k_0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ & 1 & k_0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ & & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ & & & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & & & & 1 & k_0 & 0 & 0 & \cdots & 0 & 0 \\ & & & & & 1 & 0 & 0 & \cdots & 0 & k_0 \\ & & & & & & 1 & 0 & \cdots & 0 & 0 \\ & & & & & & & 1 & \cdots & 0 & 0 \\ & & & & & & & & \ddots & \vdots & \vdots \\ & & & & & & & & & 1 & 0 \\ & & & & & & & & & & 1 \end{pmatrix}$$

where $k_0 := n-2+[a_0-a_{n-1}]_1$ and the unique k_0 appearing on n-th column is on (r,n)-entry. For each $1\leq r\leq n-2$ and $0\leq s\leq [a_0-a_{n-1}]_1+n-2$, we define $\underline{k}^{\sharp,r,s}=(k^{\sharp,r,s}_{i,j})\in\Lambda_{w_0}$ as follows:

$$k^{\sharp,r,s}_{i,j}=\begin{cases} n-2+[a_0-a_{n-1}]_1 & \text{if } 2\leq j=i+1\leq r; \\ n-2+[a_0-a_{n-1}]_1-s & \text{if } (i,j)=(r,r+1); \\ s & \text{if } (i,j)=(r,n); \\ n-2+[a_0-a_{n-1}]_1-s & \text{if } (i,j)=(r+1,n); \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we have

$$\underline{k}^{\sharp,r,0}=\underline{k}^{\sharp,r+1} \text{ and } \underline{k}^{\sharp,r,[a_0-a_{n-1}]_1+n-2}=\underline{k}^{\sharp,r} \tag{4.7.2}$$

for each $1\leq r\leq n-2$. If we represent $\underline{k}$ by a matrix in $U(\mathbf{Z})$ with (i,j)-entry given by $k_{i,j}$, then $\underline{k}^{\sharp,r,s}$ has the following form

$$\begin{pmatrix} 1 & k_0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ & 1 & k_0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ & & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ & & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & & & & 1 & k_0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ & & & & & 1 & k_0-s & 0 & 0 & \cdots & 0 & s \\ & & & & & & 1 & 0 & 0 & \cdots & 0 & k_0-s \\ & & & & & & & 1 & 0 & \cdots & 0 & 0 \\ & & & & & & & & 1 & \cdots & 0 & 0 \\ & & & & & & & & & \ddots & \vdots & \vdots \\ & & & & & & & & & & 1 & 0 \\ & & & & & & & & & & & 1 \end{pmatrix}$$

where the s appearing on n-th column is on (r,n)-entry.

We now introduce the rough idea of the proof of Theorem 4.7.17. We set

$$V_0 := \langle G(\mathbf{F}_p)\cdot S_{\underline{k}^0,w_0}v_0\rangle\subseteq \pi_0. \tag{4.7.3}$$

The first obstacle to generalize the method of Proposition 3.1.2 in [HLM] is that V_0 does not necessarily admit a structure as $\overline{G}$-representation in general. Our method to resolve this difficulty is to replace $S_{\underline{k}^0,w_0}v_0$ by $S_{\underline{k}^\sharp,w_0}v_0$. We prove in Proposition 4.7.16 that $V^\sharp$ admits a structure as

$\overline{G}$-representation and actually can be identified with a dual Weyl module $H^0(\mu_0^{w_0})$. (The notation $\mu_0^{w_0}$ will be clear in the following.) Now it remains to prove that

$$S_{\underline{k}^\sharp,w_0}v_0 \in V_0 \tag{4.7.4}$$

to deduce Theorem 4.7.17. We will prove in Proposition 4.7.8 that

$$S_{\underline{k}^{\sharp,r,s-1},w_0}v_0 \in V_0 \Longrightarrow S_{\underline{k}^{\sharp,r,s},w_0}v_0 \in V_0$$

for each $1 \leq r \leq n-2$ and $1 \leq s \leq [a_0 - a_{n-1}]_1 + n - 2$. As a result, we can thus pass from $S_{\underline{k}^0,w_0}v_0 \in V_0$ to $S_{\underline{k}^{\sharp,r},w_0}v_0 \in V_0$ for $r = n-2, n-3, \cdots, 1$. The identification $\underline{k}^\sharp = \underline{k}^{\sharp,1}$ thus gives us (4.7.4).

We firstly state three direct corollaries of Proposition 4.6.5. It is easy to check that each tuple $\underline{k}$ appearing in the following corollaries satisfies the assumption in Proposition 4.6.5.

Corollary 4.7.1. *For each $2 \leq r \leq n-1$ and $0 \leq s \leq [a_0 - a_{n-1}]_1 + n - 3$, we have*

$$\begin{aligned}
X_r^+ \bullet X_r^- \bullet S_{\underline{k}^{\sharp,r-1,s},w_0}v_0 = {} & ([a_0 - a_{n-1}]_1 + n - 2 - s)^2 \sum_{\underline{i}\in\mathbf{I}_{n-r}} \varepsilon(\underline{i}) S_{(\underline{k}^{\sharp,r-1,s})^{\underline{i},m,r,r},w_0}v_0 \\
& + ([a_{n-r} - a_{n-1}]_1 - s)([a_0 - a_{n-1}]_1 + n - 1 - s) S_{\underline{k}^{r-1,s},w_0}v_0 \\
& - ([a_0 - a_{n-1}]_1 + n - 2 - s) \sum_{\ell=2}^{n-r} (a_{n-r} - a_{\ell-1} + [a_0 - a_{n-1}]_1 + n - 2 - s) \\
& \cdot \left(\sum_{\underline{i}\in\mathbf{I}_\ell\setminus\mathbf{I}_{\ell-1}} \varepsilon(\underline{i}) S_{(\underline{k}^{\sharp,r-1,s})^{\underline{i},r,n-\ell+1,r,+},w_0}v_0 \right).
\end{aligned}$$

Corollary 4.7.2. *Fix two integers r and m such that $1 \leq m \leq r-1 \leq n-2$, and let $\underline{k} = (k_{i,j})$ be a tuple of integers in Λ_{w_0} such that*

$$k_{i,j} = \begin{cases} 0 & \text{if } r+1 \leq j \leq n-1; \\ 0 & \text{if } i \neq m \text{ and } j = r; \\ 0 & \text{if } r+1 \leq i \leq n-1 \text{ and } j = n; \\ 1 & \text{if } (i,j) = (m,r); \\ 1 & \text{if } (i,j) = (r,n). \end{cases}$$

Then we have

$$\begin{aligned}
X_r^+ \bullet X_r^- \bullet S_{\underline{k},w_0}v_0 = {} & \sum_{\underline{i}\in\mathbf{I}_{n-r}} \varepsilon(\underline{i}) S_{\underline{k}^{\underline{i},m,r,r},w_0}v_0 + 2(a_{n-r} - a_0 - n + 3) S_{\underline{k},w_0}v_0 \\
& - \sum_{\ell=2}^{n-r} (a_{n-r} - a_{\ell-1} + 1) \left(\sum_{\underline{i}\in\mathbf{I}_\ell\setminus\mathbf{I}_{\ell-1}} \varepsilon(\underline{i}) S_{\underline{k}^{\underline{i},r,n-\ell+1,r,+},w_0}v_0 \right).
\end{aligned}$$

Corollary 4.7.3. *Fix two integers r and m such that $1 \leq m \leq r-1 \leq n-2$, and let $\underline{k} = (k_{i,j})$ be a tuple of integers in Λ_{w_0} such that*

$$k_{i,j} = \begin{cases} 0 & \text{if } r \leq j \leq n-1; \\ 0 & \text{if } r \leq i \leq n-1 \text{ and } j = n. \end{cases}$$

Then we have

$$X_r^+ \bullet X_r^- \bullet S_{\underline{k},w_0}v_0 = (a_{n-r} - a_0 - n + 2) S_{\underline{k},w_0}v_0.$$

We now define the following constants in $\mathbf{F}_p$:

$$\begin{cases} \mathbf{c}_\ell & := \prod_{k=1}^{\ell-1} (a_k - a_0 - n + 2 + k)^{2^{\ell-k-1}}; \\ \mathbf{c}'_\ell & := (a_\ell - a_0 - n + 3 + \ell)\mathbf{c}_\ell \end{cases}$$

for all $1 \leq \ell \leq n-1$ where we understand $\mathbf{c}_1$ to be 1. As the tuple $(a_{n-1}, \cdots, a_0)$ is n-generic in the lowest alcove, we notice that $\mathbf{c}_\ell \neq 0 \neq \mathbf{c}'_\ell$ for all $1 \leq \ell \leq n-1$. By definition of $\mathbf{c}_k$ and $\mathbf{c}'_k$ one can also easily check that

$$\prod_{k=1}^{\ell-1} (\mathbf{c}'_k - \mathbf{c}_k) = \mathbf{c}_\ell. \tag{4.7.5}$$

We also define inductively the constants: for each $1 \leq \ell \leq n-1$

$$\mathbf{d}_{\ell,\ell'} := \begin{cases} 2(a_\ell - a_0 - n + 3) & \text{if } \ell' = 0; \\ \mathbf{c}'_{\ell'} \mathbf{d}_{\ell,\ell'-1} - (a_\ell - a_{\ell'} + 1)\mathbf{c}_{\ell'} \prod_{k=1}^{\ell'-1}(\mathbf{c}'_k - \mathbf{c}_k) & \text{if } 1 \leq \ell' \leq \ell - 1. \end{cases}$$

We further define inductively a sequence of group operators $\mathcal{Z}_\ell$ as follows:

$$\mathcal{Z}_1 := \mathbf{d}_{1,0}\mathrm{Id} - X^+_{n-1} \bullet X^-_{n-1} \in \mathbf{F}_p[G(\mathbf{F}_p)]$$

and

$$\mathcal{Z}_\ell := \mathbf{d}_{\ell,\ell-1}\mathrm{Id} - \left(\mathcal{Z}_{\ell-1} \bullet \cdots \bullet \mathcal{Z}_1 \bullet X^+_{n-\ell} \bullet X^-_{n-\ell}\right) \in \mathbf{F}_p[G(\mathbf{F}_p)]$$

for each $2 \leq \ell \leq n-2$.

Lemma 4.7.4. *For $1 \leq \ell \leq n-1$, we have the identity*

$$\mathbf{d}_{\ell,\ell-1} = (a_\ell - a_0 - n + 2)\left(\prod_{k=1}^{\ell-1} \mathbf{c}'_k\right) + \mathbf{c}'_\ell.$$

Proof. During the proof of this lemma, we will keep using the following obvious identity with two variables

$$ab = (a+1)(b-1) + a - b + 1 \tag{4.7.6}$$

By definition of $\mathbf{d}_{\ell,\ell-1}$ we know that

$$\mathbf{d}_{\ell,\ell-1} = 2(a_\ell - a_0 - n + 3)\prod_{k=1}^{\ell-1} \mathbf{c}'_k - \sum_{\ell'=1}^{\ell-1}\left((a_\ell - a_{\ell'} + 1)\mathbf{c}_{\ell'}\left(\prod_{k=1}^{\ell'-1}(\mathbf{c}'_k - \mathbf{c}_k)\right)\left(\prod_{k=\ell'+1}^{\ell-1} \mathbf{c}'_k\right)\right)$$

and therefore

$$\begin{aligned}\mathbf{d}_{\ell,\ell-1} - (a_\ell - a_0 - n + 2)\left(\prod_{k=1}^{\ell-1} \mathbf{c}'_k\right) &= (a_\ell - a_0 - n + 4)\prod_{k=1}^{\ell-1} \mathbf{c}'_k \\ &\quad - \sum_{\ell'=1}^{\ell-1}\left((a_\ell - a_{\ell'} + 1)\mathbf{c}_{\ell'}\left(\prod_{k=1}^{\ell'-1}(\mathbf{c}'_k - \mathbf{c}_k)\right)\left(\prod_{k=\ell'+1}^{\ell-1} \mathbf{c}'_k\right)\right).\end{aligned}$$

Now we prove inductively that for each $1 \leq j \leq \ell - 1$

$$\begin{aligned}\mathbf{d}_{\ell,\ell-1} - (a_\ell - a_0 - n + 2)\left(\prod_{k=1}^{\ell-1} \mathbf{c}'_k\right) &= (a_\ell - a_0 - n + 3 + j)\left(\prod_{k=1}^{j-1}(\mathbf{c}'_k - \mathbf{c}_k)\right)\left(\prod_{k=j}^{\ell-1} \mathbf{c}'_k\right) \\ &\quad - \sum_{\ell'=j}^{\ell-1}\left((a_\ell - a_{\ell'} + 1)\mathbf{c}_{\ell'}\left(\prod_{k=1}^{\ell'-1}(\mathbf{c}'_k - \mathbf{c}_k)\right)\left(\prod_{k=\ell'+1}^{\ell-1} \mathbf{c}'_k\right)\right).\end{aligned} \tag{4.7.7}$$

By the identity (4.7.6), one can easily deduce that

$$\begin{aligned}&(a_\ell - a_0 - n + 3 + j)\mathbf{c}'_j - (a_\ell - a_j + 1)\mathbf{c}_j \\ &\quad = [(a_\ell - a_0 - n + 3 + j)(a_j - a_0 - n + 3 + j) - (a_\ell - a_j + 1)]\,\mathbf{c}_j \\ &\quad = (a_\ell - a_0 - n + 4 + j)(a_j - a_0 - n + 2 + j)\mathbf{c}_j \\ &\quad = (a_\ell - a_0 - n + 4 + j)(\mathbf{c}'_j - \mathbf{c}_j).\end{aligned}$$

Hence, we get the identity:

$$\begin{aligned}\left[(a_\ell - a_0 - n + 3 + j)\mathbf{c}'_j - (a_\ell - a_j + 1)\mathbf{c}_j\right] \left(\prod_{k=j+1}^{\ell-1} \mathbf{c}'_k\right) \left(\prod_{k=1}^{j-1} (\mathbf{c}'_k - \mathbf{c}_k)\right) \\ = (a_\ell - a_0 - n + 4 + j) \left(\prod_{k=1}^{j} (\mathbf{c}'_k - \mathbf{c}_k)\right) \left(\prod_{k=j+1}^{\ell-1} \mathbf{c}'_k\right).\end{aligned} \tag{4.7.8}$$

Thus, if the equation (4.7.7) holds for j, we can deduce that it also holds for $j+1$. By taking $j = \ell - 1$ and using the equation (4.7.8) once more, we can deduce that

$$\mathbf{d}_{\ell,\ell-1} - (a_\ell - a_0 - n + 2) \left(\prod_{k=1}^{\ell-1} \mathbf{c}'_k\right) = (a_\ell - a_0 - n + 3 + \ell) \left(\prod_{k=1}^{\ell-1} (\mathbf{c}'_k - \mathbf{c}_k)\right).$$

Hence, by the equation (4.7.5), one finishes the proof. □

Proposition 4.7.5. *Fix two integers r and m such that $1 \le m \le r - 1 \le n - 2$.*

(i) *Let $\underline{k} = (k_{i,j})$ be as in Corollary 4.7.2. Then we have*

$$\mathcal{Z}_{n-r} \bullet S_{\underline{k},w_0} = \mathbf{c}_{n-r} S_{\underline{k}',w_0} \tag{4.7.9}$$

where $\underline{k}' = (k'_{i,j})$ is defined as follows:

$$k'_{i,j} := \begin{cases} 0 & \text{if } (i,j) = (m,r) \text{ or } (i,j) = (r,n); \\ 1 & \text{if } (i,j) = (m,n); \\ k_{i,j} & \text{otherwise.} \end{cases}$$

(ii) *Let $\underline{k} = (k_{i,j})$ be as in Corollary 4.7.3. Then we have*

$$\mathcal{Z}_{n-r} \bullet S_{\underline{k},w_0} = \mathbf{c}'_{n-r} S_{\underline{k},w_0}. \tag{4.7.10}$$

We prove this proposition by a series of lemmas.

Lemma 4.7.6. *Proposition 4.7.5 is true for $r = n - 1$.*

Proof. For part (i) of Proposition 4.7.5, by applying Corollary 4.7.2 to the case $r = n-1$ we deduce that

$$X^+_{n-1} \bullet X^-_{n-1} \bullet S_{\underline{k},w_0} v_0 = 2(a_1 - a_0 - n + 3) S_{\underline{k},w_0} v_0 - S_{\underline{k}^{\underline{i}_0,m,n-1,n-1},w_0} v_0$$

where $\underline{i}_0 = \{n-1, n\}$. Hence, part (i) of the proposition follows directly from the definition of $\mathcal{Z}_1$ and $\mathbf{c}_1$.

For part (ii) of Proposition 4.7.5, again by Corollary 4.7.3 to the case $r = n - 1$ we deduce that

$$X^+_{n-1} \bullet X^-_{n-1} \bullet S_{\underline{k},w_0} v_0 = (a_1 - a_0 - n + 2) S_{\underline{k},w_0} v_0.$$

Then we have

$$\mathcal{Z}_1 \bullet S_{\underline{k},w_0} v_0 = (a_1 - a_0 - n + 4) S_{\underline{k},w_0} v_0$$

and part (ii) of the proposition follows directly from the definition of $\mathbf{c}'_1$. □

Lemma 4.7.7. *Let ℓ be an integer with $2 \le \ell \le n-1$. If Proposition 4.7.5 is true for $r \ge n-\ell+1$, then it is true for $r = n - \ell$.*

Proof. We prove part (ii) first. Assume that (4.7.10) holds for $r \geq n-\ell+1$. In fact, for a Jacobi sum $S_{\underline{k},w_0}$ satisfying the conditions in the Corollary 4.7.3 for $r=n-\ell$, we have

$$X^+_{n-\ell}\bullet X^-_{n-\ell}\bullet S_{\underline{k},w_0}v_0=(a_\ell-a_0-n+2)S_{\underline{k},w_0}v_0$$

by Corollary 4.7.3. Then we can deduce

$$\mathcal{Z}_{\ell-1}\bullet\cdots\bullet\mathcal{Z}_1\bullet X^+_{n-\ell}\bullet X^-_{n-\ell}\bullet S_{\underline{k},w_0}v_0=(a_\ell-a_0-n+2)\left(\prod_{s=1}^{\ell-1}\mathbf{c}'_s\right)S_{\underline{k},w_0}v_0$$

from the inductive assumption of this lemma. Hence, by definition of $\mathcal{Z}_\ell$, we have

$$\begin{aligned}\mathcal{Z}_\ell\bullet S_{\underline{k},w_0}v_0&=\mathbf{d}_{\ell,\ell-1}S_{\underline{k},w_0}v_0-\mathcal{Z}_{\ell-1}\bullet\cdots\bullet\mathcal{Z}_1\bullet X^+_{n-\ell}\bullet X^-_{n-\ell}\bullet S_{\underline{k},w_0}v_0\\&=\left(\mathbf{d}_{\ell,\ell-1}-(a_\ell-a_0-n+2)\left(\prod_{s=1}^{\ell-1}\mathbf{c}'_s\right)\right)S_{\underline{k},w_0}v_0\\&=\mathbf{c}'_\ell S_{\underline{k},w_0}v_0\end{aligned}$$

where the last equality follows from Lemma 4.7.4.

Now we turn to part (i). Assume that (4.7.9) holds for $r\geq n-\ell+1$. We will prove inductively that for each ℓ' satisfying $0\leq\ell'\leq\ell-1$, we have

$$\begin{aligned}(4.7.11)\quad&\mathcal{Z}_{\ell'}\bullet\cdots\bullet\mathcal{Z}_1\bullet X^+_{n-\ell}\bullet X^-_{n-\ell}\bullet S_{\underline{k},w_0}v_0\\&=\mathbf{d}_{\ell,\ell'}S_{\underline{k},w_0}v_0+\left(\prod_{s=1}^{\ell'}(\mathbf{c}'_s-\mathbf{c}_s)\right)\left(\sum_{\underline{i}\in\mathbf{I}^{\ell'}_\ell}\varepsilon(\underline{i})S_{\underline{k}^{\underline{i},m,n-\ell,n-\ell},w_0}v_0\right)\\&\quad+\left(\prod_{s=1}^{\ell'}(\mathbf{c}'_s-\mathbf{c}_s)\right)\left(\sum_{h=\ell'+1}^{\ell-1}(a_\ell-a_h+1)\sum_{\underline{i}\in\mathbf{I}^{\ell'}_h\setminus\mathbf{I}^{\ell'}_{h+1}}\varepsilon(\underline{i})S_{\underline{k}^{\underline{i},n-\ell,n-h,n-\ell},w_0}v_0\right)\end{aligned}$$

where the case $\ell'=0$, namely the formula for $X^+_{n-\ell}\bullet X^-_{n-\ell}\bullet S_{\underline{k},w_0}v_0$, follows directly from Corollary 4.7.2 for $r=n-\ell$.

We begin with studying some basic properties of the index sets $\mathbf{I}^{\ell'}_h$ which are useful for our induction on ℓ' to prove (4.7.11). First of all, the set $\mathbf{I}^{\ell'}_{\ell'+1}\setminus\mathbf{I}^{\ell'}_{\ell'+2}$ has a unique element, which is precisely $\underline{i}=\{n-\ell'-1,n\}$. Furthermore, there is a natural map of sets

$$\mathrm{res}_{\ell'}:\mathbf{I}^{\ell'}_h\to\mathbf{I}^{\ell'+1}_h$$

for all $\ell'+2\leq h\leq\ell$ defined by eliminating the element $n-\ell'$ from $\underline{i}\in\mathbf{I}^{\ell'}_h$ if $n-\ell'\in\underline{i}$. In other words, for each $\underline{i}\in\mathbf{I}^{\ell'+1}_h$, we have

$$\mathrm{res}^{-1}_{\ell'}(\{\underline{i}\})=\{\underline{i},\underline{i}\cup\{n-\ell'\}\}\subseteq\mathbf{I}^{\ell'}_h.$$

We use the shortened notation

$$\underline{i}^{\ell'}:=\underline{i}\cup\{n-\ell'\}$$

for each $\underline{i}\in\mathbf{I}^{\ell'+1}_h$. Note in particular that $\varepsilon(\underline{i})=-\varepsilon(\underline{i}^{\ell'})$.

Given an arbitrary $\underline{i}\in\mathbf{I}^{\ell'+1}_h$ for $\ell'+2\leq h\leq\ell-1$, then $S_{\underline{k}^{\underline{i},n-\ell,n-h,n-\ell},w_0}$ (resp. $S_{\underline{k}^{\underline{i}^{\ell'},n-\ell,n-h,n-\ell},w_0}$) satisfies the conditions in Corollary 4.7.2 (resp. Corollary 4.7.3). As a result, by the assumption that Proposition 4.7.5 is true for $r=n-\ell'-1$, we deduce that

(4.7.12)

$$\mathcal{Z}_{\ell'+1}\bullet\left(S_{\underline{k}^{\underline{i},n-\ell,n-h,n-\ell},w_0}v_0-S_{\underline{k}^{\underline{i}^{\ell'},n-\ell,n-h,n-\ell},w_0}v_0\right)=\left(\mathbf{c}'_{\ell'+1}-\mathbf{c}_{\ell'+1}\right)S_{\underline{k}^{\underline{i},n-\ell,n-h,n-\ell},w_0}v_0.$$

Similarly, we have

$$(4.7.13)\qquad\mathcal{Z}_{\ell'+1}\bullet\left(S_{\underline{k}^{\underline{i},m,n-\ell,n-\ell},w_0}v_0-S_{\underline{k}^{\underline{i}^{\ell'},m-\ell,n-\ell},w_0}v_0\right)=\left(\mathbf{c}'_{\ell'+1}-\mathbf{c}_{\ell'+1}\right)S_{\underline{k}^{\underline{i},m,n-\ell,n-\ell},w_0}v_0$$

for each $\underline{i} \in \mathbf{I}_{\ell}^{\ell'+1}$. We also have

$$\mathcal{Z}_{\ell'+1} \bullet S_{\underline{k},w_0}v_0 = \mathbf{c}'_{\ell'+1}S_{\underline{k},w_0}v_0 \tag{4.7.14}$$

by (4.7.10) for $r = n - \ell' - 1$, and

$$\mathcal{Z}_{\ell'+1} \bullet S_{\underline{k}^{\underline{i}_0,n-\ell,n-\ell'-1,n-\ell},w_0}v_0 = \mathbf{c}_{\ell'+1}S_{\underline{k},w_0}v_0 \tag{4.7.15}$$

by (4.7.9) for $r = n - \ell' - 1$ where $\underline{i}_0 = \{n - \ell' - 1, n\}$.

Now we begin our induction and assume that (4.7.11) is true for some $0 \leq \ell' \leq \ell - 2$. Then by combining (4.7.12), (4.7.13), (4.7.14) and (4.7.15), we have

$$\begin{aligned}
&\mathcal{Z}_{\ell'+1} \bullet \cdots \bullet \mathcal{Z}_1 \bullet X^+_{n-\ell} \bullet X^-_{n-\ell} \bullet S_{\underline{k},w_0}v_0 \\
&\quad = \mathbf{d}_{\ell,\ell'}\mathcal{Z}_{\ell'+1} \bullet S_{\underline{k},w_0}v_0 + \left(\prod_{s=1}^{\ell'}(\mathbf{c}'_s - \mathbf{c}_s)\right)\mathcal{Z}_{\ell'+1} \bullet \left(\sum_{\underline{i}\in\mathbf{I}_{\ell}^{\ell'}} \varepsilon(\underline{i})S_{\underline{k}^{\underline{i},m,n-\ell,n-\ell},w_0}v_0\right) \\
&\qquad + \left(\prod_{s=1}^{\ell'}(\mathbf{c}'_s - \mathbf{c}_s)\right)\mathcal{Z}_{\ell'+1} \bullet \left(\sum_{h=\ell'+1}^{\ell-1}(a_\ell - a_h + 1)\sum_{\underline{i}\in\mathbf{I}_h^{\ell'}\setminus\mathbf{I}_{h+1}^{\ell'}} \varepsilon(\underline{i})S_{\underline{k}^{\underline{i},n-\ell,n-h,n-\ell},w_0}v_0\right)
\end{aligned}$$

which is the same as

$$\mathbf{c}'_{\ell'}\mathbf{d}_{\ell,\ell'}S_{\underline{k},w_0}v_0 + \left(\prod_{s=1}^{\ell'}(\mathbf{c}'_s - \mathbf{c}_s)\right)(X + Y + Z) \tag{4.7.16}$$

where

$$X = (a_\ell - a_{\ell'} + 1)\mathcal{Z}_{\ell'+1} \bullet S_{\underline{k}^{\underline{i}_0,n-\ell,n-\ell'-1,n-\ell},w_0}v_0,$$

$$Y = \sum_{\underline{i}\in\mathbf{I}_{\ell}^{\ell'+1}} \varepsilon(\underline{i})\mathcal{Z}_{\ell'+1} \bullet \left(S_{\underline{k}^{\underline{i},m,n-\ell,n-\ell},w_0}v_0 - S_{\underline{k}^{\underline{i}^{\ell'},m,n-\ell,n-\ell},w_0}v_0\right),$$

and

$$Z = \sum_{h=\ell'+2}^{\ell-1}(a_\ell - a_h + 1)\sum_{\underline{i}\in\mathbf{I}_h^{\ell'+1}\setminus\mathbf{I}_{h+1}^{\ell'+1}} \varepsilon(\underline{i})\mathcal{Z}_{\ell'+1} \bullet \left(S_{\underline{k}^{\underline{i},n-\ell,n-h,n-\ell},w_0}v_0 - S_{\underline{k}^{\underline{i}^{\ell'},n-\ell,n-h,n-\ell},w_0}v_0\right).$$

One can also readily check that (4.7.16) is also the same as

$$\begin{aligned}
&\left(\mathbf{c}'_{\ell'+1}\mathbf{d}_{\ell,\ell'} + \mathbf{c}_{\ell'+1}\left(\prod_{s=1}^{\ell'}(\mathbf{c}'_s - \mathbf{c}_s)\right)(a_\ell - a_{\ell'} + 1)\right)S_{\underline{k},w_0}v_0 \\
&\quad + \left(\prod_{s=1}^{\ell'+1}(\mathbf{c}'_s - \mathbf{c}_s)\right)\left(\sum_{\underline{i}\in\mathbf{I}_{\ell}^{\ell'+1}} \varepsilon(\underline{i})S_{\underline{k}^{\underline{i},m,n-\ell,n-\ell},w_0}v_0\right) \\
&\quad + \left(\prod_{s=1}^{\ell'+1}(\mathbf{c}'_s - \mathbf{c}_s)\right)\left(\sum_{h=\ell'+2}^{\ell-1}(a_\ell - a_h + 1)\sum_{\underline{i}\in\mathbf{I}_h^{\ell'+1}\setminus\mathbf{I}_{h+1}^{\ell'+1}} \varepsilon(\underline{i})S_{\underline{k}^{\underline{i},n-\ell,n-h,n-\ell},w_0}v_0\right),
\end{aligned}$$

which implies that (4.7.11) holds for $\ell' + 1$, as we have

$$\mathbf{d}_{\ell,\ell'+1} = \mathbf{c}'_{\ell'+1}\mathbf{d}_{\ell,\ell'} + \mathbf{c}_{\ell'+1}\left(\prod_{s=1}^{\ell'}(\mathbf{c}'_s - \mathbf{c}_s)\right)(a_\ell - a_{\ell'} + 1)$$

by definition.

Hence we have finished the proof of (4.7.11) for each $1 \leq \ell' \leq \ell - 1$ by induction on ℓ'. Note that the case $\ell' = \ell - 1$ for (4.7.11) is just the following

$$\mathcal{Z}_{\ell-1} \bullet \cdots \bullet \mathcal{Z}_1 \bullet X^+_{n-\ell} \bullet X^-_{n-\ell} \bullet S_{\underline{k},w_0} v_0 = \mathbf{d}_{\ell,\ell-1} S_{\underline{k},w_0} v_0 - \left(\prod_{s=1}^{\ell-1} (\mathbf{c}'_s - \mathbf{c}_s)\right) S_{\underline{k}^{\underline{i}_1,m,n-\ell,n-\ell},w_0} v_0 \tag{4.7.17}$$

where $\underline{i}_1 = \{n\}$.

Finally, (4.7.9) for $r = n - \ell$ follows from the equation (4.7.17) together with the definition of $\mathcal{Z}_\ell$ and the identity (4.7.5). □

Proof of Proposition 4.7.5. It follows easily from Lemma 4.7.6 and Lemma 4.7.7. □

Proposition 4.7.8. *For each $1 \leq r \leq n-2$ and $1 \leq s \leq [a_0 - a_{n-1}]_1 + n - 2$, if $S_{\underline{k}^{\sharp,r,s-1},w_0} v_0 \in V_0$, then $S_{\underline{k}^{\sharp,r,s},w_0} v_0 \in V_0$.*

Proof. We deduce from the same argument as (4.7.12), (4.7.13), (4.7.14) and (4.7.15) that the following equalities

$$\mathcal{Z}_{n-2-r} \bullet \cdots \bullet \mathcal{Z}_1 \bullet S_{\underline{k}^{\sharp,r,s-1},w_0} v_0 = \left(\prod_{\ell=1}^{n-2-r} \mathbf{c}'_\ell\right) S_{\underline{k}^{\sharp,r,s-1},w_0} v_0, \tag{4.7.18}$$

$$\begin{aligned}\mathcal{Z}_{n-2-r} \bullet \cdots \bullet \mathcal{Z}_1 \bullet &\left(\sum_{\underline{i} \in \mathbf{I}_{n-1-r}} \varepsilon(\underline{i}) S_{(\underline{k}^{\sharp,r,s-1})^{\underline{i},r,r+1,r+1},w_0} v_0\right) \\ &= -\left(\prod_{\ell=1}^{n-2-r} (\mathbf{c}'_\ell - \mathbf{c}_\ell)\right) S_{\underline{k}^{\sharp,r,s},w_0} v_0,\end{aligned} \tag{4.7.19}$$

and

$$\begin{aligned}\mathcal{Z}_{n-2-r} \bullet \cdots \bullet \mathcal{Z}_1 \bullet &\left(\sum_{\underline{i} \in \mathbf{I}_\ell \setminus \mathbf{I}_{\ell-1}} \varepsilon(\underline{i}) S_{(\underline{k}^{\sharp,r,s-1})^{\underline{i},r+1,n-\ell+1,r+1},w_0} v_0\right) \\ &= \mathbf{c}_\ell \left(\prod_{h=1}^{\ell-1} (\mathbf{c}'_h - \mathbf{c}_h)\right) \left(\prod_{h=\ell+1}^{n-2-r} \mathbf{c}'_h\right) S_{\underline{k}^{\sharp,r,s-1},w_0} v_0\end{aligned} \tag{4.7.20}$$

hold for each $1 \leq \ell \leq n - 2 - r$. Therefore by replacing (r, s) in Corollary 4.7.1 by $(r + 1, s - 1)$ and then using (4.7.18), (4.7.19) and (4.7.20) respectively, we can deduce that

$$\begin{aligned}&\mathcal{Z}_{n-2-r} \bullet \cdots \bullet \mathcal{Z}_1 \bullet X^+_{r+1} \bullet X^-_{r+1} \bullet S_{\underline{k}^{\sharp,r,s-1},w_0} v_0 \\ &\quad = -([a_0 - a_{n-1}]_1 + n - 1 - s)^2 \left(\prod_{\ell=1}^{n-2-r} (\mathbf{c}'_\ell - \mathbf{c}_\ell)\right) S_{\underline{k}^{\sharp,r,s},w_0} v_0 + \mathbf{C} S_{\underline{k}^{\sharp,r,s-1},w_0} v_0 \\ &\quad = -([a_0 - a_{n-1}]_1 + n - 1 - s)^2 \mathbf{c}_{n-1-r} S_{\underline{k}^{\sharp,r,s},w_0} v_0 + \mathbf{C} S_{\underline{k}^{\sharp,r,s-1},w_0} v_0\end{aligned}$$

for a certain constant $\mathbf{C} \in \mathbf{F}_p$. Note that we use the identity (4.7.5) for the last equality .

By our assumption, we know that $S_{\underline{k}^{\sharp,r,s-1},w_0} v_0 \in V_0$. Hence, we can deduce

$$S_{\underline{k}^{\sharp,r,s},w_0} v_0 \in V_0$$

since $([a_0 - a_{n-1}]_1 + n - 1 - s)^2 \mathbf{c}_{n-1-r} \neq 0$. □

Corollary 4.7.9. *We have $S_{\underline{k}^\sharp,w_0} v_0 \in V_0$.*

Proof. By (4.7.2) and Proposition 4.7.8 we deduce that

$$S_{\underline{k}^{\sharp,r}}v_0 \in V_0 \Rightarrow S_{\underline{k}^{\sharp,r-1}}v_0 \in V_0$$

for each $2 \le r \le n-1$. Then by (4.7.1) and the definition of V_0, we finish the proof. $\square$

Example 4.7.10. *We will give an example to illustrate the technical results in Proposition 4.7.5 and Proposition 4.7.8. Given a tuple $\underline{k} \in \{0,1,\cdots,p-1\}^{|\Phi^+_{w_0}|}$, we associate a matrix in $U(\mathbf{Z})$ with (i,j)-entry given by $k_{i,j}$ for all $1 \le i < j \le n$ and abuse the notation $\underline{k}$ for such a matrix. In this example, we are going to use $\underline{k}$ or the matrix in $U(\mathbf{Z})$ associated with it to represent the corresponding vector $S_{\underline{k},w_0}v_0$. We will write*

$$\underline{k} \Rightarrow \underline{k}'$$

if $S_{\underline{k}',w_0}v_0 \in \langle G(\mathbf{F}_p)\cdot S_{\underline{k},w_0}v_0\rangle$. We consider the special case $n=5$ and $r=1$ from now on, and our goal here is to illustrate the proof of

(4.7.21)

$$\underline{k}^{\sharp,1,s-1} = \begin{pmatrix} 1 & k_0-s+1 & 0 & 0 & s-1 \\ & 1 & 0 & 0 & k_0-s+1 \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix} \Rightarrow \underline{k}^{\sharp,1,s} = \begin{pmatrix} 1 & k_0-s & 0 & 0 & s \\ & 1 & 0 & 0 & k_0+s \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}$$

intuitively for all $0 \le s \le k_0$ where $k_0 = 3+[a_0-a_4]_1$. We firstly observe that

$$\mathbf{I}_1 = \{(5)\},\ \mathbf{I}_2 \setminus \mathbf{I}_1 = \{(4,5)\} \text{ and } \mathbf{I}_3 \setminus \mathbf{I}_2 = \{(3,5),(3,4,5)\}.$$

The first step towards (4.7.21) is to apply $X_2^+ \bullet X_2^-$ to $\underline{k}^{\sharp,1,s-1}$ (as a special case of Corollary 4.7.1) and obtain

$$X_2^+ \bullet X_2^- \cdot \underline{k}^{\sharp,1,s-1} = (k_0-s)^2 Y_0 + ([a_3-a_4]_1 - s)(k_0+1-s)\underline{k}^{\sharp,1,s-1} + (k_0-s)\sum_{\ell=2}^{3}(a_3-a_{\ell-1}+k_0-s)\cdot Y_\ell \tag{4.7.22}$$

where we have

$$Y_0 := -\begin{pmatrix} 1 & k_0-s & 0 & 0 & s \\ & 1 & 0 & 0 & k_0-s \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix} + \begin{pmatrix} 1 & k_0-s & 0 & 1 & s-1 \\ & 1 & 0 & 0 & k_0-s \\ & & 1 & 0 & 0 \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix} + \begin{pmatrix} 1 & k_0-s & 1 & 0 & s-1 \\ & 1 & 0 & 0 & k_0-s \\ & & 1 & 0 & 1 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix} - \begin{pmatrix} 1 & k_0-s & 1 & 0 & s-1 \\ & 1 & 0 & 0 & k_0-s \\ & & 1 & 1 & 0 \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix},$$

$$Y_2 := \begin{pmatrix} 1 & k_0-s+1 & 0 & 0 & s-1 \\ & 1 & 0 & 1 & k_0-s \\ & & 1 & 0 & 0 \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix},$$

and

$$Y_3 := \begin{pmatrix} 1 & k_0-s+1 & 0 & 0 & s-1 \\ & 1 & 1 & 0 & k_0-s \\ & & 1 & 0 & 1 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix} - \begin{pmatrix} 1 & k_0-s+1 & 0 & 0 & s-1 \\ & 1 & 1 & 0 & k_0-s \\ & & 1 & 1 & 0 \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}.$$

Note that the terms in Y_0 (resp. the terms in Y_ℓ) are indexed by $\mathbf{I}_3$ (resp. by $\mathbf{I}_\ell \setminus \mathbf{I}_{\ell-1}$ for $\ell = 2, 3$). Then we apply $\mathcal{Z}_1$ to each of Y_0, $\underline{k}^{\sharp,1,s-1}$, Y_2 and Y_3 and obtain

$$\mathcal{Z}_1 \cdot \underline{k}^{\sharp,1,s-1} = \mathbf{c}_1'\underline{k}^{\sharp,1,s-1},\ \mathcal{Z}_1 \cdot Y_2 = \mathbf{c}_1\underline{k}^{\sharp,1,s-1},$$

$$\mathcal{Z}_1 \cdot Y_3 = (\mathbf{c}_1' - \mathbf{c}_1) \cdot \begin{pmatrix} 1 & k_0-s+1 & 0 & 0 & s-1 \\ & 1 & 1 & 0 & k_0-s \\ & & 1 & 0 & 1 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}$$

and

$$\mathcal{Z}_1 \cdot Y_0 = (\mathbf{c}_1' - \mathbf{c}_1) \left(-\underline{k}^{\sharp,1,s} + \begin{pmatrix} 1 & k_0-s & 1 & 0 & s-1 \\ & 1 & 0 & 0 & k_0-s \\ & & 1 & 0 & 1 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix} \right)$$

where $\mathbf{c}_1' = a_1 - a_0 - 1$ and $\mathbf{c}_1 = 1$. Then we apply $\mathcal{Z}_2$ and obtain

$$\mathcal{Z}_2 \bullet \mathcal{Z}_1 \cdot \underline{k}^{\sharp,1,s-1} = \mathbf{c}_2'\mathbf{c}_1'\underline{k}^{\sharp,1,s-1},\ \mathcal{Z}_2 \bullet \mathcal{Z}_1 \cdot Y_2 = \mathbf{c}_2'\underline{k}^{\sharp,1,s-1},\ \mathcal{Z}_2 \bullet \mathcal{Z}_1 \cdot Y_3 = \mathbf{c}_2(\mathbf{c}_1' - \mathbf{c}_1)\underline{k}^{\sharp,1,s-1} \tag{4.7.23}$$

and

$$\mathcal{Z}_2 \bullet \mathcal{Z}_1 \cdot Y_0 = -(\mathbf{c}_2' - \mathbf{c}_2)(\mathbf{c}_1' - \mathbf{c}_1)\underline{k}^{\sharp,1,s} \tag{4.7.24}$$

where we have $\mathbf{c}_2 = a_1 - a_0 - 2$ and $\mathbf{c}_2' = (a_2 - a_0)(a_1 - a_0 - 2)$. By combining (4.7.22), (4.7.23) and (4.7.24), we deduce that

$$\mathcal{Z}_2 \bullet \mathcal{Z}_1 \bullet X_2^+ \bullet X_2^- \cdot \underline{k}^{\sharp,1,s-1} = \mathbf{C}\underline{k}^{\sharp,1,s-1} - (k_0 + 1 - s)^2\mathbf{c}_3\underline{k}^{\sharp,1,s}$$

for $\mathbf{c}_3 = (a_1 - a_0 - 2)^2(a_2 - a_0 - 1)$ and a certain constant $\mathbf{C} \in \mathbf{F}_p$, which implies (4.7.21). If we consider the subspace V of π_0 spanned by the various $\underline{k}$ (namely $S_{\underline{k},w_0}v_0$) appearing in (4.7.22), then $\mathcal{Z}_1$ and $\mathcal{Z}_2 \bullet \mathcal{Z}_1$ induce maps in $\mathrm{End}_{\mathbf{F}_p}(V)$. In fact, the image of $\mathcal{Z}_1$ is spanned by

$$\underline{k}^{\sharp,1,s},\ \underline{k}^{\sharp,1,s-1},\ \begin{pmatrix} 1 & k_0-s+1 & 0 & 0 & s-1 \\ & 1 & 1 & 0 & k_0-s \\ & & 1 & 0 & 1 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & k_0-s & 1 & 0 & s-1 \\ & 1 & 0 & 0 & k_0-s \\ & & 1 & 0 & 1 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}$$

while the image of $\mathcal{Z}_2 \bullet \mathcal{Z}_1$ is simply spanned by $\underline{k}^{\sharp,1,s}$ and $\underline{k}^{\sharp,1,s-1}$.

Remark 4.7.11. *If we view the procedure of applying a group operator of the form*

$$\mathbf{C}\mathrm{Id} - X_r^+ \bullet X_r^-$$

(for some $2 \leq r \leq n-1$ and a certain constant $\mathbf{C} \in \mathbf{F}_p$) as an elementary operation, then $\mathcal{Z}_\ell$ is the composition of $2^{\ell-1}$ such elementary operations by definition. In particular, we need to apply such elementary operations 2^{n-2-r} times in the proof of Proposition 4.7.8. Such complexity is hidden in the inductive definition of $\mathcal{Z}_\ell$ for $1 \leq \ell \leq n-2$.

We write β for $\sum_{r=1}^{n-1} \alpha_r$ to lighten the notation.

Lemma 4.7.12. *Given a Jacobi sum $S_{\underline{k},w_0}$, we have*

$$X_{\beta,k_{1,n}} \bullet S_{\underline{k},w_0} = (-1)^{k_{1,n}+1} S_{\underline{k}',w_0}$$

where $\underline{k}' = (k'_{i,j})$ satisfies $k'_{1,n} = 0$ and $k'_{i,j} = k_{i,j}$ otherwise.

Proof. This is a special case of Lemma 4.1.2 when $\alpha_0 = \beta$ and $m = k_{1,n}$. □

From now on, whenever we want to view the notation $\mu_0^{w_0}$ as a weight, namely to fix a lift of $\mu_0^{w_0} \in X(T)/(p-1)X(T)$ into $X_1^{\mathrm{reg}}(T)$, we always mean

$$\mu_0^{w_0} = (a_0 + p - 1, a_{n-2}, \cdots, a_1, a_{n-1} - p + 1) \in X(T).$$

In particular, we have

$$(1, n) \cdot \mu_0^{w_0} + p\beta = \mu^*.$$

We recall the operators $X_{\beta,k}^{\mathrm{alg}}$ from the beginning of Section 4.

Lemma 4.7.13. *For $1 \leq r \leq n-1$, we have the following equalities on $H^0(\mu_0^{w_0})_{\mu^*}$:*

$$X_{\beta,k} = -X_{\beta,k}^{\mathrm{alg}}$$

for all $1 \leq k \leq p-1$.

Proof. Note that we have

$$\mu_0^{w_0} - (\mu^* + k\beta) = ([a_0 - a_{n-1}]_1 + n - 2 - k, 0, \cdots, 0, k - ([a_0 - a_{n-1}]_1 + n - 2)).$$

Therefore $\mu_0^{w_0} - (\mu^* + k\beta) \notin \sum_{\alpha\in\Phi^+} \mathbf{Z}_{\geq 0}\alpha$ as long as $k > [a_0 - a_{n-1}]_1 + n - 2$. As $(a_{n-1}, \cdots, a_0)$ is assumed to be n-generic in the lowest alcove throughout this section, we deduce that

$$\mu_0^{w_0} - (\mu^* + k\beta) \notin \sum_{\alpha\in\Phi^+} \mathbf{Z}_{\geq 0}\alpha \text{ for all } k \geq p-1. \tag{4.7.25}$$

Note by the definition (4.0.3) that the image of $X_{\beta,k}^{\mathrm{alg}}$ lies inside $H^0(\mu_0^{w_0})_{\mu^*+k\beta}$, which is zero by (4.7.25) assuming $k \geq p-1$. Hence we deduce that

$$X_{\beta,k}^{\mathrm{alg}} = 0 \text{ on } H^0(\mu_0^{w_0})_{\mu^*} \text{ for all } k \geq p-1.$$

Then the conclusion of this lemma follows from the equality (4.1.4). □

We have a natural embedding $H^0(\mu_0^{w_0}) \hookrightarrow \pi_0$ by the definition of algebraic induction and parabolic induction. Recall that we have defined U_1 in Example 4.1.8.

Lemma 4.7.14. *We have*

$$\mathbf{F}_p[S_{\underline{k}^\sharp,w_0} v_0] = H^0(\mu_0^{w_0})_{\mu^*}^{\overline{U}_1}.$$

In particular,

$$V^\sharp \subseteq H^0(\mu_0^{w_0}).$$

Proof. It follows from Corollary 4.5.3 that

$$\dim_{\mathbf{F}_p} H^0(\mu_0^{w_0})_{\mu^*}^{\overline{U}_1} = 1,$$

and this space is generated by $v_{\{n-1\},\underline{m}^\sharp}^{\mathrm{alg}}$ where

$$\underline{m}^\sharp = (m_1^\sharp, \cdots, m_{n-1}^\sharp) := (0, \cdots, 0, [a_0 - a_{n-1}]_1 + n - 2). \tag{4.7.26}$$

We now need to identify the vector $v_{\{n-1\},\underline{m}^\sharp}^{\mathrm{alg}}$ with certain linear combination of Jacobi sums. Note that by Corollary 4.5.3 we have

$$v_{\{n-1\},\underline{m}^\sharp}^{\mathrm{alg}} = D_n^{a_{n-1}-p+1} D_{n-1}^{a_1-a_0-n+2} (D_{\{n-1\},n-1})^{[a_0-a_{n-1}]_1+n-2} D_1^{[a_0-a_{n-2}]_1} \prod_{i=2}^{n-2} D_i^{a_{n-i}-a_{n-i-1}}.$$

Given a matrix $A \in G(\mathbf{F}_p)$, then $D_i(A) \neq 0$ for all $1 \leq i \leq n-1$ if and only if

$$A \in B(\mathbf{F}_p)w_0B(\mathbf{F}_p),$$

and thus the support of $v^{\mathrm{alg}}_{\{n-1\},\underline{m}^\sharp}$ is contained in $B(\mathbf{F}_p)w_0B(\mathbf{F}_p)$. As a result, according to Proposition 4.1.4, we know that $v^{\mathrm{alg}}_{\{n-1\},\underline{m}^\sharp}$ is a linear combination of vectors of the form

$$S_{\underline{k},w_0}v_0.$$

As $v^{\mathrm{alg}}_{\{n-1\},\underline{m}^\sharp}$ is $\overline{U}_1$-invariant, and in particular $U_1(\mathbf{F}_p)$-invariant, then by Proposition 4.1.11 we know that it has the form

$$\sum_{\underline{k}} C_{\underline{k}} S_{\underline{k},w_0} v_0 \tag{4.7.27}$$

where we sum over tuples $\underline{k}$ satisfying $k_{1,n} = [a_0 - a_{n-1}]_1 + n - 2$, $k_{1,j} = 0$ or $p-1$ for $2 \leq j \leq n-1$ and $k_{i,j} = 0$ for all $2 \leq i < j \leq n$, and $C_{\underline{k}} \in \mathbf{F}_p$ is a certain constant for each tuple $\underline{k}$.

Finally, note that

$$u_\beta(t)\ v^{\mathrm{alg}}_{\{n-1\},\underline{m}^\sharp} = D_n^{a_{n-1}-p+1} D_{n-1}^{a_1-a_0-n+2} (D_{\{n-1\},n-1} + tD_{n-1})^{[a_0-a_{n-1}]_1+n-2} D_1^{[a_0-a_{n-2}]_1} \prod_{i=2}^{n-2} D_i^{a_{n-i}-a_{n-i-1}}$$

is a polynomial of t with degree $[a_0 - a_{n-1}] + n - 2$, we conclude that

$$X^{\mathrm{alg}}_{\beta,[a_0-a_{n-1}]_1+n-2}\ v^{\mathrm{alg}}_{\{n-1\},\underline{m}^\sharp} = v^{\mathrm{alg}}_{\{n-1\},\underline{0}}$$

where $\underline{0}$ is the $(n-1)$-tuple with all entries zero.

By Lemma 4.7.13 and the fact that

$$\mathbf{F}_p[v^{\mathrm{alg}}_{\{n-1\},\underline{0}}] = \mathbf{F}_p[S_{\underline{0},w_0}v_0] = \pi_0^{U(\mathbf{F}_p),\mu_0^{w_0}},$$

we deduce that

$$X_{\beta,[a_0-a_{n-1}]_1+n-2}\ v^{\mathrm{alg}}_{\{n-1\},\underline{m}^\sharp} = C' S_{\underline{0},w_0} v_0$$

for some constant $C' \in \mathbf{F}_p^\times$. By Lemma 4.7.12 and the linear independence of Jacobi sums proved in Proposition 4.1.4, we know that only the vector $C_{\underline{k}^\sharp} S_{\underline{k}^\sharp,w_0} v_0$ can appear in the sum (4.7.27). In other words, we have shown that

$$v^{\mathrm{alg}}_{\{n-1\},\underline{m}^\sharp} = C'' S_{\underline{k}^\sharp,w_0} v_0$$

for some constant $C'' \in \mathbf{F}_p^\times$, and thus we finish the proof. □

Lemma 4.7.15. *The dual Weyl module $H^0(\mu_0^{w_0})$ is uniserial of length two with socle $F(\mu_0^{w_0})$ and cosocle $F(\mu^*)$.*

Proof. By [Jan03] Proposition II 2.2 we know that $\mathrm{soc}_{\overline{G}}\left(H^0(\mu_0^{w_0})\right)$ is irreducible and can be identified with $F(\mu_0^{w_0})$ (which is in fact the definition of $F(\mu_0^{w_0})$). Therefore it suffices to show that $H^0(\mu_0^{w_0})$ has only two Jordan–Hölder factor $F(\mu_0^{w_0})$ and $F(\mu^*)$, each of which has multiplicity one.

By [Jan03] II 2.13 (2) it is harmless for us to replace $H^0(\mu_0^{w_0})$ by the Weyl module $V(\mu_0^{w_0})$ (defined in [Jan03] II 2.13) and show that $V(\mu_0^{w_0})$ has only two Jordan–Hölder factor $F(\mu_0^{w_0})$ and $F(\mu^*)$ and each of them has multiplicity one. As

$$\begin{cases} p < \left\langle \mu_0^{w_0}, (\sum_{i=1}^{n-1}\alpha_i)^\vee \right\rangle < 2p; \\ 0 < \left\langle \mu_0^{w_0}, (\sum_{i=1}^{n-2}\alpha_i)^\vee \right\rangle < p; \\ 0 < \left\langle \mu_0^{w_0}, (\sum_{i=2}^{n-1}\alpha_i)^\vee \right\rangle < p, \end{cases}$$

we deduce that the only dominant alcove lying below the one containing $\mu_0^{w_0}$ is the lowest p-restricted alcove. In particular, the only dominant weight which is linked to and strictly smaller than $\mu_0^{w_0}$ is μ^*.

By [Jan03] Proposition II 8.19, we know the existence of a filtration of subrepresentation

$$V(\mu_0^{w_0}) \supseteq V_1(\mu_0^{w_0}) \supseteq \cdots$$

such that the following equality in Grothendieck group holds

$$\sum_{i>0} V_i(\mu_0^{w_0}) = F(\mu^*).$$

This equality implies that

$$V_1(\mu_0^{w_0}) = F(\mu^*)$$

and

$$V_i(\mu_0^{w_0}) = 0 \text{ for all } i \geq 2.$$

By [Jan03] II 8.19 (2) we also know that

$$V(\mu_0^{w_0})/V_1(\mu_0^{w_0}) \cong F(\mu_0^{w_0}),$$

and thus we have shown that

$$V(\mu_0^{w_0}) = F(\mu_0^{w_0}) + F(\mu^*)$$

in the Grothendieck group. $\square$

Proposition 4.7.16. *We have*

$$V^\sharp = H^0(\mu_0^{w_0}).$$

Proof. By Lemma 4.7.15, we have the natural surjection

$$H^0(\mu_0^{w_0}) \twoheadrightarrow F(\mu^*)$$

which induces a morphism

$$H^0(\mu_0^{w_0})_{\mu_*} \to F(\mu^*)_{\mu_*}.$$

Now we consider $H^0(\mu_0^{w_0})$ as a $\overline{L}_1$-representation where $L_1 \cong \mathbb{G}_m \times \mathrm{GL}_{n-1}$ is the standard Levi subgroup of G which contains U_1 as a maximal unipotent subgroup. We denote the set of $\lambda \in X(T)$ which is dominant while viewed as a weight of L_1 by $X_{L_1}(T)_+$. Then we use the notation $H^0_{L_1}(\lambda)$ for the dual Weyl module of $\overline{L}_1$ which is defined via the same way as the dual Weyl module of $\overline{G}$ determined by a weight in $X(T)_+$ (cf. the beginning of Section 4). The dual Weyl module $H^0(\mu_0^{w_0})$ is the mod p reduction of a lattice $V_{\mathbf{Z}_p}$ in the unique irreducible algebraic representation $V_{\mathbf{Q}_p}$ of G such that $\left(V_{\mathbf{Q}_p}^U\right)_{\mu_0^{w_0}} \neq 0$. As the category of finite dimensional algebraic representations of L_1 in characteristic 0 is semisimple, $V_{\mathbf{Q}_p}$ decomposes into a direct sum of characteristic 0 irreducible representations of L_1. More precisely, we have the decomposition

$$V_{\mathbf{Q}_p}|_{L_1} = \bigoplus_{\substack{\lambda \in X_{L_1}(T)_+ \\ (V_{\mathbf{Q}_p})_\lambda^{U_1} \neq 0}} m_\lambda V_{L_1}(\lambda)$$

where $V_{L_1}(\lambda)$ is the unique (up to isomorphism) irreducible algebraic representation of L_1 such that $\left(V_{L_1}(\lambda)^{U_1}\right)_\lambda \neq 0$ and

$$m_\lambda := \dim_{\mathbf{Q}_p}\left(V_{\mathbf{Q}_p}^{U_1}\right)_\lambda.$$

Therefore in the Grothendieck group of algebraic representations of $\overline{L}_1$ over $\mathbf{F}_p$, we have

$$[H^0(\mu_0^{w_0})]|_{\overline{L}_1} = \bigoplus_{\substack{\lambda \in X_{L_1}(T)_+ \\ H^0(\mu_0^{w_0})_\lambda^{\overline{U}_1} \neq 0}} m_\lambda [H^0_{L_1}(\lambda)] \tag{4.7.28}$$

as by Corollary 4.5.3 $H^0(\mu_0^{w_0})^{\overline{U}_1}$ is the mod p reduction of $V_{\mathbf{Z}_p}^{U_1}$ and $V_{\mathbf{Z}_p}^{U_1}\otimes_{\mathbf{Z}_p}\mathbf{Q}_p = V_{\mathbf{Q}_p}^{U_1}$.

We use the notation $\widetilde{W}^{L_1}$ for the affine Weyl group associated with the group L_1. We say that

$$\mu^*\uparrow_{L_1}\lambda$$

if there exists $\widetilde{w}\in\widetilde{W}^{L_1}$ such that

$$\lambda=\widetilde{w}\cdot\mu^* \text{ and } \mu^*\leq\lambda.$$

Assume that there exists a $\lambda\in X_{L_1}(T)_+$ such that $\mu^*\uparrow_{L_1}\lambda$ and that $H^0(\mu_0^{w_0})^{\overline{U}_1}_{\lambda}\neq 0$. We denote by $v^{\mathrm{alg}}_{\{n-1\},\underline{m}}$ the vector in $H^0(\mu_0^{w_0})^{\overline{U}_1}_{\lambda}\neq 0$ given by Corollary 4.5.3. We note that by Corollary 4.5.3 the vector in $H^0(\mu_0^{w_0})^{\overline{U}_1}_{\mu^*}$ is $v^{\mathrm{alg}}_{\{n-1\},\underline{m}^\sharp}$ (see (4.7.26)). As $\mu^*\uparrow_{L_1}\lambda$, we must firstly have $\sum_{i=1}^{n-1}m_i=[a_0-a_{n-1}]_1+n-2$. By the last statement in Corollary 4.5.3, we have

$$\lambda=\left(a_0+p-1-\sum_{i=1}^{n-1}m_i, a_{n-2}+m_1,\cdots,a_1+m_{n-2},a_{n-1}-p+1+m_{n-1}\right) \tag{4.7.29}$$
$$=(a_{n-1}-n+2,a_{n-2}+m_1,\cdots,a_1+m_{n-2},a_{n-1}-p+1+m_{n-1}).$$

Recall $\eta=(n-1,n-2,\cdots,1,0)$. We notice that $\mu^*-\eta$ lies in the lowest p-restricted $\overline{L}_1$-alcove in the sense that

$$0<\langle\mu^*,\alpha^\vee\rangle<p \text{ for all } \alpha\in\Phi^+_{L_1} \tag{4.7.30}$$

where $\Phi^+_{L_1}$ is the set of positive roots of L_1 naturally viewed as a subset of Φ^+.

As we assume that $(a_{n-1},\cdots,a_0)$ is n-generic, it is easy to see the following

$$\begin{cases} a_{n-2}+m_1-(a_{n-1}-p+1+m_{n-1})\leq p+1+a_{n-2}-a_{n-1}+m_1<2p;\\ a_{n-2}+m_1-(a_1+m_{n-2})\leq a_{n-2}+m_1-a_1\leq[a_0-a_1]_1<p;\\ a_{n-3}+m_2-(a_{n-1}-p+1+m_{n-1})\leq[a_{n-3}-a_{n-1}]_1+m_2\leq[a_{n-2}-a_{n-1}]_1<p,\end{cases}$$

so that we know that $\lambda-\eta$ lies in either the lowest $\overline{L}_1$-alcove in the sense of (4.7.30) (if we replace μ^* by λ) or the p-restricted $\overline{L}_1$-alcove described by the conditions

$$\begin{cases} p & < & \left\langle\lambda,\left(\sum_{i=2}^{n-1}\alpha_i\right)^\vee\right\rangle & < & 2p\\ 0 & < & \left\langle\lambda,\left(\sum_{i=2}^{n-2}\alpha_i\right)^\vee\right\rangle & < & p\\ 0 & < & \left\langle\lambda,\left(\sum_{i=3}^{n-1}\alpha_i\right)^\vee\right\rangle & < & p\end{cases}$$

and

$$0<\langle\lambda,\alpha^\vee\rangle<p \text{ for all } \alpha\in\Delta_{L_1}$$

where $\Delta_{L_1}:=\{\alpha_i\mid 2\leq i\leq n-1\}$ is the set of simple positive roots in $\Phi^+_{L_1}$.

In the first case, if $\lambda-\eta$ lies in the lowest $\overline{L}_1$-alcove, as we assume that $\mu^*\uparrow_L\lambda$, we must have $\lambda=\mu^*$; in the second case, we must have

$$\lambda=(2,n)\cdot\mu^*+p\left(\sum_{i=2}^{n-1}\alpha_i\right)=(a_{n-1}-n+2,a_0+p,a_{n-3},\cdots,a_1,a_{n-2}+n-2-p)$$

which means by (4.7.29) that

$$\underline{m}=(m_1,\cdots,m_{n-1})=([a_0-a_{n-2}]_1+1,0,\cdots,0,a_{n-2}-a_{n-1}+n-3).$$

This implies $a_{n-2}-a_{n-1}+n-1=m_{n-1}\geq 0$, which is a contradiction to the n-generic assumption on $(a_{n-1},\cdots,a_0)$. Therefore we must have $\lambda=\mu^*$. Hence we deduce by (4.7.28) and the strong linkage principle [Jan03] II 2.12 (1) that $F^{L_1}(\mu^*)$ (see the beginning of Section 5 for notation) has multiplicity one in $\mathrm{JH}_{\overline{L}_1}(H^0(\mu_0^{w_0})|_{\overline{L}_1})$ and is actually a direct summand.

On the other hand, as $F^{L_1}(\mu^*)$ is obviously an $\overline{L}_1$-subrepresentation of $F(\mu^*)$, we know that the surjection of $\overline{G}$-representation $H^0(\mu_0^{w_0}) \twoheadrightarrow F(\mu^*)$ induces an isomorphism of $\overline{L}_1$-representation on the direct summand $F^{L_1}(\mu^*)$ on both sides with multiplicity one, by restriction from $\overline{G}$ to $\overline{L}_1$. In particular, we know that the map

$$H^0(\mu_0^{w_0})_{\mu^*}^{\overline{U}_1} \to F(\mu^*)_{\mu^*}$$

is a bijection, and therefore the composition

$$V^\sharp \hookrightarrow H^0(\mu_0^{w_0}) \twoheadrightarrow F(\mu^*)$$

is non-zero as

$$H^0(\mu_0^{w_0})_{\mu^*}^{\overline{U}_1} = \mathbf{F}_p[v^{\mathrm{alg}}_{\{n-1\},\underline{m}^\sharp}] = \mathbf{F}_p[S_{\underline{k}^\sharp,w_0}v_0]$$

by Lemma 4.7.14. Hence we obtain a surjection

$$V^\sharp \twoheadrightarrow F(\mu^*), \tag{4.7.31}$$

which implies that the injection

$$V^\sharp \hookrightarrow H^0(\mu_0^{w_0})$$

must be an isomorphism as it induces surjection on cosocle according to Lemma 4.7.15 and (4.7.31). The proof is thus finished. □

Theorem 4.7.17. *Assume that $(a_{n-1}, \cdots, a_0)$ is n-generic in the lowest alcove (cf. Definition 4.1.1). Then $H^0(\mu_0^{w_0}) \subseteq V_0$. In particular, we have*

$$F(\mu^*) \in \mathrm{JH}(V_0).$$

Proof. The first inclusion is a direct consequence of Proposition 4.7.16 together with Corollary 4.7.9. The second inclusion follows from the first as we have $F(\mu^*) \in \mathrm{JH}(H^0(\mu_0^{w_0}))$. □

Before we end this section, we need several remarks to summarize the proof, and to clarify the necessity for all the constructions.

Remark 4.7.18. *If we assume that for all $2 \leq k \leq n-2$*

$$[a_0 - a_{n-1}]_1 + n - 2 < a_k - a_{k-1}, \tag{4.7.32}$$

then we can actually show that

$$S_{\underline{k}^0,w_0}v_0 \in H^0(\mu_0^{w_0})_{\mu^*}^{[\overline{U},\overline{U}]}$$

using Corollary 4.1.10 and the case $s = n-1$ of Proposition 4.5.1, and thus

$$V_0 = H^0(\mu_0^{w_0}).$$

Moreover, under the condition (4.7.32), we can even prove that the set

$$\{S_{\underline{k},w_0}v_0 \mid \underline{k} \in \Lambda_{w_0}\}$$

forms a basis for $H^0(\mu_0^{w_0})_{\mu^*}$.

On the other hand, if we have

$$[a_0 - a_{n-1}]_1 + n - 2 \geq a_k - a_{k-1}$$

for some $2 \leq k \leq n-2$, then we can show that

$$F(\mu_0^{s_k w_0}) \in \mathrm{JH}(V_0)$$

which means that the inclusion

$$H^0(\mu_0^{w_0}) \subseteq V_0$$

is actually strict.

In fact, through the proof of Proposition 4.7.8, the subrepresentation of π_0 generated by $S_{\underline{k}^\sharp,r,s}v_0$ is shrinking if r is fixed and s is growing. Therefore the subrepresentation of π_0 generated by $\overline{S}_{\underline{k}^\sharp,r}v_0$

shrinks as r decreases. Finally, we succeeded in shrinking from V_0 to $V^\sharp$ which can be identified with $H^0(\mu_0^{w_0})$.

Remark 4.7.19. *We need to emphasize that the choice of the operators X_r^+ and X_r^- for $2 \leq r \leq n-1$ are crucial. For example, the operator*

$$\sum_{t\in\mathbf{F}_p} t^{p-2} w_0 u_{\alpha_r}(t) w_0 \in \mathbf{F}_p[G(\mathbf{F}_p)]$$

for some $2 \leq r \leq n-2$ does not work in general. The reason is that, as one can check by explicit computation, applying such operator to $S_{\underline{k}w_0}v_0$ for some $\underline{k} \in \Lambda_{w_0}$ will generally give us a huge linear combination of Jacobi sum operators. From our point of view, it is basically impossible to compute such a huge linear combination explicitly and systematically. Instead, as stated in Proposition 4.6.5, our operators X_r^+ and X_r^- can be computed systematically, even though the computation is still complicated.

The motivation of the choice of operators X_r^+ and X_r^- can be roughly explained as follows. First of all, we need one 'weight raising operator' X^+ and one 'weight lowering operator' X^-. These are two operators lying in a subalgebra $\mathbf{F}_p\langle X^+, X^-\rangle$ of $\mathbf{F}_p[G(\mathbf{F}_p)]$ such that

$$\mathbf{F}_p\langle X^+, X^-\rangle \cong \mathbf{F}_p[\mathrm{GL}_2(\mathbf{F}_p)].$$

We start with the vector $S_{\underline{k},w_0}v_0$ for some $\underline{k} \in \Lambda_{w_0}$. We apply the operator X^- once and then X^+ once, the result is a vector with the same $T(\mathbf{F}_p)$-eigencharacter μ^*. We observe that $S_{\underline{k},w_0}v_0$ is in general not an eigenvector of the operator $X^+ \bullet X^-$ because the representation π_0, after restricting from $\mathbf{F}_p[G(\mathbf{F}_p)]$ to $\mathbf{F}_p\langle X^+, X^-\rangle$, is highly non-semisimple. The naive expectation is that we just take the difference

$$X^+ \bullet X^- \bullet S_{\underline{k},w_0}v_0 - cS_{\underline{k},w_0}v_0$$

for some constant $c \in \mathbf{F}_p$, and then repeat the procedure by applying some other operators similar to X^+ and X^-.

The case $n = 3$ is easy. In the case $n = 4$, the operator

$$\sum_{t\in\mathbf{F}_p} t^{p-2} w_0 u_{\alpha_2}(t) w_0 \in \mathbf{F}_p[\mathrm{GL}_4(\mathbf{F}_p)]$$

is not well behaved as we explained in this remark, and therefore we are forced to use our X_2^- to replace $\sum_{t\in\mathbf{F}_p} t^{p-2} w_0 u_{\alpha_2}(t) w_0$.

Now we consider the general case, and it is possible for us to carry on an induction step. We have an increasing sequence of subgroups of $\overline{G}$

$$\overline{P}_{\{n-1\}} \subsetneq \overline{P}_{\{n-2,n-1\}} \subsetneq \cdots \subsetneq \overline{P}_{\{2,\cdots,n-1\}}$$

and

$$\overline{L}_{\{n-1\}} \subsetneq \overline{L}_{\{n-2,n-1\}} \subsetneq \cdots \subsetneq \overline{L}_{\{2,\cdots,n-1\}}$$

where $\overline{P}_{\{r,\cdots,n-1\}}$ is the standard parabolic subgroup corresponding to the simple roots α_k for $r \leq k \leq n-1$ and $\overline{L}_{\{r,\cdots,n-1\}}$ is its standard Levi subgroup. Technically speaking, constructing the vector $S_{\underline{k}^{\sharp,r+1},w_0}v_0$ (for some $1 \leq r \leq n-2$) from $S_{\underline{k}^0,w_0}v_0$ should be reduced to Corollary 4.7.9 when we replace $\overline{G}$ by its Levi subgroup $\overline{L}_{\{r+1,\cdots,n-1\}}$. In other words, to construct $S_{\underline{k}^{\sharp,r+1},w_0}v_0$ from $S_{\underline{k}^0,w_0}v_0$ we only need the operators

$$X_k^+, X_k^- \in \mathbf{F}_p[\overline{L}_{\{r+2,\cdots,n-1\}}(\mathbf{F}_p)] \subsetneq \mathbf{F}_p[\overline{L}_{\{r+1,\cdots,n-1\}}(\mathbf{F}_p)]$$

for all $r+2 \leq k \leq n-1$.

In order to construct $S_{\underline{k}^{\sharp,r},w_0}v_0$ from $S_{\underline{k}^{\sharp,r+1},w_0}v_0$, we only need to prove Proposition 4.7.8. Then we summarize the proof of Proposition 4.7.8 as the following: for some $a \in \mathbf{F}_p^\times$ and $b \in \mathbf{F}_p$

$$X_{r+1}^+ \bullet X_{r+1}^- \bullet S_{\underline{k}^{\sharp,r,s-1},w_0}v_0 \equiv aS_{\underline{k}^{\sharp,r,s},w_0}v_0 + bS_{\underline{k}^{\sharp,r,s-1},w_0}v_0 + \textit{error terms}$$

and the error terms can be killed by combinations of the operators X_k^+, X_k^- for $r+2 \leq k \leq n-1$.

4.8. **Main results in characteristic p.** In this section, we summary our main results on certain Jacobi sum operators in characteristic p.

We recall two important Jacobi sum operators $\mathcal{S}_n$ and $\mathcal{S}'_n$ from (4.4.11) and recall from (4.7.3) that V_0 is the sub-representations of π_0 generated by

$$\mathcal{S}_{\underline{k}^0,w_0}\left(\pi_0^{U(\mathbf{F}_p),\mu_0}\right).$$

We also define V_1 and V'_1 as the sub-representations of π_0 generated by

$$\mathcal{S}_n\left(\pi_0^{U(\mathbf{F}_p),\mu_1}\right) \text{ and } \mathcal{S}'_n\left(\pi_0^{U(\mathbf{F}_p),\mu'_1}\right)$$

respectively.

The following theorem, which we usually call the *non-vanishing theorem*, is a technical heart on the local automorphic side. The proofs of this non-vanishing theorem as well as the next theorem, which we usually call the *multiplicity one theorem*, have occupied the previous sections.

Theorem 4.8.1. *Assume that $(a_{n-1},\cdots,a_0)$ is n-generic in the lowest alcove.*

Then we have

$$V_1 = V'_1 = V_0$$

and

$$F(\mu^*) \in \mathrm{JH}(V_0).$$

Proof. This is an immediate consequence of Corollary 4.4.8 and Theorem 4.7.17. □

We also have the following multiplicity one result.

Theorem 4.8.2. *Assume that $(a_{n-1},\cdots,a_0)$ is $2n$-generic in the lowest alcove.*

Then $F(\mu^*)$ has multiplicity one in π_0.

Proof. This is a special case of Corollary 4.3.7: replace $\mu_\pi^{0,n-1}$ with μ^*. □

Corollary 4.8.3. *Assume that $(a_{n-1},\cdots,a_0)$ is $2n$-generic in the lowest alcove and that τ is an $\mathcal{O}_E$-lattice in $\widetilde{\pi}_0^\circ \otimes_{\mathcal{O}_E} E$ such that*

$$\mathrm{soc}_{G(\mathbf{F}_p)}\left(\tau \otimes_{\mathcal{O}_E} \mathbf{F}\right) = F(\mu^*).$$

Then we have

$$\mathcal{S}_n\left(\left(\tau \otimes_{\mathcal{O}_E} \mathbf{F}\right)^{U(\mathbf{F}_p),\mu_1}\right) \neq 0 \quad \textit{and} \quad \mathcal{S}'_n\left(\left(\tau \otimes_{\mathcal{O}_E} \mathbf{F}\right)^{U(\mathbf{F}_p),\mu'_1}\right) \neq 0.$$

Proof. Such a τ is unique up to homothety by Theorem 4.8.2. By multiplying a suitable power of ϖ_E, we may assume that

$$\widetilde{\pi}_0^\circ \subsetneq \tau \text{ and } \widetilde{\pi}_0^\circ \nsubseteq \varpi\tau,$$

and thus we have a non-zero morphism

$$\pi_0 \to \tau \otimes_{\mathcal{O}_E} \mathbf{F}$$

whose image is the unique quotient of π_0 with socle $F(\mu^*)$. We now finish the proof by applying Theorem 4.8.1. □

Remark 4.8.4. *Theorem 4.8.1 and Corollary 4.8.3 can be generalized to the case when μ^* is replaced by any weight lying sufficiently deep in an arbitrary p-restricted alcove except the highest one. The crucial points here are the $[U(\mathbf{F}_p),U(\mathbf{F}_p)]$-invariance of $\mathcal{S}_n$ (resp. $\mathcal{S}'_n$) and that τ (in Corollary 4.8.3) is one of the simplest lattices of $\widetilde{\pi}_0^\circ \otimes_{\mathcal{O}_E} E$ apart from those coming from parabolic inductions from $B(\mathbf{F}_p)$.*

5. Mod p local-global compatibility

In this section, we state and prove our main results on mod p local-global compatibility, which is a global application of our local results of Sections 3 and 4. In the first two sections, we recall some necessary known results on algebraic automorphic forms and Serre weights, for which we closely follow [EGH15], [HLM], and [BLGG].

We first fix some notation for the whole section. Let $P \supseteq B$ be an arbitrary standard parabolic subgroup and N its unipotent radical. We denote the opposite parabolic by $P^- := w_0 P w_0$ with corresponding unipotent radical $N^- := w_0 N w_0$. We fix a standard choice of Levi subgroup $L := P \cap P^- \subseteq G$. We denote the positive roots of L defined by the pair $(B \cap L, T)$ by Φ_L^+. We use

$$X_L(T)_+ := \{\lambda \in X(T) \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Phi_L^+\}$$

to denote the set of dominant weights with respect to the pair $(B \cap L, T)$. We denote the Weyl group of L by W^L and identify it with a subgroup of W. The longest Weyl element in W^L is denoted by w_0^L. We define the affine Weyl group $\widetilde{W}^L$ of L as the semi-direct product of W^L and $X(T)$ with respect to the natural action of W^L on $X(T)$. Therefore $\widetilde{W}^L$ has a natural embedding into $\widetilde{W}$. We define the groups $\overline{G}$, $\overline{P}$, $\overline{L}$, $\cdots$ to be the base change of G, P, L, $\cdots$ to $\mathbf{F}_p$, respectively.

We also need to define several open compact subgroups of $L(\mathbf{Q}_p)$. We define

$$K^L := L(\mathbf{Z}_p),$$

and via the mod p reduction map

$$\mathrm{red}^L : K^L = L(\mathbf{Z}_p) \twoheadrightarrow L(\mathbf{F}_p)$$

we also define $K^L(1)$, $I^L(1)$, and I^L as follows:

$$\begin{aligned} K^L(1) := (\mathrm{red}^L)^{-1}(1) \quad &\subseteq \ I^L(1) := (\mathrm{red}^L)^{-1}(U(\mathbf{F}_p) \cap L(\mathbf{F}_p)) \\ &\subseteq \ I^L := (\mathrm{red}^L)^{-1}(B(\mathbf{F}_p) \cap L(\mathbf{F}_p)). \end{aligned}$$

For any dominant weight $\lambda \in X(T)_+$, we let

$$H_L^0(\lambda) := \left(\mathrm{Ind}_{\overline{B} \cap \overline{L}}^{\overline{L}} w_0^L \lambda\right)^{\mathrm{alg}}_{/\mathbf{F}_p}$$

be the associated dual Weyl module of L. We also write $F^L(\lambda) := \mathrm{soc}_{\overline{L}}\left(H_L^0(\lambda)\right)$ for its irreducible socle as an algebraic representation of $\overline{L}$. Through a similar argument presented at the beginning of Section 4, the notation $F^L(\lambda)$ is well defined as an irreducible representation of $L(\mathbf{F}_p)$ if $\lambda \in T(\mathbf{F}_p)$ is p-regular, namely lies in the image of $X_1^{\mathrm{reg}}(T) \to X(T)/(p-1)X(T)$. We will sometimes abuse the notation $F^L(\lambda)$ for $F^L(\lambda) \otimes_{\mathbf{F}_p} \mathbf{F}$ or $F^L(\lambda)$ for $F^L(\lambda) \otimes_{\mathbf{F}_p} \overline{\mathbf{F}}_p$ in the literature. We will emphasize the abuse of the notation $F^L(\lambda)$ each time we do so.

We introduce some specific standard parabolic subgroups of G. Fix integers i_0 and j_0 such that $0 \leq j_0 < j_0 + 1 < i_0 \leq n-1$, and let i_1 and j_1 be the integers determined by the equation

$$i_0 + i_1 = j_0 + j_1 = n - 1. \tag{5.0.1}$$

We let $P_{i_1,j_1} \supset B$ be the standard parabolic subgroup of $G = \mathrm{GL}_n$ corresponding to the subset $\{\alpha_k \mid j_0 + 1 \leq k \leq i_0\}$ of Δ. By specifying the notation for general P to P_{i_1,j_1}, we can define $P^-_{i_1,j_1}$, L_{i_1,j_1}, N_{i_1,j_1} and $N^-_{i_1,j_1}$. We can naturally embed $\mathrm{GL}_{j_1-i_1+1}$ into G with its image denoted by G_{i_1,j_1} such that $L_{i_1,j_1} = G_{i_1,j_1} T$:

$$\mathrm{GL}_{j_1-i_1+1} \xrightarrow{\sim} G_{i_1,j_1} \hookrightarrow L_{i_1,j_1} \hookrightarrow P_{i_1,j_1} \hookrightarrow G.$$

We define T_{i_1,j_1} to be the maximal tori of G_{i_1,j_1} that is contained in T, and define $X(T_{i_1,j_1})$ to be the character group of T_{i_1,j_1}. If i_1 and j_1 are clear from the context (or equivalently i_0 and j_0 are clear) then we often write P, P^- L, N, and N^- for P_{i_1,j_1}, $P^-_{i_1,j_1}$, L_{i_1,j_1}, N_{i_1,j_1}, and $N^-_{i_1,j_1}$, respectively.

5.1. **The space of algebraic automorphic forms.** Let $F/\mathbf{Q}$ be a CM field with maximal totally real subfield F^+. We write c for the generator of $\mathrm{Gal}(F/F^+)$, and let S_p^+ (resp. S_p) be the set of places of F^+ (resp. F) above p. For v (resp. w) a finite place of F^+ (resp. F) we write k_v (resp. k_w) for the residue field of F_v^+ (resp. F_w).

From now on, we assume that

- F/F^+ is unramified at all finite places;
- p splits completely in F.

Note that the first assumption above excludes $F^+ = \mathbf{Q}$. We also note that the second assumption is not essential in this section, but it is harmless since we are only interested in $G_{\mathbf{Q}_p}$-representations in this paper. Every place v of F^+ above p further decomposes and we often write $v = ww^c$ in F.

There exists a reductive group G_{n/F^+} satisfying the following properties (cf. [BLGG], Section 2):

- G_n is an outer form of GL_n with $G_{n/F} \cong \mathrm{GL}_{n/F}$,
- G_n is a quasi-split at any finite place of F^+;
- $G_n(F_v^+) \simeq U_n(\mathbf{R})$ for all $v|\infty$.

By [CHT08], Section 3.3, G_n admits an integral model $\mathcal{G}_n$ over $\mathcal{O}_{F^+}$ such that $\mathcal{G}_n \times_{\mathcal{O}_{F^+}} \mathcal{O}_{F_v^+}$ is reductive if v is a finite place of F^+ which splits in F. If v is such a place and w is a place of F above v, then we have an isomorphism

$$\iota_w : \mathcal{G}_n(\mathcal{O}_{F_v^+}) \xrightarrow{\sim} \mathcal{G}_n(\mathcal{O}_{F_w}) \xrightarrow{\sim} \mathrm{GL}_n(\mathcal{O}_{F_w}).$$

We fix this isomorphism for each such place v of F^+.

Define $F_p^+ := F^+ \otimes_{\mathbf{Q}} \mathbf{Q}_p$ and $\mathcal{O}_{F^+,p} := \mathcal{O}_{F^+} \otimes_{\mathbf{Z}} \mathbf{Z}_p$. If W is an $\mathcal{O}_E$-module endowed with an action of $\mathcal{G}_n(\mathcal{O}_{F^+,p})$ and $U \subset G_n(\mathbf{A}_{F^+}^{\infty,p}) \times \mathcal{G}_n(\mathcal{O}_{F^+,p})$ is a compact open subgroup, the space of algebraic automorphic forms on G_n of level U and coefficients in W, which is also an $\mathcal{O}_E$-module, is defined as follows:

$$S(U,W) := \left\{ f : G_n(F^+)\backslash G_n(\mathbf{A}_{F^+}^\infty) \to W \mid f(gu) = u_p^{-1} f(g) \ \forall\, g \in G_n(\mathbf{A}_{F^+}^\infty), u \in U \right\}$$

with the usual notation $u = u^p u_p$ for $u \in U$.

We say that the level U is *sufficiently small* if

$$t^{-1} G_n(F^+) t \cap U$$

has finite order prime to p for all $t \in G_n(\mathbf{A}_{F^+}^\infty)$. We say that U is *unramified* at a finite place v of F^+ if it has a decomposition

$$U = \mathcal{G}_n(\mathcal{O}_{F_v^+}) U^v$$

for some compact open $U^v \subset G_n(\mathbf{A}_{F^+}^{\infty,v})$. If w is a finite place of F, then we say, by abuse of notation, that w is an unramified place for U or U is unramified at w if U is unramified at $w|_{F^+}$.

For a compact open subgroup U of $G_n(\mathbf{A}_{F^+}^{\infty,p}) \times \mathcal{G}_n(\mathcal{O}_{F^+,p})$, we let $\mathcal{P}_U$ denote the set consisting of finite places w of F such that

- $w|_{F^+}$ is split in F,
- $w \notin S_p$,
- U is unramified at w.

For a subset $\mathcal{P} \subseteq \mathcal{P}_U$ of finite complement and closed with respect to complex conjugation we write $\mathbf{T}^{\mathcal{P}} = \mathcal{O}_E[T_w^{(i)},\ w \in \mathcal{P},\ i \in \{0, 1, \cdots, n\}]$ for the universal Hecke algebra on $\mathcal{P}$, where the Hecke operator $T_w^{(i)}$ acts on $S(U,W)$ via the usual double coset operator

$$\iota_w^{-1} \left[\mathrm{GL}_n(\mathcal{O}_{F_w}) \begin{pmatrix} \varpi_w \mathrm{Id}_i & 0 \\ 0 & \mathrm{Id}_{n-i} \end{pmatrix} \mathrm{GL}_n(\mathcal{O}_{F_w}) \right]$$

where ϖ_w is a uniformizer of $\mathcal{O}_{F_w}$ and Id_i is the identity matrix of size i. The Hecke algebra $\mathbf{T}^{\mathcal{P}}$ naturally acts on $S(U,W)$.

Recall that we assume that p splits completely in F. Following [EGH15], Section 7.1 we consider the subset $(\mathbf{Z}^n_+)^{S_p}_0$ consisting of dominant weights $\underline{a} = (\underline{a}_w)_w$ where $\underline{a}_w = (a_{1,w}, a_{2,w}, \cdots, a_{n,w})$ satisfying

$$a_{i,w} + a_{n+1-i,w^c} = 0 \tag{5.1.1}$$

for all $w \in S_p$ and $1 \leq i \leq n$. We let

$$W_{\underline{a}_w} := M_{\underline{a}_w}(\mathcal{O}_{F_w}) \otimes_{\mathcal{O}_{F_w}} \mathcal{O}_E$$

where $M_{\underline{a}_w}(\mathcal{O}_{F_w})$ is the $\mathcal{O}_{F_w}$-specialization of the dual Weyl module associated to $\underline{a}_w$ (cf. [EGH15], Section 4.1.1); by condition (5.1.1), one deduces an isomorphism of $\mathcal{G}_n(\mathcal{O}_{F^+_v})$-representations $W_{\underline{a}_w} \circ \iota_w \cong W_{\underline{a}_{w^c}} \circ \iota_{w^c}$. Therefore, by letting $W_{\underline{a}_v} := W_{\underline{a}_w} \circ \iota_w$ for any place $w|v$, the $\mathcal{O}_E$-representation of $\mathcal{G}_n(\mathcal{O}_{F^+,p})$

$$W_{\underline{a}} := \bigotimes_{v|p} W_{\underline{a}_v}$$

is well-defined.

For a weight $\underline{a} \in (\mathbf{Z}^n_+)^{S_p}_0$, let us write $S_{\underline{a}}(\overline{\mathbf{Q}}_p)$ to denote the inductive limit of the spaces $S(U, W_{\underline{a}}) \otimes_{\mathcal{O}_E} \overline{\mathbf{Q}}_p$ over the compact open subgroups $U \subset G_n(\mathbf{A}^{\infty,p}_{F^+}) \times \mathcal{G}_n(\mathcal{O}_{F^+,p})$. (Note that the transition maps are induced, in a natural way, from the inclusions between levels U.) Then $S_{\underline{a}}(\overline{\mathbf{Q}}_p)$ has a natural left action of $G_n(\mathbf{A}^\infty_{F^+})$ induced by right translation of functions.

We briefly recall the relation between the space $\mathcal{A}$ of classical automorphic forms and the previous spaces of algebraic automorphic forms in the particular case which is relevant to us. Fix an isomorphism $\imath : \overline{\mathbf{Q}}_p \xrightarrow{\sim} \mathbf{C}$ for the rest of the paper. As we did for the $\mathcal{O}_{F_w}$-specialization of the dual Weyl modules, we define a finite dimensional $G_n(F^+ \otimes_{\mathbf{Q}} \mathbf{R})$-representation $\sigma_{\underline{a}} \cong \bigoplus_{v|\infty} \sigma_{\underline{a}_v}$ with $\mathbf{C}$-coefficients. (We refer to [EGH15], Section 7.1.4 for the precise definition of $\sigma_{\underline{a}}$.)

Lemma 5.1.1 ([EGH15], Lemma 7.1.6)**.** *The isomorphism $\imath : \overline{\mathbf{Q}}_p \xrightarrow{\sim} \mathbf{C}$ induces an isomorphism of smooth $G_n(\mathbf{A}^\infty_{F^+})$-representations*

$$S_{\underline{a}}(\overline{\mathbf{Q}}_p) \otimes_{\overline{\mathbf{Q}}_p, \imath} \mathbf{C} \xrightarrow{\imath} \mathrm{Hom}_{G_n(F^+ \otimes_{\mathbf{Q}} \mathbf{R})}(\sigma^\vee_{\underline{a}}, \mathcal{A})$$

for any $\underline{a} \in (\mathbf{Z}^n_+)^{S_p}_0$.

The following theorem guarantees the existence of Galois representations attached to automorphic forms on the unitary group G_n. We let $|\ |^{\frac{1-n}{2}} : F^\times \to \overline{\mathbf{Q}}_p^\times$ denote the unique square root of $|\ |^{1-n}$ whose composite with $\iota : \overline{\mathbf{Q}}_p \xrightarrow{\sim} \mathbf{C}$ takes positive values.

Theorem 5.1.2 ([EGH15], Theorem 7.2.1)**.** *Let Π be an irreducible $G_n(\mathbf{A}^\infty_{F^+})$-subrepresentation of $S_{\underline{a}}(\overline{\mathbf{Q}}_p)$.*

Then there exists a continuous semisimple representation

$$r_\Pi : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$$

such that

(i) $r^c_\Pi \otimes \varepsilon^{n-1} \cong r^\vee_\Pi$;

(ii) *for each place w above p, the representation $r_\Pi|_{G_{F_w}}$ is de Rham with Hodge–Tate weights*

$$\mathrm{HT}(r_\Pi|_{G_{F_w}}) = \{a_{1,w} + (n-1), a_{2,w} + (n-2), \cdots, a_{n,w}\};$$

(iii) *if $w|p$ is a place of F and $v := w|_{F^+}$ splits in F, then*

$$\mathrm{WD}(r_\Pi|_{G_{F_w}})^{\mathrm{F-ss}} \cong \mathrm{rec}_w((\Pi_v \circ \iota_w^{-1}) \otimes |\cdot|^{\frac{1-n}{2}}).$$

We note that the fact that (iii) holds without semi-simplification on the automorphic side is one of the main results of [Cara14]. We also note that property (iii) says that the restriction to G_{F_w} is compatible with the local Langlands correspondence at w, which is denoted by rec_w.

5.2. **Serre weights and potentially crystalline lifts.** In this section, we recall the relation of Serre weights and potentially crystalline lifts via (inertial) local Langlands correspondence.

Definition 5.2.1. *A* Serre weight *for $\mathcal{G}_n$ is an isomorphism class of an irreducible smooth $\overline{\mathbf{F}}_p$-representation V of $\mathcal{G}_n(\mathcal{O}_{F^+,p})$. If v is a place of F^+ above p, then a* Serre weight at v *is an isomorphism class of an irreducible $\overline{\mathbf{F}}_p$-smooth representation V_v of $\mathcal{G}_n(\mathcal{O}_{F_v^+})$. Finally, if w is a place of F above p, a* Serre weight at w *is an isomorphism class of an irreducible $\overline{\mathbf{F}}_p$-smooth representation V_w of $\mathrm{GL}_n(\mathcal{O}_{F_w})$.*

We will often say a Serre weight for a Serre weight for $\mathcal{G}_n$ if $\mathcal{G}_n$ is clear from the context. A smooth representation defined over a finite extension of $\mathbf{F}_p$ is often called a Serre weight if it is absolutely irreducible. Note that if V_v is a Serre weight at v, there is an associated Serre weight at w above v defined by $V_v \circ \iota_w^{-1}$.

As explained in [EGH15], Section 7.3, a Serre weight V admits an explicit description in terms of $\mathrm{GL}_n(k_w)$-representations. More precisely, let w be a place of F above p and write $v := w|_{F^+}$. For any n-tuple of integers $\underline{a}_w := (a_{1,w}, a_{2,w}, \cdots, a_{n,w}) \in \mathbf{Z}_+^n$, that is p-restricted (i.e., $0 \leq a_{i,w} - a_{i+1,w} \leq p-1$ for $i = 1, 2, \cdots, n-1$), we consider the Serre weight $F(\underline{a}_w) := F(a_{1,w}, a_{2,w}, \cdots, a_{n,w})$, as defined in [EGH15], Section 4.1.2. It is an irreducible $\overline{\mathbf{F}}_p$-representation of $\mathrm{GL}_n(k_w)$ and of $\mathcal{G}_n(k_v)$ via the isomorphism ι_w. Note that $F(a_{1,w}, a_{2,w}, \cdots, a_{n,w})^\vee \circ \iota_{w^c} \cong F(a_{1,w}, a_{2,w}, \cdots, a_{n,w}) \circ \iota_w$ as $\mathcal{G}_n(k_v)$-representations, i.e. $F(\underline{a}_{w^c}) \circ \iota_{w^c} \cong F(\underline{a}_w) \circ \iota_w$ if $a_{i,w} + a_{n+1-i,w^c} = 0$ for all $1 \leq i \leq n$. Hence, if $\underline{a} = (\underline{a}_w)_w \in (\mathbf{Z}_+^n)_0^{S_p}$ that is p-restricted, then we can set $F_{\underline{a}_v} := F(\underline{a}_w) \circ \iota_w$ for $w|v$. We also set

$$F_{\underline{a}} := \bigotimes_{v|p} F_{\underline{a}_v}$$

which is a Serre weight for $\mathcal{G}_n(\mathcal{O}_{F^+,p})$. From [EGH15], Lemma 7.3.4 if V is a Serre weight for $\mathcal{G}_n$, there exists a p-restricted weight $\underline{a} = (\underline{a}_w)_w \in (\mathbf{Z}_+^n)_0^{S_p}$ such that V has a decomposition $V \cong \bigotimes_{v|p} V_v$ where the V_v are Serre weights at v satisfying $V_v \circ \iota_w^{-1} \cong F(\underline{a}_w)$.

Recall that we write $\mathbf{F}$ for the residue field of E.

Definition 5.2.2. *Let $\overline{r} : G_F \to \mathrm{GL}_n(\mathbf{F})$ be an absolutely irreducible continuous Galois representation and let V be a Serre weight for $\mathcal{G}_n$. We say that $\overline{r}$ is* automorphic of weight V *(or that V is a Serre weight of $\overline{r}$) if there exists a compact open subgroup U in $G_n(\mathbf{A}_F^{\infty,p}) \times \mathcal{G}_n(\mathcal{O}_{F^+,p})$ unramified above p and a cofinite subset $\mathcal{P} \subseteq \mathcal{P}_U$ such that $\overline{r}$ is unramified at each place of $\mathcal{P}$ and*

$$S(U, V)_{\mathfrak{m}_{\overline{r}}} \neq 0$$

where $\mathfrak{m}_{\overline{r}}$ is the kernel of the system of Hecke eigenvalues $\overline{\alpha} : \mathbf{T}^{\mathcal{P}} \to \mathbf{F}$ associated to $\overline{r}$, i.e.

$$\det\left(1 - \overline{r}^\vee(\mathrm{Frob}_w)X\right) = \sum_{j=0}^{n} (-1)^j (\mathbf{N}_{F/\mathbf{Q}}(w))^{\binom{j}{2}} \overline{\alpha}(T_w^{(j)}) X^j$$

for all $w \in \mathcal{P}$.

We write $W(\overline{r})$ for the set of automorphic Serre weights of $\overline{r}$. Let w be a place of F above p and $v = w|_{F^+}$. We also write $W_w(\overline{r})$ for the set of Serre weights $F(\underline{a}_w)$ such that

$$(F(\underline{a}_w) \circ \iota_w) \otimes \left(\bigotimes_{v' \in S_p^+ \setminus \{v\}} V_{v'} \right) \in W(\overline{r})$$

where $V_{v'}$ are Serre weights of $\mathcal{G}_n(\mathcal{O}_{F_{v'}^+})$ for all $v' \in S_p^+ \setminus \{v\}$. We often write $W(\overline{r}|_{G_{F_w}})$ and $W_w(\overline{r}|_{G_{F_w}})$ for $W(\overline{r})$ and $W_w(\overline{r})$ respectively, when the given $\overline{r}|_{G_{F_w}}$ is clearly a restriction of an automorphic representation $\overline{r}$ to G_{F_w}.

Fix a place w of F above p and let $v = w|_{F^+}$. We also fix a compact open subgroup U of $G_n(\mathbf{A}_F^{\infty,p}) \times \mathcal{G}_n(\mathcal{O}_{F^+,p})$ which is sufficiently small and unramified at all places above p. We may write $U = \mathcal{G}_n(\mathcal{O}_{F_v^+}) \times U^v$. If W' is an $\mathcal{O}_E$-module with an action of $\prod_{v' \in S_p^+ \setminus \{v\}} \mathcal{G}_n(\mathcal{O}_{F_{v'}^+})$, we define

$$S(U^v, W') := \varinjlim_{U_v} S(U^v \cdot U_v, W')$$

where the limit runs over all compact open subgroups U_v of $\mathcal{G}_n(\mathcal{O}_{F_v^+})$, endowing W' with a trivial $\mathcal{G}_n(\mathcal{O}_{F_v^+})$-action. Note that $S(U^v, W')$ has a smooth action of $\mathcal{G}_n(F_v^+)$ (given by right translation) and hence of $\mathrm{GL}_n(F_w)$ via ι_w. We also note that $S(U^v, W')$ has an action of $\mathbf{T}^{\mathcal{P}}$ commuting with the smooth action of $\mathcal{G}_n(F_v^+)$, where $\mathcal{P}$ is a cofinite subset of $\mathcal{P}_U$.

Lemma 5.2.3 ([EGH15], Lemma 7.4.3)**.** *Let U be a compact open subgroup of $G_n(\mathbf{A}_F^{\infty,p}) \times \mathcal{G}_n(\mathcal{O}_{F^+,p})$ which is sufficiently small and unramified at all places above p, and $\mathcal{P}$ a cofinite subset of $\mathcal{P}_U$. Fix a place w of F above p and let $v = w|_{F^+}$. Let $V \cong \bigotimes_{v' \in S_p^+} V_{v'}$ be a Serre weight for $\mathcal{G}_n$. Then there is a natural isomorphism of $\mathbf{T}^{\mathcal{P}}$-modules*

$$\mathrm{Hom}_{\mathcal{G}_n(\mathcal{O}_{F_v^+})}\left(V_v^{\vee},\ S(U^v, V')\right) \xrightarrow{\sim} S(U, V)$$

where $V' := \bigotimes_{v' \in S_p^+ \setminus \{v\}} V_{v'}$.

We now recall some formalism related to Deligne–Lusztig representations from Section 4.2. Let w be a place of F above p. For a positive integer m, let $k_{w,m}/k_w$ be an extension satisfying $[k_{w,m} : k_w] = m$, and let $\mathbb{T}$ be an F-stable maximal torus in GL_{n/k_w} where F is the Frobenius morphism. We have an identification from [Her09], Lemma 4.7

$$\mathbb{T}(k_w) \xrightarrow{\sim} \prod_j k_{w,n_j}^{\times}$$

where $n \geq n_j > 0$ and $\sum_j n_j = n$; the isomorphism is unique up to $\prod_j \mathrm{Gal}(k_{w,n_j}/k_w)$-conjugacy. In particular, any character $\theta : \mathbb{T}(k_w) \to \overline{\mathbf{Q}}_p^{\times}$ can be written as $\theta = \otimes_j \theta_j$ where $\theta_j : k_{w,n_j}^{\times} \to \overline{\mathbf{Q}}_p^{\times}$ is a character.

Given an F-stable maximal torus $\mathbb{T}$ and a primitive character θ, we consider the Deligne-Lusztig representation $R_{\mathbb{T}}^{\theta}$ of $\mathrm{GL}_n(k_w)$ over $\overline{\mathbf{Q}}_p$ defined in Section 4.2. Recall from Section 4.2 that $\Theta(\theta_j)$ is a cuspidal representation of $\mathrm{GL}_{n_j}(k_w)$ associated to the primitive character θ_j, we have

$$R_{\mathbb{T}}^{\theta} \cong (-1)^{n-r} \cdot \mathrm{Ind}_{P_{\underline{n}}(k_w)}^{\mathrm{GL}_n(k_w)}(\otimes_j \Theta(\theta_j))$$

where $P_{\underline{n}}$ is the standard parabolic subgroup containing the Levi $\prod_j \mathrm{GL}_{n_j}$ and r denotes the number of its Levi factors.

Let $F_{w,m} := W(k_{w,m})[\frac{1}{p}]$ for a positive integer m. We consider θ_j as a character on $\mathcal{O}_{F_{w,n_j}}^{\times}$ by inflation and we define the following Galois type $\mathrm{rec}(\theta) : I_{F_w} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ as follows:

$$\mathrm{rec}(\theta) := \bigoplus_{j=1}^{r} \left(\bigoplus_{\sigma \in \mathrm{Gal}(k_{w,n_j}/k_w)} \sigma\left(\theta_j \circ \mathrm{Art}_{F_{w,n_j}}^{-1}\right) \right)$$

where θ_j is a primitive character on $k_{w,n_j}^{\times}$ of niveau n_j for each $j = 1, \cdots, r$. Recall that $\mathrm{Art}_{F_{w,n_j}} : F_{w,n_j}^{\times} \to W_{F_{w,n_j}}^{ab}$ is the isomorphism of local class field theory, normalized by sending the uniformizers to the geometric Frobenii.

We quickly review the inertial local Langlands correspondence. Recall that we write $\mathrm{rec}_{\mathbf{Q}_p}$ for the local Langlands correspondence for $\mathrm{GL}_n(\mathbf{Q}_p)$ (cf. Theorem 5.1.2).

Theorem 5.2.4 ([CEGGPS], Theorem 3.7 and [LLL], Proposition 2.3.4)**.** *Let $\tau : I_{\mathbf{Q}_p} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ be a Galois type. Then there exists a finite dimensional irreducible smooth $\overline{\mathbf{Q}}_p$-representation $\sigma(\tau)$ of $\mathrm{GL}_n(\mathbf{Z}_p)$ such that if π is any irreducible smooth $\overline{\mathbf{Q}}_p$-representation of $\mathrm{GL}_n(\mathbf{Q}_p)$ then $\pi|_{\mathrm{GL}_n(\mathbf{Z}_p)}$ contains a unique copy of $\sigma(\tau)$ as a subrepresentation if and only if $\mathrm{rec}_{\mathbf{Q}_p}(\pi)|_{I_{\mathbf{Q}_p}} \cong \tau$ and $N = 0$ on $\mathrm{rec}_{\mathbf{Q}_p}(\pi)$.*

Moreover, if $\tau \cong \oplus_{j=1}^r \tau_j$ and the τ_j are pairwise distinct, then $\sigma(\tau) \cong R_{\mathbb{T}}^{\theta}$ and $\tau \cong \mathrm{rec}(\theta)$ for a maximal torus $\mathbb{T}$ in $\mathrm{GL}_{n/\mathbf{F}_p}$ and a primitive character $\theta : \mathbb{T}(\mathbf{F}_p) \to \overline{\mathbf{Q}}_p^{\times}$.

The following theorem provides a connection between Serre weights and potentially crystalline lifts, which will be useful for the main result, Theorem 5.6.2.

Theorem 5.2.5 ([LLL], Proposition 4.2.5)**.** *Let w be a place of F above p, $\mathbb{T}$ a maximal torus in GL_{n/k_w}, $\theta = \bigotimes_{j=1}^r \theta_j : \mathbb{T}(k_w) \to \overline{\mathbf{Q}}_p^{\times}$ a primitive character such that θ_j are pairwise distinct, and V_w a Serre weight at w for a Galois representation $\overline{r} : G_F \to \mathrm{GL}_n(\mathbf{F})$.*

Assume that V_w is a Jordan-Hölder constituent in the mod p reduction of the Deligne–Lusztig representation $R_{\mathbb{T}}^{\theta}$ of $\mathrm{GL}_n(k_w)$. Then $\overline{r}|_{G_{F_w}}$ has a potentially crystalline lift with Hodge–Tate weights $\{-(n-1), -(n-2), \cdots, 0\}$ and Galois type $\mathrm{rec}(\theta)$.

For a given automorphic Galois representation $\overline{r} : G_F \to \mathrm{GL}_n(\mathbf{F})$, it is quite difficult to determine if a given Serre weight is a Serre weight of $\overline{r}$. Thanks to the work of [BLGG], we have the following theorem, in which we refer the reader to [BLGG] for the unfamiliar terminology.

Theorem 5.2.6 ([BLGG], Theorem 4.1.9)**.** *Assume that if n is even then so is $\frac{n[F^+:\mathbf{Q}]}{2}$, that $\zeta_p \notin F$, and that $\overline{r} : G_F \to \mathrm{GL}_n(\mathbf{F})$ is an absolutely irreducible representation with split ramification. Assume further that there is a RACSDC automorphic representation Π of $\mathrm{GL}_n(\mathbf{A}_F)$ such that*

- *$\overline{r} \simeq \overline{r}_{\Pi}$;*
- *For each place $w|p$ of F, $r_{\Pi}|_{G_{F_w}}$ is potentially diagonalizable;*
- *$\overline{r}(G_{F(\zeta_p)})$ is adequate.*

If $\underline{a} = (\underline{a}_w)_w \in (\mathbf{Z}_+^n)_0^{S_p}$ and for each $w \in S_p$ $\overline{r}|_{G_{F_w}}$ has a potentially diagonalizable crystalline lift with Hodge–Tate weights $\{a_{1,w}+(n-1), a_{2,w}+(n-2), \cdots, a_{n-1,w}+1, a_{n,w}\}$, then a Jordan–Hölder factor of $W_{\underline{a}} \otimes_{\mathbf{Z}_p} \mathbf{F}$ is a Serre weight of $\overline{r}$.

5.3. **Weight elimination and automorphy of a Serre weight.** In this section, we state our main Conjecture for weight elimination (Conjecture 5.3.1) which will be a crucial assumption in the proof of Theorem 5.6.2. We also prove the automorphy of a certain obvious Serre weight under the assumptions of Taylor–Wiles type.

Throughout this section, we assume that $\overline{\rho}_0$ is always a restriction of an automorphic representation $\overline{r} : G_F \to \mathrm{GL}_n(\mathbf{F})$ to G_{F_w} for a fixed place w above p and is generic (cf. Definition 3.0.5). Recall that for $0 \leq j_0 < j_0+1 < i_0 \leq n-1$ we have defined a tuple of integers $(r_{n-1}^{i_0,j_0}, \cdots, r_1^{i_0,j_0}, r_0^{i_0,j_0})$ in (3.7.1), which determines the Galois types as in (1.1.2). In many cases, we will consider the dual of our Serre weights, so that we define a pair of integers (i_1, j_1) by the equation (5.0.1). We also let

$$b_k := -c_{n-1-k}$$

for all $0 \leq k \leq n-1$. We will keep the notation (i_1, j_1) and b_k for the rest of the paper.

For the rest of this section, we are mainly interested in the following characters of $T(\mathbf{F}_p)$: let

$$\mu^{\square} := (b_{n-1}, \cdots, b_0)$$

and

$$\mu^{\square, i_1, j_1} := (y_{n-1}, y_{n-2}, \cdots, y_1, y_0)$$

where

$$y_j = \begin{cases} b_j & \text{if } j \notin \{j_1, i_1\}; \\ b_{i_1} - j_1 + i_1 + 1 & \text{if } j = j_1; \\ b_{j_1} + j_1 - i_1 - 1 & \text{if } j = i_1. \end{cases}$$

As $\overline{\rho}_0$ is generic, each of the characters above is p-regular and thus uniquely determines a p-restricted weight up to a twist in $(p-1)X_0(T)$, and, by abuse of notation, we write $\mu^{\square}$, $\mu^{\square,i_1,j_1}$ for those corresponding p-restricted weights, respectively. We will clarify the twist in $(p-1)X_0(T)$ whenever necessary. We also define a principal series representation

$$\pi_*^{i_1,j_1} := \mathrm{Ind}_{B(\mathbf{F}_p)}^{G(\mathbf{F}_p)}(\mu^{\square,i_1,j_1})^{w_0}. \tag{5.3.1}$$

We now state necessary results of weight elimination to our proof of the main results, Theorem 5.6.2, in this paper.

Conjecture 5.3.1. *Let $\overline{r} : G_F \to \mathrm{GL}_n(\mathbf{F})$ be a continuous automorphic Galois representation with $\overline{r}|_{G_{F_w}} \cong \overline{\rho}_0$ as in (3.0.1). Fix a pair of integers (i_0, j_0) such that $0 \le j_0 < j_0+1 < i_0 \le n-1$, and assume that $\overline{\rho}_{i_0,j_0}$ is Fontaine–Laffaille generic and that $\mu^{\square,i_1,j_1}$ is $2n$-generic.*

Then we have

$$W_w(\overline{r}) \cap \mathrm{JH}((\pi_*^{i_1,j_1})^\vee) \subseteq \{F(\mu^{\square})^\vee, F(\mu^{\square,i_1,j_1})^\vee\}.$$

In an earlier version of this paper, we prove Conjecture 5.3.1 for $n \le 5$. But our method is rather elaborate to execute for general n. But Bao V. Le Hung pointed out that one can prove Conjecture 5.3.1 by constructing certain potentially crystalline deformation rings, and a proof of the conjecture will appear in our forthcoming paper [LHMPQ].

Finally, we prove the automorphy of the Serre weight $F(\mu^{\square})^\vee$.

Proposition 5.3.2. *Keep the assumptions and notation of Conjecture 5.3.1. Assume further that if n is even then so is $\frac{n[F^+:\mathbf{Q}]}{2}$, that $\zeta_p \notin F$, that $\overline{r} : G_F \to \mathrm{GL}_n(\mathbf{F})$ is an irreducible representation with split ramification, and that there is a RACSDC automorphic representation Π of $\mathrm{GL}_n(\mathbf{A}_F)$ such that*

- $\circ$ *$\overline{r} \simeq \overline{r}_\Pi$;*
- $\circ$ *for each place $w'|p$ of F, $r_\Pi|_{G_{F_{w'}}}$ is potentially diagonalizable;*
- $\circ$ *$\overline{r}(G_{F(\zeta_p)})$ is adequate.*

Then

$$\{F(\mu^{\square})^\vee\} \subseteq W_w(\overline{r}) \cap \mathrm{JH}((\pi_*^{i_1,j_1})^\vee).$$

Proof. We prove that $F(\mu^{\square})^\vee = F(c_{n-1}, c_{n-2}, \cdots, c_0) \in W_w(\overline{r})$ as well as $F(\mu^{\square})^\vee \in \mathrm{JH}((\pi_*^{i_1,j_1})^\vee)$. Note that $(c_{n-1}, \cdots, c_0)$ is in the lowest alcove as $\overline{\rho}_0$ is generic, so that by Theorem 5.2.6 it is enough to show that $\overline{\rho}_0$ has a potentially diagonalizable crystalline lift with Hodge–Tate weights $\{c_{n-1} + (n-1), \cdots, c_1 + 1, c_0\}$. Since $\overline{\rho}_0$ is generic, by [BLGGT], Lemma 1.4.3 it is enough to show that $\overline{\rho}_0$ has an ordinary crystalline lift with those Hodge–Tate weights. The existence of such a crystalline lift is immediate by [GHLS], Proposition 2.1.10. On the other hand, we have $F(\mu^{\square})^\vee \in \mathrm{JH}((\pi_*^{i_1,j_1})^\vee$ which is a direct corollary of Theorem 5.5.2. Therefore, we conclude that $F(\mu^{\square})^\vee \in W_w(\overline{r}) \cap \mathrm{JH}((\pi_*^{i_1,j_1})^\vee)$. $\square$

5.4. **Some application of Morita theory.** In this section, we will recall standard results from Morita theory to prove Corollary 5.4.3. We fix here an arbitrary finite group H and a finite dimensional irreducible E-representation V of H. By Proposition 16.16 in [CR90], we know that for any $\mathcal{O}_E$-lattice $V^\circ \subseteq V$, the set $\mathrm{JH}_{\mathbf{F}[H]}(V^\circ \otimes_{\mathcal{O}_E} \mathbf{F})$ depends only on V and is independent of the choice of V°, and thus we will use the notation $\mathrm{JH}_{\mathbf{F}[H]}(\overline{V})$ from now on where $\overline{V} = V^\circ \otimes_{\mathcal{O}_E} \mathbf{F}$ for a randomly chosen V°. We may assume that E is sufficiently large such that E (resp. its residual field $\mathbf{F}$) is a splitting field of V (resp. $\mathrm{JH}_{\mathbf{F}[H]}(\overline{V})$). Let $\mathcal{C}$ be the category of all finitely generated $\mathcal{O}_E$-modules with an H-action which are isomorphic to subquotients of $\mathcal{O}_E$-lattices in $V^{\oplus k}$ for some $k \geq 1$. Then the irreducible objects of $\mathcal{C}$ are just elements of $\mathrm{JH}_{\mathbf{F}[H]}(\overline{V})$. If $\sigma \in \mathrm{JH}_{\mathbf{F}[H]}(\overline{V})$ has multiplicity one in $\overline{V}$, then there is an $\mathcal{O}_E$-lattice V^σ (unique up to homothety by following the proof of Lemma 4.4.1 of [EGS15] as it actually requires only the multiplicity one of σ in our notation) such that

$$\mathrm{cosoc}_H(V^\sigma \otimes_{\mathcal{O}_E} \mathbf{F}) = \sigma.$$

By considering an $\mathcal{O}_E$-lattice in the E-dual of V with the $\mathbf{F}$-dual of σ as cosocle and then taking $\mathcal{O}_E$-dual of this lattice, we reach another $\mathcal{O}_E$-lattice V_σ in V, which is the unique (up to homethety), such that

$$\mathrm{soc}_H(V_\sigma \otimes_{\mathcal{O}_E} \mathbf{F}) = \sigma.$$

By repeating the proof of Lemma 2.3.1, Lemma 2.3.2 and Proposition 2.3.3 in [Le15], we deduce the following.

Proposition 5.4.1. *If σ has multiplicity one in $\overline{V}$, then the lattice V^σ is a projective object in $\mathcal{C}$.*

Note that the proof of Proposition 2.3.3 in [Le15] requires only that the multiplicity of σ in $\overline{V}$ is one, rather than the much stronger condition that each constituent of $\overline{V}$ has multiplicity one.

Corollary 5.4.2. *Let Σ be a subset of $\mathrm{JH}_{\mathbf{F}[H]}(\overline{V})$ such that each $\sigma \in \Sigma$ has multiplicity one in $\overline{V}$. If an $\mathcal{O}_E$-lattice $V^\circ \subseteq V$ satisfies*

$$\mathrm{cosoc}_H(V^\circ \otimes_{\mathcal{O}_E} \mathbf{F}) = \bigoplus_{\sigma\in\Sigma} \sigma \tag{5.4.1}$$

then we have a surjection

$$\bigoplus_{\sigma\in\Sigma} V^\sigma \twoheadrightarrow V^\circ. \tag{5.4.2}$$

Proof. By (5.4.1) we have a surjection

$$V^\circ \twoheadrightarrow \bigoplus_{\sigma\in\Sigma} \sigma.$$

By Proposition 5.4.1 we know that $\bigoplus_{\sigma\in\Sigma} V^\sigma$ is a projective object in $\mathcal{C}$. By the definition of V^σ we know that there is a surjection

$$\bigoplus_{\sigma\in\Sigma} V^\sigma \twoheadrightarrow \bigoplus_{\sigma\in\Sigma} \sigma$$

which can be lifted by projectivity to (5.4.2). □

Note in particular that (5.4.2) implies automatically the surjection

$$\bigoplus_{\sigma\in\Sigma} V^\sigma \otimes_{\mathcal{O}_E} \mathbf{F} \twoheadrightarrow V^\circ \otimes_{\mathcal{O}_E} \mathbf{F}. \tag{5.4.3}$$

Corollary 5.4.3. *Let Σ be a subset of $\mathrm{JH}_{\mathbf{F}[H]}(\overline{V})$ such that each $\sigma \in \Sigma$ has multiplicity one in $\overline{V}$. If an $\mathcal{O}_E$-lattice $V_\circ \subseteq V$ satisfies*

$$\mathrm{soc}_H(V_\circ \otimes_{\mathcal{O}_E} \mathbf{F}) = \bigoplus_{\sigma\in\Sigma} \sigma$$

then we have an injection

$$V_\circ \otimes_{\mathcal{O}_E} \mathbf{F} \hookrightarrow \bigoplus_{\sigma\in\Sigma} V_\sigma \otimes_{\mathcal{O}_E} \mathbf{F}.$$

Proof. This is simply the $\mathbf{F}$-dual of (5.4.3). □

5.5. **Generalization of Section 4.** In this section, we fix a pair of integers (i_0, j_0) satisfying $0 \leq j_0 < j_0 + 1 < i_0 \leq n-1$, and determine (i_1, j_1) by the equation (5.0.1). We will use the shortened notation P (resp. N, L, $P^- \cdots$) for P_{i_1,j_1} (resp. N_{i_1,j_1}, L_{i_1,j_1}, $P^-_{i_1,j_1}$, $\cdots$) as introduced at the beginning of Section 5. Proposition 5.5.5 is crucial for the proof of Theorem 5.6.2. We assume throughout this section that $\mu^{\square,i_1,j_1}$ is $2n$-generic (cf. Definition 4.1.1).

We start this section by defining some weights and Jacobi sum operators which will play a crucial role for our main results, Theorem 5.6.2. Let

$$\mu_1^{i_1,j_1} := (x^1_{n-1}, x^1_{n-2}, \cdots, x^1_1, x^1_0) \ \text{ and } \ \mu_1^{i_1,j_1,\prime} := (x^{1,\prime}_{n-1}, x^{1,\prime}_{n-2}, \cdots, x^{1,\prime}_1, x^{1,\prime}_0)$$

where

$$x^1_j = \begin{cases} b_{n+i_1-j} & \text{if } n-j_1+i_1+1 \leq j \leq n-1; \\ b_{j+j_1-i_1-1} & \text{if } i_1+2 \leq j \leq n-j_1+i_1; \\ b_{j_1}+j_1-i_1-1 & \text{if } j = i_1+1; \\ b_{i_1}-j_1+i_1+1 & \text{if } j = i_1; \\ b_j & \text{if } 0 \leq j \leq i_1-1 \end{cases}$$

and

$$x^{1,\prime}_j = \begin{cases} b_{j_1-1-j} & \text{if } 0 \leq j \leq j_1-i_1-2; \\ b_{j-j_1+i_1+1} & \text{if } j_1-i_1-1 \leq j \leq j_1-2; \\ b_{j_1}+j_1-i_1-1 & \text{if } j = j_1; \\ b_{i_1}-j_1+i_1+1 & \text{if } j = j_1-1; \\ b_j & \text{if } j_1+1 \leq j \leq n-1. \end{cases}$$

We also fix certain two elements in the Weyl group W:

$$w_1^{i_1,j_1} := (s_{n-3-i_1}\cdots s_1)^{j_1-i_1-1} \in W \ \text{ and } \ w_1^{i_1,j_1,\prime} := (s_{n-j_1+2}\cdots s_{n-1})^{j_1-i_1-1} \in W,$$

and further define two more weights

$$\mu^{i_1,j_1} := (\mu_1^{i_1,j_1})^{w_1^{i_1,j_1}} \ \text{ and } \ \mu^{i_1,j_1,\prime} := (\mu_1^{i_1,j_1,\prime})^{w_1^{i_1,j_1,\prime}}.$$

More precisely, μ^{i_1,j_1} and $\mu^{i_1,j_1,\prime}$ can be written as follow:

$$\mu^{i_1,j_1} = (x_{n-1}, x_{n-2}, \cdots, x_1, x_0) \ \text{ and } \ \mu^{i_1,j_1,\prime} = (x'_{n-1}, x'_{n-2}, \cdots, x'_1, x'_0)$$

where

$$x_j = \begin{cases} b_j & \text{if } j > j_1 \text{ or } i_1 > j; \\ b_{j_1+i_1+1-j} & \text{if } j_1 \geq j > i_1+1; \\ b_{j_1}+j_1-i_1-1 & \text{if } j = i_1+1; \\ b_{i_1}-j_1+i_1+1 & \text{if } j = i_1 \end{cases}$$

and

$$x'_j = \begin{cases} b_j & \text{if } j > j_1 \text{ or } i_1 > j; \\ b_{j_1+i_1-1-j} & \text{if } j_1-1 > j \geq i_1; \\ b_{j_1}+j_1-i_1-1 & \text{if } j = j_1; \\ b_{i_1}-j_1+i_1+1 & \text{if } j = j_1-1. \end{cases}$$

Note that if we let

$$w^{i_1,j_1} := s_{n-j_1}\cdots s_{n-i_1-2} \in W^L \ \text{ and } \ w^{i_1,j_1,\prime} := s_{n-i_1-1}\cdots s_{n-j_1+1} \in W^L$$

then we have

$$(\mu^{i_1,j_1})^{w^{i_1,j_1}} = (\mu^{\square,i_1,j_1})^{w_0^L} = (\mu^{i_1,j_1,\prime})^{w^{i_1,j_1,\prime}}.$$

Recall that w_0^L is defined at the beginning of Section 5 and that $\mu^{\square,i_1,j_1}$ is defined in Section 5.3.

We now define certain mod p Jacobi sum operators:

$$\mathcal{S}_1^{i_1,j_1} := S_{\underline{0},w_1^{i_1,j_1}} \ \text{ and } \ \mathcal{S}_1^{i_1,j_1,\prime} := S_{\underline{0},w_1^{i_1,j_1,\prime}}.$$

We further define

$$\mathcal{S}^{i_1,j_1} := S_{\underline{k}^{i_1,j_1},w_0^L} \ \text{ and } \ \mathcal{S}^{i_1,j_1,\prime} := S_{\underline{k}^{i_1,j_1,\prime},w_0^L}$$

where $\underline{k}^{i_1,j_1} = (k^{i_1,j_1}_{i,j})_{i,j} \in \{0,\cdots,p-1\}^{|\Phi^+_{w_0^L}|}$ and $\underline{k}^{i_1,j_1,\prime} = (k^{i_1,j_1,\prime}_{i,j})_{i,j} \in \{0,\cdots,p-1\}^{|\Phi^+_{w_0^L}|}$ satisfy

$$k^{i_1,j_1}_{i,j} := \begin{cases} [b_{i_1} - b_{n-i}]_1 & \text{if } n-j_1+1 \le i = j-1 \le n-i_1-1; \\ i_1 - j_1 + 1 + [b_{i_1} - b_{j_1}]_1 & \text{if } i = j-1 = n-j_1; \\ 0 & \text{if } j \ge i+2 \end{cases}$$

and

$$k^{i_1,j_1,\prime}_{i,j} := \begin{cases} [b_{n-1-i} - b_{j_1}]_1 & \text{if } n-j_1 \le i = j-1 \le n-i_1-2; \\ i_1 - j_1 + 1 + [b_{i_1} - b_{j_1}]_1 & \text{if } i = j-1 = n-i_1-1; \\ 0 & \text{if } j \ge i+2. \end{cases}$$

We now consider characteristic 0 lifts of the mod p Jacobi sum operators above.

$$\widehat{\mathcal{S}}^{i_1,j_1} := \left(\sum_{A \in U_{w_0^L}(\mathbf{F}_p)} \left(\prod_{\ell=n-j_1}^{n-i_1-1} \lceil A_{\ell,\ell+1} \rceil^{k^{i_1,j_1}_{\ell,\ell+1}} \right) \lceil A \rceil \right) w_0^L$$

and

$$\widehat{\mathcal{S}}^{i_1,j_1,\prime} := \left(\sum_{A \in U_{w_0^L}(\mathbf{F}_p)} \left(\prod_{\ell=n-j_1}^{n-i_1-1} \lceil A_{\ell,\ell+1} \rceil^{k^{i_1,j_1,\prime}_{\ell,\ell+1}} \right) \lceil A \rceil \right) w_0^L.$$

We also let

$$\widehat{\mathcal{S}}^{i_1,j_1}_0 := \left(\sum_{A \in U_{w_0^L}(\mathbf{F}_p)} \left(\prod_{\ell=n-j_1}^{n-i_1-1} \lceil A_{\ell,\ell+1} \rceil^{k^{i_1,j_1,0}_{\ell,\ell+1}} \right) \lceil A \rceil \right) w_0^L$$

where $\underline{k}^{i_1,j_1,0} = (k^{i_1,j_1,0}_{i,j})_{i,j} \in \{0,\cdots,p-1\}^{|\Phi^+_{w_0^L}|}$ satisfies

$$k^{i_1,j_1,0}_{i,j} := \begin{cases} i_1 - j_1 + 1 + [b_{i_1} - b_{j_1}]_1 & \text{if } n-j_1 \le i = j-1 \le n-i_1-1; \\ 0 & \text{if } j \ge i+2. \end{cases}$$

Note that $\widehat{\mathcal{S}}^{i_1,j_1}$, $\widehat{\mathcal{S}}^{i_1,j_1,\prime}$, $\widehat{\mathcal{S}}^{i_1,j_1}_0$ are Teichmüller lifts of $\mathcal{S}^{i_1,j_1}$, $\mathcal{S}^{i_1,j_1,\prime}$, $S_{\underline{k}^{i_1,j_1,0},w_0^L}$, respectively. We will also consider the Teichmüller lifts of $\mathcal{S}^{i_1,j_1}_1$ and $\mathcal{S}^{i_1,j_1,\prime}_1$ as follows:

$$\widehat{\mathcal{S}}^{i_1,j_1}_1 := \left(\sum_{A \in U_{w_1^{i_1,j_1}}(\mathbf{F}_p)} \lceil A \rceil \right) w_1^{i_1,j_1} \quad \text{and} \quad \widehat{\mathcal{S}}^{i_1,j_1,\prime}_1 := \left(\sum_{A \in U_{w_1^{i_1,j_1,\prime}}(\mathbf{F}_p)} \lceil A \rceil \right) w_1^{i_1,j_1,\prime}.$$

We recall the operator $\Xi_n \in G(\mathbf{Q}_p)$ from (4.4.1). Note that $\widetilde{\mu}^{i_1,j_1}_1 : T(\mathbf{F}_p) \to \mathcal{O}_E^\times$ is the Teichmüller lift of μ^{i_1,j_1}_1. We also recall $\kappa^{(1)}_n$, $\kappa^{(2)}_n$ (cf. (4.4.10)), κ_n (cf. (4.4.14)), ε^* (cf. (4.4.13)), and $\mathcal{P}_n$ (cf. (4.4.12)), whose definitions are completely determined by fixing the data n and $(a_{n-1},\cdots,a_0)$. We define $\kappa^{(1)}_{i_1,j_1}$, $\kappa^{(2)}_{i_1,j_1}$, $\kappa_{i_1,j_1} \in \mathbf{Z}_p^\times$, $\varepsilon^{i_1,j_1} = \pm 1$ and $\mathcal{P}_{i_1,j_1} \in \mathbf{Z}_p^\times$ by replacing n and $(a_{n-1},\cdots,a_1,a_0)$ by $j_1 - i_1 + 1$ and $(b_{j_1} + j_1 - i_1 - 1, b_{j_1-1}, \cdots, b_{i_1+1}, b_{i_1} - j_1 + i_1 + 1)$ respectively with b_k as at the beginning of Section 5.3.

Proposition 5.5.1. *Assume that $\mu^{\square,i_1,j_1}$ is $2n$-generic. Let*

$$\Pi^{i_1,j_1} := \mathrm{Ind}^{G(\mathbf{Q}_p)}_{B(\mathbf{Q}_p)} \chi^{i_1,j_1}$$

be a tamely ramified principal series where $\chi^{i_1,j_1} = \chi^{i_1,j_1}_{n-1} \otimes \cdots \otimes \chi^{i_1,j_1}_0 : T(\mathbf{Q}_p) \to E^\times$ is a smooth character satisfying $\chi\,|_{T(\mathbf{Z}_p)} \cong \widetilde{\mu}^{i_1,j_1}_1$. Then we have the identity

$$\widehat{\mathcal{S}}^{i_1,j_1,\prime} \bullet \widehat{\mathcal{S}}^{i_1,j_1,\prime}_1 \bullet (\Xi_n)^{j_1-i_1-1} = p^{(j_1-i_1-1)(i_1+1)} \kappa_{i_1,j_1} \left(\prod_{k=n-j_1+i_1+1}^{n-1} \chi^{i_1,j_1}_k(p) \right) \widehat{\mathcal{S}}^{i_1,j_1} \bullet \widehat{\mathcal{S}}^{i_1,j_1}_1$$

on the 1*-dimensional space* $(\Pi^{i_1,j_1})^{I(1),\widetilde{\mu}_1^{i_1,j_1}}$.

Proof. By Lemma 4.4.2 we know that

$$(\Xi_n)^{j_1-i_1-1}\bullet U_n^{j_1-i_1-1}=\widehat{\mathcal{S}}_{(w^*)^{j_1-i_1-1}}.$$

Then by Lemma 4.4.1 and the fact

$$\ell(w_1^{i_1,j_1,\prime})+\ell((w^*)^{j_1-i_1-1})=\ell(w_1^{i_1,j_1,\prime}(w^*)^{j_1-i_1-1}(w_1^{i_1,j_1})^{-1})+\ell(w_1^{i_1,j_1})+2(j_1-i_1-1)i_1$$

we deduce that

$$\widehat{\mathcal{S}}_{w_1^{i_1,j_1,\prime}}\bullet\widehat{\mathcal{S}}_{(w^*)^{j_1-i_1-1}}=p^{(j_1-i_1-1)i_1}\widehat{\mathcal{S}}_{w_1^{i_1,j_1,\prime}(w^*)^{j_1-i_1-1}(w_1^{i_1,j_1})^{-1}}\bullet\widehat{\mathcal{S}}_{w_1^{i_1,j_1}}.$$

Therefore it remains to show that

$$\widehat{\mathcal{S}}^{i_1,j_1,\prime}\bullet\widehat{\mathcal{S}}_{w_1^{i_1,j_1,\prime}(w^*)^{j_1-i_1-1}(w_1^{i_1,j_1})^{-1}}=p^{j_1-i_1-1}\kappa_{i_1,j_1}\widehat{\mathcal{S}}^{i_1,j_1}$$

on the 1-dimensional space

$$(\Pi^{i_1,j_1})^{I(1),\widetilde{\mu}^{i_1,j_1}}=\widehat{\mathcal{S}}_{w_1^{i_1,j_1}}\left((\Pi^{i_1,j_1})^{I(1),\widetilde{\mu}_1^{i_1,j_1}}\right).$$

We observe by Lemma 4.4.1 that

$$\widehat{\mathcal{S}}_{w^{i_1,j_1,\prime}}\bullet\widehat{\mathcal{S}}_{w_1^{i_1,j_1,\prime}(w^*)^{j_1-i_1-1}(w_1^{i_1,j_1})^{-1}}=p^{j_1-i_1-1}\widehat{\mathcal{S}}_{w^{i_1,j_1}}$$

and therefore by composing $\widehat{\mathcal{S}}_0^{i_1,j_1}$ it remains to show that

$$\widehat{\mathcal{S}}_0^{i_1,j_1}\bullet\widehat{\mathcal{S}}_{w^{i_1,j_1,\prime}}=p^{j_1-i_1-1}(\kappa_{i_1,j_1}^{(2)})^{-1}\widehat{\mathcal{S}}^{i_1,j_1,\prime}\tag{5.5.1}$$

on $(\Pi^{i_1,j_1})^{I(1),\widetilde{\mu}^{i_1,j_1,\prime}}$ and

$$\widehat{\mathcal{S}}_0^{i_1,j_1}\bullet\widehat{\mathcal{S}}_{w^{i_1,j_1}}=p^{j_1-i_1-1}(\kappa_{i_1,j_1}^{(1)})^{-1}\widehat{\mathcal{S}}^{i_1,j_1}\tag{5.5.2}$$

on $(\Pi^{i_1,j_1})^{I(1),\widetilde{\mu}^{i_1,j_1}}$. But these can be checked by the same argument as in Corollary 4.4.7. □

We state here a generalization of the Theorem 4.8.2. Recall the definition of $\pi_*^{i_1,j_1}$ from (5.3.1).

Theorem 5.5.2. *The constituent* $F(\mu^\square)$ *has multiplicity one in* $\pi_*^{i_1,j_1}$.

Proof. This is Corollary 4.3.7 if we replace $\mu_\pi^{i_1,j_1}$ by $\mu^\square$. □

We define a characteristic 0 principal series

$$(\widetilde{\pi}_*^{i_1,j_1})^\circ:=\mathrm{Ind}_{B(\mathbf{F}_p)}^{G(\mathbf{F}_p)}(\widetilde{\mu}^{\square,i_1,j_1})^{w_0}$$

which is an $\mathcal{O}_E$-lattice in $(\widetilde{\pi}_*^{i_1,j_1})^\circ\otimes_{\mathcal{O}_E}E$.

Lemma 5.5.3. (i) *For* $\mu\in\{\mu^{i_1,j_1},\mu^{i_1,j_1,\prime},\mu_1^{i_1,j_1},\mu_1^{i_1,j_1,\prime}\}$, *we have*

$$\dim_{\mathbf{F}_p}(\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),\mu}=1.$$

(ii) *We have the following non-vanishing results:*

$$\mathcal{S}^{i_1,j_1}\left((\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),\mu^{i_1,j_1}}\right)=\mathcal{S}^{i_1,j_1,\prime}\left((\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),\mu^{i_1,j_1,\prime}}\right)\neq 0.$$

(iii) *We also have the following non-vanishing results:*

$$\mathcal{S}_1^{i_1,j_1}\left((\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),\mu_1^{i_1,j_1}}\right)=(\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),\mu^{i_1,j_1}}$$

and

$$\mathcal{S}_1^{i_1,j_1}\left((\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),\mu_1^{i_1,j_1,\prime}}\right)=(\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),\mu^{i_1,j_1,\prime}}.$$

Proof. The statement (i) is immediate by Bruhat decomposition (4.0.6).

Now we prove (ii). According to Lemma 4.4.1, (5.5.1) and (5.5.2) and Lemma 4.4.6, we deduce by mod p reduction with respect to the lattice $(\widetilde{\pi}_*^{i_1,j_1})^\circ$ that

$$\begin{aligned}\mathcal{S}^{i_1,j_1}\left((\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),\mu^{i_1,j_1}}\right) &= \mathcal{S}^{i_1,j_1,\prime}\left((\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),\mu^{i_1,j_1,\prime}}\right)\\ &= S_{\underline{k}^{i_1,j_1,0},w_0^L}\left((\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),(\mu^{\square,i_1,j_1})^{w_0^L}}\right).\end{aligned}$$

If we abuse the notation $\underline{k}^{i_1,j_1,0}$ for the tuple in $\{0,\cdots,p-1\}^{|\Phi^+_{w_0}|}$ satisfying

$$\underline{k}_\alpha^{i_1,j_1,0} = 0 \text{ for all } \alpha \notin \Phi^+_{w_0^L}$$

then by mod p reduction of first possibility of Proposition 4.4.3 we deduce that

$$S_{\underline{k}^{i_1,j_1,0},w_0^L} \bullet S_{\underline{0},w_0^L w_0} = S_{\underline{k}^{i_1,j_1,0},w_0}$$

on the 1-dimensional subspace $(\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),(\mu^{\square,i_1,j_1})^{w_0}}$. Thus we finish the proof of (ii) by

$$S_{\underline{k}^{i_1,j_1,0},w_0}\left((\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),(\mu^{\square,i_1,j_1})^{w_0}}\right) \neq 0$$

which follows from Proposition 4.1.4.

Finally we prove (iii). We only prove the first equality in (iii) as the same proof works for the second equality. By Lemma 4.1.6 we know that

$$S_{\underline{0},(w^{i_1,j_1})^{-1}w_0^L w_0}\left((\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),(\mu^{\square,i_1,j_1})^{w_0}}\right) = (\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),\mu^{i_1,j_1}}$$

and

$$S_{\underline{0},(w^{i_1,j_1}w_1^{i_1,j_1})^{-1}w_0^L w_0}\left((\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),(\mu^{\square,i_1,j_1})^{w_0}}\right) = (\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),\mu_1^{i_1,j_1}}.$$

Therefore it remains to show that

$$\mathcal{S}_1^{i_1,j_1} \bullet S_{\underline{0},(w^{i_1,j_1}w_1^{i_1,j_1})^{-1}w_0^L w_0} = S_{\underline{0},(w^{i_1,j_1})^{-1}w_0^L w_0}$$

on the 1-dimensional subspace $(\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),(\mu^{\square,i_1,j_1})^{w_0}}$, which follows from the mod p reduction of Lemma 4.4.1 and the fact that

$$\ell(w_1^{i_1,j_1}) + \ell((w^{i_1,j_1}w_1^{i_1,j_1})^{-1}w_0^L w_0) = \ell((w^{i_1,j_1})^{-1}w_0^L w_0).$$

This completes the proof. $\square$

We define V^{i_1,j_1} and $V^{i_1,j_1,\prime}$ to be the subrepresentations of $\pi_*^{i_1,j_1}$ generated by

$$\mathcal{S}^{i_1,j_1}\left((\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),\mu^{i_1,j_1}}\right) \quad\text{and}\quad \mathcal{S}^{i_1,j_1,\prime}\left((\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),\mu^{i_1,j_1,\prime}}\right)$$

respectively. Similarly, we define $V_0^{i_1,j_1}$ as the subrepresentation of $\pi_*^{i_1,j_1}$ generated by

$$S_{\underline{k}^{i_1,j_1,0}}\left((\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),(\mu^{\square,i_1,j_1})^{w_0}}\right).$$

Lemma 5.5.4. *We have*

$$V^{i_1,j_1} = V^{i_1,j_1,\prime} = V_0^{i_1,j_1} \tag{5.5.3}$$

and

$$F(\mu^\square) \in \mathrm{JH}(V_0^{i_1,j_1}). \tag{5.5.4}$$

Proof. The equality (5.5.3) follows directly from the proof of (ii) of Lemma 5.5.3.

We define a new tuple $\underline{k}^{i_1,j_1,\sharp} = (k^{i_1,j_1,\sharp}_{i,j})_{i,j} \in \{0, \cdots, p-1\}^{|\Phi^+_{w_0}|}$ defined by

$$k^{i_1,j_1,\sharp}_{i,j} := \begin{cases} i_1 - j_1 + 1 + [b_{i_1} - b_{j_1}]_1 & \text{if } (i,j) = (n-j_1, n-i_1); \\ 0 & \text{otherwise.} \end{cases}$$

We also define $V^{i_1,j_1,\sharp}$ to be the subrepresentation of π^{i_1,j_1}_* generated by

$$S_{\underline{k}^{i_1,j_1,\sharp},w_0}\left((\pi^{i_1,j_1}_*)^{U(\mathbf{F}_p),(\mu^{\square,i_1,j_1})^{w_0}}\right).$$

By Proposition 4.6.5 and the same method in the proof of Proposition 4.7.8 we deduce that

$$V^{i_1,j_1,\sharp} \subseteq V^{i_1,j_1}_0. \tag{5.5.5}$$

By abuse of notation we view $\mu^{\square,i_1,j_1}$ as a fixed weight in $X_1(T)$, and then there exists $\mu^{\square,\prime} \in X_+(T)$ such that

$$\mu^{\square,\prime} \equiv \mu^{\square} \pmod{(p-1)X(T)} \ \text{ and } \ \mu^{\square,\prime} = (n-i_1, n-j_1)\cdot\mu^{\square,i_1,j_1} + p\sum_{r=n-j_1}^{n-i_1-1}\alpha_r.$$

We define $\overline{U}^{i_1,j_1}_1$ to be the unipotent subgroup of $\overline{L}$ generated by $\overline{U}_{\alpha_r}$ for $n-j_1+1 \leq r \leq n-i_1-1$ and then define

$$\overline{U}^{i_1,j_1} := \overline{U}^{i_1,j_1}_1 \cdot \overline{N}.$$

By a direct generalization of proof of Lemma 4.7.14, we can show that

$$S_{\underline{k}^{i_1,j_1,\sharp},w_0}\left((\pi^{i_1,j_1}_*)^{U(\mathbf{F}_p),(\mu^{\square,i_1,j_1})^{w_0}}\right) = H^0(\mu^{\square,i_1,j_1})^{\overline{U}^{i_1,j_1}}_{\mu^{\square,\prime}}.$$

We define $V^{i_1,j_1}_{\mathrm{alg}}$ to be the $\overline{G}$-subrepresentation of $H^0(\mu^{\square,i_1,j_1})$ generated by $H^0(\mu^{\square,i_1,j_1})^{\overline{U}^{i_1,j_1}}_{\mu^{\square,\prime}}$ and by definition we have

$$(V^{i_1,j_1}_{\mathrm{alg}})^{\overline{N}} \hookrightarrow H^0(\mu^{\square,i_1,j_1})^{\overline{N}} \ \text{ and } \ (V^{i_1,j_1}_{\mathrm{alg}})^{\overline{U}^{i_1,j_1}}_{\mu^{\square,\prime}} = H^0(\mu^{\square,i_1,j_1})^{\overline{U}^{i_1,j_1}}_{\mu^{\square,\prime}}. \tag{5.5.6}$$

We have natural identification (cf the beginning of Section 5 for definition of $H^0_L(\mu^{\square,i_1,j_1})$)

$$H^0(\mu^{\square,i_1,j_1})^{\overline{N}} \cong H^0_L(\mu^{\square,i_1,j_1}) \ \text{ and } \ H^0(\mu^{\square,i_1,j_1})^{\overline{U}^{i_1,j_1}} \cong H^0_L(\mu^{\square,i_1,j_1})^{\overline{U}^{i_1,j_1}_1}. \tag{5.5.7}$$

By applying Lemma 4.7.15 and the proof of Proposition 4.7.16 to the Levi L, we deduce that $H^0_L(\mu^{\square,i_1,j_1})$ is uniserial of length two with socle $F^L(\mu^{\square,i_1,j_1})$ and cosocle $F^L(\mu^{\square,\prime})$ and that

$$H^0_L(\mu^{\square,i_1,j_1})^{\overline{U}^{i_1,j_1}_1}_{\mu^{\square,\prime}} \xrightarrow{\sim} F^L(\mu^{\square,\prime})_{\mu^{\square,\prime}}. \tag{5.5.8}$$

Combine (5.5.6), (5.5.7) and (5.5.8) we deduce the surjection of representations of $\overline{L}$

$$(V^{i_1,j_1}_{\mathrm{alg}})^{\overline{N}} \twoheadrightarrow F^L(\mu^{\square,\prime}) \cong H^0_L(\mu^{\square,\prime}) \cong H^0(\mu^{\square,\prime})^{\overline{N}}$$

and thus a non-zero morphism

$$(V^{i_1,j_1}_{\mathrm{alg}}) \to H^0(\mu^{\square,\prime}) \text{ and } (V^{i_1,j_1}_{\mathrm{alg}})^{\overline{U}^{i_1,j_1}}_{\mu^{\square,\prime}} \xrightarrow{\sim} H^0(\mu^{\square,\prime})^{\overline{U}}_{\mu^{\square,\prime}} \xleftarrow{\sim} F(\mu^{\square,\prime})^{\overline{U}}_{\mu^{\square,\prime}}$$

by coinduction for algebraic representation from $\overline{P}$ to $\overline{G}$. In particular we know that

$$F(\mu^{\square,\prime}) \in \mathrm{JH}_{\overline{G}}\left(V^{i_1,j_1}_{\mathrm{alg}}\right).$$

Now we restrict the action of $\overline{G}$ to $G(\mathbf{F}_p)$ and observe the injections

$$V^{i_1,j_1,\sharp} \hookrightarrow V^{i_1,j_1}_{\mathrm{alg}}|_{G(\mathbf{F}_p)} \ \text{ and } \ F(\mu^{\square}) \hookrightarrow F(\mu^{\square,\prime})|_{G(\mathbf{F}_p)}$$

which induces

$$S_{\underline{k}^{i_1,j_1,\sharp},w_0}\left((\pi^{i_1,j_1}_*)^{U(\mathbf{F}_p),(\mu^{\square,i_1,j_1})^{w_0}}\right) = (V^{i_1,j_1,\sharp})^{U^{i_1,j_1}(\mathbf{F}_p),\mu^{\square}} = (V^{i_1,j_1}_{\mathrm{alg}})^{\overline{U}^{i_1,j_1}}_{\mu^{\square,\prime}}$$

and
$$F(\mu^{\square})^{U(\mathbf{F}_p),\mu^{\square}} = (F(\mu^{\square,\prime})|_{G(\mathbf{F}_p)})^{U(\mathbf{F}_p),\mu^{\square}} = F(\mu^{\square,\prime})_{\mu^{\square,\prime}}.$$
Hence, we deduce that
$$F(\mu^{\square}) \in \mathrm{JH}_{G(\mathbf{F}_p)}\left(V^{i_1,j_1,\sharp}\right)$$
which together with (5.5.5) finishes the proof of (5.5.4). $\square$

Proposition 5.5.5. *Let τ be an $\mathcal{O}_E$-lattice in $(\widetilde{\pi}_*^{i_1,j_1})^\circ \otimes_{\mathcal{O}_E} E$ satisfying*
$$\mathrm{soc}_{G(\mathbf{F}_p)}(\tau \otimes_{\mathcal{O}_E} \mathbf{F}) \hookrightarrow F(\mu^{\square}) \oplus F(\mu^{\square,i_1,j_1}).$$

(i) *For $\mu \in \{\mu^{i_1,j_1}, \mu^{i_1,j_1,\prime}, \mu_1^{i_1,j_1}, \mu_1^{i_1,j_1,\prime}\}$, we have*
$$\dim_{\mathbf{F}}(\tau \otimes_{\mathcal{O}_E} \mathbf{F})^{U(\mathbf{F}_p),\mu} = 1.$$

(ii) *We have the non-vanishing results for $\mathcal{S}^{i_1,j_1}$ and $\mathcal{S}^{i_1,j_1,\prime}$:*
$$\mathcal{S}^{i_1,j_1}\left((\tau \otimes_{\mathcal{O}_E} \mathbf{F})^{U(\mathbf{F}_p),\mu^{i_1,j_1}}\right) = \mathcal{S}^{i_1,j_1,\prime}\left((\tau \otimes_{\mathcal{O}_E} \mathbf{F})^{U(\mathbf{F}_p),\mu^{i_1,j_1,\prime}}\right) \neq 0.$$

(iii) *We also have the non-vanishing results for $\mathcal{S}_1^{i_1,j_1}$ and $\mathcal{S}_1^{i_1,j_1,\prime}$:*
$$\mathcal{S}_1^{i_1,j_1}\left((\tau \otimes_{\mathcal{O}_E} \mathbf{F})^{U(\mathbf{F}_p),\mu_1^{i_1,j_1}}\right) = (\tau \otimes_{\mathcal{O}_E} \mathbf{F})^{U(\mathbf{F}_p),\mu^{i_1,j_1}}$$
and
$$\mathcal{S}_1^{i_1,j_1,\prime}\left((\tau \otimes_{\mathcal{O}_E} \mathbf{F})^{U(\mathbf{F}_p),\mu_1^{i_1,j_1,\prime}}\right) = (\tau \otimes_{\mathcal{O}_E} \mathbf{F})^{U(\mathbf{F}_p),\mu^{i_1,j_1,\prime}}.$$

Proof. We can easily deduce (i) from
$$\dim_E((\widetilde{\pi}_*^{i_1,j_1})^\circ \otimes_{\mathcal{O}_E} E)^{U(\mathbf{F}_p),\widetilde{\mu}^{i_1,j_1}} = \dim_E((\widetilde{\pi}_*^{i_1,j_1})^\circ \otimes_{\mathcal{O}_E} E)^{U(\mathbf{F}_p),\widetilde{\mu}^{i_1,j_1,\prime}} = 1$$
and Frobenius reciprocity as $F(\mu^{i_1,j_1})$, $F(\mu^{i_1,j_1,\prime})$, $F(\mu_1^{i_1,j_1})$ and $F(\mu_1^{i_1,j_1,\prime})$ all have multiplicity one in $\tau \otimes_{\mathcal{O}_E} \mathbf{F}$.

We define $\pi_\flat^{i_1,j_1}$ as the mod p reduction of $(\widetilde{\pi}_*^{i_1,j_1})^\circ \otimes_{\mathcal{O}_E} E$ with respect to the unique (up to homothety) $\mathcal{O}_E$-lattice such that
$$\mathrm{soc}_{G(\mathbf{F}_p)}\left(\pi_\flat^{i_1,j_1}\right) = F(\mu^{\square}).$$
Then we deduce from Corollary 5.4.3 that there exists an injection
$$\tau \otimes_{\mathcal{O}_E} \mathbf{F} \hookrightarrow \pi_*^{i_1,j_1} \oplus \pi_\flat^{i_1,j_1}$$
Note that we have
$$\left(\pi_*^{i_1,j_1} \oplus \pi_\flat^{i_1,j_1}\right)^{U(\mathbf{F}_p),\mu} = (\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),\mu} \oplus (\pi_\flat^{i_1,j_1})^{U(\mathbf{F}_p),\mu} \tag{5.5.9}$$
for $\mu \in \{\mu^{i_1,j_1}, \mu^{i_1,j_1,\prime}, \mu_1^{i_1,j_1}, \mu_1^{i_1,j_1,\prime}\}$.

The equality of two spaces in (ii) is true because both of them can be identified with
$$S_{\underline{k}^{i_1,j_1,0},w_0}\left((\tau \otimes_{\mathcal{O}_E} \mathbf{F})^{U(\mathbf{F}_p),(\mu^{\square,i_1,j_1})^{w_0}}\right)$$
by the same argument as in the proof of (ii) of Lemma 5.5.3. Therefore we only need to show that $\mathcal{S}^{i_1,j_1}$ (resp. $\mathcal{S}^{i_1,j_1,\prime}$) gives rise to a bijection from $\left(\pi_*^{i_1,j_1} \oplus \pi_\flat^{i_1,j_1}\right)^{U(\mathbf{F}_p),\mu^{i_1,j_1}}$ (resp. from $\left(\pi_*^{i_1,j_1} \oplus \pi_\flat^{i_1,j_1}\right)^{U(\mathbf{F}_p),\mu^{i_1,j_1}}$) to its image. According to (ii) of Lemma 5.5.3 and (5.5.9) we only need to show that
$$\mathcal{S}^{i_1,j_1}\left((\pi_\flat^{i_1,j_1})^{U(\mathbf{F}_p),\mu^{i_1,j_1}}\right) \neq 0 \text{ and } \mathcal{S}^{i_1,j_1,\prime}\left((\pi_\flat^{i_1,j_1})^{U(\mathbf{F}_p),\mu^{i_1,j_1,\prime}}\right) \neq 0$$
which follows from Lemma 5.5.4 by definition of $\pi_\flat^{i_1,j_1}$.

We have a unique (up to scalar) non-zero morphism

$$\pi_*^{i_1,j_1} \to \pi_\flat^{i_1,j_1} \tag{5.5.10}$$

which by Lemma 5.5.4 induces isomorphisms

$$(\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),\mu} \xrightarrow{\sim} (\pi_\flat^{i_1,j_1})^{U(\mathbf{F}_p),\mu}$$

for $\mu \in \{\mu^{i_1,j_1}, \mu^{i_1,j_1,\prime}\}$, and hence (iii) follows from (iii) of Lemma 5.5.3 by considering the image of (iii) of Lemma 5.5.3 under (5.5.10) inside $\pi_\flat^{i_1,j_1}$. □

Corollary 5.5.6. *Let τ be an $\mathcal{O}_E$-lattice in $(\widetilde{\pi}_*^{i_1,j_1})^\circ \otimes_{\mathcal{O}_E} E$ satisfying*

$$\mathrm{soc}_{G(\mathbf{F}_p)}\left(\tau \otimes_{\mathcal{O}_E} \mathbf{F}\right) \hookrightarrow F(\mu^\square) \oplus F(\mu^{\square,i_1,j_1}).$$

Then we have

$$\mathcal{S}^{i_1,j_1} \bullet \mathcal{S}_1^{i_1,j_1}\left((\tau \otimes_{\mathcal{O}_E} \mathbf{F})^{U(\mathbf{F}_p),\mu_1^{i_1,j_1}}\right) = \mathcal{S}^{i_1,j_1,\prime} \bullet \mathcal{S}_1^{i_1,j_1,\prime}\left((\tau \otimes_{\mathcal{O}_E} \mathbf{F})^{U(\mathbf{F}_p),\mu_1^{i_1,j_1,\prime}}\right) \neq 0.$$

5.6. **Main results.** In this section, we state and prove our main results on mod p local-global compatibility. Throughout this section, $\overline{\rho}_0$ is always assumed to be a restriction of a global representation $\overline{r} : G_F \to \mathrm{GL}_n(\mathbf{F})$ to G_{F_w} for a fixed place w of F above p. Let $v := w|_{F^+}$, and assume further that $\overline{r}$ is automorphic of a Serre weight $V = \bigotimes_{v'} V_{v'}$ with $V_w := V_v \circ \iota_w^{-1} \cong F(\mu^\square)^\vee$. We may write $V_{v'} \circ \iota_{w'}^{-1} \cong F(\underline{a}_{w'})^\vee$ for a dominant weight $\underline{a}_{w'} \in \mathbf{Z}_+^n$ where w' is a place of F above v', and define

$$V' := \bigotimes_{v' \neq v} V_{v'} \quad \text{and} \quad \widetilde{V}' := \bigotimes_{v' \neq v} W_{\underline{a}_{v'}}. \tag{5.6.1}$$

From now on, we also assume that $\underline{a}_{w'}$ *is in the lowest alcove for each place w' of F above p*, so that

$$V' \cong \widetilde{V}' \otimes_{\mathcal{O}_E} \mathbf{F}.$$

Let U be a compact open subgroup of $G_n(\mathbf{A}_F^{\infty,p}) \times \mathcal{G}_n(\mathcal{O}_{F^+,p})$, which is sufficiently small and unramified at all places above p, such that $S(U,V)[\mathfrak{m}_{\overline{r}}] \neq 0$ where $\mathfrak{m}_{\overline{r}}$ is the maximal ideal of $\mathbf{T}^{\mathcal{P}}$ attached to $\overline{r}$ for a cofinite subset $\mathcal{P}$ of $\mathcal{P}_U$.

We fix a pair of integers (i_0, j_0) such that $0 \leq j_0 < j_0 + 1 < i_0 \leq n-1$, and determine a pair of integers (i_1, j_1) by the equation (5.0.1). We also define

$$\begin{cases} M & := & S(U^v, \widetilde{V}')_{\mathfrak{m}_{\overline{r}}}; \\ M^{i_1,j_1} & := & S(U^v, \widetilde{V}')_{\mathfrak{m}_{\overline{r}}}^{I(1),\widetilde{\mu}_1^{i_1,j_1}}. \end{cases}$$

Note that M^{i_1,j_1} is a free $\mathcal{O}_E$-module of finite rank as M is a smooth admissible representation of $G(\mathbf{Q}_p)$ which is ϖ_E-torsion free. For any $\mathcal{O}_E$-algebra A, we write $M_A^{i_1,j_1}$ for $M^{i_1,j_1} \otimes_{\mathcal{O}_E} A$. We similarly define M_A.

Let $\mathbf{T}^{i_1,j_1}$ be the $\mathcal{O}_E$-module that is the image of $\mathbf{T}^{\mathcal{P}}$ in $\mathrm{End}_{\mathcal{O}_E}(M^{i_1,j_1})$. Then $\mathbf{T}^{i_1,j_1}$ is a local $\mathcal{O}_E$-algebra with the maximal ideal $\mathfrak{m}_{\overline{r}}$, where, by abuse of notation, we write $\mathfrak{m}_{\overline{r}} \subseteq \mathbf{T}^{i_1,j_1}$ for the image of $\mathfrak{m}_{\overline{r}}$ of $\mathbf{T}^{\mathcal{P}}$. As the level U is sufficiently small, by passing to a sufficiently large E as in the proof of Theorem 4.5.2 of [HLM], we may assume that $\mathbf{T}_E^{i_1,j_1} \cong E^r$ for some $r > 0$. For any $\mathcal{O}_E$-algebra A we write $\mathbf{T}_A^{i_1,j_1}$ for $\mathbf{T}^{i_1,j_1} \otimes_{\mathcal{O}_E} A$.

We have $M_E^{i_1,j_1} = \bigoplus_{\mathfrak{p}} M_E^{i_1,j_1}[\mathfrak{p}_E]$, where the sum runs over the minimal primes $\mathfrak{p}$ of $\mathbf{T}^{i_1,j_1}$ and $\mathfrak{p}_E := \mathfrak{p}\mathbf{T}_E^{i_1,j_1}$. Note that $\mathbf{T}_E^{i_1,j_1}/\mathfrak{p}_E \cong E$ for any such $\mathfrak{p}$. By abuse of notation, we also write $\mathfrak{p}$ (resp. $\mathfrak{p}_E$) for its inverse image in $\mathbf{T}^{\mathcal{P}}$ (resp. $\mathbf{T}_E^{\mathcal{P}}$).

Definition 5.6.1. *A non-zero vector $v^{i_1,j_1} \in M_{\mathbf{F}}^{i_1,j_1}$ is said to be* primitive *if there exists a vector $\widehat{v}^{i_1,j_1} \in M^{i_1,j_1}[\mathfrak{p}]$ that lifts v^{i_1,j_1}, for certain minimal prime $\mathfrak{p}$ of $\mathbf{T}^{i_1,j_1}$.*

Note that the $G(\mathbf{Q}_p)$-subrepresentation of M_E generated by a lift $\widehat{v}^{i_1,j_1}$ of a primitive element v^{i_1,j_1} is irreducible and actually lies in $M_E[\mathfrak{p}_E]$.

Now we can state our main results in this paper. Recall that by $\overline{\rho}_0$ we always mean an n-dimensional ordinary representation of $G_{\mathbf{Q}_p}$ as described in (3.0.1).

Theorem 5.6.2. *Fix a pair of integers (i_0, j_0) satisfying $0 \leq j_0 < j_0 + 1 < i_0 \leq n-1$, and let (i_1, j_1) be a pair of integers such that $i_0 + i_1 = j_0 + j_1 = n-1$. We also let $\overline{r} : G_F \to \mathrm{GL}_n(\mathbf{F})$ be an irreducible automorphic representation with $\overline{r}|_{G_{F_w}} \cong \overline{\rho}_0$. Assume that*

- *$\mu^{\square,i_1,j_1}$ is $2n$-generic;*
- *$\overline{\rho}_{i_0,j_0}$ is Fontaine–Laffaille generic.*

Assume further that

$$\{F(\mu^{\square})^\vee\} \subseteq W_w(\overline{r}) \cap \mathrm{JH}((\pi_*^{i_1,j_1})^\vee) \subseteq \{F(\mu^{\square})^\vee, F(\mu^{\square,i_1,j_1})^\vee\}. \tag{5.6.2}$$

Then there exists a primitive vector in $S(U^v, V')[\mathfrak{m}_{\overline{r}}]^{I(1),\mu_1^{i_1,j_1}}$. Moreover, for each primitive vector $v^{i_1,j_1} \in S(U^v, V')[\mathfrak{m}_{\overline{r}}]^{I(1),\mu_1^{i_1,j_1}}$ we have $\mathcal{S}^{i_1,j_1} \bullet \mathcal{S}_1^{i_1,j_1} v^{i_1,j_1} \neq 0$ and

$$\mathcal{S}^{i_1,j_1,\prime} \bullet \mathcal{S}^{i_1,j_1,\prime} \bullet (\Xi_n)^{j_1-i_1-1} v^{i_1,j_1} = \varepsilon^{i_1,j_1} \mathcal{P}_{i_1,j_1}(b_{n-1}, \cdots, b_0) \cdot \mathrm{FL}_n^{i_0,j_0}(\overline{r}|_{G_{F_w}}) \cdot \mathcal{S}^{i_1,j_1} \bullet \mathcal{S}_1^{i_1,j_1} v^{i_1,j_1}$$

where

$$\varepsilon^{i_1,j_1} = \prod_{k=i_1+1}^{j_1-1} (-1)^{b_{i_1} - b_k - j_1 + i_1 + 1}$$

and

$$\mathcal{P}_{i_1,j_1}(b_{n-1}, \cdots, b_0) = \prod_{k=i_1+1}^{j_1-1} \prod_{j=1}^{j_1-i_1-1} \frac{b_k - b_{j_1} - j}{b_{i_1} - b_k - j} \in \mathbf{Z}_p^\times.$$

Remark 5.6.3. *The right inclusion of (5.6.2) is just Conjecture 5.3.1, which becomes a theorem in* [LHMPQ] *(cf. Remark 1.3.2). We also give an evidence for the left inclusion of (5.6.2) in Proposition 5.3.2 under some assumption of Taylor–Wiles type. As a result, the condition (5.6.2) can be removed under some standard Taylor–Wiles conditions.*

Remark 5.6.4. *If M^{i_1,j_1} is free as $\mathbf{T}^{i_1,j_1}$-module, then all vectors in $S(U^v, V')[\mathfrak{m}_{\overline{r}}]^{I(1),\mu_1^{i_1,j_1}}$ are primitive. As a result, one needs such a freeness result to remove the "primitive" condition. Under a stronger generic condition (compared to our Fontaine-Laffaille generic conditions), it should be possible to use results from* [LHMPQ] *to improve* (5.6.2) *to be an equality*

$$W_w(\overline{r}) \cap \mathrm{JH}((\pi_*^{i_1,j_1})^\vee) = \{F(\mu^{\square})^\vee\}$$

in which case one is able to prove the freeness result mentioned above through the technique in Section 5 of [HLM] *under some standard global assumption. It is also possible to prove a freeness result over some enlarged Hecke algebra as in Section 5 of* [HLM]*, at least if* $(i_1, j_1) = (0, n-1)$.

Proof. We firstly point out that $M^{i_1,j_1} \neq 0$, as $S(U, (F(\mu^{\square})^\vee \circ \iota_w) \otimes V')_{\mathfrak{m}_{\overline{r}}} \neq 0$ and $F(\mu^{\square})$ is a factor of $\overline{\mathrm{Ind}_I^K \widetilde{\mu}_1^{i_1,j_1}} = \mathrm{Ind}_{B(\mathbf{F}_p)}^{G(\mathbf{F}_p)} \mu_1^{i_1,j_1}$.

Picking an embedding $E \hookrightarrow \overline{\mathbf{Q}}_p$, as well as an isomorphism $\iota : \overline{\mathbf{Q}}_p \xrightarrow{\sim} \mathbb{C}$, we see that

$$M_{\overline{\mathbf{Q}}_p}^{i_1,j_1} \cong \bigoplus_{\Pi} m(\Pi) \cdot \Pi_v^{I(1),\widetilde{\mu}_1^{i_1,j_1}} \otimes (\Pi^{\infty,v})^{U^v}, \tag{5.6.3}$$

where the sum runs over irreducible representations $\Pi \cong \Pi_\infty \otimes \Pi_v \otimes \Pi^{\infty,v}$ of $G_n(\mathbb{A}_{F^+})$ over $\overline{\mathbb{Q}}_p$ such that $\Pi \otimes_\iota \mathbb{C}$ is a cuspidal automorphic representation of multiplicity $m(\Pi) \in \mathbf{Z}_{>0}$ with $\Pi_\infty \otimes_\iota \mathbb{C}$ being determined by the algebraic representation $(\widetilde{V}')^\vee$ and with associated Galois representation r_Π lifting $\overline{r}^\vee$ (cf. Lemma 5.1.1).

We write δ for the modulus character of $B(\mathbf{Q}_p)$:

$$\delta :=|\ |^{n-1} \otimes |\ |^{n-2} \otimes \cdots \otimes |\ | \otimes 1$$

where $|\ |$ is the (unramified) norm character sending p to p^{-1}. For any Π contributing to (5.6.3), we have

(i) $\Pi_v \cong \mathrm{Ind}_{B(\mathbf{Q}_p)}^{G(\mathbf{Q}_p)}(\psi \otimes \delta)$ for some smooth character

$$\psi = \psi_{n-1} \otimes \psi_{n-2} \otimes \cdots \otimes \psi_1 \otimes \psi_0$$

of $T(\mathbf{Q}_p)$ such that $\psi|_{T(\mathbf{Z}_p)} = \widetilde{\mu}_1^{i_1,j_1}|_{T(\mathbf{Z}_p)}$, where ψ_k are the smooth characters of $\mathbf{Q}_p^\times$.

(ii) $r_\Pi^\vee|_{G_{F_w}}$ is a potentially crystalline lift of $\overline{r}$ with Hodge–Tate weights $\{-(n-1), -(n-2), \cdots, -1, 0\}$ and $\mathrm{WD}(r_\Pi^\vee|_{G_{F_w}})^{\mathrm{F-ss}} \cong \oplus_{k=0}^{n-1}\psi_k^{-1}$.

Here, part (i) follows from [EGH15], Propositions 2.4.1 and 7.4.4, and part (ii) follows from classical local-global compatibility (cf. Theorem 5.1.2). Moreover, by Corollary 3.7.3, we have

$$\mathrm{FL}_n^{i_0,j_0}(\overline{\rho}_0) = \overline{\frac{\prod_{k=j_0+1}^{i_0-1} \psi_{i_1+1+k}(p)}{p^{\frac{(i_0+j_0)(i_0-j_0-1)}{2}}}}. \tag{5.6.4}$$

(Note that we may identify ψ_{i_1+1+k} with Ω_k^{-1} for $j_0 < k < i_0$, where Ω_k is defined in Corollary 3.7.3.)

Now we pick an arbitrary primitive vector $v^{i_1,j_1} \in M_{\mathbf{F}}^{i_1,j_1}[\mathfrak{m}_{\overline{r}}]$ with a lift $\widehat{v}^{i_1,j_1} \in M^{i_1,j_1}[\mathfrak{p}]$. We set

$$\tau_E := \langle K\widehat{v}^{i_1,j_1}\rangle_E \subseteq M_E[\mathfrak{p}_E] \text{ and } \tau := \langle K\widehat{v}^{i_1,j_1}\rangle \subseteq M[\mathfrak{p}],$$

and thus τ is an $\mathcal{O}_E$-lattice in τ_E. Note that $M_E^{i_1,j_1}[\mathfrak{p}_E] \otimes_E \overline{\mathbf{Q}}_p$ is a direct summand of (5.6.3) where Π runs over a subset of automorphic representations in (5.6.3). The same argument as in the paragraph above (4.5.7) of [HLM] using Cebotarev density theorem shows that the local component Π_v of each Π occurring in this direct summand does not depend on Π.

By the definition of τ, we obtain non-zero morphisms

$$\tau \otimes_{\mathcal{O}_E} \mathbf{F} \to M[\mathfrak{p}] \otimes_{\mathcal{O}_E} \mathbf{F} \to M_{\mathbf{F}}[\mathfrak{m}_{\overline{r}}] \tag{5.6.5}$$

as $\mathfrak{p} + \varpi_E \mathbf{T}^{\mathcal{P}} = \mathfrak{m}_{\overline{r}}$. We denote the image of $\tau \otimes_{\mathcal{O}_E} \mathbf{F}$ under the composition (5.6.5) by V and note that it can be naturally identified with $\langle Kv^{i_1,j_1}\rangle_{\mathbf{F}}$ according to the definition of τ. By the assumption (5.6.2) (cf. Conjecture 5.3.1), we deduce that

$$\mathrm{JH}\left(\mathrm{soc}_{G(\mathbf{F}_p)}\left(M_{\mathbf{F}}[\mathfrak{m}_{\overline{r}}]\right)\right) \subseteq \{F(\mu^{\square}), F(\mu^{\square,i_1,j_1})\}$$

and therefore by (5.6.5) we have

$$\mathrm{JH}\left(\mathrm{soc}_{G(\mathbf{F}_p)}(V)\right) \subseteq \{F(\mu^{\square}), F(\mu^{\square,i_1,j_1})\}.$$

We know that there exists an $\mathcal{O}_E$-lattice $\tau' \subseteq \tau_E$ such that

$$\mathrm{soc}_{G(\mathbf{F}_p)}(V) \cong \mathrm{soc}_{G(\mathbf{F}_p)}\left(\tau' \otimes_{\mathcal{O}_E} \mathbf{F}\right).$$

Moreover, we have a saturated inclusion $\tau \hookrightarrow \tau'$ which induces a morphism

$$\tau \otimes_{\mathcal{O}_E} \mathbf{F} \to \tau' \otimes_{\mathcal{O}_E} \mathbf{F}$$

whose image is isomorphic to V. It follows from Proposition 5.5.5 that we necessarily have isomorphisms of $\mathbf{F}$-lines

$$\left(\tau \otimes_{\mathcal{O}_E} \mathbf{F}\right)^{U(\mathbf{F}_p),\mu^{i_1,j_1}} \xrightarrow{\sim} V^{U(\mathbf{F}_p),\mu^{i_1,j_1}} \xrightarrow{\sim} \left(\tau' \otimes_{\mathcal{O}_E} \mathbf{F}\right)^{U(\mathbf{F}_p),\mu^{i_1,j_1}}.$$

Hence, by Corollary 5.5.6 and the fact that

$$V^{U(\mathbf{F}_p),\mu^{i_1,j_1}} = \mathbf{F}[v^{i_1,j_1}] \subseteq M_{\mathbf{F}}[\mathfrak{m}_{\overline{r}}],$$

we deduce that

$$\mathcal{S}^{i_1,j_1} \bullet \mathcal{S}_1^{i_1,j_1} v^{i_1,j_1} \neq 0. \tag{5.6.6}$$

On the other hand, we have the following equality by Proposition 5.5.1

$$\widehat{\mathcal{S}}^{i_1,j_1,\prime} \bullet \widehat{\mathcal{S}}_1^{i_1,j_1,\prime} \bullet (\Xi_n)^{j_1-i_1-1}\widehat{v}^{i_1,j_1} = \kappa_{i_1,j_1}\left(\frac{\prod_{k=j_0+1}^{i_0-1}\psi_{i_1+1+k}(p)}{p^{\frac{(i_0+j_0)(i_0-j_0-1)}{2}}}\right)\widehat{\mathcal{S}}^{i_1,j_1} \bullet \widehat{\mathcal{S}}_1^{i_1,j_1}\widehat{v}^{i_1,j_1}. \tag{5.6.7}$$

By taking mod p reduction of (5.6.7) we deduce from (5.6.4) that

$$\mathcal{S}^{i_1,j_1,\prime} \bullet \mathcal{S}_1^{i_1,j_1,\prime} \bullet (\Xi_n)^{j_1-i_1-1}v^{i_1,j_1} = \varepsilon^{i_1,j_1}\mathcal{P}_{i_1,j_1}(b_{n-1},\cdots,b_0)\cdot \mathrm{FL}_n^{i_0,j_0}(\overline{r}|_{G_{F_w}})\cdot \mathcal{S}^{i_1,j_1} \bullet \mathcal{S}_1^{i_1,j_1}v^{i_1,j_1}.$$

This equation together with (5.6.6) finishes the proof. □

Corollary 5.6.5. *Keep the notation of Theorem 5.6.2 and assume that each assumption in Theorem 5.6.2 holds for all (i_0,j_0) such that $0 \le j_0 < j_0+1 < i_0 \le n-1$. Assume further that M^{i_1,j_1} is free over $\mathbf{T}^{i_1,j_1}$ for all pair (i_1,j_1) (cf. Remark 5.6.4).*

Then the structure of $S(U^v,V')[\mathfrak{m}_{\overline{r}}]$ as a smooth admissible $\mathbf{F}$-representation of $G(\mathbf{Q}_p)$ determines $\overline{\rho}_0$ up to isomorphism.

Proof. We follow the notation in Section 3.4 of [BH15]. As $\overline{\rho}_0$ is ordinary, we can view it as a morphism

$$\overline{\rho}_0 : G_{\mathbf{Q}_p} \to \widehat{B}(\mathbf{F}) \subseteq \widehat{G}(\mathbf{F})$$

where $\widehat{B}$ (resp. $\widehat{G}$) is the dual group of B (resp. G). The local class field theory gives us a bijection between smooth characters of $\mathbf{Q}_p^\times$ and the smooth characters of the Weil group of $\mathbf{Q}_p$ in characteristic 0. This bijection restricts to a bijection between smooth characters of $\mathbf{Q}_p^\times$ and smooth characters of $G_{\mathbf{Q}_p}$ both with values in $\mathcal{O}_E^\times$. Taking mod p reduction and then taking products we reach a bijection between smooth $\mathbf{F}$-characters of $T(\mathbf{Q}_p)$ and $\mathrm{Hom}(G_{\mathbf{Q}_p},\widehat{T}(\mathbf{F}))$. We can therefore define $\chi_{\overline{\rho}_0}$ as the character of $T(\mathbf{Q}_p)$ corresponding to the composition

$$\widehat{\chi}_{\overline{\rho}_0} : G_{\mathbf{Q}_p} \to \widehat{B}(\mathbf{F}) \twoheadrightarrow \widehat{T}(\mathbf{F}).$$

In [BH15], a closed subgroup $C_{\overline{\rho}_0} \subseteq B$ (at the beginning of Section 3.2) and a subset $W_{\overline{\rho}_0}$ ((2) before Lemma 2.3.6) of W is defined.

As we are assuming that $\overline{\rho}_0$ is maximally non-split, we observe that $C_{\overline{\rho}_0} = B$ and $W_{\overline{\rho}_0} = \{1\}$ in our case. Therefore by the definition of $\Pi^{ord}(\overline{\rho}_0)$ in [BH15] before Definition 3.4.3, we know that it is indecomposable with socle

$$\mathrm{Ind}_{B^-(\mathbf{Q}_p)}^{G(\mathbf{Q}_p)}\chi_{\overline{\rho}_0}\cdot(\omega^{-1}\circ\theta)$$

where $\theta \in X(T)$ is a twist character defined after Conjecture 3.1.2 in [BH15] which can be chosen to be η in our notation. Then as a Corollary of Theorem 4.4.7 in [BH15], we deduce that $S(U^v,V')[\mathfrak{m}_{\overline{r}}]$ determines $\chi_{\overline{\rho}_0}$ and hence $\widehat{\chi}_{\overline{\rho}_0}$.

Now, we know that $\overline{\rho}_0$ is determined by the Fontaine–Laffaille parameters

$$\{\mathrm{FL}_n^{i_0,j_0}(\overline{\rho}_0) \in \mathbb{P}^1(\mathbf{F}) \mid 0 \le j_0 < j_0+1 < i_0 \le n-1\}$$

and $\widehat{\chi}_{\overline{\rho}_0}$, up to isomorphism. Our conclusion thus follows from Theorem 5.6.2 together with Remark 5.6.4. □

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