# ANALYTIC ESTIMATES FOR THE CHEBOTAREV DENSITY THEOREM AND THEIR APPLICATIONS

by

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# Abstract

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In this thesis, we study the distribution of prime ideals within the Chebotarev Density Theorem. The theorem states that the Artin symbols attached to prime ideals are equidistributed within the Galois group of a given Galois extension.

We exhibit field-uniform unconditional bounds with explicit constants for the least prime ideal in the Chebotarev Density Theorem, that is, the prime ideal of least norm with a specified Artin symbol. Moreover, we provide a new upper bound for the number of prime ideals with a specified Artin symbol which is valid for a wide range and sharp, short of precluding a putative Siegel zero. To achieve these results, we establish explicit statistical information on the zeros of Hecke $L$-functions and the Dedekind zeta function. Our methods were inspired by works of Linnik, Heath-Brown, and Maynard in the classical case and the papers of Lagarias–Odlyzko, Lagarias–Montgomery–Odlyzko, and Weiss in the Chebotarev setting.

We include applications for primes represented by certain binary quadratic forms, congruences of coefficients for modular forms, and the group structure of elliptic curves reduced modulo a prime. In particular, we establish the best known unconditional upper bounds for the least prime represented by a positive definite primitive binary quadratic form and for the Lang–Trotter conjectures on elliptic curves.

*To my parents, without whom I would not have begun.*

*To Mubnii, without whom I would not have finished.*

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# Chapter 1

# Introduction

*"Of course it is happening inside your head... but why on earth should that mean that it is not real?"*

– Albus Dumbledore.

## 1.1 Primes in an arithmetic progression

In his breakthrough 1837 manuscript, Dirichlet proved that there are infinitely many primes in any given arithmetic progression. Riemann, in his seminal 1859 paper, proceeded to outline a remarkable strategy to asymptotically count such primes and, by the end of the 19th century, the Prime Number Theorem (PNT) for Arithmetic Progressions (APs) was established with the works of Hadamard and de la Vallée–Poussin. It states that primes are equidistributed amongst arithmetic progressions; that is, for $(a, q) = 1$, the prime counting function defined by

$$\pi(x; q, a) := \#\{p \leq x : p \equiv a \,(\mathrm{mod}\, q)\}$$

satisfies

$$\pi(x; q, a) \sim \frac{1}{\varphi(q)} \mathrm{Li}(x) \tag{1.1}$$

as $x \to \infty$, where $\varphi(q) = \#(\mathbb{Z}/q\mathbb{Z})^\times$ is the Euler totient function and $\mathrm{Li}(x) = \int_2^x \frac{1}{\log t} dt$ is the logarithmic integral. Recall $\mathrm{Li}(x) \sim \frac{x}{\log x}$ as $x \to \infty$.

The remarkable arguments leading to (1.1) and its predecessors relied on a deep analytic understanding of functions associated to Dirichlet characters and their zeros. A Dirichlet character $\chi \,(\mathrm{mod}\, q)$ is a completely multiplicative $q$-periodic function on the integers $n \in \mathbb{Z}$ taking complex roots of unity as values for $(n, q) = 1$ and zero otherwise. Given a Dirichlet character

$\chi \,(\mathrm{mod}\, q)$, the Dirichlet $L$-function associated to it is given by

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} = \prod_{p} \Big(1 - \frac{\chi(p)}{p^s}\Big)^{-1} \tag{1.2}$$

for $\mathrm{Re}\{s\} > 1$. Here the product is over all primes $p$. Of special importance is the principal[1] character $\chi = \chi_0$ which satisfies $\chi_0(n) = 1$ for all $(n,q) = 1$ and equals 0 otherwise. In the special case $q = 1$, the principal character $\chi_0$ is identically unity and its Dirichlet $L$-function is the famous Riemann zeta function given by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p} \Big(1 - \frac{1}{p^s}\Big)^{-1} \tag{1.3}$$

for $\mathrm{Re}\{s\} > 1$. It is well-known that Dirichlet $L$-functions $L(s,\chi)$ can be analytically continued to the entire complex plane, except for a simple pole at $s = 1$ when $\chi = \chi_0$ is trivial. In fact, they satisfy a functional equation relating $L(s,\chi)$ to $L(1-s,\overline{\chi})$ which yields a symmetry of their zeros about the critical line $\mathrm{Re}\{s\} = 1/2$. As demonstrated by Dirichlet and many others, the distribution of their zeros is intimately related to the distribution of primes in arithmetic progressions. The zeros of Dirichlet $L$-functions either lie in the critical strip $0 < \mathrm{Re}\{s\} < 1$ (which are referred to as non-trivial zeros) or at certain non-positive integers (which are referred to as trivial zeros). It is the non-trivial zeros which are deeply mysterious and dictate the behaviour of $\pi(x;q,a)$.

With a more refined understanding of these zeros, the Siegel-Walfisz theorem (1936) quantifies the error term in (1.1) asserting that, for any $\epsilon > 0$ and $x \geq \exp(O_\epsilon(q^\epsilon))$,

$$\pi(x;q,a) = \frac{1}{\varphi(q)} \mathrm{Li}(x) + O\big(x \exp(-\sqrt{c_1 \log x})\big) \tag{1.4}$$

for some constant $c_1 = c_1(\epsilon) > 0$. However, for any $\epsilon < 1/2$, the constant $c_1$ and implied constants depending on $\epsilon$ are *ineffective*; that is, they are not effectively computable. The source of this drawback is a putative real zero $\beta_1$ of a Dirichlet $L$-function $L(s,\chi_1)$ attached to the quadratic Dirichlet character $\chi_1 \,(\mathrm{mod}\, q)$ and this zero could conceivably be exceedingly close to $s = 1$. We refer to the zero $\beta_1$ as an *exceptional zero*[2]. In general, if many zeros of a Dirichlet $L$-function happen to live near the edge of the critical strip at $\mathrm{Re}\{s\} = 1$, then $\pi(x;q,a)$ could behave erratically. Given the symmetry of zeros about the critical line, the ideal conjectured scenario is the Generalized Riemann Hypothesis (GRH) which states that all

[1] It is also referred to as the trivial character.

[2] Other sources may refer to it as a Siegel zero (or Landau–Siegel zero), but we will later make a distinction with this terminology.

the non-trivial zeros $\rho$ of a Dirichlet $L$-function satisfy $\mathrm{Re}\{\rho\} = 1/2$. Assuming GRH, the error term in (1.4) drastically improves to $O(x^{1/2}\log x)$ with an effective implied constant and the asymptotic for $\pi(x;q,a)$ is valid for $x \geq q^2 \log^4 q$. This strong conditional range suggests there is much left to be desired from the unconditional range in (1.4) which is both exponential in the modulus $q$ and ineffective. Furthermore, the range of validity in (1.4) is prohibitive for many applications. Unfortunately, since 1936, there has been little progress towards improving the valid range of $x$ in (1.4) while maintaining the asymptotic for $\pi(x;q,a)$.

One may seek to relax the precision of our estimate for $\pi(x;q,a)$ in hopes of enhancing the range of $x$. For example, an estimate of the form

$$\frac{1}{\varphi(q)}\mathrm{Li}(x) \ll \pi(x;q,a) \ll \frac{1}{\varphi(q)}\mathrm{Li}(x)$$

for a range of $x$ which is polynomial in $q$ would be tremendously useful; roughly speaking, for smaller values of $x$, can one bound $\pi(x;q,a)$ within a constant factor of its asymptotic size? As we shall see, the desired lower bound is overly optimistic and unattainable with current methods (due to the possible existence of an exceptional zero) but we can obtain a weaker variant of it. The upper bound, on the other hand, has been established in a very precise form.

### Linnik's theorem

A lower bound for $\pi(x;q,a)$ is intimately related to bounding the least prime in an arithmetic progression $a \pmod q$. For $(a,q)=1$, define

$$P(a,q) = \min\{p \text{ prime} : p \equiv a \pmod q\}. \tag{1.5}$$

The best known lower bounds for $\max\limits_{a} P(a,q)$ are due to Granville and Pomerance [GP90]. For upper bounds, (1.4) trivially gives an ineffective unconditional estimate for $P(a,q)$ which is exponential in $q$. In a spectacular breakthrough, Linnik [Lin44a, Lin44b] established the first non-trivial unconditional upper bound on $P(a,q)$ which is polynomial in $q$.

**Theorem** (Linnik)**.** *Let $(a,q)=1$. For some absolute constant $L>0$,*

$$P(a,q) \ll q^L, \tag{1.6}$$

*where the implied constant is absolute and effectively computable.*

The constant $L$ is known as Linnik's constant. The Generalized Riemann Hypothesis implies any fixed $L>2$ is admissible in (1.6) and conjecturally $L>1$ holds. Recently, Lamzouri, Li, and Soundararajan [LLS15] made this GRH bound explicit, showing that

$$P(a,q) \le (\varphi(q)\log q)^2 \tag{1.7}$$

for $q \ge 4$. Unconditional bounds for Linnik's constant have a long history beginning with Pan [Pan57] at $L = 10,000$ and the current world record sitting at $L = 5$ by Xylouris [Xyl11b] based on suggestions in the landmark paper of Heath-Brown [HB92] (see the references therein for a detailed list of prior works). While elementary proofs of Linnik's theorem exist [FI10, GHS], the records for $L$ have thus far been based on Linnik's original proof which revolves around deep statistical information on the zeros of Dirichlet $L$-functions.

We recall the modern approach to proving Linnik's bound on the least prime in an arithmetic progression. In order to obtain small explicit values of $L$ in (1.6), one typically requires three principles [IK04, Chapter 18]; we cite explicit versions which are recorded in [HB92, Section 1]:

1) **A zero-free region**: If $q$ is sufficiently large, then the product $\prod_{\chi \,(\mathrm{mod}\, q)} L(s,\chi)$ has at most one zero in the region

$$s = \sigma + it, \qquad \sigma \ge 1 - \frac{0.10367}{\log(q(2+|t|))}. \tag{1.8}$$

If such an exceptional zero exists, then it is real and simple and it corresponds with a non-trivial real character.

2) **A "log-free" zero density estimate**: If $q$ is sufficiently large, $\epsilon > 0$, and we define $N(\sigma,T,\chi) = \#\{\rho = \beta + i\gamma : L(\rho,\chi) = 0, |\gamma| \le T, \beta \ge \sigma\}$, then

$$\sum_{\chi \,(\mathrm{mod}\, q)} N(\sigma,T,\chi) \ll_\epsilon (qT)^{(\frac{12}{5}+\epsilon)(1-\sigma)}, \qquad T \ge 1. \tag{1.9}$$

3) **Deuring–Heilbronn phenomenon**: If $q$ is sufficiently large, $\lambda_1 > 0$ is sufficiently small, $\epsilon > 0$, and the exceptional zero $\beta_1$ in the region (1.8) exists and equals $1 - \lambda_1/\log q$, then $\prod_{\chi \,(\mathrm{mod}\, q)} L(s,\chi)$ has no other zeros in the region

$$\sigma \ge 1 - \frac{(\frac{2}{3} - \epsilon)\log(1/\lambda_1)}{\log(q(2+|t|))}. \tag{1.10}$$

As mentioned earlier, upper bounds for $P(a,q)$ are connected with lower bounds for $\pi(x;q,a)$. The most recent such estimate is due to Maynard [May13] who showed for $x > q^8$ and $q$

sufficiently large that

$$\pi(x;q,a) \gg \frac{\log q}{\sqrt{q}} \cdot \frac{1}{\varphi(q)} \mathrm{Li}(x), \tag{1.11}$$

where the implied constant is effectively computable. As with prior versions of (1.11), the valid range of $x$ constitutes a significant improvement over the range of $x$ in (1.4). Linnik's theorem with $L = 5$ and (1.11) represent the best available unconditional lower bounds for $\pi(x;q,a)$ with current techniques.

### Brun–Titchmarsh theorem

Now, we turn to upper bounds for $\pi(x;q,a)$. Titchmarsh [Tit30] made the first major development by using Brun's sieve to show for $x > q$ and $\theta = \frac{\log q}{\log x}$ that

$$\pi(x;q,a) \ll \frac{1}{1-\theta} \frac{1}{\varphi(q)} \mathrm{Li}(x). \tag{1.12}$$

The implied constant can be made precise and has been estimated by many authors. Using the large sieve, Montgomery and Vaughan [MV73] established the strongest such result uniform over all $x > q$ and obtained the following:

**Theorem** (Brun–Titchmarsh theorem)**.** *Let* $\theta = \frac{\log q}{\log x}$. *For* $x > q$*,*

$$\pi(x;q,a) < C(\theta) \frac{1}{\varphi(q)} \mathrm{Li}(x), \tag{1.13}$$

*where* $C(\theta) = 2/(1-\theta)$.

The range of $x > q$ is best possible since trivially $\pi(x;q,a) \leq 1$ for $x \leq q$. The constant 2 in $C(\theta)$ is also best possible, short of precluding an exceptional zero. Thus, subsequent authors have instead improved the $1/(1-\theta)$ factor for various ranges of $\theta$ including, for example, Motohashi [Mot74], Goldfeld [Gol75], Iwaniec [Iwa82], and Friedlander–Iwaniec [FI97]. The aforementioned works made progress on (1.12) from advances in sieve theory and exponential sums. However, the recent work of Maynard [May13] avoids sieve methods entirely and relies on information about Dirichlet $L$-functions inspired by the three principles of Linnik's approach.

**Theorem** (Maynard)**.** *For* $x > q^8$ *and* $q$ *sufficiently large,*

$$\pi(x;q,a) < \frac{2}{\varphi(q)} \mathrm{Li}(x). \tag{1.14}$$

*Equivalently,* $C(\theta) = 2$ *for* $0 < \theta < 1/8$.

His proof builds on Heath-Brown's analysis in [HB92], using a log-free zero density estimate for Dirichlet $L$-functions and a delicate analysis of exceptional zeros. This alternate approach will be crucial for our purposes. Ultimately, the range of $x$ in the Brun-Titchmarsh theorem (and its descendants) constitutes a substantial improvement over the Siegel–Walfisz theorem (1.4). It remains the best upper bound of $\pi(x; q, a)$ for small $x$.

## 1.2 Primes in the Chebotarev Density Theorem

The setting of this thesis will be a vast generalization of the Prime Number Theorem for Arithmetic Progressions. Let $L/F$ be a Galois extension of number fields with Galois group $G := \mathrm{Gal}(L/F)$. For a prime ideal $\mathfrak{p}$ of $F$ unramified in $L$, let $[\frac{L/F}{\mathfrak{p}}]$ denote the conjugacy class of Frobenius automorphisms in $G$ above $\mathfrak{p}$; we refer to it as the Artin symbol of $\mathfrak{p}$. For a conjugacy class $C \subseteq G$, define for $x > 1$

$$\pi_C(x, L/F) := \#\Big\{\mathfrak{p} : \mathrm{N}^F_{\mathbb{Q}}\mathfrak{p} < x, \mathfrak{p} \text{ prime ideal of } F \text{ unramified in } L, \Big[\frac{L/F}{\mathfrak{p}}\Big] = C\Big\}, \quad (1.15)$$

where $\mathrm{N}^F_{\mathbb{Q}}$ is the absolute norm of $F$ over $\mathbb{Q}$. We are interested in the growth of the prime counting function $\pi_C(x, L/F)$. Established in 1926 [Tsc26], the Chebotarev Density Theorem (CDT) states that the Artin symbols of primes ideals of $F$ are equidistributed in $G$; namely,

$$\pi_C(x, L/F) \sim \frac{|C|}{|G|}\mathrm{Li}(x) \quad (1.16)$$

as $x \to \infty$. The special case $F = \mathbb{Q}$ and $L = \mathbb{Q}(e^{2\pi i/q})$ reduces to the Prime Number Theorem for Arithmetic Progressions in the form of (1.1). The CDT is tremendously powerful in a wide variety of applications such as the distribution of primes ideals, binary quadratic forms, elliptic curves, $\ell$-adic representations, and modular forms.

Analogous to the PNT for APs, proving the CDT requires knowledge about the distributions of zeros of $L$-functions attached to the extension $L/F$. In particular, for the number field $L$, one must analyze the Dedekind zeta function of $L$ given by

$$\zeta_L(s) = \sum_{\mathfrak{N}} (\mathrm{N}^L_{\mathbb{Q}}\mathfrak{N})^{-s} = \prod_{\mathfrak{P}} \Big(1 - \frac{1}{(\mathrm{N}^L_{\mathbb{Q}}\mathfrak{P})^s}\Big)^{-1} \quad (1.17)$$

for $\mathrm{Re}\{s\} > 1$. Here the sum is over integral ideals $\mathfrak{N}$ of $L$ and the product is over prime ideals $\mathfrak{P}$ of $L$. As with Dirichlet $L$-functions, the Dedekind zeta function $\zeta_L(s)$ satisfies a functional equation and has analytic continuation to the entire complex plane with a simple pole at $s = 1$. Further, its non-trivial zeros lie in the critical strip $0 < \mathrm{Re}\{s\} < 1$ and its trivial zeros are at certain non-positive integers. The Dedekind zeta function also suffers from a putative exceptional zero; that is, we cannot eliminate the existence of a real non-trivial zero $\beta_1$ which is exceedingly close to $s = 1$. Conjecturally, such zeros do not exist and, moreover, the Generalized Riemann Hypothesis posits that all of the non-trivial zeros lie on the critical line $\mathrm{Re}\{s\} = 1/2$.

Lagarias and Odlyzko [LO77] gave an unconditional field-uniform version of the CDT with an error term, which generalizes the Siegel–Walfisz theorem (1.4). In particular,

$$\pi_C(x, L/F) = \frac{|C|}{|G|}\mathrm{Li}(x) + O\Big(x \exp\Big(-c_2\sqrt{\frac{\log x}{n_L}}\Big)\Big) \tag{1.18}$$

for $x \geq \exp(10 n_L (\log D_L)^2)$, where $n_L = [L : \mathbb{Q}]$ is the degree of $L/\mathbb{Q}$ and $D_L = |\mathrm{disc}(L/\mathbb{Q})|$ is the absolute value of the absolute discriminant of $L$. In the above form, the implied constants in (1.18) are ineffective but can be made effective and absolute by using results of Stark [Sta74] and enlarging the range of $x$. Assuming GRH, Lagarias and Odlyzko also showed that the error term in (1.18) may be significantly improved to $O(x^{1/2}\log(D_L x^{n_L}))$ for $x \gg (\log D_L)^2$. While (1.18) is unsurprisingly far from this expected truth, like (1.4), its range of validity can be restrictive in many applications. For instance, in the special case of PNT for APs, (1.18) implies $\pi(x; q, a)$ attains its asymptotic for $x \geq e^{O(q^3 \log^2 q)}$ which is far worse than Siegel–Walfisz (1.4). This predicament underlies the motivating question of this thesis:

*Can one estimate $\pi_C(x, L/F)$ within an absolute constant of its asymptotic size $\frac{|C|}{|G|}\mathrm{Li}(x)$ for a range of $x$ which is superior to* (1.18)*?*

For example, an estimate of the form

$$\frac{|C|}{|G|}\mathrm{Li}(x) \ll \pi_C(x, L/F) \ll \frac{|C|}{|G|}\mathrm{Li}(x),$$

for small values of $x$ (polynomial in $D_L$, say) would be extremely desirable. As we shall see, there has already been some progress towards answering this question. However, just as with $\pi(x; q, a)$, the quoted lower bound is overly optimistic (we will settle for a slightly weaker variant) whereas a suitable upper bound is attainable. We will first review the surrounding literature and then state our main results in Sections 1.3 and 1.4

**Least prime ideal**

First, we consider lower bounds for $\pi_C(x, L/F)$. In analogy with (1.5), these are intimately related to bounding the prime ideal of least norm with Artin symbol equal to the conjugacy class $C$. In other words, if we define

$$P(C, L/F) := \min\Big\{\mathrm{N}^F_{\mathbb{Q}}\mathfrak{p} : \mathfrak{p} \text{ degree 1 prime ideal of } F \text{ unramified in } L, \Big[\frac{L/F}{\mathfrak{p}}\Big] = C\Big\}, \tag{1.19}$$

then we are interested in providing an upper bound for $P(C, L/F)$. We sometimes refer to this quantity as the *least prime ideal*. Note the condition that $\mathfrak{p}$ is degree 1 (equivalently, $\mathrm{N}^F_{\mathbb{Q}}\mathfrak{p}$ is a rational prime) is unnecessary but interesting for certain applications. Assuming GRH for the Dedekind zeta function of $L$, Lagarias and Odlyzko [LO77] showed that

$$P(C, L/F) \ll (\log D_L)^2. \tag{1.20}$$

Bach and Sorenson [BS96] have since provided an explicit version of this bound. These estimates specialize to (1.7) up to quality of the implied constant. Additionally assuming Artin's holomorphy conjecture, V.K. Murty [Mur00] proved a further refinement of (1.20) which nicely depends on the size of the conjugacy class $C$.

The first non-trivial unconditional upper bound for $P(C, L/F)$ is due to Lagarias–Montgomery–Odlyzko [LMO79], wherein they showed

$$P(C, L/F) \ll D_L^{B_1} \tag{1.21}$$

for some absolute effectively computable constant $B_1 > 0$. Compared with what is implied by (1.18), this is a remarkable improvement. Their proof was modelled after Linnik's classical approach but the analysis of the Dedekind zeta function of $L$ only required two of the three principles: a zero-free region and Deuring–Heilbronn phenomenon. The establishment of the latter was through a pioneering application of power sums. However, unlike Linnik's constant for arithmetic progressions and (1.11) for $\pi(x; q, a)$, no explicit value of $B_1 > 0$ has been computed before and no corresponding quantitative lower bound has yet been established for $\pi_C(x, L/F)$ in the range $x \gg D_L^{B_1}$. Furthermore, (1.21) implies $P(a, q) \ll q^{B_1 q}$ which is a far cry from Linnik's theorem (1.6); in fact, the bound implied by Siegel–Walfisz (1.4) is better.

By exploiting some class field theory within $L/F$, one can obtain further improvement over (1.21) in many cases and also recover Linnik's theorem. Let $H \subseteq G$ be an abelian subgroup such that $H \cap C$ is non-empty. Let $K = L^H$ be the subfield of $L$ fixed by $H$. By class field theory, the characters of $H = \mathrm{Gal}(L/K)$ are Hecke characters $\chi$ of $K$ and therefore have an associated $K$-integral ideal $\mathfrak{f}_\chi$ called the conductor of $\chi$. Thus, we may define the *maximum conductor* of $L/K$ to be

$$\mathcal{Q} = \mathcal{Q}(L/K) = \max\{\mathrm{N}^K_{\mathbb{Q}}\mathfrak{f}_\chi : \chi \in \widehat{\mathrm{Gal}}(L/K)\}. \tag{1.22}$$

Roughly speaking, $\mathcal{Q}$ is a measure of the ramification occurring in the abelian extension $L/K$. For example, $\mathcal{Q} = 1$ implies $L$ is an unramified extension of $K$. The first result on $P(C, L/F)$ utilizing $\mathcal{Q}$ follows[3] from the works of Fogels [Fog62b, Fog62c, Fog62a]; namely,

$$P(C, L/F) \ll (D_K \mathcal{Q})^{C(n_K)}, \tag{1.23}$$

where $C(n_K) > 0$ is some constant depending only on $n_K = [K : \mathbb{Q}]$. The main drawback of this bound is its lack of complete field-uniformity, especially its unclear and unsatisfactory dependence of the exponent on the degree of $K$ over $\mathbb{Q}$. A couple of decades later, Weiss [Wei83, Wei80] amended these issues and proved

$$P(C, L/F) \ll \left(n_K^{n_K} D_K \mathcal{Q}\right)^{B_2} \tag{1.24}$$

for some absolute and effectively computable constant $B_2 > 0$. As with (1.21), no explicit value of $B_2 > 0$ has yet been calculated. This gap is one of the major objectives of this thesis. To see how (1.24) compares to (1.21), observe[4] that if $H$ is a cyclic subgroup of $G$, then

$$D_L^{1/|H|} \leq D_K \mathcal{Q} \leq D_L^{1/\varphi(|H|)}.$$

Therefore, if the $n_K^{n_K}$ term is negligible in (1.24) then a large cyclic subgroup $H$ intersecting the conjugacy class $C$ is expected to yield savings in (1.24) over (1.21). We will elaborate further on this comparison of (1.21) and (1.24) following Theorem 1.3.2. As one last example, if $F = \mathbb{Q}$ and $L = \mathbb{Q}(e^{2\pi i/q})$, then one may take $H$ to be the full Galois group $G \cong (\mathbb{Z}/q\mathbb{Z})^\times$, in which case $K = F = \mathbb{Q}$ and $\mathcal{Q}(L/K) = q$. Thus, Weiss proves a bound on $P(C, L/F)$ which provides a "continuous transition" from (1.6) to (1.21). In particular, Linnik's theorem (1.6) follows from (1.24).

The proof of (1.24) is again fundamentally motivated by Linnik's approach in the case of arithmetic progressions, requiring an intense study of Hecke $L$-functions and their zeros. For a Hecke character $\chi$ of $K$, the Hecke $L$-function of $\chi$ is given by

$$L(s, \chi, K) = \prod_{\mathfrak{p}} \left(1 - \frac{\chi(\mathfrak{p})}{\mathrm{N}_{\mathbb{Q}}^{K} \mathfrak{p}^s}\right)^{-1} \tag{1.25}$$

for $\mathrm{Re}\{s\} > 1$, where the product is over prime ideals $\mathfrak{p}$ of $K$. In the case $K = \mathbb{Q}$, these are precisely Dirichlet $L$-functions. They satisfy the same type of analytic properties with all

[3] Fogels actually bounds $P(C, L/K)$ when $L/K$ is abelian, but his results can be used to give the claimed estimate.

[4] See [BS96, Lemma 4.2] for a proof of the upper bound; the lower bound holds for all $H$ and follows from the conductor-discriminant formula.

their non-trivial zeros lying in the critical strip $0 < \mathrm{Re}\{s\} < 1$. By class field theory, our understanding of the distribution of Artin symbols in $L/F$ is dictated by our knowledge of the distribution of the zeros of Hecke $L$-functions of $K$ (see Section 2.5 for details). To prove (1.24), Weiss utilized non-explicit analogues of Linnik's three principles for Hecke $L$-functions and, most importantly, proved a field-uniform log-free zero density estimate for Hecke $L$-functions. To be specific, he applied[5]:

1) **A zero-free region**: The product $\prod_\chi L(s,\chi,K)$ over Hecke characters $\chi$ attached to $L/K$ have at most one zero in the region

$$s = \sigma + it, \qquad \sigma \geq 1 - \frac{c_1}{\log(D_K \mathcal{Q} n_K^{n_K}) + n_K \log(2+|t|)}, \tag{1.26}$$

for some absolute constant $c_1 > 0$. If such an exceptional zero exists, then it is real and simple and it corresponds with a (possibly trivial) real character.

2) **A "log-free" zero density estimate**: For a Hecke character $\chi$, if we define $N(\sigma,T,\chi) = \#\{\rho = \beta + i\gamma : L(\rho,\chi,K) = 0, |\gamma| \leq T, \beta \geq \sigma\}$ then, for some absolute constant $c_2 > 0$,

$$\sum_\chi N(\sigma,T,\chi) \ll (D_K \mathcal{Q} n_K^{n_K} T^{n_K})^{c_2(1-\sigma)}, \qquad T \geq 1, \tag{1.27}$$

where the sum is over Hecke characters $\chi$ attached to $L/K$.

3) **Deuring–Heilbronn phenomenon**: If $\lambda_1 > 0$ is sufficiently small and the exceptional zero $\beta_1$ in the region (1.26) exists and equals $1 - \lambda_1/\log(D_K \mathcal{Q} n_K^{n_K})$, then the product $\prod_\chi L(s,\chi,K)$ over Hecke characters $\chi$ attached to $L/K$ has no other zeros in the region

$$\sigma \geq 1 - \frac{c_3 \log(1/\lambda_1)}{\log(D_K \mathcal{Q} n_K^{n_K}) + n_K \log(2+|t|))} \tag{1.28}$$

for some absolute constant $c_3 > 0$.

Weiss established (1.27) and (1.28) while (1.26) is contained in [LMO79]. He actually proved estimates [Wei83, Theorem 5.2] which imply a quantitative lower bound for $\pi_C(x,L/F)$ (see Theorem 1.3.2 for an explicit variant). In summary, we have adequate field-uniform lower bounds for $\pi_C(x,L/F)$ but, in contrast with Linnik's theorem and primes in arithmetic progressions, none of the existing results have explicit versions.

[5]The appearance of $n_K^{n_K}$ is not actually necessary for principles 1 and 3 but we keep it for simplicity.

**Brun–Titchmarsh analogues**

Next, we consider upper bounds for $\pi_C(x, L/F)$ which extend the range of (1.18). Here far less is known. There are a number of results estimating variants or special cases of $\pi_C(x, L/F)$, such as [Hux68, Sch70] for counting prime integers in the ring of integers of a number field and [HL94] for counting prime ideals lying in ray classes, but these lack complete field-uniformity or do not directly estimate the distribution of Artin symbols. It seems the only existing field-uniform upper bound for $\pi_C(x, L/F)$ comes from the foundational paper of Lagarias–Montgomery–Odlyzko [LMO79] in which they showed

$$\pi_C(x, L/F) \ll \frac{|C|}{|G|}\mathrm{Li}(x) \tag{1.29}$$

for $\log x \gg \log D_L \log\log D_L \log\log\log(e^{20} D_L)$. The proof avoids sieve methods and uses basic information about the zeros of the Dedekind zeta function of $L$. While (1.29) is a significant refinement over the range $\log x \gg n_L(\log D_L)^2$ provided by the effective Chebotarev Density Theorem (1.18), it remains prohibitive for use in applications. For example, in the case of arithmetic progressions, it implies the Brun-Titchmarsh theorem (1.12) for $x \geq \exp(O(q \log^2 q))$. This is worse than the effective asymptotic given by Siegel–Walfisz. Moreover, there has been no explicit computation of the implied constant in (1.29). One expects it to be close to $2$ due to the possibility of a real exceptional zero, just as in (1.13). These deficiencies suggest there is much left to be desired for upper bounds of $\pi_C(x, L/F)$ beyond (1.29).

## 1.3 Analytic estimates

We may now state the main results of this thesis. Recall

$$n_L = [L : \mathbb{Q}], \qquad D_L = |\mathrm{disc}(L/\mathbb{Q})|,$$

and $\mathcal{Q} = \mathcal{Q}(L/K)$ is defined by (1.22).

### 1.3.1 Primes in the Chebotarev Density Theorem

We begin with lower bounds for $\pi_C(x, L/F)$. We establish the first explicit value of $B_1$ in (1.21) and a corresponding quantitative lower bound for $\pi_C(x, L/F)$.

**Theorem 1.3.1.** *Let $L/F$ be a Galois extension of number fields with Galois group $G$ and let*

*$C \subseteq G$ be a conjugacy class. Then*

$$\pi_C(x, L/F) \gg \frac{1}{D_L^{19}} \frac{|C|}{|G|} \mathrm{Li}(x)$$

*for $x \geq D_L^{35}$ and $D_L$ sufficiently large. In particular,*

$$P(C, L/F) \ll D_L^{35}.$$

**Remark.**

- In several cases, one can reduce the exponent $B_1 = 35$ by straightforward modifications. For example, one can take

$$B_1 = \begin{cases} 32 & \text{if } L \text{ has a tower of normal extensions with base } \mathbb{Q}, \\ 24.1 & \text{if } n_L = o(\log D_L) \text{ as } D_L \to \infty, \\ 7.5 & \text{if } \zeta_L(s) \text{ does not have a real zero } \beta_1 = 1 - \frac{\lambda_1}{\log D_L} \text{ satisfying } \lambda_1 = o(1), \end{cases}$$

  where $\zeta_L(s)$ is the Dedekind zeta function of $L$. See the remark at the end of Section 7.2.4 for details.

- Note Theorem 1.3.1 improves over [Zam17] wherein the constant $B_1 = 40$ is shown to be admissible. This improvement stems from a minor adjustment which can be found in the proof of Theorem 6.1.2; namely, we discard some of the real non-trivial zeros in a certain power sum estimate.

Theorem 1.3.1 is an explicit variant of [LMO79, Theorem 1.1] though the quantitative lower bound in Theorem 1.3.1 is not contained in [LMO79]. Its proof is motivated by its predecessor in conjunction with the powerful techniques pioneered by Heath-Brown [HB92] in the classical case of arithmetic progressions. In particular, we required explicit versions of the zero-free region for the Dedekind zeta function (due to Kadiri [Kad12]) and the Deuring–Heilbronn phenomenon. For the latter principle, we use an explicit variant due to Kadiri–Ng [KN12] but their result is not intended to repel zeros deep into the critical strip. Hence, we carefully used the power sum method founded in [LMO79] to obtain a fully equipped Deuring–Heilbronn phenomenon for the Dedekind zeta function. See Chapter 6 for details. After the completion[6] of Theorem 1.3.1, the author was informed[7] by Kadiri and Ng of their unpublished work [KN] in the case $F = \mathbb{Q}$ in which they obtain an upper bound of $D_L^{40}$ for $P(C, L/\mathbb{Q})$.

[6]An earlier version of [Zam17] was posted to the arXiv in August 2015.
[7]private communication, January 2016.

In joint work with Jesse Thorner, our second main result is the first explicit value of $B_2$ in (1.24) and a corresponding quantitative lower bound for $\pi_C(x, L/F)$.

**Theorem 1.3.2** (Thorner–Z.). *Let $L/F$ be a Galois extension of number fields with Galois group $G$ and let $C \subseteq G$ be a conjugacy class. Let $H \subseteq G$ be an abelian subgroup such that $H \cap C$ is nonempty, $K = L^H$ be the subfield of $L$ fixed by $H$, and $\mathcal{Q} = \mathcal{Q}(L/K)$ be defined by* (1.22). *Then*

$$\pi_C(x, L/F) \gg \frac{1}{D_K^5 \mathcal{Q}^4 n_K^{3n_K}} \frac{|C|}{|G|} \mathrm{Li}(x)$$

*for $x \geq D_K^{694} \mathcal{Q}^{521} + D_K^{232} \mathcal{Q}^{367} n_K^{290 n_K}$ and $D_K \mathcal{Q} n_K^{n_K}$ sufficiently large. In particular,*

$$P(C, L/F) \ll D_K^{694} \mathcal{Q}^{521} + D_K^{232} \mathcal{Q}^{367} n_K^{290 n_K}. \tag{1.30}$$

**Remarks.**

- Theorem 1.3.2 immediately implies that $P(a, q) \ll q^{521}$. For historical context, this is slightly better than Jutila's bound [Jut70] on $P(a, q)$ established in 1970, which was over 25 years after Linnik's original theorem.

- If $n_K \leq 2(\log D_K)/\log\log D_K$, then $P(C, L/F) \ll D_K^{694} \mathcal{Q}^{521}$. Situations where $n_K > 2(\log D_K)/\log\log D_K$ are rare; the largest class of known examples involve infinite $p$-class tower extensions, which were first studied by Golod and Shafarevich [GS64].

- If $L/K$ is unramified, then $\mathcal{Q} = 1$ and $D_K = D_L^{1/|H|}$. Thus,

  $$P(C, L/F) \ll D_L^{694/|H|} + D_L^{232/|H|} n_K^{290 n_K}.$$

  If additionally $n_K \leq 2(\log D_K)/\log\log D_K$, this gives

  $$P(C, L/F) \ll D_L^{694/|H|}$$

  which improves over Theorem 1.3.1 when $|H| \geq 18$.

- In independent work of the author, we consider the case when the degree $n_K$ is absolutely bounded. For $n_K \leq 10^{49}$, we obtain a further numerical improvement on the exponents in (1.30). See Theorem 7.4.1 for details.

- See Theorem 10.1.4 for an alternate formulation.

Theorem 1.3.2 is an explicit variant of [Wei83, Theorem 5.2]. The quantitative lower bound in Theorem 1.3.2 could conceivably be sharpened to match [Wei83, Theorem 5.2] with some additional effort. The proof is inspired by Weiss's approach combined with the innovations of Heath-Brown [HB92] in the classical case of arithmetic progressions. We naturally required explicit versions of Linnik's three principles (namely, (1.26), (1.27), (1.28)) for Hecke $L$-functions. Some information about their zero-free regions is due to Ahn and Kwon [AK14] and Kadiri [Kad12], but this limited scope is the extent of pre-existing results. Thus, another major contribution of this thesis is the explicit estimates for the zeros of Hecke $L$-functions. See Section 1.3.2 for an overview and the beginnings of Chapters 4 to 6 for details on these new results.

Now, in comparison with the world record value for Linnik's constant in (1.6), the exponents appearing in (1.30) may seem unusually large. This difference chiefly originates from the log-free zero density estimate and its proof which uses Turán power sums. This method is numerically less efficient than those employed in the classical case of arithmetic progressions, but it has seemingly been the only way to obtain the desired field uniformity. See Section 5.1, especially Section 5.1.1, for a more detailed explanation of this numerical deficiency.

Next, we direct our attention to new upper bounds of $\pi_C(x, L/F)$ established in joint work with Jesse Thorner. Using the log-free zero density estimates in Chapter 5, we prove:

**Theorem 1.3.3** (Thorner–Z.)**.** *Let $L/F$ be a Galois extension of number fields with Galois group $G$. Let $C$ be any conjugacy class of $G$ and let $H$ be an abelian subgroup of $G$ such that $H \cap C$ is non-empty. If $K$ is the subfield of $L$ fixed by $H$ and $\mathcal{Q} = \mathcal{Q}(L/K)$ is given by* (1.22)*, then*

$$\pi_C(x, L/F) \ll \frac{|C|}{|G|}\mathrm{Li}(x),$$

*provided that*

$$x \gg D_K^{246}\mathcal{Q}^{185} + D_K^{82}\mathcal{Q}^{130}n_K^{246n_K}. \tag{1.31}$$

**Remarks.**

- For the valid range of $x$, one can minimize the exponents of $D_K$ and $\mathcal{Q}$ at the expense of a less desirable dependence on $n_K^{n_K}$ and vice versa. In particular, the same upper bound for $\pi_C(x, L/F)$ holds when

$$x \gg D_K^{164}\mathcal{Q}^{123} + D_K^{55}\mathcal{Q}^{87}n_K^{68n_K} + D_K^{2}\mathcal{Q}^{2}{n_K}^{14{,}000n_K}. \tag{1.32}$$

  See the remarks at the end of Section 8.2.1 for details.

- See Theorem 10.1.5 for an alternate formulation.

Our result always gives an improvement over (1.29). Choosing $H$ to be the cyclic group generated by a fixed element of $C$, we have that $D_L^{1/|H|} \leq D_K\mathcal{Q} \leq D_L^{1/\varphi(|H|)}$ (see [Wei83, Section 6]). Moreover, by the classical work of Minkowski and the conductor-discriminant formula (2.21), we have that $n_K \ll \log D_K \leq \frac{1}{|H|}\log D_L$. Therefore, Theorem 1.3.3 holds when $\log x \gg \frac{1}{\varphi(|H|)}(\log D_L)(\log\log D_L)$. This improves (1.29) especially when $H$ is a large. One usually obtains further savings. For most fields $K$, it seems reasonable to expect $n_K \ll (\log D_K)/\log\log D_K$ holds in light of Minkowski's bound for $n_K$. In this case, Theorem 1.3.3 holds for $\log x \gg \log(D_K\mathcal{Q})$ or rather $\log x \gg \frac{1}{\varphi(|H|)}\log D_L$.

Building on [May13], we obtain an implied constant that is essentially sharp (short of precluding the existence of an exceptional zero) when $x$ is sufficiently large in terms of $L/F$.

**Theorem 1.3.4** (Thorner–Z.). *Let $L/F$ be a Galois extension of number fields with Galois group $G$ and let $C$ be any conjugacy class of $G$. Let $H$ be an abelian subgroup of $G$ such that $H \cap C$ is non-empty. If $K$ is the subfield of $L$ fixed by $H$ and $\mathcal{Q} = \mathcal{Q}(L/K)$ is given by* (1.22)*, then*

$$\pi_C(x, L/F) < \left\{2 + O\left(n_K x^{-\frac{1}{166n_K+327}}\right)\right\}\frac{|C|}{|G|}\mathrm{Li}(x)$$

*for*

$$x \gg D_K^{695}\mathcal{Q}^{522} + D_K^{232}\mathcal{Q}^{367}n_K^{290n_K}, \tag{1.33}$$

*provided that $D_K\mathcal{Q}n_K^{n_K}$ is sufficiently large. If any of the following conditions also hold, then the error term can be omitted:*

- *$K$ has a tower of normal extensions over $\mathbb{Q}$.*
- *$(2n_K)^{2n_K^2} \ll D_K\mathcal{Q}^{1/2}$.*
- *$x \gg n_K^{334n_K^2}$.*

The source of our improvements in Theorems 1.3.3 and 1.3.4 over (1.29) stem from further exploiting the decomposition of the Dedekind zeta function of $L$ as a product of Hecke $L$-functions of $K$. This allows us to apply the powerful log-free zero density estimate and more efficiently estimate certain sums over non-trivial zeros. The proofs are inspired by Maynard's [May13] $L$-function and "Linnik-type" approach to the Brun–Titchmarsh theorem. Consequently, we carefully apply the same explicit estimates for Hecke $L$-functions used in the proof of Theorem 1.3.2 and perform a similarly delicate analysis in the case of an exceptional zero.

### 1.3.2 Distribution of zeros of Hecke $L$-functions

We summarize the key results on the zeros of Hecke $L$-functions in this thesis which make explicit principles (1.26), (1.27), and (1.28). Additional results and a more detailed discussion for each principle can be found at the beginning of Chapters 4 to 6.

**Theorem 1.3.5.** *Suppose $L/K$ is an abelian extension and $\mathcal{Q} = \mathcal{Q}(L/K)$ is given by* (1.22). *If $D_K \mathcal{Q} n_K^{n_K}$ is sufficiently large then the product of Hecke $L$-functions $\prod_\chi L(s,\chi,K)$ of Hecke characters $\chi$ attached to $L/K$ has at most one zero, counting with multiplicity, in the rectangle*

$$\sigma \geq 1 - \frac{0.0875}{\log(D_K \mathcal{Q} n_K^{n_K})}, \qquad |t| \leq 1,$$

*where $s = \sigma + it \in \mathbb{C}$. If this exceptional zero exists, then it is a simple real zero and its associated character is real.*

**Corollary 1.3.6.** *Suppose $D_K n_K^{n_K}$ is sufficiently large. The Dedekind zeta function $\zeta_K(s)$ has at most 1 zero, counting with multiplicity, in the rectangle*

$$\sigma \geq 1 - \frac{0.0875}{\log(D_K n_K^{n_K})}, \qquad |t| \leq 1,$$

*where $s = \sigma + it$. If this exceptional zero exists, it is real.*

Theorem 1.3.5 and Corollary 1.3.6 are both improvements over [AK14, Kad12] when $n_K = o(\log D_K / \log\log D_K)$, which is often a mild assumption as $n_K = O(\log D_K)$ unconditionally. See Section 4.1 for a stronger theorem (cf. Theorem 4.1.1) and additional details. Next, for a Hecke character $\chi$ of a number field $K$, define

$$N(\sigma, T, \chi) = \#\{\rho = \beta + i\gamma : L(\rho, \chi, K) = 0, |\gamma| \leq T, \beta \geq \sigma\}.$$

We prove the first explicit version of a "log-free" zero density estimate for Hecke $L$-functions.

**Theorem 1.3.7** (Thorner–Z.). *Suppose $L/K$ is an abelian extension and $\mathcal{Q} = \mathcal{Q}(L/K)$ is given by* (1.22). *For $T \geq 1$ and $0 \leq \sigma \leq 1$,*

$$\sum_\chi N(\sigma, T, \chi) \ll (D_K \mathcal{Q} n_K^{n_K} T^{n_K+1})^{162(1-\sigma)}.$$

*where the sum is over Hecke characters $\chi$ attached to $L/K$.*

See Sections 5.1 and 7.3.1 for a stronger theorem (cf. Theorem 5.1.1 and Theorem 7.3.6) and further discussion. Finally, we prove an explicit version of Deuring–Heilbronn phenomenon for Hecke $L$-functions.

**Theorem 1.3.8** (Thorner–Z.). *Suppose $L/K$ is an abelian extension, $\mathcal{Q} = \mathcal{Q}(L/K)$ is given by* (1.22)*, and $D_K \mathcal{Q} n_K^{n_K}$ is sufficiently large. Assume a real Hecke character $\psi$ of $L/K$ has a* *real zero $\frac{1}{2} \leq \beta_1 < 1$. Then the product $\prod_\chi L(s,\chi,K)$ over Hecke characters $\chi$ attached to $L/K$ has no other zeros in the region*

$$\sigma \geq 1 - \frac{\log\left(\frac{c_3}{(1-\beta_1)\log(D_K \mathcal{Q} n_K^{n_K} T^{n_K})}\right)}{61\log(D_K \mathcal{Q} n_K^{n_K} T^{n_K})},$$

*where $c_3 > 0$ is an absolute effective sufficiently small constant.*

See Section 6.1 for a stronger theorem (cf. Theorem 6.1.1 which has no $n_K^{n_K}$ dependence) and further discussion. See also Section 4.1 (cf. Theorem 4.1.3) for other variants of Deuring–Heilbronn which we refer to as "zero repulsion".

## 1.4 Applications

Finally, we list the key applications of the results from Section 1.3.

### 1.4.1 Binary quadratic forms

Let us review the classical theory of primitive (integral) binary quadratic forms with negative discriminant and their connections with the Chebotarev Density Theorem. The results from Section 1.3 allow us to deduce new consequences for such forms. We follow much of the notation and conventions of [Cox89].

Let $D \geq 1$ be a positive integer. Let $Q(X,Y) = aX^2 + bXY + cY^2 \in \mathbb{Z}[X,Y]$ be a binary quadratic form with discriminant $b^2 - 4ac = -D$. The form is primitive if its coefficients $a,b,c$ are relatively prime. A matrix $g = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in GL_2(\mathbb{Z})$ naturally acts on such forms via

$$(g \cdot Q)(X,Y) = Q(pX + qY, rX + sY).$$

This gives an equivalence relation between primitive binary quadratic forms with discriminant $-D$. Two forms are said to be equivalent if they differ by a transformation in $\mathrm{GL}_2(\mathbb{Z})$. Two forms are said to be properly equivalent[8] if they differ by a transformation in $\mathrm{SL}_2(\mathbb{Z})$. By the beautiful composition laws and genus theory of Gauss, the set of such forms, up to proper equivalence, form a finite abelian group, say $\mathrm{Cl}(-D)$. Let $h(-D)$ be the size of this group; that is, $h(-D)$ is the number of primitive binary quadratic forms with discriminant $-D$, up to proper equivalence.

[8] Sometimes we may refer to this as $\mathrm{SL}_2$-equivalence.

We say that an integer $m$ is represented by $Q(X,Y)$ if there exists $(X,Y) \in \mathbb{Z}^2$ such that $Q(X,Y) = m$. Our central focus is on primes $p$ represented by $Q(X,Y)$. For $x > 1$, define

$$\pi_Q(x) = \#\{p \leq x : p \nmid D,\ p \text{ is represented by } Q(X,Y)\}.$$

Amazingly, $\pi_Q(x)$ is an instance of a Chebotarev prime counting function $\pi_C(x, L/\mathbb{Q})$ for a particular number field $L$.

**Theorem.** *Let $D \geq 1$ be a positive integer. Let $Q(X,Y) = aX^2 + bXY + cY^2 \in \mathbb{Z}[X,Y]$ be a primitive binary quadratic form with discriminant $-D$. Let $K = \mathbb{Q}(\sqrt{-D})$ and write $D = f^2 D_K$. There exists a number field $L$, called the ring class field of the order of discriminant $-D$ in $K$, such that:*

*(i)* *$L$ is abelian over $K$ and Galois over $\mathbb{Q}$.*

*(ii)* *$\mathcal{Q} = \mathcal{Q}(L/K) \leq f$ where the maximum conductor $\mathcal{Q}$ is given by* (1.22).

*(iii)* *$\varphi : \mathrm{Cl}(-D) \xrightarrow{\sim} \mathrm{Gal}(L/K)$ is an isomorphism.*

*(iv)* *Let $C$ be the conjugacy class in $\mathrm{Gal}(L/\mathbb{Q})$ containing the class of $Q(X,Y)$ mapped under $\varphi$. Then $\#C = 1$ if $Q(X,Y)$ is properly equivalent to its opposite $Q(X,-Y)$ and $\#C = 2$ otherwise.*

*(v)* *A prime $p \nmid D$ is represented by $Q(X,Y)$ if and only if $[\frac{L/\mathbb{Q}}{p}] = C$. In particular,*

$$\pi_Q(x) = \pi_C(x, L/\mathbb{Q}).$$

*(vi)* *$[L : \mathbb{Q}] = 2h(-D)$ and $D_L \leq D^{h(-D)}$*

*(vii)* *$h(-D) \ll_\epsilon D^{1/2+\epsilon}$ for $\epsilon > 0$.*

*Proof.* This celebrated theorem is the culmination of many classical results; parts (i)–(v) can be deduced from the arguments in [Cox89, Theorem 9.12], for example. For (vi), note that $[L : K] = h(-D)$ since $h(-D) = \#\mathrm{Cl}(-D) = \#\mathrm{Gal}(L/K) = [L : K]$ by (iii). Moreover, by the conductor-discriminant formula (2.21) and (ii), we have that

$$\log D_L = \sum_\chi \log D_\chi \leq [L : K] \log(D_K \mathcal{Q}) \leq h(-D) \log D.$$

For (vii), let $h_K$ denote the (broad) class number of $K$ so, by classical estimates involving the class number formula, we have that $h_K \ll D_K^{1/2+\epsilon}$. Thus, by [Cox89, Theorem 7.24],

$$h(-D) \leq h_K f \prod_{p|f}\left(1+\frac{1}{p}\right) \ll_\epsilon (D_K f^2)^{1/2+\epsilon} \ll_\epsilon D^{1/2+\epsilon}.$$

□

We will use the above well-known theorem repeatedly without reference. Now, by the Chebotarev Density Theorem, it follows that

$$\pi_Q(x) \sim \frac{\delta_Q}{h(-D)}\mathrm{Li}(x) \tag{1.34}$$

as $x \to \infty$, where $\delta_Q = 1/2$ if $Q$ is properly equivalent to its opposite and $\delta_Q = 1$ otherwise. Under the effective Chebotarev Density Theorem, (1.34) holds for $\log x \gg (\log D)^2$ provided the Dedekind zeta function $\zeta_L(s)$ does not have a real exceptional zero; otherwise, the effective range is even worse. On the Generalized Riemann Hypothesis (GRH), the asymptotic (1.34) holds for $x \gg D^{1+\epsilon}$.

There have been a few results in the literature on $\pi_Q(x)$ beyond (1.34). As a consequence of Weiss's result [Wei83, Theorem 5.2], it is known that

$$\pi_Q(x) \gg \frac{1}{D^{1/2}} \cdot \frac{1}{h(-D)}\mathrm{Li}(x) \tag{1.35}$$

for $\log x \gg \log D$ or equivalently for $x \gg D^{O(1)}$. Thus, the least prime $p$ represented by $Q(X,Y)$ satisfies $p \ll D^{O(1)}$; this result was originally proven by Fogels [Fog62b] and was also observed by Kowalski and Michel [KM02]. There has been no explicit constant in place of the $O(1)$, unlike the many works on Linnik's theorem for the least prime in an arithmetic progression.

Recently, Ditchen [Dit13] established very strong estimates for $\pi_Q(x)$ on the average distribution of primes represented by binary quadratic forms. His results emulate the spectacular theorems of Bombieri–Vinogradov and Barban–Davenport–Halberstam on primes in arithmetic progressions. Roughly speaking, he obtains a GRH-quality estimate for $\pi_Q(x)$ on average over fundamental discriminants $-D \not\equiv 0 \pmod 8$ provided $D \leq x^{3/20-\epsilon}$. Ditchen obtains a similarly strong result [Dit13, Theorem 1.2] by averaging over form classes $[Q] \in \mathrm{Cl}(-D)$ as well. These yield average bounds for the least prime $p$ represented by $Q(X,Y)$. Informally speaking, he showed forms with fundamental discriminant $-D \not\equiv 0 \pmod 8$ represent some prime $p$ satisfying

$$p \ll_\delta \begin{cases} D^{20/3+\delta} & \text{on average over discriminants } D, \\ D^{3+\delta} & \text{on average over discriminants } D \text{ and form classes.} \end{cases} \tag{1.36}$$

We exhibit an unconditional explicit bound for the least such prime $p$ represented by $Q(X,Y)$. This is an explicit variant of (1.35).

**Corollary 1.4.1.** *Let $D \geq 1$ be an integer and let $Q(X,Y) \in \mathbb{Z}[X,Y]$ be a primitive binary quadratic form of discriminant $-D$. For $D$ sufficiently large and $x \gg D^{455}$,*

$$\pi_Q(x) \gg \frac{1}{D^5} \cdot \frac{1}{h(-D)} \mathrm{Li}(x).$$

*In particular, there exists a prime $p \nmid D$ represented by $Q(X,Y)$ satisfying*

$$p \ll D^{455}.$$

*Proof.* This is an immediate consequence of Theorem 7.4.1. □

**Remarks.**

- Theorem 1.3.2 implies $p \ll D^{694}$, so the above represents an improvement over this original bound.

- With a more careful analysis in Theorem 1.3.2 when $n_K = 2$, the $D^{-5}$ in the lower bound can likely be improved to $D^{-1/2}$ which would agree with (1.35).

From the results of Chapter 9, we also obtain a substantially better bound (in an exceptional case) for the least prime represented by $Q(X,Y)$.

**Corollary 1.4.2.** *Let $D \geq 1$ be an integer and let $Q(X,Y) \in \mathbb{Z}[X,Y]$ be a primitive binary quadratic form of discriminant $-D$. Let $L$ be the ring class field of the order in $K = \mathbb{Q}(\sqrt{-D})$ of discriminant $-D$. Let $C$ be the element of $\mathrm{Gal}(L/K)$ corresponding to $Q(X,Y)$.*

*Suppose $\psi \in \widehat{\mathrm{Gal}(L/K)}$ is a real Hecke character such that $L(s,\psi,L/K)$ has a real zero*

$$\beta = 1 - \frac{1}{\eta \log(n_K^{n_K} D_K \mathcal{Q})},$$

*where $\eta \geq 20$. Let $\delta > 0$ be arbitrary. If $\psi(C) = 1$ and $\eta \geq \eta(\delta)$ then there exists a prime*

*$p \nmid D$ such that $p$ is represented by $Q(X,Y)$ and*

$$p \ll_\delta \begin{cases} D^{9.5+\delta} & \textit{if } \psi \textit{ is quadratic}, \\ D^{6+\delta} & \textit{if } \psi \textit{ is principal}. \end{cases}$$

*All implied constants are effective.*

**Remarks.**

- As per Remark 3 following Theorem 9.1.1, one can sharpen the bound in Corollary 1.4.2 to
$$p \ll_\delta \begin{cases} D^{7+\delta} & \text{if } \psi \text{ is quadratic}, \\ D^{4+\delta} & \text{if } \psi \text{ is principal}, \end{cases}$$
but the implied constants are rendered ineffective.

- One may indirectly compare Corollary 1.4.2 to (1.36). The quality of exponents are fairly similar.

In the opposite direction, one may seek an upper bound for $\pi_Q(x)$ within an absolute constant factor of its asymptotic size (1.34). A result of Lagarias–Montgomery–Odlyzko [LMO79, Theorem 1.5] implies that
$$\pi_Q(x) \ll \frac{1}{h(-D)}\mathrm{Li}(x)$$
for $x \geq e^{O_\epsilon(D^{1/2+\epsilon})}$. As far as the author is aware, this was the only upper bound of its kind. While this range of $x$ improves over the effective Chebotarev Density Theorem (1.34), it remains very far from the GRH range of $x \gg_\epsilon D^{1+\epsilon}$. We prove an unconditional improvement.

**Corollary 1.4.3** (Thorner–Z.). *Let $D \geq 1$ be an integer and let $Q(X,Y) \in \mathbb{Z}[X,Y]$ be a primitive binary quadratic form of discriminant $-D$. For $x \gg D^{164}$,*

$$\pi_Q(x) \ll \frac{1}{h(-D)}\mathrm{Li}(x).$$

*Proof.* This is an immediate consequence of Theorem 1.3.3. Note we have applied Theorem 1.3.3 with the range (1.32). □

Inspired by the classical Brun-Titchmarsh theorem, we are also able to deduce a more precise upper bound for $\pi_Q(x)$.

**Corollary 1.4.4** (Thorner–Z.). *Let $D \geq 1$ be an integer and let $Q(X,Y) \in \mathbb{Z}[X,Y]$ be a primitive binary quadratic form of discriminant $-D$. For $x \gg D^{695}$ and $D$ sufficiently large,*

$$\pi_Q(x) < 2\frac{\delta_Q}{h(-D)}\mathrm{Li}(x).$$

*Proof.* This is an immediate consequence of Theorem 1.3.4 . □

Up to the quality of exponent, the unconditional ranges in Corollaries 1.4.3 and 1.4.4 are commensurate with the GRH range $x \gg_\epsilon D^{1+\epsilon}$. Furthermore, Corollary 1.4.4 is within a factor of 2 of the asymptotic (1.34), which is best possible short of precluding a Siegel zero.

### 1.4.2 Elliptic curves and modular forms

We now consider applications to the study of elliptic curves and modular forms. Let $E/\mathbb{Q}$ be an elliptic curve without complex multiplication (CM), and let $N_E$ be the conductor of $E$. The order and group structure of $E(\mathbb{F}_p)$, the group of $\mathbb{F}_p$-rational points on $E$, frequently appears when doing arithmetic over $E$. We are interested in understanding the distribution of values and divisibility properties of $\#E(\mathbb{F}_p)$.

V. K. Murty [Mur94] and Li [Li12] proved unconditional and GRH-conditional bounds on the least prime that does not split completely in a number field. This yields bounds on the least prime $p \nmid \ell N_E$ such that $\ell \nmid \#E(\mathbb{F}_p)$, where $\ell \geq 11$ is prime. As an application of Theorem 1.3.2, we prove a complementary result on the least $p \nmid \ell N_E$ such that $\ell \mid \#E(\mathbb{F}_p)$. To state the result, we define $\omega(N_E) = \#\{p : p \mid N_E\}$ and $\mathrm{rad}(N_E) = \prod_{p \mid N_E} p$.

**Theorem 1.4.5** (Thorner–Z.). *Let $E/\mathbb{Q}$ be an non-CM elliptic curve of conductor $N_E$, and let $\ell \geq 11$ be prime. There exists a prime $p \nmid \ell N_E$ such that $\ell \mid \#E(\mathbb{F}_p)$ and*

$$p \ll \ell^{(5000+1600\omega(N_E))\ell^2}\mathrm{rad}(N_E)^{1900\ell^2}.$$

**Remark.** The proof is easily adapted to allow for elliptic curves over other number fields; we omit further discussion for brevity.

One of the first significant results in the study of the distribution of values of $\#E(\mathbb{F}_p)$ is due to Hasse, who proved that if $p \nmid N_E$, then $|p+1-\#E(\mathbb{F}_p)| < 2\sqrt{p}$. For a given prime $\ell$, the distribution of the primes $p$ such that $\#E(\mathbb{F}_p) \equiv p+1 \pmod{\ell}$ can also be studied using the mod $\ell$ Galois representations associated to $E$.

**Theorem 1.4.6** (Thorner–Z.). *Let $E/\mathbb{Q}$ be a non-CM elliptic curve of squarefree conductor $N_E$, and let $\ell \geq 11$ be prime. There exists a prime $p \nmid \ell N_E$ such that $\#E(\mathbb{F}_p) \equiv p+1 \pmod{\ell}$*

*and*

$$p \ll \left(\ell^{900\omega(N_E)+4100}\mathrm{rad}(N_E)^{1800}\right)^{\ell+1}.$$

Theorem 1.4.6 will follow from a more general result on congruences for the Fourier coefficients of certain holomorphic cuspidal modular forms. Let

$$f(z) = \sum_{n=1}^{\infty} a_f(n)e^{2\pi inz} \tag{1.37}$$

be a cusp form of integral weight $k_f \geq 2$, level $N_f \geq 1$, and nebentypus $\chi_f$. Suppose further that $f$ is a normalized eigenform for the Hecke operators. We call such a cusp form $f$ a newform; for each newform $f$, the map $n \mapsto a_f(n)$ is multiplicative. Suppose that $a_f(n) \in \mathbb{Z}$ for all $n \geq 1$. In this case, $\chi_f$ is trivial when $f$ does not have CM, and $\chi_f$ is a nontrivial real character when $f$ does have CM. Furthermore, when $k_f = 2$, $f$ is the newform associated to an isogeny class of elliptic curves $E/\mathbb{Q}$. In this case, $N_f = N_E$, and for any prime $p \nmid N_E$, we have that $a_f(p) = p + 1 - \#E(\mathbb{F}_p)$.

**Theorem 1.4.7** (Thorner–Z.). *Let $f(z) = \sum_{n=1}^{\infty} a_f(n)e^{2\pi inz} \in \mathbb{Z}[[e^{2\pi iz}]]$ be a non-CM newform of even integral weight $k_f \geq 2$, level $N_f$, and trivial nebentypus. Let $\ell \geq 3$ be a prime such that* (10.3) *holds and* $\gcd(k_f - 1, \ell - 1) = 1$. *For any progression $a \pmod{\ell}$, there exists a prime $p \nmid \ell N_f$ such that $a_f(p) \equiv a \pmod{\ell}$ and*

$$p \ll \left(\ell^{900\omega(N_f)+4100}\mathrm{rad}(N_f)^{1800}\right)^{\ell+1}.$$

**Remarks.**

- Equation (10.3) is a fairly mild condition regarding whether the modulo $\ell$ reduction of a certain representation is surjective. This condition is satisfied by all but finitely many choices of $\ell$. See Section 8.2.3 for further details.

- The proofs of Theorems 1.4.5 to 1.4.7 are easily adapted to allow composite moduli $\ell$ as well as elliptic curves and modular forms with CM. Moreover, the proofs can be easily modified to study the mod $\ell$ distribution of the trace of Frobenius for elliptic curves over number fields other than $\mathbb{Q}$. We omit further discussion for brevity.

- Using Theorem 1.3.1, the least prime $p$ such that $a_f(p) \equiv a \pmod{\ell}$ satisfies the bound $p \ll \ell^{120\ell^3(1+\omega(N_f))}\mathrm{rad}(N_f)^{40(\ell^3-1)}$ for any choice of $a$. Thus, Theorem 1.4.7 is an improvement for large $\ell$.

- If $r_{24}(n)$ is the number of representations of $n$ as a sum of 24 squares, then $691r_{24}(p) = 16(p^{11}+1)+33152\tau(p)$, where Ramanujan's function $\tau(n)$ is the $n$-th Fourier coefficient of $\Delta(z)$, the unique non-CM newform of weight 12 and level 1. If $\ell \neq 691$ is such that (10.3) holds for $f(z) = \Delta(z)$, then by Theorem 1.4.7, there exists $p \neq \ell$ such that $691r_{24}(p) \equiv 16(p^{11}+1) \pmod{\ell}$ and $p \ll \ell^{4100(\ell+1)}$.

Next, we use Theorem 1.3.3 to improve the best unconditional upper bounds for two outstanding conjectures of Lang and Trotter [LT76]. Let

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z}$$

be a newform as in (1.37) with integral coefficients $a_f(n)$. Further, suppose that $f$ does not have complex multiplication and hence the nebentypus of $f$ is trivial. Fix $a \in \mathbb{Z}$, and let

$$\pi_f(x,a) = \#\{p \leq x : a_f(p) = a\}. \tag{1.38}$$

Lang and Trotter conjectured that as $x \to \infty$, we have that

$$\pi_f(x,a) \sim c_{f,a} \begin{cases} \dfrac{\sqrt{x}}{\log x} & \text{if } k_f = 2, \\ 1 & \text{if } k_f \geq 4, \end{cases}$$

where $c_{f,a} \geq 0$ is a certain constant depending on $f$ and $a$ alone.

In the special case where $k_f = 2$, Elkies [Elk91] proved that $\pi_f(x,0) \ll_{N_f} x^{3/4}$. In all other cases, Serre proved in 1981 that

$$\pi_f(x,a) \ll_{N_f} \frac{x}{(\log x)^{1+\delta}}$$

for any $\delta < 1/4$; following the ideas of M. R. Murty, V. K. Murty, and Saradha [MMS88], Wan [Wan90] improved the range of $\delta$ in 1990 to any $\delta < 1$. This was further sharpened by V. K. Murty [Mur97] in 1997; he proved[9] that

$$\pi_f(x,a) \ll_{N_f} \frac{x(\log\log x)^3}{(\log x)^2}. \tag{1.39}$$

Using Theorem 1.3.3, we give a modest improvement[10].

[9] Theorem 5.1 of [Mur97] actually claims a stronger result, but a step in the proof seems not to be justified. The best that the argument appears to give is what we have stated above; see the end of Section 9.1 in [TZ17a] for further discussion.

[10] Note that we recover the claimed result [Mur97, Theorem 5.1].

**Theorem 1.4.8** (Thorner–Z.). *Let $f$ be a newform of even integral weight $k_f \geq 2$, level $N_f$, and trivial nebentypus with integral coefficients. If $\pi_f(x,a)$ is given by* (1.38)*, then*

$$\pi_f(x,a) \ll_{N_f} \frac{x(\log\log x)^2}{(\log x)^2}.$$

We also consider a different (but closely related) conjecture of Lang and Trotter regarding the Frobenius fields of an elliptic curve. Let $E/\mathbb{Q}$ be an elliptic curve of conductor $N_E$ without complex multiplication. For a prime $p \nmid N$, let $\Pi_p$ be the Frobenius endomorphism of $E/\mathbb{F}_p$. Defining $a_E(p) = p + 1 - \#E(\mathbb{F}_p)$, we have that $\Pi_p^2 - a_E(p)\Pi_p + p = 0$. By Hasse, we know that $|a_E(p)| < 2\sqrt{p}$, so $\mathbb{Q}(\Pi_p)$ in $\operatorname{End}(E/\mathbb{F}_p) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an imaginary quadratic field. For a fixed imaginary quadratic field $k$ with absolute discriminant $D_k$, let

$$\pi_E(x,k) = \#\{p \leq x : \mathbb{Q}(\Pi_p) \cong k\}. \tag{1.40}$$

Lang and Trotter conjectured that as $x \to \infty$,

$$\pi_E(x,k) \sim c_{E,k}\frac{\sqrt{x}}{\log x},$$

where $c_{E,k} > 0$ is a certain constant depending on $E$ and $k$ alone. Using the square sieve, Cojocaru, Fouvry, and M. R. Murty [CFM05] proved that

$$\pi_E(x,k) \ll_{N_E,k} \frac{x(\log\log x)^{13/12}}{(\log x)^{25/24}}.$$

Using V. K. Murty's version of the Chebotarev Density Theorem and Serre's method of mixed representations (see [Ser81]), Zywina [Zyw15] improved this bound to

$$\pi_E(x,k) \ll_{N_E,k} \frac{x(\log\log x)^2}{(\log x)^2}. \tag{1.41}$$

Using Theorem 1.3.3, we establish a modest improvement to (1.41).

**Theorem 1.4.9** (Thorner–Z.). *Let $E/\mathbb{Q}$ be an elliptic curve of conductor $N_E$ and let $k$ be a fixed imaginary quadratic number field. If $\pi_E(x,k)$ is defined by* (1.40) *then*

$$\pi_E(x,k) \ll_{N_E,k} \frac{x\log\log x}{(\log x)^2}.$$

## 1.5 Conventions and organization

### Conventions

We will employ Vinogradov's notation and big-O notation. That is,

- $f \ll g$ or $f = O(g)$ implies there is an absolute constant $C > 0$ such that $|f| \leq C \cdot g$.
- $f \asymp g$ if and only $f \ll g$ and $g \ll f$.
- $f = o(g)$ if and only if $\frac{f}{g} \to 0$ as some parameter, say $x$, goes to infinity.
- $f \sim g$ if and only if $\frac{f}{g} \to 1$ as some parameter, say $x$, goes to infinity.

We also adhere to the convention that all implied constants in all asymptotic inequalities (e.g. $f \ll g$ or $f = O(g)$) are absolute with respect to all parameters, unless otherwise specified. If an implied constant depends on a parameter, such as $\epsilon$, then we use $\ll_\epsilon$ and $O_\epsilon$ to denote that the implied constant depends at most on $\epsilon$. All implied constants will be effectively computable, unless otherwise specified.

The sets $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ will respectively denote the integers, rational numbers, real numbers, and complex numbers.

### Organization

For the reader who wishes to proceed quickly to the proofs of the main theorems:

- Theorem 1.3.1 is proven in Section 7.2.
- Theorem 1.3.2 is proven in Section 7.3.
- Theorem 1.3.3 is proven in Section 8.2.1.
- Theorem 1.3.4 is proven in Sections 8.2.2 and 8.2.3.
- Theorem 1.3.5 is a consequence of Theorem 4.1.1, proven in Section 4.4.
- Theorem 1.3.7 is a consequence of Theorem 5.1.1, proven in Chapter 5.
- Theorem 1.3.8 is a consequence of Theorem 6.1.1, proven in Chapter 6.
- Corollary 1.4.2 is a consequence of Theorem 9.1.1, proven in Chapter 9.
- Theorems 1.4.5 to 1.4.7 are proven in Section 8.2.3.
- Theorems 1.4.8 and 1.4.9 are proven in Section 10.3.

We also briefly describe the organization and contents of the chapters.

- Chapter 2 consists of background material and notation for Hecke $L$-functions, elementary estimates, Artin $L$-functions, and Deuring's reduction. All subsequent chapters rely on the information here. If we refer to a chapter as "self–contained" then we are not precluding the use of results from Chapter 2.

- Chapter 3 specifies some key notation and identifies certain zeros of Hecke $L$-functions. Further, we establish several different "explicit inequalities" related to Hecke $L$-functions by involving classical arguments, higher derivatives, and smooth weights. The results therein form the technical crux of all subsequent proofs and applications in Chapter 4.

- Chapter 4 is a continuation of Chapter 3, establishing explicit zero-free regions and zero repulsion of Hecke $L$-functions. Chapters 3 and 4 are chiefly inspired by the work of Heath-Brown [HB92].

- Chapter 5 contains the proof of log-free zero density estimates. It is self-contained aside from a crucial application of Lemma 3.2.4. The overall strategy follows Weiss [Wei83] but requires a more careful analysis.

- Chapter 6 is on the Deuring–Heilbronn phenomenon for Hecke $L$-functions and the Dedekind zeta function. It is self-contained and uses power sums as the main tool. The arguments are motivated by the proof of [LMO79, Theoreom 5.1].

- Chapter 7 contains proofs of two of the main results of this thesis (Theorems 1.3.1 and 1.3.2). It amasses the results of Chapters 4 to 6 along with ideas of Heath-Brown [HB92] to address the least prime ideal problem in two different ways.

- Chapter 8 contains proofs of two of the main results of this thesis (Theorems 1.3.3 and 1.3.4). As in Chapter 7, we combine the results of Chapters 4 to 6 to give upper bounds for the number of prime ideals with a prescribed Artin symbol. The approach here is influenced by Maynard [May13] as well as the arguments of Chapter 7.

- Chapter 9 is a self-contained piece on an exceptional case of the "least prime ideal" problem. The methods are entirely different, employing sieve techniques inspired by Heath-Brown [HB90] and Friedlander–Iwaniec [FI10, Chapter 24].

- Chapter 10 contains the applications of our main theorems to elliptic curves and modular forms, including the Lang–Trotter conjectures. The arguments therein borrow from a variety of sources including works of Serre [Ser81], Murty–Murty–Saradha [MMS88], V.K. Murty [Mur97], and Zywina [Zyw15].

## 1.6 Joint work and published material

### Joint work

Parts of this thesis were produced in collaboration with Jesse Thorner [TZ17b, TZ17a]. We have made these joint contributions clear in the statement of the theorems in Chapter 1. For the remainder of this thesis, we will no longer continue making these distinctions. Instead, we shall specify here which parts of this thesis are a product of joint work:

- parts of Chapter 2; most of the contents here is elementary or background material.
- Chapter 5
- parts of Chapter 6; namely, Theorem 6.1.1 and Sections 6.2.2 and 6.3.1.
- parts of Chapter 7; namely, Section 7.3 and parts of Section 7.1.
- Chapter 8
- Chapter 10

### Published material

Parts of this thesis are based on published or accepted material:

- [Zam16a] (doi:10.1016/j.jnt.2015.10.003) is related to Chapters 3 and 4.
- [Zam16b] (doi:10.1142/S1793042116501335) is related to Chapter 9.
- [Zam17] (doi:10.7169/facm/1651) is related to Chapters 6 and 7.
- [TZ17b] (accepted) is related to Chapters 5 to 7 and 10.
- [TZ17a] (doi:10.1093/imrn/rnx031) is related to Chapters 8 and 10.

# Chapter 2

# Background

*"Sometimes I'll start a sentence and I don't even know where it's going. I just hope I find it along the way."*

– Michael Scott, The Office.

In this chapter, we establish notation and recall basic facts regarding Hecke characters, $L$-functions, arithmetic sums, prime ideal counting functions, Artin $L$-functions, and Deuring's reduction. The necessary analytic and algebraic number theory material can be found in [IK04, Neu99]. The contents here will be used throughout this thesis.

## 2.1 Hecke characters and congruence class groups

The notation here is motivated by the discussion in [Wei83, Section 1]. Let $K$ be a number field of degree $n_K = [K : \mathbb{Q}]$ with ring of integers $\mathcal{O}_K$. Let $D_K$ denote the absolute value of the discriminant of $K$ over $\mathbb{Q}$ and $\mathrm{N} = \mathrm{N}_{\mathbb{Q}}^{K}$ denote the absolute field norm of $K$ over $\mathbb{Q}$. For an integral ideal $\mathfrak{q}$ of $K$, let $I(\mathfrak{q})$ be the group of fractional ideals of $K$ relatively prime to $\mathfrak{q}$ and let $P_{\mathfrak{q}}$ be the group of principal ideals $(\alpha)$ of $K$ such that $\alpha$ is totally positive and $\alpha \equiv 1 \pmod{\mathfrak{q}}$. The narrow ray class group of $K$ modulo $\mathfrak{q}$ is given by $\mathrm{Cl}(\mathfrak{q}) = I(\mathfrak{q})/P_{\mathfrak{q}}$. A subgroup $H$, or $H \pmod{\mathfrak{q}}$, of $\mathrm{Cl}(\mathfrak{q})$ will be referred to as a congruence class group of $K$ modulo $\mathfrak{q}$. Abusing notation, we will also regard $H$ as a subgroup of $I(\mathfrak{q})$ containing $P_{\mathfrak{q}}$.

Characters of $\mathrm{Cl}(\mathfrak{q})$ are Hecke characters and denoted $\chi \pmod{\mathfrak{q}}$ or simply $\chi$ when the modulus is understood. The notation $\chi \pmod{H}$ refers to a character $\chi \pmod{\mathfrak{q}}$ satisfying $\chi(H) = 1$. Properly speaking, the domain of $\chi$ is the quotient group $\mathrm{Cl}(\mathfrak{q})$ but, for notational convenience, we pullback the domain of $\chi$ to $I(\mathfrak{q})$ and then extend it to all of $I(\mathcal{O}_K)$ by zero. In other words, $\chi(\mathfrak{n})$ is a multiplicative function on all integral ideals $\mathfrak{n} \subseteq \mathcal{O}$ and $\chi(\mathfrak{n}) = 0$ for $(\mathfrak{n}, \mathfrak{q}) \neq 1$.

The trivial character $\chi_0 \pmod{\mathfrak{q}}$ is also referred to as the principal character and satisfies $\chi_0(\mathfrak{n}) = 1$ for all $(\mathfrak{n}, \mathfrak{q}) \neq 1$. We distinguish this character with an additional piece of notation:

$$E_0(\chi) := \begin{cases} 1 & \text{if } \chi \text{ is principal,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

For $\mathfrak{m}$ dividing $\mathfrak{n}$, the natural inclusion $I(\mathfrak{n}) \hookrightarrow I(\mathfrak{m})$ induces a surjective homomorphism $I(\mathfrak{n})/P_{\mathfrak{n}} \to I(\mathfrak{m})/P_{\mathfrak{m}}$ through which a character $\chi \pmod{\mathfrak{m}}$ induces a character $\tilde{\chi} \pmod{\mathfrak{n}}$. Similarly, a congruence class group $H \pmod{\mathfrak{m}}$ induces a congruence class group $\tilde{H} \pmod{\mathfrak{n}}$. A Hecke character $\chi \pmod{\mathfrak{m}}$ (resp. congruence class group $H$) is primitive if it cannot be induced, except by itself.

For a Hecke character $\chi \pmod{\mathfrak{q}}$, let $\chi^* \pmod{\mathfrak{f}_\chi}$ be the unique primitive character inducing $\chi$. The conductor of $\chi$ is the integral ideal $\mathfrak{f}_\chi$. Similarly, for a congruence class group $H \pmod{\mathfrak{q}}$, let $H^* \pmod{\mathfrak{f}_H}$ be the unique primitive congruence class group inducing $H$. The conductor of $H$ is the integral ideal $\mathfrak{f}_H$. It is well known that

$$\mathfrak{f}_H = \operatorname{lcm}\{\mathfrak{f}_\chi : \chi \pmod{H}\}.$$

We require analytic measures of congruence class groups $H$ and Hecke characters $\chi$. For a Hecke character $\chi \pmod{\mathfrak{q}}$, denote

$$D_\chi := D_K \mathrm{N}\mathfrak{f}_\chi,$$

and for a congruence class group $H \pmod{\mathfrak{q}}$, denote

$$h_H := [I(\mathfrak{q}) : H], \qquad Q_H := \max\{\mathrm{N}\mathfrak{f}_\chi : \chi \pmod{H}\}, \tag{2.2}$$

which we refer to as the class number of $H$ and the maximum analytic conductor of $H$ respectively. Observe that the quantity $D_\chi$ depends only on the primitive character $\chi^*$ and the quantities $h_H$ and $Q_H$ depend only on the primitive congruence class group $H^*$. For simplicity, we will often write $Q = Q_H$ since we will usually retain the same $H$ throughout our arguments.

## 2.2 Hecke $L$-functions

The Hecke $L$-function associated to a Hecke character $\chi \pmod{\mathfrak{q}}$ is given by

$$L(s,\chi) = L(s,\chi,K) = \sum_{\mathfrak{n} \subseteq \mathcal{O}} \chi(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{-s} = \prod_{\mathfrak{p}} \Big(1 - \frac{\chi(\mathfrak{p})}{(\mathrm{N}\mathfrak{p})^s}\Big)^{-1} \qquad \text{for } \sigma > 1,$$

where $s = \sigma + it \in \mathbb{C}$. Throughout this thesis, we will retain the convention that the complex variable $s$ may be written as $s = \sigma + it$ for $\sigma, t \in \mathbb{R}$. Moreover, unless otherwise specified, we shall henceforth refer to Hecke characters as characters. We will also usually suppress the dependence of $L(s, \chi, K)$ on the number field $K$ when it is understood.

If $\chi$ is the primitive principal character then its Hecke $L$-function is the Dedekind zeta function of $K$, which is defined as

$$\zeta_K(s) = \sum_{\mathfrak{n} \subseteq \mathcal{O}} (\mathrm{N}\mathfrak{n})^{-s} = \prod_{\mathfrak{p}} \Big(1 - \frac{1}{(\mathrm{N}\mathfrak{p})^s}\Big)^{-1} \qquad \text{for } \sigma > 1. \tag{2.3}$$

In this section, we record classical facts about $L(s, \chi)$.

### Functional Equation

Let $\chi \pmod{\mathfrak{f}_\chi}$ be a primitive character. Recall that the *gamma factor of* $\chi$ is given by

$$\gamma_\chi(s) := \Big[\pi^{-s/2} \Gamma\Big(\frac{s}{2}\Big)\Big]^{a(\chi)} \cdot \Big[\pi^{-\frac{s+1}{2}} \Gamma\Big(\frac{s+1}{2}\Big)\Big]^{b(\chi)}, \tag{2.4}$$

where $\Gamma(s)$ is the Gamma function and $a(\chi), b(\chi)$ are certain non-negative integers satisfying

$$a(\chi) + b(\chi) = [K : \mathbb{Q}] = n_K. \tag{2.5}$$

The *completed $L$-function of* $L(s, \chi)$ is defined to be

$$\xi(s, \chi) := [s(1-s)]^{E_0(\chi)} D_\chi^{s/2} \gamma_\chi(s) L(s, \chi). \tag{2.6}$$

With an appropriate choice of $a(\chi)$ and $b(\chi)$, it is well-known that $\xi(s, \chi)$ is an entire function satisfying the functional equation

$$\xi(s, \chi) = w(\chi) \cdot \xi(1 - s, \overline{\chi}), \tag{2.7}$$

where $w(\chi) \in \mathbb{C}$ is the global root number having absolute value 1. The zeros of $\xi(s, \chi)$ are the *non-trivial zeros* $\rho$ of $L(s, \chi)$ and are known to satisfy $0 < \mathrm{Re}\{\rho\} < 1$. The *trivial zeros* $\omega$ of $L(s, \chi)$ are given by

$$\operatorname*{ord}_{s=\omega} L(s, \chi) = \begin{cases} a(\chi) - E_0(\chi) & \text{if } \omega = 0, \\ b(\chi) & \text{if } \omega = -1, -3, -5, \dots, \\ a(\chi) & \text{if } \omega = -2, -4, -6, \dots, \end{cases} \tag{2.8}$$

and arise as poles of the gamma factor of $\chi$. Since $\xi(s,\chi)$ is entire of order 1, it admits a Hadamard product factorization given by

$$\xi(s,\chi) = e^{A(\chi)+B(\chi)s} \prod_{\rho} \Big(1 - \frac{s}{\rho}\Big) e^{s/\rho}. \tag{2.9}$$

If $\chi$ is trivial then $L(s,\chi)$ is the Dedekind zeta function of $K$. Of course, the above information still holds but we shall sometimes use separate notation to distinguish this case. The *completed Dedekind zeta function* $\xi_K(s)$ is given by

$$\xi_K(s) = s(1-s) D_K^{s/2} \gamma_K(s) \zeta_K(s), \tag{2.10}$$

where $\gamma_K$ is the *gamma factor of* $K$ defined by

$$\gamma_K(s) = \Big[\pi^{-\frac{s}{2}} \Gamma\big(\frac{s}{2}\big)\Big]^{r_1+r_2} \cdot \Big[\pi^{-\frac{s+1}{2}} \Gamma\big(\frac{s+1}{2}\big)\Big]^{r_2}. \tag{2.11}$$

Here $r_1 = r_1(K)$ and $2r_2 = 2r_2(K)$ are respectively the number of real and complex embeddings of $K$. It is well-known that $\xi_K(s)$ is entire and satisfies the functional equation

$$\xi_K(s) = \xi_K(1-s). \tag{2.12}$$

We refer to its zeros as the *non-trivial zeros* $\rho$ of $\zeta_K(s)$, which are known to lie in the strip $0 < \mathrm{Re}\{s\} < 1$. The *trivial zeros* $\omega$ of $\zeta_K(s)$ occur at certain non-positive integers arising from poles of the gamma factor of $K$; namely,

$$\operatorname*{ord}_{s=\omega} \zeta_K(s) = \begin{cases} r_1 + r_2 - 1 & \text{if } \omega = 0, \\ r_2 & \text{if } \omega = -1, -3, -5, \dots, \\ r_1 + r_2 & \text{if } \omega = -2, -4, -6, \dots. \end{cases} \tag{2.13}$$

See [LO77, Section 5] for further details on these facts.

### Explicit Formula

Using the Hadamard product for $\xi(s,\chi)$, one may derive an explicit formula for the logarithmic derivative of $L(s,\chi)$.

**Lemma 2.2.1.** *Let $\chi$ be a primitive Hecke character. Then*

$$-\frac{L'}{L}(s,\chi) = \frac{E_0(\chi)}{s-1} + \frac{E_0(\chi)}{s} + \frac{1}{2}\log D_\chi + \frac{\gamma_\chi'}{\gamma_\chi}(s) - B(\chi) - \sum_{\rho} \Big(\frac{1}{s-\rho} + \frac{1}{\rho}\Big),$$

*identically for all $s \in \mathbb{C}$. The constant $B(\chi) \in \mathbb{C}$ depends only on $\chi$ and the conditionally convergent sum is over all zeros $\rho$ of $\xi(s,\chi)$. Moreover,*

$$\mathrm{Re}\{B(\chi)\} = -\frac{1}{2}\sum_{\rho}\Big(\frac{1}{1-\rho} + \frac{1}{\rho}\Big) = -\sum_{\rho}\mathrm{Re}\frac{1}{\rho} < 0.$$

*Proof.* See [LO77, Section 5] for a proof. Note "$\sum_{\rho}$" denotes "$\lim_{T\to\infty} \sum_{|\mathrm{Im}\rho|\le T}$". □

## 2.3 Elementary $L$-function estimates

In this section, we state elementary estimates of $L$-functions beginning with well-known bounds for the Dedekind zeta function and the convexity bound for Hecke $L$-functions.

**Lemma 2.3.1.** *For $\sigma > 1$,*

$$\begin{aligned} \zeta_K(\sigma) &\le \zeta(\sigma)^{n_K} \le \Big(\frac{\sigma}{\sigma-1}\Big)^{n_K}, \\ \log\zeta_K(\sigma) &\le n_K \log\Big(\frac{\sigma}{\sigma-1}\Big), \\ -\frac{\zeta_K'}{\zeta_K}(\sigma) &\le -n_K\frac{\zeta'}{\zeta}(\sigma) \le \frac{n_K}{\sigma-1}. \end{aligned}$$

*where $\zeta(s) = \zeta_{\mathbb{Q}}(s)$ is the classical Riemann zeta function.*

*Proof.* For the first inequality, observe that

$$\zeta_K(\sigma) = \prod_{\mathfrak{p}}(1-(\mathrm{N}\mathfrak{p})^{-\sigma})^{-1} = \prod_{p}\prod_{(p)\subseteq\mathfrak{p}}(1-(\mathrm{N}\mathfrak{p})^{-\sigma})^{-1} \le \prod_{p}(1-p^{-\sigma})^{-n_K} = \zeta(\sigma)^{n_K}$$

and note $\zeta(\sigma) \le \big(\frac{\sigma}{\sigma-1}\big)$ from [MV07, Corollary 1.14]. The second inequality follows easily from the first. The third inequality follows by an argument similar to that of the first and additionally noting $-\frac{\zeta'}{\zeta}(\sigma) < \frac{1}{\sigma-1}$ by [Lou92, Lemma (a)] for example. □

**Lemma 2.3.2** (Rademacher)**.** *Let $\delta \in (0, \frac{1}{2})$ and $\chi$ be a primitive Hecke character. Then*

$$|L(s,\chi)| \ll \Big|\frac{s+1}{s-1}\Big|^{E_0(\chi)} \zeta_{\mathbb{Q}}(1+\delta)^{n_K}\Big(\frac{D_\chi}{(2\pi)^{n_K}}(1+|s|)^{n_K}\Big)^{(1-\sigma+\delta)/2}$$

*uniformly in the region*

$$-\delta \le \sigma \le 1+\delta.$$

*Proof.* This is a version of [Rad60, Theorem 5] which has been simplified for our purposes. In his notation, the constants $v_q, a_p, a_{p+r_2}, v_p$ are all zero for characters of $\mathrm{Cl}(\mathfrak{q})$. Recall that $\zeta_{\mathbb{Q}}(\,\cdot\,)$ is the classical Riemann zeta function. $\square$

When applying the above convexity result, we may sometimes require bounds for the Gamma function $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt$ in a vertical strip; for instance, from [MV07, Appendix C], we have that

$$\Gamma(s) \ll_\delta e^{-|t|} \tag{2.14}$$

uniformly in the region $-1+\delta \leq \mathrm{Re}\{s\} \leq 2$ with $|s| \geq \delta$. A more accurate bound for $\Gamma(s)$ in such a region has exponent $-\frac{\pi}{2}+\delta$ instead of $-1$. However, this detail does not affect our calculations so we choose this weaker bound for simplicity.

Next, we record some bounds related to $\gamma_\chi(s)$ defined in (2.4).

**Lemma 2.3.3.** *Let $s = \sigma + it$ with $\sigma > 1$ and $t \in \mathbb{R}$. Then*

*(i)* $\mathrm{Re}\{\frac{\Gamma'}{\Gamma}(s)\} \leq \log|s| + \sigma^{-1}$

*(ii)* $\mathrm{Re}\{\frac{\gamma_\chi'}{\gamma_\chi}(s)\} \leq \frac{n_K}{2}(\log(|s|+1) + \sigma^{-1} - \log\pi)$

*In particular, for $1 < \sigma \leq 6.2$ and $|t| \leq 1$,*

$$\mathrm{Re}\{\frac{\gamma_\chi'}{\gamma_\chi}(s)\} \leq 0$$

*Proof.* The first estimate follows from [OS97, Lemma 4]. The second estimate is a straightforward consequence of the first combined with the definition of $\gamma_\chi(s)$ in (2.4). The third estimate follows from [AK14, Lemma 3]. $\square$

**Lemma 2.3.4.** *Let $\chi$ be a primitive Hecke character. If $\mathrm{Re}\{s\} \geq 1/8$, then*

$$\frac{\gamma_\chi'}{\gamma_\chi}(s) \ll n_K \log(2+|s|).$$

*Proof.* See [LO77, Lemma 5.3]. $\square$

**Lemma 2.3.5.** *Let $k \geq 1$ and $\chi$ be a Hecke character. Then*

$$\frac{1}{k!}\frac{d^k}{ds^k}\frac{\gamma_\chi'}{\gamma_\chi}(s) \ll n_K$$

*uniformly for $s$ satisfying $\mathrm{Re}\{s\} > 1$.*

*Proof.* Denote $\psi^{(k)}(\,\cdot\,) = \frac{d^k}{ds^k}\frac{\Gamma'}{\Gamma}(\,\cdot\,)$. From (2.4), we have that

$$\frac{d^k}{ds^k}\frac{\gamma_\chi'}{\gamma_\chi}(s) = \frac{a(\chi)}{2^{k+1}} \cdot \psi^{(k)}\Big(\frac{s}{2}\Big) + \frac{b(\chi)}{2^{k+1}} \cdot \psi^{(k)}\Big(\frac{s+1}{2}\Big). \tag{2.15}$$

Since $a(\chi) + b(\chi) = n_K$, it suffices to bound $\psi^{(k)}(z)$ for $\mathrm{Re}\{z\} > 1/2$. From the well-known logarithmic derivative of the Gamma function (see [MV07, (C.10)] for example), observe

$$\Big|\frac{\psi^{(k)}(z)}{k!}\Big| = \Big|(-1)^k \sum_{n=1}^{\infty} \frac{1}{(n+z)^{k+1}}\Big| \leq 2^{k+1} \sum_{n \text{ odd}} \frac{1}{n^{k+1}} = (2^{k+1}-1)\zeta_{\mathbb{Q}}(k+1)$$

for $\mathrm{Re}\{z\} > 1/2$. This yields the result when combined with (2.15) as $\zeta_{\mathbb{Q}}(k+1) \leq \zeta_{\mathbb{Q}}(2) = \pi^2/6$. □

**Lemma 2.3.6.** *Let $\chi$ be a Hecke character (not necessarily primitive) of a number field $K$ and $k \geq 1$ be a positive integer. Then*

$$(-1)^{k+1}\frac{d^k}{ds^k}\frac{L'}{L}(s,\chi) = \sum_{\mathfrak{p}}\sum_{m=1}^{\infty}(\log \mathrm{N}\mathfrak{p})\chi(\mathfrak{p})\frac{(\log \mathrm{N}\mathfrak{p}^m)^k}{(\mathrm{N}\mathfrak{p}^m)^s} = \frac{\delta(\chi)k!}{(s-1)^{k+1}} - \sum_{\omega}\frac{k!}{(s-\omega)^{k+1}}$$

*for $\mathrm{Re}\{s\} > 1$, where the first sum is over prime ideals $\mathfrak{p}$ of $K$ and the second sum is over all zeros $\omega$ of $L(s,\chi)$, including trivial ones, counted with multiplicity.*

*Proof.* By standard arguments, this follows from the Hadamard product (2.9) of $\xi(s,\chi)$ and the Euler product of $L(s,\chi)$. See [LMO79, Equations (5.2) and (5.3)], for example. □

We end this subsection with a classical explicit bound on the number of zeros of $L(s,\chi)$ in a circle. See [LMO79, Lemma 2.2] for a non-explicit version.

**Lemma 2.3.7.** *Let $\chi$ be a Hecke character. Let $s = \sigma + it$ with $\sigma > 1$ and $t \in \mathbb{R}$. For $r > 0$, denote*

$$N_\chi(r;s) := \#\{\rho = \beta + i\gamma : 0 < \beta < 1, L(\rho,\chi) = 0, |s-\rho| \leq r\}. \tag{2.16}$$

*If $0 < r \leq 1$, then*

$$N_\chi(r;s) \leq \{4\log D_K + 2\log \mathrm{N}\mathfrak{f}_\chi + 2n_K \log(|t|+3) + 4 + 4E_0(\chi)\} \cdot r + 4 + 4E_0(\chi).$$

*Proof.* Without loss, we may assume $\chi$ is primitive. Observe $N_\chi(r;s) \leq N_\chi(r;1+it) \leq N_\chi(2r;1+r+it)$ so it suffices to bound the latter quantity. Now, if $s_0 = 1+r+it$, notice

$$N_\chi(2r;s_0) \leq 4r \sum_{|s_0-\rho|\leq 2r} \mathrm{Re}\Big\{\frac{1}{s_0-\rho}\Big\} \leq 4r\sum_{\rho}\mathrm{Re}\Big\{\frac{1}{s_0-\rho}\Big\}.$$

Applying Lemmas 2.2.1 and 2.3.3 twice and noting $\mathrm{Re}\left\{\frac{L'}{L}(s_0,\chi)\right\} \leq -\frac{\zeta_K'}{\zeta_K}(1+r)$ via their respective Euler products, the above is

$$\begin{aligned}&\leq 4r\left(\mathrm{Re}\left\{\frac{L'}{L}(s_0,\chi)\right\} + \frac{1}{2}\log D_\chi + \mathrm{Re}\left\{\frac{\gamma_\chi'}{\gamma_\chi}(s_0)\right\} + E_0(\chi)\mathrm{Re}\left\{\frac{1}{s_0} + \frac{1}{s_0-1}\right\}\right)\\ &\leq \{4\log D_K + 2\log \mathrm{N}\mathfrak{f}_\chi + 2n_K\log(|t|+3) + 4 + 4E_0(\chi)\}\cdot r + 4 + 4E_0(\chi)\end{aligned}$$

as $D_\chi = D_K \mathrm{N}\mathfrak{f}_\chi$. For the details on estimating $-\frac{\zeta_K'}{\zeta_K}(1+r)$, see Lemma 2.4.3. □

In Chapter 3, we will improve the bound in Lemma 2.3.7 by exhibiting an explicit inequality involving the logarithmic derivative of $L(s,\chi)$.

## 2.4 Arithmetic sums

Here we estimate various basic arithmetic sums over integral and prime ideals of $K$ and consequently we must define some additional quantities related to the Dedekind zeta function $\zeta_K(s)$, given by (2.3). It is well-known that $\zeta_K(s)$ has a simple pole at $s=1$. Thus, we may define

$$\kappa_K := \operatorname*{Res}_{s=1}\zeta_K(s) \quad \text{and} \quad \gamma_K := \kappa_K^{-1}\lim_{s\to 1}\left(\zeta_K(s) - \frac{\kappa_K}{s-1}\right), \tag{2.17}$$

so the Laurent expansion of $\zeta_K(s)$ at $s=1$ is given by

$$\zeta_K(s) = \frac{\kappa_K}{s-1} + \kappa_K\gamma_K + O_K(|s-1|).$$

We refer to $\gamma_K$ as the Euler-Kronecker constant of $K$, which was introduced by Ihara [Iha06]. For more details on $\gamma_K$, see also [Iha10, Mur11] for example.

**Lemma 2.4.1.** *For $x>0$ and $\epsilon>0$,*

$$\Big|\sum_{\mathrm{N}\mathfrak{n}<x}\frac{1}{\mathrm{N}\mathfrak{n}}\Big(1-\frac{\mathrm{N}\mathfrak{n}}{x}\Big)^{n_K} - \kappa_K\Big(\log x - \sum_{j=1}^{n_K}\frac{1}{j}\Big) - \kappa_K\gamma_K\Big| \ll_\epsilon \left(n_K^{n_K}D_K\right)^{1/4+\epsilon}x^{-1/2}.$$

*Proof.* Without loss, $\epsilon<1/2$. The quantity we wish to bound equals

$$\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\zeta_K(s+1)\frac{x^s}{s}\frac{n_K!}{\prod_{j=1}^{n_K}(s+j)}ds = \frac{n_K!}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\zeta_K(s+1)\frac{\Gamma(s)}{\Gamma(n_K+1+s)}x^s ds.$$

Using Lemma 2.3.2, Stirling's formula, and $\zeta_{\mathbb{Q}}(1+\epsilon)^{n_K} \ll e^{O_\epsilon(n_K)}$, the result follows. □

**Corollary 2.4.2.** *Let $\epsilon > 0$ be arbitrary. If $x \geq 3\big(n_K^{n_K} D_K\big)^{1/2+\epsilon}$ then*

$$\sum_{\mathrm{N}\mathfrak{n}<x} \frac{1}{\mathrm{N}\mathfrak{n}} \geq \{1 - (1+2\epsilon)^{-1} + O_\epsilon((\log x)^{-1})\} \cdot \kappa_K \log x.$$

*Proof.* It suffices to assume that $\kappa_K \geq 1/\log x$. From Lemma 2.4.1, it follows that

$$\frac{1}{\kappa_K} \sum_{\mathrm{N}\mathfrak{n}<x} \frac{1}{\mathrm{N}\mathfrak{n}} \geq \log x - \sum_{j=1}^{n_K} \frac{1}{j} + \gamma_K + O_\epsilon\left(x^{-\frac{\epsilon}{8}} \log x\right),$$

by our assumption on $x$. By [Iha06, Proposition 3],

$$\gamma_K \geq -\frac{1}{2} \log D_K + \frac{\gamma_{\mathbb{Q}} + \log 2\pi}{2} \cdot n_K - 1$$

where $\gamma_{\mathbb{Q}} = 0.5772\ldots$ is Euler's constant. Since $\sum_{1\leq j\leq n_K} j^{-1} \leq \log n_K + 1$,

$$\begin{aligned}\frac{1}{\kappa_K} \sum_{\mathrm{N}\mathfrak{n}<x} \frac{1}{\mathrm{N}\mathfrak{n}} &\geq (\log x)\{1 + O_\epsilon(x^{-\epsilon/8})\} - \frac{1}{2}\log D_K + \frac{\gamma_{\mathbb{Q}} + \log 2\pi}{2} \cdot n_K - \log n_K - 2 \\ &\geq (\log x)\{1 - \tfrac{1}{1+2\epsilon} + O_\epsilon((\log x)^{-1})\},\end{aligned}$$

by our assumption on $x$. □

Taking the logarithmic derivative of $\zeta_K(s)$ yields in the usual way

$$-\frac{\zeta_K'}{\zeta_K}(s) = \sum_{\mathfrak{n}\subseteq\mathcal{O}_K} \frac{\Lambda_K(\mathfrak{n})}{(\mathrm{N}\mathfrak{n})^s} \tag{2.18}$$

for $\mathrm{Re}\{s\} > 1$, where $\Lambda_K(\,\cdot\,)$ is the von Mangoldt $\Lambda$-function of the field $K$ defined by

$$\Lambda_K(\mathfrak{n}) = \begin{cases} \log \mathrm{N}\mathfrak{p} & \text{if } \mathfrak{n} \text{ is a power of a prime ideal } \mathfrak{p}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.19}$$

Using this identity, we prove an elementary lemma.

**Lemma 2.4.3.** *For $y \geq 3$ and $0 < r < 1$,*

*(i)* $-\dfrac{\zeta_K'}{\zeta_K}(1+r) = \displaystyle\sum_{\mathfrak{n}} \frac{\Lambda_K(\mathfrak{n})}{\mathrm{N}\mathfrak{n}^{1+r}} \leq \frac{1}{2}\log D_K + \frac{1}{r} + 1.$

*(ii)* $\displaystyle\sum_{\mathrm{N}\mathfrak{n}\leq y} \frac{\Lambda_K(\mathfrak{n})}{\mathrm{N}\mathfrak{n}} \leq e \log(eD_K^{1/2} y).$

*Proof.* Part (i) follows from Lemmas 2.2.1 and 2.3.3, (2.18), and the fact that $\mathrm{Re}\{(1+r-\rho)^{-1}\} \geq 0$. Part (ii) follows from (i) by taking $r = \frac{1}{\log y}$. □

We will need a lemma to transfer from imprimitive characters to primitive ones.

**Lemma 2.4.4.** *Let* $\mathfrak{m}$ *be an integral ideal. Then, for* $\epsilon > 0$,

$$\sum_{\mathfrak{p}|\mathfrak{m}} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}} \leq \sqrt{n_K \log \mathrm{N}\mathfrak{m}} \leq \frac{1}{2}\Big(\frac{n_K}{\epsilon} + \epsilon \log \mathrm{N}\mathfrak{m}\Big)$$

*where the sum is over prime ideals* $\mathfrak{p}$ *dividing* $\mathfrak{m}$.

*Proof.* The second inequality follows from $(x+y)/2 \geq \sqrt{xy}$ for $x, y \geq 0$. It suffices to prove the first estimate. Write $\mathfrak{m} = \prod_{i=1}^r \mathfrak{p}_i^{e_i}$ in its unique ideal factorization where $\mathfrak{p}_i$ are distinct prime ideals and $e_i \geq 1$. Denote $q_i = \mathrm{N}\mathfrak{p}_i$ and $a_m = \#\{i : q_i = m\}$. Observe that $a_m = 0$ unless $m$ is a power of a rational prime $p$. Since the principal ideal $(p)$ factors into at most $n_K$ prime ideals in $K$, it follows $a_m \leq n_K$ for $m \geq 1$. Thus, by Cauchy-Schwarz,

$$\begin{aligned}
\sum_{\mathfrak{p}|\mathfrak{m}} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}} = \sum_{i=1}^r \frac{\log q_i}{q_i} &\leq \Big(\sum_{i=1}^r \frac{\log q_i}{q_i^2}\Big)^{1/2}\Big(\sum_{i=1}^r \log q_i\Big)^{1/2} \\
&= \Big(\sum_{m\geq 1} a_m \frac{\log m}{m^2}\Big)^{1/2}\Big(\sum_{i=1}^r \log q_i\Big)^{1/2} \\
&\leq n_K^{1/2}\Big(\sum_{m\geq 1} \frac{\log m}{m^2}\Big)^{1/2}\Big(\sum_{i=1}^r e_i \log q_i\Big)^{1/2} \\
&= \Big(\sum_{m\geq 1} \frac{\log m}{m^2}\Big)^{1/2}\sqrt{n_K \log \mathrm{N}\mathfrak{m}}.
\end{aligned}$$

Since $\sum_{m\geq 1} \frac{\log m}{m^2} < 1$, the result follows. □

We record a lemma involving some simple sums over prime ideals.

**Lemma 2.4.5.** *Let* $a \in (0,1], \delta > 0$ *be arbitrary and* $\mathfrak{d}$ *be an integral ideal of* $K$. *Then*

*(i)* $\displaystyle\sum_{\mathfrak{p}} \frac{1}{(\mathrm{N}\mathfrak{p})^{1+\delta}} \ll_\delta n_K$

*(ii)* $\displaystyle\sum_{\mathfrak{p}|\mathfrak{d}} \frac{1}{(\mathrm{N}\mathfrak{p})^{a}} \ll n_K^{a/2}(\log \mathrm{N}\mathfrak{d})^{1-a/2}$

*(iii)* $\displaystyle\sum_{\mathfrak{p}|\mathfrak{d}} \frac{1}{(\mathrm{N}\mathfrak{p})^{a}} \ll \delta^{-2/a+1} n_K + \delta \log \mathrm{N}\mathfrak{d}$

*Proof.* For (i), observe that

$$\sum_{\mathfrak{p}} \frac{1}{(\mathrm{N}\mathfrak{p})^{1+\delta}} \leq n_K \sum_p \frac{1}{p^{1+\delta}} \ll_\delta n_K,$$

where the latter sum is over rational primes $p$. For (ii), using Hölder's inequality, we see that

$$\sum_{\mathfrak{p}|\mathfrak{d}} \frac{1}{(\mathrm{N}\mathfrak{p})^a} \leq \Big(\sum_{\mathfrak{p}|\mathfrak{d}} 1\Big)^{1-a}\Big(\sum_{\mathfrak{p}|\mathfrak{d}} \frac{1}{\mathrm{N}\mathfrak{p}}\Big)^a.$$

Bounding the first sum by $\log \mathrm{N}\mathfrak{d}$ and the second sum by Lemma 2.4.4 yields the desired result. Statement (iii) follows easily from (ii) by considering whether $n_K \leq \delta^{2/a}\log \mathrm{N}\mathfrak{d}$ or not. □

Next, we desire a bound for $h_H$ in terms of $n_K, D_K$, and $Q = Q_H$.

**Lemma 2.4.6.** *Let $H$ be a congruence class group of $K$. For $\epsilon > 0$, $h_H \leq e^{O_\epsilon(n_K)} D_K^{1/2+\epsilon} Q^{1+\epsilon}$.*

*Proof.* Observe, by the definitions of $Q$ and $\mathfrak{f}_H$ in Section 2.1, that for a Hecke character $\chi \,(\mathrm{mod}\, H)$ we have $\mathfrak{f}_\chi \mid \mathfrak{f}_H$ and $\mathrm{N}\mathfrak{f}_\chi \leq Q$. Hence,

$$h_H = \sum_{\chi\,(\mathrm{mod}\,H)} 1 \leq \sum_{\substack{\mathrm{N}\mathfrak{f}\leq Q\\ \mathfrak{f}\,|\,\mathfrak{f}_H}} \sum_{\chi\,(\mathrm{mod}\,\mathfrak{f})} 1 = \sum_{\substack{\mathrm{N}\mathfrak{f}\leq Q\\ \mathfrak{f}\,|\,\mathfrak{f}_H}} \#\mathrm{Cl}(\mathfrak{f}).$$

Recall the classical bound $\#\mathrm{Cl}(\mathfrak{f}) \leq 2^{n_K} h_K \mathrm{N}\mathfrak{f}$ where $h_K$ is the class number of $K$ (in the broad sense) from [Mil13, Theorem 1.7], for example. Bounding the class number using Minkowski's bound (see [Wei83, Lemma 1.12] for example), we deduce that

$$h_H \leq \sum_{\substack{\mathrm{N}\mathfrak{f}\leq Q\\ \mathfrak{f}\,|\,\mathfrak{f}_H}} e^{O_\epsilon(n_K)} D_K^{1/2+\epsilon} \mathrm{N}\mathfrak{f} \leq e^{O_\epsilon(n_K)} D_K^{1/2+\epsilon} Q^{1+\epsilon} \sum_{\mathfrak{f}\,|\,\mathfrak{f}_H} \frac{1}{(\mathrm{N}\mathfrak{f})^\epsilon}.$$

For the remaining sum, notice $\sum_{\mathfrak{f}\,|\,\mathfrak{f}_H} (\mathrm{N}\mathfrak{f})^{-\epsilon} \leq \prod_{\mathfrak{p}|\mathfrak{f}_H} (1-\mathrm{N}\mathfrak{p}^{-\epsilon})^{-1} \leq e^{O(\omega(\mathfrak{f}_H))}$, where $\omega(\mathfrak{f}_H)$ is the number of prime ideals $\mathfrak{p}$ dividing $\mathfrak{f}_H$. From [Wei83, Lemma 1.13], we have $\omega(\mathfrak{f}_H) \ll O_\epsilon(n_K) + \epsilon \log(D_K Q)$ whence the desired estimate follows after rescaling $\epsilon$. □

[Wei83, Lemma 1.16] is comparable with Lemma 2.4.6 but $Q^{1+\epsilon}$ is replaced by $\mathrm{N}\mathfrak{f}_H$. The relative size of these quantities is not immediately clear, so we end this section with a comparison between $Q$ and $\mathrm{N}\mathfrak{f}_H$.

**Lemma 2.4.7.** *Let $H$ be a congruence class group of $K$. Then $Q \leq \mathrm{N}\mathfrak{f}_H \leq Q^2$.*

**Remark.** The lower bound is achieved when $H = P_{\mathfrak{f}_H}$. We did not investigate the tightness of the upper bound as this estimate will be sufficient our purposes.

*Proof.* The arguments here are motivated by [Wei83, Lemma 1.13]. Without loss, we may assume $H$ is primitive. Since $Q = Q_H = \max\{\mathrm{N}\mathfrak{f}_\chi : \chi\,(\mathrm{mod}\,H)\}$ and $\mathfrak{f}_H = \mathrm{lcm}\{\mathfrak{f}_\chi : \chi\,(\mathrm{mod}\,H)\}$, the lower bound is immediate. For the upper bound, consider any $\mathfrak{m} \mid \mathfrak{f}_H$. Let $H_\mathfrak{m}$ denote the image of $H$ under the map $I(\mathfrak{f}_H)/P_{\mathfrak{f}_H} \to I(\mathfrak{m})/P_\mathfrak{m}$. This induces a map $I(\mathfrak{f}_H)/H \to$

$I(\mathfrak{m})/H_\mathfrak{m}$, which, since $H$ is primitive, must have non-trivial kernel. Hence, characters of $I(\mathfrak{m})/H_\mathfrak{m}$ induce characters of $I(\mathfrak{f}_H)/H$.

Now, for $\mathfrak{p} \mid \mathfrak{f}_H$, choose $e = e_\mathfrak{p} \geq 1$ maximum satisfying $\mathfrak{p}^e \mid \mathfrak{f}_H$. Define $\mathfrak{m}_\mathfrak{p} := \mathfrak{f}_H \mathfrak{p}^{-1}$ and consider the induced map $I(\mathfrak{f}_H)/H \to I(\mathfrak{m}_\mathfrak{p})/H_{\mathfrak{m}_\mathfrak{p}}$ with kernel $V_\mathfrak{p}$. Since $H$ is primitive, $V_\mathfrak{p}$ must be non-trivial and hence $\#V_\mathfrak{p} \geq 2$. Observe that the characters $\chi$ of $I(\mathfrak{f}_H)/H$ such that $\mathfrak{p}^e \nmid \mathfrak{f}_\chi$ are exactly those which are trivial on $V_\mathfrak{p}$ and hence are $\frac{h_H}{\#V_\mathfrak{p}}$ in number. For a given $\mathfrak{p}$, this yields the following identity:

$$\frac{h_H}{2} \leq h_H\Big(1 - \frac{1}{\#V_\mathfrak{p}}\Big) = \sum_{\substack{\chi \, (\mathrm{mod}\, H) \\ \mathfrak{p}^{e_\mathfrak{p}} \| \mathfrak{f}_\chi}} 1.$$

Multiplying both sides by $\log(\mathrm{N}\mathfrak{p}^{e_\mathfrak{p}})$ and summing over $\mathfrak{p} \mid \mathfrak{f}_H$, we have that

$$\frac{1}{2} h_H \log \mathrm{N}\mathfrak{f}_H = \frac{h_H}{2} \sum_{\mathfrak{p} \mid \mathfrak{f}_H} \log(\mathrm{N}\mathfrak{p}^{e_\mathfrak{p}}) \leq \sum_{\mathfrak{p} \mid \mathfrak{f}_H} \sum_{\substack{\chi \, (\mathrm{mod}\, H) \\ \mathfrak{p}^{e_\mathfrak{p}} \| \mathfrak{f}_\chi}} \log \mathrm{N}\mathfrak{p}^{e_\mathfrak{p}} \leq \sum_{\chi \, (\mathrm{mod}\, H)} \log \mathrm{N}\mathfrak{f}_\chi \leq h_H \log Q.$$

Comparing both sides, we deduce $\mathrm{N}\mathfrak{f}_H \leq Q^2$, as desired. $\square$

## 2.5 Artin $L$-functions and Deuring's reduction

This section can be safely ignored until Chapters 7 and 8. Let $L/F$ be a Galois extension of number fields with Galois group $G := \mathrm{Gal}(L/F)$ and let $C$ be a conjugacy class of $G$. This section consists of preliminary material required for counting prime ideals $\mathfrak{p}$ of $F$ with Artin symbol $[\frac{L/F}{\mathfrak{p}}] = C$. A similar discussion can be found in [LMO79, Section 3]. For the number field $F$, we will use the following notation throughout this section:

- $\mathcal{O}_F$ is the ring of integers of $F$.
- $n_F = [F : \mathbb{Q}]$ is the degree of $F/\mathbb{Q}$.
- $D_F = |\mathrm{disc}(F/\mathbb{Q})|$ is the absolute value of the absolute discriminant of $F$.
- $\mathrm{N} = \mathrm{N}^F_\mathbb{Q}$ is the absolute field norm of $F$.
- $\zeta_F(s)$ is the Dedekind zeta function of $F$.
- $\mathfrak{p}$ is a prime ideal of $F$.
- $\mathfrak{n}$ is an integral ideal of $F$.

- $\Lambda_F(\mathfrak{n})$ is the von Mangoldt $\Lambda$-function for $F$ given by

$$\Lambda_F(\mathfrak{n}) = \begin{cases} \log \mathrm{N}_{\mathbb{Q}}^F \mathfrak{p} & \text{if } \mathfrak{n} \text{ is a power of a prime ideal } \mathfrak{p}, \\ 0 & \text{otherwise.} \end{cases}$$

### Artin $L$-functions

Let us briefly recall the definition of an Artin $L$-function from [MM97, Chapter 2, Section 2]. For each prime ideal $\mathfrak{p}$ of $F$, and a prime ideal $\mathfrak{P}$ of $L$ lying above $\mathfrak{p}$, define the decomposition group $D_{\mathfrak{P}}$ to be $\mathrm{Gal}(L_{\mathfrak{P}}/F_{\mathfrak{p}})$, where $L_{\mathfrak{P}}$ (resp. $F_{\mathfrak{p}}$) is the completion of $L$ (resp. $F$) at $\mathfrak{P}$ (resp. $\mathfrak{p}$). Let $k_{\mathfrak{P}}$ (resp. $k_{\mathfrak{p}}$) denote the residue field of $L_{\mathfrak{P}}$ (resp. $F_{\mathfrak{p}}$). We have a map $D_{\mathfrak{P}}$ to $\mathrm{Gal}(k_{\mathfrak{P}}/k_{\mathfrak{p}})$ (the Galois group of the residue field extension), which is surjective by Hensel's lemma. The kernel of this map is the inertia group $I_{\mathfrak{P}}$. Thus, we have the exact sequence

$$1 \to I_{\mathfrak{P}} \to D_{\mathfrak{P}} \to \mathrm{Gal}(k_{\mathfrak{P}}/k_{\mathfrak{p}}) \to 1.$$

The group $\mathrm{Gal}(k_{\mathfrak{P}}/k_{\mathfrak{p}})$ is cyclic with generator $x \mapsto x^{\mathrm{N}\mathfrak{p}}$, where $\mathrm{N}\mathfrak{p}$ is the cardinality of $k_{\mathfrak{p}}$. We can choose an element $\sigma_{\mathfrak{P}} \in D_{\mathfrak{P}}$ whose image in $\mathrm{Gal}(k_{\mathfrak{P}}/k_{\mathfrak{p}})$ is this generator. We call $\sigma_{\mathfrak{P}}$ a Frobenius element at $\mathfrak{P}$; it is well-defined modulo $I_{\mathfrak{P}}$. We have that $I_{\mathfrak{P}}$ is trivial for all unramified $\mathfrak{p}$, and for these $\mathfrak{p}$, $\sigma_{\mathfrak{P}}$ is well-defined. For $\mathfrak{p}$ unramified, we denote by $\sigma_{\mathfrak{p}}$ the conjugacy class of Frobenius elements at primes $\mathfrak{P}$ above $\mathfrak{p}$; in this case, note that $\sigma_{\mathfrak{p}} = [\frac{L/F}{\mathfrak{p}}]$.

Let $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be a representation of $G$, and let $\psi$ denote its character. Let $V$ be the underlying complex vector space on which $\rho$ acts, and let $V^{I_{\mathfrak{P}}}$ be the subspace of $V$ on which $I_{\mathfrak{P}}$ acts trivially. We now define their local Euler factors to be

$$L_{\mathfrak{p}}(s, \psi, L/F) = \begin{cases} \det(I_n - \rho(\sigma_{\mathfrak{p}})\mathrm{N}\mathfrak{p}^{-s})^{-1} & \text{if } \mathfrak{p} \text{ is unramified in } L, \\ \det(I_n - \rho(\sigma_{\mathfrak{P}}) \mid_{V^{I_{\mathfrak{P}}}} \mathrm{N}\mathfrak{p}^{-s})^{-1} & \text{if } \mathfrak{p} \text{ is ramified in } L, \end{cases}$$

where $I_n$ is the $n \times n$ identity matrix. This is well-defined for all $\mathfrak{p}$, which allows us to define the Artin $L$-function

$$L(s, \psi, L/F) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(s, \psi, L/F)$$

for $\mathrm{Re}\{s\} > 1$. It is well-known to be analytic and non-zero for $\mathrm{Re}\{s\} > 1$.

### Some class field theory

Let $A$ be any abelian subgroup of $G = \mathrm{Gal}(L/F)$ and let $K = L^A$ be the subfield of $L$ fixed by $A$. We describe some properties of the associated 1-dimensional Artin $L$-functions

$L(s,\chi,L/K)$. First, from [Hei67] for example, note that

$$\zeta_L(s) = \prod_{\chi \in \hat{A}} L(s,\chi,L/K), \tag{2.20}$$

where the product is over the Artin characters $\chi$ of $A = \mathrm{Gal}(L/K)$. From the above, one can deduce the conductor-discriminant formula, which states

$$\log D_L = \sum_{\chi} \log D_{\chi}. \tag{2.21}$$

We wish to elaborate on the relationship between the $L$-functions in (2.20) and the Hecke $L$-functions defined in Section 2.2.

By the fundamental theorem of class field theory, there is an integral ideal $\mathfrak{f} = \mathfrak{f}_{L/K}$ attached to the extension $L/K$ and a surjective homomorphism $\varphi : I(\mathfrak{f}) \to \mathrm{Gal}(L/K)$, where $I(\mathfrak{f})$ is the group of fractional ideals of $K$ relatively prime to $\mathfrak{f}$. Hence, $I(\mathfrak{f})/H$ is isomorphic to $\mathrm{Gal}(L/K)$ where $H = \ker\varphi$. From this isomorphism, we obtain a natural correspondence between the 1-dimensional Artin characters $\chi$ of $\mathrm{Gal}(L/K)$ and the Hecke characters $\tilde{\chi} \pmod{H}$ of $I(\mathfrak{f})/H$. In particular, they satisfy

$$\tilde{\chi}^*(\mathfrak{P}) = \chi\Big(\Big[\frac{L/K}{\mathfrak{P}}\Big]\Big) \tag{2.22}$$

for all prime ideals $\mathfrak{P} \subseteq \mathcal{O}_K$ unramified in $L$. We emphasize that $\tilde{\chi}^*$ is the primitive Hecke character inducing $\tilde{\chi}$. Furthermore, under this correspondence, we have that

$$\prod_{\chi \in \hat{A}} L(s,\chi,L/K) = \prod_{\tilde{\chi} \pmod{H}} L(s,\tilde{\chi}^*,K). \tag{2.23}$$

In particular, the 1-dimensional Artin $L$-function $L(s,\chi,L/K)$ is equal to a certain *primitive* Hecke $L$-function $L(s,\tilde{\chi}^*,K)$. While $L(s,\tilde{\chi},K)$ is not necessarily primitive for any given Hecke character $\tilde{\chi} \pmod{H}$, its $L$-function $L(s,\tilde{\chi},K)$ equals its primitive counterpart $L(s,\tilde{\chi}^*,K)$ up to a finite number of local Euler factors. Thus, the two $L$-functions have the same non-trivial zeros, counted with multiplicity. Hence, each 1-dimensional Artin $L$-function $L(s,\chi,L/K)$ has the same non-trivial zeros, counted with multiplicity, as a corresponding (not necessarily primitive) Hecke $L$-function $L(s,\tilde{\chi},K)$ for some $\tilde{\chi} \pmod{H}$. By (2.20) and (2.23), this implies that

$$\prod_{\tilde{\chi} \pmod{H}} L(s,\tilde{\chi},K) \tag{2.24}$$

has the same non-trivial zeros, counted with multiplicity, as $\prod_{\chi \in \hat{A}} L(s,\chi,L/K)$.

**Prime ideal counting function**

For a conjugacy class $C \subseteq G$, let $g_C \in C$ be arbitrary. Define

$$Z_C(s) := -\frac{|C|}{|G|} \sum_{\psi} \overline{\psi}(g_C) \frac{L'}{L}(s, \psi, L/F), \tag{2.25}$$

where $\psi$ runs over irreducible characters of $G$ and $L(s, \psi, L/F)$ is the associated Artin $L$-function. Note the definition of $Z_C(s)$ does not depend on the choice of $g_C$ since $\psi$ is the trace of the representation $\rho$ and $g_C$ is conjugate to any other choice. By orthogonality relations for characters (see [Hei67, Section 3] for example),

$$Z_C(s) = \sum_{\mathfrak{n} \subseteq \mathcal{O}_F} \Lambda_F(\mathfrak{n}) \Theta_C(\mathfrak{n}) (\mathrm{N}\mathfrak{n})^{-s}, \tag{2.26}$$

where $\Theta_C(\mathfrak{n})$ is supported on integral ideals $\mathfrak{n}$ which are powers of a prime ideal; in particular, for prime ideals $\mathfrak{p}$ unramified in $L$ and $m \geq 1$,

$$\Theta_C(\mathfrak{p}^m) = \begin{cases} 1 & \text{if } [\frac{L/F}{\mathfrak{p}}]^m \subseteq C, \\ 0 & \text{otherwise,} \end{cases} \tag{2.27}$$

and $0 \leq \Theta_C(\mathfrak{p}^m) \leq 1$ if $\mathfrak{p}$ ramifies in $L$. This discussion and definition of $\Theta_C(\,\cdot\,)$ is also contained in [LMO79, Section 3]. Thus, by (1.15), we have that

$$\pi_C(x, L/F) = \sum_{\substack{\mathrm{N}\mathfrak{p} < x \\ \mathfrak{p} \text{ unramified in } L}} \Theta_C(\mathfrak{p}) \tag{2.28}$$

for $x > 1$. In Chapters 7 and 8, we will be concerned with a prime ideal counting function which is naturally related to $\pi_C(x, L/F)$ and is given by

$$\psi_C(x, L/F) := \sum_{\mathrm{N}\mathfrak{n} < x} \Lambda_F(\mathfrak{n}) \Theta_C(\mathfrak{n}). \tag{2.29}$$

Observe, by (2.26) and Mellin inversion, that

$$\psi_C(x, L/F) = \int_{2-i\infty}^{2+i\infty} Z_C(s) \frac{x^s}{s} ds. \tag{2.30}$$

This property motivates the use of $Z_C(s)$ in our analytic arguments. Next, we record a basic lemma relating $\pi_C(x, L/F)$ with $\psi_C(x, L/F)$ for use in Chapter 8. In that scenario, we will only be interested in an upper bound for $\pi_C(x, L/F)$, so we give a simpler statement that suffices for our purposes.

**Lemma 2.5.1.** *If $x > x_0 > 3$, then*

$$\pi_C(x, L/F) \leq \frac{\psi_C(x, L/F)}{\log x} + \int_{x_0}^{x} \frac{\psi_C(t, L/F)}{t \log^2 t} dt + O(n_F x_0).$$

*Proof.* For simplicity, write $\psi_C(t)$ in place of $\psi_C(t, L/F)$. For $t > 1$, define

$$\tilde{\pi}_C(t) := \sum_{\mathrm{N}\mathfrak{p} < t} \Theta_C(\mathfrak{p}), \qquad \theta_C(t) := \sum_{\mathrm{N}\mathfrak{p} < t} \Theta_C(\mathfrak{p}) \log \mathrm{N}\mathfrak{p},$$

where the sums are over all prime ideals $\mathfrak{p}$ of $F$ and $\Theta_C(\mathfrak{p})$ is given by (2.27). First, observe that, by (2.27) and (2.28), the only difference between $\tilde{\pi}_C(x)$ and $\pi_C(x, L/F)$ is the contribution from the prime ideals $\mathfrak{p}$ of $F$ ramified in $L$. Since $0 \leq \Theta_C(\mathfrak{p}) \leq 1$ for such prime ideals, we observe that

$$\pi_C(x, L/F) \leq \tilde{\pi}_C(x), \tag{2.31}$$

so it suffices to estimate $\tilde{\pi}_C(x)$. Using partial summation, we see that if $3 < x_0 < x$, then

$$\tilde{\pi}_C(x) = \frac{\theta_C(x)}{\log x} + \int_{x_0}^{x} \frac{\theta_C(t)}{t \log^2 t} dt + \tilde{\pi}_C(x_0). \tag{2.32}$$

Since there are at most $n_F$ prime ideals above a rational prime $p$, observe that

$$\tilde{\pi}_C(x_0) \leq \sum_{p < x_0} \sum_{\mathfrak{p} | (p)} 1 \leq n_F \sum_{p < x_0} 1 \ll \frac{n_F x_0}{\log x_0} \ll n_F x_0. \tag{2.33}$$

Moreover, $\theta_C(t) \leq \psi_C(t)$ for all $t > 1$. Combining these observations with (2.31) and (2.32) yields the desired result. $\square$

### Deuring's reduction

In general, Artin $L$-functions $L(s, \psi, L/F)$ are only known to be meromorphic in the half-plane $\mathrm{Re}\{s\} > 1$. Thus, $Z_C(s)$ is meromorphic in $\mathrm{Re}\{s\} > 1$. However, we will need $Z_C(s)$ to be meromorphically continued to the entire complex plane in order to execute standard arguments involving contour integrals like (2.30). To do so, we must enact Deuring's reduction.

Let $A$ be any abelian subgroup of $G = \mathrm{Gal}(L/F)$ such that $A \cap C$ is non-empty. From the definition of $Z_C(s)$ in (2.25), we may assume without loss of generality that $g_C \in A \cap C$. If

$K = L^A$ is the fixed field of $A$ then by [Hei67, Lemma 4],

$$Z_C(s) = -\frac{|C|}{|G|}\sum_{\chi}\overline{\chi}(g_C)\frac{L'}{L}(s,\chi,L/K), \tag{2.34}$$

where the sum runs over irreducible characters $\chi$ of $A$, which are necessarily 1-dimensional since $A$ is abelian. By class field theory, the Artin $L$-function $L(s,\chi,L/K)$ is a certain *primitive* Hecke $L$-function. Therefore, (2.34) implies $Z_C(s)$ is meromorphic in the entire complex plane. This concludes Deuring's reduction.

### Notational convention

One may wish to notationally distinguish a 1-dimensional Artin $L$-function $L(s,\chi,L/K)$ with the primitive Hecke $L$-function $L(s,\tilde{\chi}^*,K)$ associated to it by class field theory. However, throughout this thesis, we will frequently make no such distinction and abuse notation. We will often treat $L(s,\chi,L/K)$ as a primitive Hecke $L$-function with conductor $\mathfrak{f}_\chi \subseteq \mathcal{O}_K$.

# Chapter 3

# Explicit inequalities for Hecke $L$-functions

> *"You may encounter many defeats, but you must not be defeated. In fact, it may be necessary to encounter the defeats, so you can know who you are, what you can rise from, how you can still come out of it."*
>
> – Maya Angelou.

In this chapter, we establish several different explicit inequalities related to the zeros of Hecke $L$-functions by involving classical arguments, higher derivatives, and smooth weights. The notation and results build the foundations for Chapter 4.

## 3.1 Zero-free gap and labelling of zeros

Let $H \pmod{\mathfrak{q}}$ be an arbitrary congruence class group of the number field $K$. The main goal of this section is to show that there is a thin rectangle inside the critical strip above which there is a zero-free gap for

$$\prod_{\chi \pmod{H}} L(s, \chi). \tag{3.1}$$

This zero-free gap is necessary for the proof of Lemma 3.4.3, which is a crucial component for later sections.

Let $\nu(x)$ and $\eta(x)$ be fixed increasing functions for $x \in [1, \infty)$ such that

$$\begin{aligned} &\nu(x) \in [4, \infty), \qquad \nu(x) \gg \log(x+4), \\ &\eta(x) \in [2, \infty), \qquad \eta(x) \to \infty \text{ as } x \to \infty, \qquad \text{and } \tfrac{x}{\eta(x)\log(x+1)} \text{ is increasing.} \end{aligned} \tag{3.2}$$

One could take $\eta(x) = \frac{1}{2}\log x + 2$, for example. Denote

$$\begin{aligned} \mathcal{L} &:= \log D_K + \tfrac{3}{4}\log Q + n_K \cdot \nu(n_K), \\ \mathcal{L}^* &:= \log D_K + \tfrac{3}{4} \cdot \log Q, \\ \mathcal{T} &:= (\mathcal{L}^*)^{1/\eta(n_K)\log(n_K+1)} + \nu(n_K), \end{aligned} \tag{3.3}$$

where $Q = Q_H$ is defined by (2.2). Similarly, for a Hecke character $\chi \pmod{H}$ with conductor $\mathfrak{f}_\chi$, define

$$\begin{aligned} \mathcal{L}_\chi &:= \log D_\chi + n_K \cdot \nu(n_K), \\ \mathcal{L}^*_\chi &:= \log D_\chi, \\ \mathcal{L}_0 &:= \mathcal{L}_{\chi_0} = \log D_K + n_K \cdot \nu(n_K). \end{aligned} \tag{3.4}$$

Note that $D_\chi \leq D_K Q$ for any $\chi \pmod{H}$ by definition of $Q$.

For the remainder of Chapters 3 and 4, we shall maintain this notation because these quantities will be ubiquitous in all of our estimates. Moreover, all implicit constants will be independent of the number field $K$, the congruence class group $H$, and all Hecke characters $\chi$, and will only implicitly depend on the choice of $\nu$ and $\eta$.

First, we record some simple relationships between the quantities defined in (3.3) and (3.4).

**Lemma 3.1.1.** *Let $\chi \pmod{H}$ be arbitrary. For the quantities defined in* (3.3) *and* (3.4)*, all of the following hold:*

*(i)* $4 \leq \mathcal{T} \leq \mathcal{L}$.

*(ii)* $n_K \log \mathcal{T} = o(\mathcal{L})$.

*(iii)* $\mathcal{L}^* + n_K \log \mathcal{T} \leq \mathcal{L} + o(\mathcal{L})$ *and* $\mathcal{L}^*_\chi + n_K \log \mathcal{T} \leq \mathcal{L}_\chi + o(\mathcal{L})$.

*(iv)* $\mathcal{T} \to \infty$ *as* $\mathcal{L} \to \infty$.

*(v)* $a\mathcal{L}_0 + b\mathcal{L}_\chi \leq (a+b)\mathcal{L}$ *for all* $0 \leq b \leq 3a$.

*Proof.* Statements (i) and (iii) follow easily from (ii) and the definitions of $\mathcal{T}, \mathcal{L}$ and $\mathcal{L}^*$. For (ii), observe that

$$n_K \log \mathcal{T} \leq \frac{n_K \log \mathcal{L}^*}{\eta(n_K)\log(n_K+1)} + n_K \log \nu(n_K) + n_K \log 2.$$

The second and third terms are $o(\mathcal{L})$ as $\nu(n_K)$ is increasing. For the first term, note that $\frac{n_K}{\eta(n_K)\log(n_K+1)}$ is increasing as a function $n_K$ by (3.2). Thus, substituting the upper bound

$n_K = O(\log D_K) = O(\mathcal{L}^*)$ from Minkowski's theorem, we deduce that

$$n_K \log \mathcal{T} \ll \frac{\mathcal{L}^*}{\eta(\mathcal{L}^*)} + o(\mathcal{L}) = o(\mathcal{L}),$$

since $\eta(x) \to \infty$ as $x \to \infty$ and $\mathcal{L}^* \leq \mathcal{L}$. For (iv), if $n_K$ is bounded, then necessarily $\mathcal{L}^* \to \infty$ in which case both $\mathcal{T}$ and $\mathcal{L}$ approach infinity. Otherwise, if $n_K \to \infty$, then both $\mathcal{T}$ and $\mathcal{L}$ approach infinity since $\nu(x) \to \infty$ as $x \to \infty$. For (v), the claim follows from the definition of $\mathcal{L}$ and the fact that $D_\chi \leq D_K Q$ for all $\chi \pmod{H}$. □

Next, we establish a zero-free gap which motivates the choice of $\mathcal{L}$ and its related quantities.

**Lemma 3.1.2.** *Let $T_\star \geq 1$ be fixed and let $C_0 > 0$ be a sufficiently large absolute constant and let $\mathcal{T}$ be defined as in* (3.3)*. For $\mathcal{L}$ sufficiently large, there exists a positive integer $T_0 = T_0(H)$ such that $T_\star \leq T_0 \leq \frac{\mathcal{T}}{10}$ and $\prod_{\chi \pmod{H}} L(s,\chi)$ has no zeros in the region*

$$1 - \frac{\log\log \mathcal{T}}{C_0 \mathcal{L}} \leq \sigma \leq 1, \quad T_0 \leq |t| \leq 10 T_0.$$

*Proof.* For $0 \leq \alpha \leq 1$ and $T \geq 0$, denote

$$N_H(\alpha, T) = \sum_{\chi \pmod{H}} \#\{\rho \in \mathbb{C} \mid L(\rho,\chi) = 0, \quad \alpha \leq \beta \leq 1, \quad 0 \leq |\gamma| \leq T\},$$

where we count zeros with multiplicity. By [Wei83, Theorem 4.3], for $c_6 \leq \alpha \leq 1 - \frac{c_7}{\mathcal{L} + n_K \log T}$,

$$N_H(\alpha, T) \ll (n_K^{n_K} h_H D_K Q T^{n_K})^{c_8(1-\alpha)}$$

for some absolute constants $0 < c_6 < 1, c_7 > 0, c_8 > 0$ and provided $T$ and $\mathcal{L}$ are sufficiently large. By Lemma 3.1.1 and Lemma 2.4.6, observe $h_H \leq e^{O(\mathcal{L})}$ and $n_K \log T \ll \mathcal{L}$ for $T \leq \mathcal{T}$. Moreover, $n_K^{n_K} D_K Q \leq e^{O(\mathcal{L})}$ since $\nu(x) \gg \log(x+4)$ by (3.2). It follows that, for $T = \mathcal{T}$ and $\mathcal{T}$ sufficiently large,

$$N_H(\alpha, \mathcal{T}) \ll (e^{\mathcal{L}} \cdot \mathcal{T}^{n_K})^{c_9(1-\alpha)} \tag{3.5}$$

for $c_6 \leq \alpha \leq 1 - \frac{c_{10}}{\mathcal{L}}$ and some absolute constants $c_9 > 0$ and $c_{10} > 0$.

Now suppose, for a contradiction, that no such $T_0$ exists. Setting $\alpha = 1 - \frac{\log\log \mathcal{T}}{C_0 \mathcal{L}}$, it follows that every region

$$\alpha \leq \sigma \leq 1, \quad 10^j \leq |t| \leq 10^{j+1},$$

for $J_\star \leq j < J$, where $J := \lfloor \frac{\log \mathcal{T}}{\log 10} \rfloor$ and $J_\star = \lceil \frac{\log T_\star}{\log 10} \rceil$, contains at least one zero of

$\prod_{\chi \,(\mathrm{mod}\, H)} L(s,\chi)$. Hence,

$$N_H(\alpha,\mathcal{T}) \geq J - J_\star \gg \log \mathcal{T},$$

since $T_\star$ is fixed and $\mathcal{T}$ is sufficiently large. On the other hand, by (3.5), our choice of $\alpha$ implies

$$N_H(\alpha,\mathcal{T}) \ll \exp\Big(\frac{c_9}{C_0}\Big(\log\log\mathcal{T} + \frac{n_K \log\mathcal{T}\log\log\mathcal{T}}{\mathcal{L}}\Big)\Big).$$

From Lemma 3.1.1, we have $n_K \log\mathcal{T} = o(\mathcal{L})$ so for some absolute constant $c_{11} > 0$,

$$N_H(\alpha,\mathcal{T}) \ll \exp\Big(\frac{c_{11}}{C_0}\log\log\mathcal{T}\Big) \ll (\log\mathcal{T})^{\frac{c_{11}}{C_0}}.$$

Upon taking $C_0 = 2c_{11}$, we obtain a contradiction for $\mathcal{T}$ sufficiently large. From Lemma 3.1.1, we may equivalently ask that $\mathcal{L}$ is sufficiently large. □

### 3.1.1 Labeling of zeros

Using the zero-free gap from Lemma 3.1.2, we label important "bad" zeros of

$$\prod_{\chi \,(\mathrm{mod}\, H)} L(s,\chi).$$

These zeros will be referred to throughout Chapters 3 and 4. A typical zero of $L(s,\chi)$ will be denoted $\rho = \beta + i\gamma$ or $\rho_\chi = \beta_\chi + i\gamma_\chi$ when necessary.

*Worst Zero of each Character*

Let $T_\star \geq 1$ be a fixed quantity throughout Chapters 3 and 4; consequently, the condition that $\mathcal{L}$ is sufficiently large also depends implicitly on $T_\star$ throughout Chapters 3 and 4. Consider the rectangle

$$\mathcal{R} = \mathcal{R}_H := \{s \in \mathbb{C} : 1 - \frac{\log\log\mathcal{T}}{C_0\mathcal{L}} \leq \sigma \leq 1, \quad |t| \leq T_0\}$$

for $T_0 = T_0(H) \in [T_\star, \frac{\mathcal{T}}{10}]$ and $C_0 > 0$ defined by Lemma 3.1.2. Denote $\mathcal{Z}$ to be the multiset of zeros of $\prod_{\chi \,(\mathrm{mod}\, H)} L(s,\chi)$ contained in $\mathcal{R}$. Choose finitely many zeros $\rho_1, \rho_2, \ldots$ from $\mathcal{Z}$ as follows:

1. Pick $\rho_1$ such that $\beta_1$ is maximal, and let $\chi_1$ be the corresponding character. Remove all zeros of $L(s,\chi_1)$ and $L(s,\overline{\chi_1})$ from $\mathcal{Z}$.

2. Pick $\rho_2$ such that $\beta_2$ is maximal, and let $\chi_2$ be the corresponding character. Remove all zeros of $L(s,\chi_2)$ and $L(s,\overline{\chi_2})$ from $\mathcal{Z}$.

$\vdots$

Continue in this fashion until $\mathcal{Z}$ has no more zeros to choose. It follows that if $\chi \neq \chi_i, \overline{\chi_i}$ for $1 \leq i < k$, then by Lemma 3.1.2 every zero $\rho$ of $L(s,\chi)$ satisfies:

$$\mathrm{Re}(\rho) \leq \mathrm{Re}(\rho_k) \quad \text{or} \quad |\mathrm{Im}(\rho)| \geq 10T_0. \tag{3.6}$$

For convenience of notation, denote

$$\rho_k = \beta_k + i\gamma_k, \quad \beta_k = 1 - \frac{\lambda_k}{\mathcal{L}}, \quad \gamma_k = \frac{\mu_k}{\mathcal{L}}.$$

*Second Worst Zero of the Worst Character*

Suppose $L(s,\chi_1)$ has a zero $\rho' \neq \rho_1, \overline{\rho_1}$ in the rectangle $\mathcal{R}$, or possibly a repeated real zero $\rho' = \rho_1$. Choose $\rho'$ with $\mathrm{Re}(\rho')$ maximal and write

$$\rho' = \beta' + i\gamma', \quad \beta' = 1 - \frac{\lambda'}{\mathcal{L}}, \quad \gamma' = \frac{\mu'}{\mathcal{L}}.$$

## 3.2 Classical explicit inequality

We may now prove an inequality for $-\mathrm{Re}\{\frac{L'}{L}(s,\chi)\}$ based on a bound for $L(s,\chi)$ in the critical strip and a type of Jensen's formula employed by Heath-Brown in [HB92, Section 3]. First, we introduce an estimate designed to deal with non-primitive characters.

**Lemma 3.2.1.** *Assume $H$ is a primitive congruence class group and $\chi \pmod{H}$ is induced by the character $\chi^*$. For $\epsilon > 0$,*

$$\frac{L'}{L}(s,\chi) = \frac{L'}{L}(s,\chi^*) + O\Big(\frac{n_K}{\epsilon} + \epsilon\mathcal{L}^*\Big),$$

*uniformly in the range $\sigma > 1$.*

*Proof.* Since $H$ is primitive, notice $\chi \pmod{H}$ is a Hecke character modulo $\mathfrak{f}_H$. Hence, using the Euler products of the respective $L$-functions,

$$\Big|\frac{L'}{L}(s,\chi) - \frac{L'}{L}(s,\chi^*)\Big| \leq \sum_{\mathfrak{p}|\mathfrak{f}_H}\sum_{j\geq 1} \frac{\log \mathrm{N}\mathfrak{p}}{(\mathrm{N}\mathfrak{p})^j} \leq 2\sum_{\mathfrak{p}|\mathfrak{f}_H} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}}.$$

The desired result then follows from Lemmas 2.4.4 and 2.4.7. $\square$

Second, we rewrite the convexity bound for Hecke $L$-functions in a convenient form.

**Lemma 3.2.2.** *Assume $H$ is a primitive congruence class group and $\chi \pmod{H}$ is induced by the primitive character $\chi^*$. There exists an absolute constant $\phi > 0$ such that for $\epsilon > 0$,*

$$\left|\frac{s-1}{s+1}\right|^{E_0(\chi)} \cdot |L(s,\chi^*)| \le \exp\left\{2\phi \log(D_\chi \tau^{n_K})(1-\sigma+\epsilon) + O_\epsilon(n_K)\right\}$$

*uniformly in the region*

$$-\epsilon \le \sigma \le 1+\epsilon, \qquad \tau = |t|+3.$$

*In particular, one may take $\phi = \dfrac{1}{4}$.*

*Proof.* Combining Lemmas 2.3.1 and 2.3.2 yields the desired result. □

Any improvement on the constant $\phi$ in Lemma 3.2.2 will have a wide-reaching effect on the mains results of this thesis. For example, the Lindelöf hypothesis for Hecke $L$-functions gives $\phi = \epsilon$. *For the remainder of Chapters 3 and 4 , we set*

$$\phi := \frac{1}{4}.$$

We may now establish the main result of this section.

**Proposition 3.2.3.** *Let $\chi$ be a primitive Hecke character. For any $0 < \epsilon < 1/4$ and any $0 < \delta < \epsilon$,*

$$\begin{aligned} -\mathrm{Re}\left\{\frac{L'}{L}(s,\chi)\right\} \le{}& (\phi + \tfrac{1}{\pi}\epsilon + 4\epsilon^2 + 5\epsilon^{10}) \log(D_\chi \tau^{n_K}) + (4\epsilon^2 + 80\epsilon^{10}) \log D_K \\ &+ \mathrm{Re}\left\{\frac{E_0(\chi)}{s-1}\right\} - \sum_{|1+it-\rho| \le \delta} \mathrm{Re}\left\{\frac{1}{s-\rho}\right\} + O_\epsilon(n_K), \end{aligned} \tag{3.7}$$

*and*

$$-\,\mathrm{Re}\left\{\frac{L'}{L}(s,\chi)\right\} \le (\phi + \tfrac{1}{\pi}\epsilon + 5\epsilon^{10}) \log(D_\chi \tau^{n_K}) + \mathrm{Re}\left\{\frac{E_0(\chi)}{s-1}\right\} + O_\epsilon(n_K), \tag{3.8}$$

*uniformly in the region*

$$1 < \sigma \le 1+\epsilon, \qquad t \in \mathbb{R},$$

*where $\tau = |t|+3$.*

**Remark.**

- When $K = \mathbb{Q}$, Heath-Brown [HB92, Lemma 3.1] showed a similar inequality with $\phi = \frac{1}{6}$ instead of $\phi = \frac{1}{4}$ by leveraging Burgess' estimate for character sums. Our arguments are largely motivated by his result, but we include a few modifications. Note that Heath-Brown's notation for $\phi$ differs by a factor of 2 with our notation.

- Li [Li12] proved a similar result for the Dedekind zeta function with $t = 0$. Using a different approach involving Stechkin's trick, Kadiri and Ng [KN12] established an analogous estimate for the Dedekind zeta function with $\phi = \frac{1}{2}(1 - \frac{1}{\sqrt{5}}) \approx 0.27$.

- When applying Proposition 3.2.3, we often use the notation in (3.3) and (3.4) in which case we will use, without mention, that
$$(\phi + \tfrac{1}{\pi}\epsilon + 4\epsilon^2 + 5\epsilon^{10}) \log D_\chi + (4\epsilon^2 + 80\epsilon^{10}) \log D_K \leq (\phi + 4\epsilon)\mathcal{L}_\chi$$
for $0 < \epsilon < 1/4$.

*Proof.* Let $r = r(\epsilon) \in (0, \frac{1}{10})$ be a parameter to be specified later. Choose $R = R_{s,\chi}(r) < 1$ such that the circles $|w - s| = R$ and $|w - 1| = r$ are disjoint and $L(w, \chi)$ has no zeros on the circle $|w - s| = R$. From these choices, one may take $R \in (R_1 - r, R_1)$ with $R_1 = 1$ or $R_1 = 1 - 4r$.

Apply [HB92, Lemma 3.2] with $f(z) = (\frac{z-1}{z+1})^{E_0(\chi)} L(z, \chi)$ and $a = s$ to deduce
$$-\operatorname{Re}\Big\{\frac{L'}{L}(s,\chi)\Big\} = \frac{E_0(\chi)}{s-1} - \sum_{|s-\rho|<R} \operatorname{Re}\Big\{\frac{1}{s-\rho} - \frac{s-\rho}{R^2}\Big\} - J + O(1), \tag{3.9}$$
since $\frac{E_0(\chi)}{s+1} = O(1)$ and where
$$J := \frac{1}{\pi R}\int_0^{2\pi} (\cos\theta) \cdot \log\Big|\Big(\frac{s + Re^{i\theta} - 1}{s + Re^{i\theta} + 1}\Big)^{E_0(\chi)} \cdot L(s + Re^{i\theta}, \chi)\Big| d\theta.$$
Since $|w - s| = R$ and $|w - 1| = r$ are disjoint and $R < 1$, one can verify
$$\Big|\frac{s + Re^{i\theta} - 1}{s + Re^{i\theta} + 1}\Big| \asymp_\epsilon 1,$$
as $r = r(\epsilon)$ depends only on $\epsilon$. Thus,
$$J = \frac{1}{\pi R}\int_0^{2\pi} (\cos\theta) \cdot \log|L(s + Re^{i\theta}, \chi)| d\theta + O_\epsilon(1) = \tilde{J} + O_\epsilon(1),$$
say. We require a lower bound for $\tilde{J}$ so we divide the contribution of the integral into three

separate intervals depending on the sign of $\cos\theta$; that is,

$$\tilde{J} = J_1 + J_2 + J_3 = \int_0^{\pi/2} + \int_{\pi/2}^{3\pi/2} + \int_{3\pi/2}^{2\pi}.$$

- For $\theta \in [0, \pi/2]$, by Lemma 2.3.1 it follows that

$$|\log L(s + Re^{i\theta}, \chi)| \leq \log \zeta_K(\sigma + R\cos\theta) \leq n_K \log\Big(\frac{1}{\sigma - 1 + R\cos\theta}\Big).$$

  Thus, on the interval $I_1 := [0, \frac{\pi}{2} - (\sigma - 1)]$, as $\sigma - 1 \geq 0$, the contribution to $J_1$ is

$$\ll_\epsilon n_K \int_0^{\pi/2} (\cos\theta) \log(1/R\cos\theta) d\theta \ll_\epsilon n_K.$$

  On the interval $I_2 := [\frac{\pi}{2} - (\sigma - 1), \frac{\pi}{2}]$, as $\cos\theta \geq 0$, the contribution to $J_1$ is

$$\ll_\epsilon n_K \log(1/(\sigma - 1)) \int_{I_2} (\cos\theta) d\theta \ll_\epsilon n_K(\sigma - 1)\log(1/(\sigma - 1)) \ll_\epsilon n_K,$$

  because $0 \leq \sigma - 1 \leq \epsilon$ and $x\log(1/x)$ is bounded as $x \to 0^+$.

- For $\theta \in [\pi/2, 3\pi/2]$, notice

$$0 < \sigma - 1 \leq \sigma + R\cos\theta \leq \sigma \leq 1 + \epsilon,$$

  as $R < 1$. Hence, by Lemma 3.2.2,

$$\begin{aligned}\log|L(s + Re^{i\theta}, \chi)| &\leq 2\phi\log(D_\chi \tau^{n_K})(1 - \sigma - R\cos\theta + \epsilon) + O_\epsilon(n_K) \\ &\leq 2\phi\log(D_\chi \tau^{n_K})(-R\cos\theta + \epsilon) + O_\epsilon(n_K).\end{aligned}$$

  This implies that

$$\begin{aligned}&\int_{\pi/2}^{3\pi/2} (\cos\theta) \cdot \log|L(s + Re^{i\theta}, \chi)| d\theta \\ &\qquad \geq 2\phi\log(D_\chi \tau^{n_K}) \int_{\pi/2}^{3\pi/2} \big(-R\cos^2\theta + \epsilon\cos\theta\big) d\theta + O_\epsilon(n_K) \\ &\qquad = \phi\log(D_\chi \tau^{n_K})\big(-\pi R - 4\epsilon\big) + O_\epsilon(n_K).\end{aligned}$$

- For $\theta \in [3\pi/2, 2\pi]$, we similarly obtain the same contribution as $\theta \in [0, \pi/2]$.

Combining all contributions, we have that

$$J \geq -\Big(\phi + \frac{\epsilon}{\pi R}\Big) \log(D_\chi \tau^{n_K}) + O_\epsilon(n_K), \tag{3.10}$$

since $\phi = 1/4$. For the sum over zeros in (3.9), notice that we may arbitrarily discard zeros from the sum since, for $|s - \rho| < R$,

$$\mathrm{Re}\Big\{\frac{1}{s-\rho} - \frac{s-\rho}{R^2}\Big\} = (\sigma - \beta)\Big(\frac{1}{|s-\rho|^2} - \frac{1}{R^2}\Big) \geq 0. \tag{3.11}$$

Substituting (3.10) and the above observation to every zero $\rho$ in (3.9) yields (3.8) with $\frac{1}{\pi}\epsilon + 5\epsilon^{10}$ replaced by $\frac{1}{\pi R}\epsilon$. Taking $R \to R_1$ and $r(\epsilon) = \epsilon^{10}$ yields (3.8) whether $R_1 = 1$ or $R_1 = 1 - 4r$.

To establish (3.7), we continue our argument and notice by (3.11) that we may restrict our sum over zeros from $|s - \rho| < R$ to a smaller circle within it: $|1 + it - \rho| \leq r$ for any $0 < r < R - \epsilon$. From our previous observation, we may discard zeros outside of this smaller circle. As $R > 9/10$, we have that $\delta < \epsilon < 1/4 < R - \epsilon$ so we may take $r = \delta$ for the radius of the smaller circle.

Now, from Lemma 2.3.7,

$$N_\chi(\delta, 1 + it) = \#\{\rho : |1 + it - \rho| \leq \delta\} \leq 2\log(D_K D_\chi \tau^{n_K})\delta + O(1).$$

Further, for such zeros $\rho$ satisfying $|1 + it - \rho| \leq \delta$, notice that

$$\mathrm{Re}\{s - \rho\} = \sigma - \beta \leq \epsilon + \delta \leq 2\epsilon,$$

implying

$$\sum_{|1+it-\rho|\leq\delta} \mathrm{Re}\Big\{\frac{s-\rho}{R^2}\Big\} \leq \frac{4\epsilon^2}{R^2}\log(D_K D_\chi \tau^{n_K}) + O(1), \tag{3.12}$$

as $\delta < \epsilon$. Combining (3.9), (3.10), and (3.12) and similarly taking $R \to R_1$ establishes (3.7). $\square$

Using Proposition 3.2.3, we may improve upon Lemma 2.3.7.

**Lemma 3.2.4.** *Let $\chi$ be a Hecke character and $0 < r < \epsilon < 1/4$. If $s = \sigma + it$ with $1 < \sigma < 1 + \epsilon$ and $N_\chi(r; s)$ by* (2.16)*, then*

$$N_\chi(r; s) \leq \Phi\big(2\log D_K + \log \mathrm{N}\mathfrak{f}_\chi + n_K \log(|t| + 3) + O_\epsilon(n_K)\big) \cdot r + 4 + 4\delta(\chi),$$

*where $\Phi = 1 + \frac{4}{\pi}\epsilon + 16\epsilon^2 + 340\epsilon^{10}$.*

*Proof.* Analogous to Lemma 2.3.7 except we bound $N_\chi(r; 1+it)$ instead of $N_\chi(2r; 1+r+it)$ and further, we apply Proposition 3.2.3 in place of Lemmas 2.2.1 and 2.3.3. $\square$

## 3.3 Polynomial explicit inequality

By including higher derivatives of $-\frac{L'}{L}(s,\chi)$, the goal of this section is establish a generalization of the "classical explicit inequality" based on techniques in [HB92, Section 4]. Let $\chi \pmod{H}$ be a Hecke character. For a polynomial $P(X) = \sum_{k=1}^{d} a_k X^k \in \mathbb{R}[X]$ of degree $d \geq 1$, define a real-valued function

$$\mathcal{P}(s,\chi) = \mathcal{P}(s,\chi;P) := \sum_{\mathfrak{n}\subseteq\mathcal{O}} \frac{\Lambda_K(\mathfrak{n})}{(\mathrm{N}\mathfrak{n})^{\sigma}} \Big( \sum_{k=1}^{d} a_k \frac{\big((\sigma-1)\log \mathrm{N}\mathfrak{n}\big)^{k-1}}{(k-1)!} \Big) \cdot \mathrm{Re}\Big\{ \frac{\chi(\mathfrak{n})}{(\mathrm{N}\mathfrak{n})^{it}} \Big\} \tag{3.13}$$

for $\sigma > 1$ and where $\Lambda_K(\,\cdot\,)$ is the von Mangoldt $\Lambda$-function on integral ideals of $\mathcal{O}_K$ defined by (2.5). From the classical formula

$$-\frac{L'}{L}(s,\chi) = \sum_{\mathfrak{n}\subseteq\mathcal{O}} \Lambda_K(\mathfrak{n})\chi(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{-s} \qquad \text{for } \sigma > 1,$$

it is straightforward to deduce that

$$\mathcal{P}(s,\chi) = \sum_{k=1}^{d} a_k(\sigma-1)^{k-1} \cdot \mathrm{Re}\Big\{ \frac{(-1)^k}{(k-1)!} \frac{d^{k-1}}{ds^{k-1}} \frac{L'}{L}(s,\chi) \Big\} \quad \text{for } \sigma > 1. \tag{3.14}$$

To prove an explicit inequality using $\mathcal{P}(s,\chi)$, we reduce the problem to primitive characters.

**Lemma 3.3.1.** *Assume $H$ is a primitive congruence class group. Let $\chi \pmod{H}$ be induced from the primitive character $\chi^* \pmod{\mathfrak{f}_\chi}$. Let $P(X) \in \mathbb{R}[X]$ be a polynomial with $P(0) = 0$. Then, for $\epsilon > 0$,*

$$\mathcal{P}(s,\chi) = \mathcal{P}(s,\chi^*) + O_P(\epsilon^{-1} n_K + \epsilon\mathcal{L}),$$

*uniformly in the region*

$$1 < \sigma \leq 1 + \frac{100}{\mathcal{L}}.$$

*Proof.* Since $H$ is primitive, notice $\chi \pmod{H}$ is a Hecke character modulo $\mathfrak{f}_H$. Denote $d = \deg P$. Observe that

$$\Big|\mathcal{P}(s,\chi) - \mathcal{P}(s,\chi^*)\Big| \ll_P \sum_{(\mathfrak{n},\mathfrak{f}_H)\neq 1} \frac{\Lambda_K(\mathfrak{n})}{\mathrm{N}\mathfrak{n}} \Big(\frac{\log \mathrm{N}\mathfrak{n}}{\mathcal{L}}\Big)^{d-1} \ll_P \sum_{\mathfrak{p}|\mathfrak{f}_H} \sum_{j\geq 1} \frac{\log \mathrm{N}\mathfrak{p}}{(\mathrm{N}\mathfrak{p})^j} \cdot \Big(\frac{j\log \mathrm{N}\mathfrak{p}}{\mathcal{L}}\Big)^{d-1}.$$

For $\mathfrak{p} \mid \mathfrak{f}_H$, note $\log \mathrm{N}\mathfrak{p} \leq \log \mathrm{N}\mathfrak{f}_H \leq 2\log Q \ll \mathcal{L}$ by Lemma 2.4.7 and (3.3). Thus, the above is

$$\ll_P \sum_{\mathfrak{p} \mid \mathfrak{f}_H} \sum_{j \geq 1} \frac{j^{d-1} \log \mathrm{N}\mathfrak{p}}{(\mathrm{N}\mathfrak{p})^j} \ll_P \sum_{\mathfrak{p} \mid \mathfrak{f}_H} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}}.$$

The desired result then follows from Lemma 2.4.4 and Lemma 2.4.7. □

**Proposition 3.3.2.** *Assume $H$ is a primitive congruence class group. Let $\chi \pmod{H}$ and $\epsilon > 0$ be arbitrary. Suppose the polynomial $P(X) = \sum_{k=1}^d a_k X^k$ of degree $d \geq 1$ has non-negative real coefficients. There exists $\delta = \delta(\epsilon, P) > 0$ such that*

$$\tfrac{1}{\mathcal{L}} \cdot \mathcal{P}(s,\chi) \leq \mathrm{Re}\left\{ \frac{P\left(\frac{\sigma-1}{s-1}\right)}{\sigma - 1} E_0(\chi) - \sum_{|1+it-\rho_\chi| \leq \delta} \frac{P\left(\frac{\sigma-1}{s-\rho_\chi}\right)}{\sigma-1} \right\} \cdot \frac{1}{\mathcal{L}} + a_1 \phi \frac{\mathcal{L}_\chi}{\mathcal{L}} + \epsilon \tag{3.15}$$

*uniformly in the region*

$$1 + \frac{1}{\mathcal{L}\log\mathcal{L}} \leq \sigma \leq 1 + \frac{100}{\mathcal{L}}, \qquad |t| \leq \mathcal{T},$$

*provided $\mathcal{L}$ is sufficiently large depending on $\epsilon$ and $P$.*

*Proof.* Let $\chi^* \pmod{\mathfrak{f}_\chi}$ be the primitive character inducing $\chi \pmod{H}$. From Lemma 3.3.1 and the observation that $n_K = o(\mathcal{L})$, it follows that

$$\tfrac{1}{\mathcal{L}}\mathcal{P}(s,\chi) = \tfrac{1}{\mathcal{L}}\mathcal{P}(s,\chi^*) + \epsilon$$

for $\mathcal{L}$ sufficiently large depending on $\epsilon$ and $P$. Thus, it suffices to show (3.15) with $\mathcal{P}(s,\chi^*)$ instead of $\mathcal{P}(s,\chi)$. Define

$$P_2(X) := \sum_{k=2}^d a_k X^k = P(X) - a_1 X.$$

Using Lemmas 2.2.1 and 2.3.5, we see for $k \geq 2$ and $\sigma > 1$ that

$$\begin{aligned} \frac{(-1)^k}{(k-1)!} \frac{d^{k-1}}{ds^{k-1}} \frac{L'}{L}(s,\chi^*) &= \frac{E_0(\chi)}{(s-1)^k} - \sum_{\rho_\chi} \frac{1}{(s-\rho_\chi)^k} + \frac{E_0(\chi)}{s^k} - \frac{(-1)^k}{(k-1)!} \frac{d^{k-1}}{ds^{k-1}} \frac{L'_\infty}{L_\infty}(s,\chi^*) \\ &= \frac{E_0(\chi)}{(s-1)^k} - \sum_{\rho_\chi} \frac{1}{(s-\rho_\chi)^k} + O(n_K). \end{aligned}$$

Substituting these formulae into $\mathcal{P}(s,\chi^*;P_2)$ defined via (3.14), it follows for $\sigma > 1$ that

$$\mathcal{P}(s,\chi^*;P_2) = \frac{1}{\sigma-1}\sum_{k=2}^{d} a_k \mathrm{Re}\Big\{ \Big(\frac{\sigma-1}{s-1}\Big)^k E_0(\chi) - \sum_{\rho_\chi} \Big(\frac{\sigma-1}{s-\rho_\chi}\Big)^k \Big\} + O_P(n_K). \tag{3.16}$$

Without loss, suppose $\epsilon < 1/4$ and obtain $\delta = \delta(\epsilon)$ from Proposition 3.2.3. Since $\sigma < 1 + \frac{100}{\mathcal{L}}$, we see by the zero density estimate [LMO79, Lemma 2.1] and Lemma 3.1.1 that

$$\begin{aligned}\sum_{|1+it-\rho_\chi|\geq\delta} \Big|\frac{\sigma-1}{s-\rho_\chi}\Big|^k &\ll \Big(\frac{100}{\mathcal{L}}\Big)^k \sum_{|1+it-\rho_\chi|\geq\delta} \frac{1}{|s-\rho_\chi|^k} \ll_\delta \Big(\frac{100}{\mathcal{L}}\Big)^k \sum_{\rho_\chi} \frac{1}{1+|t-\gamma_\chi|^2} \\ &\ll_\delta \Big(\frac{100}{\mathcal{L}}\Big)^k \cdot (\mathcal{L}^* + n_K \log \mathcal{T}) \\ &\ll_\delta \frac{(100)^k}{\mathcal{L}^{k-1}} \ll_{\delta,P} \frac{1}{\mathcal{L}}.\end{aligned}$$

Hence,

$$\frac{1}{\sigma-1}\sum_{k=2}^{d} a_k \mathrm{Re}\Big\{ \sum_{|1+it-\rho_\chi|\geq\delta} \Big(\frac{\sigma-1}{s-\rho_\chi}\Big)^k \Big\} \ll_{\epsilon,P} \log \mathcal{L},$$

since $\sigma > 1 + \frac{1}{\mathcal{L}\log\mathcal{L}}$ and $\delta$ depends only on $\epsilon$. Removing this contribution in (3.16) implies

$$\begin{aligned}\mathcal{P}(s,\chi^*;P_2) &= \frac{1}{\sigma-1}\sum_{k=2}^{d} a_k \mathrm{Re}\Big\{ \Big(\frac{\sigma-1}{s-1}\Big)^k E_0(\chi) - \sum_{|1+it-\rho_\chi|\leq\delta} \Big(\frac{\sigma-1}{s-\rho_\chi}\Big)^k \Big\} \\ &\quad + O_{\epsilon,P}(n_K + \log \mathcal{L}) \\ &= \mathrm{Re}\Big\{ \frac{P_2(\frac{\sigma-1}{s-1})\}}{\sigma-1} E_0(\chi) - \sum_{|1+it-\rho_\chi|\leq\delta} \frac{P_2(\frac{\sigma-1}{s-\rho_\chi})}{\sigma-1} \Big\} + O_{\epsilon,P}(n_K + \log\mathcal{L}).\end{aligned}$$

For the linear polynomial $P_1(X) := a_1 X$, we apply Proposition 3.2.3 directly to find that

$$\begin{aligned}\mathcal{P}(s,\chi^*;P_1) \leq a_1\big(\phi + 4\epsilon\big)\mathcal{L}_\chi + \mathrm{Re}\Big\{ \frac{P_1\big(\frac{\sigma-1}{s-1}\big)}{\sigma-1} E_0(\chi) - \sum_{|1+it-\rho_\chi|\leq\delta} \frac{P_1\big(\frac{\sigma-1}{s-\rho_\chi}\big)}{\sigma-1} \Big\} \\ + O_{\epsilon,P}(n_K \log \mathcal{T})\end{aligned}$$

for $\mathcal{L}$ sufficiently large depending on $\epsilon$. Finally, from (3.14), we see that $\mathcal{P}(s,\chi^*;P) = \mathcal{P}(s,\chi^*;P_1) + \mathcal{P}(s,\chi^*;P_2)$ since $P = P_1 + P_2$, so combining the above inequality with the previous equation, we conclude that

$$\mathcal{P}(s,\chi^*) \le a_1\big(\phi+4\epsilon\big)\mathcal{L}_\chi + \mathrm{Re}\Bigg\{\frac{P\big(\frac{\sigma-1}{s-1}\big)}{\sigma-1}E_0(\chi) - \sum_{|1+it-\rho_\chi|\le\delta}\frac{P\big(\frac{\sigma-1}{s-\rho_\chi}\big)}{\sigma-1}\Bigg\} + O_{\epsilon,P}(n_K\log\mathcal{T}+\log\mathcal{L}).$$

Dividing both sides by $\mathcal{L}$ and taking $\mathcal{L}$ sufficiently large depending on $\epsilon$ and $P$, the errors may be made arbitrarily small by Lemma 3.1.1. Rescaling $\epsilon$ yields the desired result. □

Proposition 3.3.2 will be utilized in many contexts but typically we want to restrict the sum over zeros $\rho$ to a few specified zeros. To do so, we impose an additional condition on $P(X)$.

**Definition 3.3.3.** A polynomial $P(X)\in\mathbb{R}_{\ge 0}[X]$ is *admissible* if $P(0)=0$ and

$$\mathrm{Re}\Big\{P\Big(\frac{1}{z}\Big)\Big\}\ge 0 \qquad \text{when } \mathrm{Re}\{z\}\ge 1.$$

Now we establish the desired general lemma.

**Lemma 3.3.4.** *Let $\epsilon>0$ and $0<\lambda<100$ be arbitrary, and let $s=\sigma+it$ with*

$$\sigma = 1+\frac{\lambda}{\mathcal{L}}, \qquad |t|\le\mathcal{T}.$$

*Let $\chi \pmod H$ be an arbitrary Hecke character and let $\mathcal{Z} := \{\tilde\rho_1,\tilde\rho_2,\dots,\tilde\rho_J\}$ be a finite multiset of zeros of $L(s,\chi)$ (called the* <u>*extracted zeros*</u>*), where*

$$\tilde\rho_j = \tilde\beta_j + i\tilde\gamma_j = \Big(1-\frac{\tilde\lambda_j}{\mathcal{L}}\Big) + i\cdot\frac{\tilde\mu_j}{\mathcal{L}}, \qquad 1\le j\le J.$$

*Suppose $P(X)=\sum_{k=1}^d a_kX^k$ is an admissible polynomial. Then*

$$\frac{\lambda}{\mathcal{L}}\cdot\mathcal{P}(s,\chi) \le \mathrm{Re}\Bigg\{E_0(\chi)P\Big(\frac{\lambda}{\lambda+i\mu}\Big) - \sum_{j=1}^J P\Big(\frac{\lambda}{\lambda+\tilde\lambda_j+i(\mu-\tilde\mu_j)}\Big)\Bigg\} + a_1\lambda\phi\tfrac{\mathcal{L}_\chi}{\mathcal{L}}+\epsilon$$

*for $\mathcal{L}$ sufficiently large depending only on $\epsilon$, $P$, and $J$.*

*Proof.* From Proposition 3.3.2 and the admissibility of $P$, it follows that

$$\tfrac{\lambda}{\mathcal{L}}\mathcal{P}(s,\chi) \le a_1\lambda\phi\tfrac{\mathcal{L}_\chi}{\mathcal{L}}+\epsilon+\mathrm{Re}\Bigg\{E_0(\chi)P\Big(\frac{\lambda}{\lambda+i\mu}\Big) - \sum_{\substack{|1+it-\rho_\chi|\le\delta\\ \rho_\chi\in\mathcal{Z}}} P\Big(\frac{\lambda}{\lambda+\lambda_\chi+i(\mu-\mu_\chi)}\Big)\Bigg\} \tag{3.17}$$

for some $\delta = \delta(\epsilon, P)$ and $\mathcal{L}$ sufficiently large depending on $\epsilon, P$ and $\lambda$. Note the admissibility of $P$ was used to restrict the sum over zeros further by throwing out $\rho_\chi \notin \mathcal{Z}$ satisfying $|1 + it - \rho_\chi| \leq \delta$. For the remaining sum, consider $\tilde{\rho}_j \in \mathcal{Z}$. If $|1+it-\tilde{\rho}_j| \geq \delta$, then $|\tilde{\mu}_j - \mu| \gg_\delta \mathcal{L}$ or $\tilde{\lambda}_j \gg_\delta \mathcal{L}$. As $P(0) = 0$, it follows that

$$\mathrm{Re}\Big\{ P\Big( \frac{\lambda}{\lambda + \tilde{\lambda}_j + i(\mu - \tilde{\mu}_j)} \Big) \Big\} \ll_{\epsilon, P} \mathcal{L}^{-1}.$$

Hence, in the sum over zeros in (3.17), we may include each extracted zero $\tilde{\rho}_j$ with error at most $O_{\epsilon,P}(\mathcal{L}^{-1})$. This implies

$$\sum_{\substack{|1+it-\rho_\chi| \leq \delta \\ \rho_\chi \in \mathcal{Z}}} \mathrm{Re}\Big\{ P\Big( \frac{\lambda}{\lambda + \lambda_\chi + i(\mu - \mu_\chi)} \Big) \Big\} = \sum_{j=1}^{J} \mathrm{Re}\Big\{ P\Big( \frac{\lambda}{\lambda + \tilde{\lambda}_j + i(\mu - \tilde{\mu}_j)} \Big) \Big\} + O_{\epsilon,P}(\mathcal{L}^{-1} J).$$

Using this estimate in (3.17) and taking $\mathcal{L}$ sufficiently large depending on $\epsilon, P$ and $J$, we have the desired result after rescaling $\epsilon$. ☐

During computations, we will employ Lemma 3.3.4 with $P(X) = P_4(X)$ as given in [HB92]. That is, for the remainder of Chapters 3 and 4, denote

$$P_4(X) := X + X^2 + \tfrac{4}{5}X^3 + \tfrac{2}{5}X^4. \tag{3.18}$$

We establish a key property of $P_4(X)$ in Lemma 3.3.6 using the following observation.

**Lemma 3.3.5.** *Let $V, W \geq 0$ be arbitrary and $m \geq 1$ be a positive integer. Define*

$$G_m(x,y,z) := V \cdot \frac{x^m}{(x^2+z^2)^m} + W \cdot \frac{y^m}{(y^2+z^2)^m} - \frac{1}{(1+z^2)^m}$$

*for $x, y, z \in \mathbb{R}$. If $x, y \geq 1$ then*

$$G_m(x,y,z) \geq 0 \quad \textit{provided} \quad \frac{V}{x^m} + \frac{W}{y^m} \geq 1.$$

*Proof.* Notice

$$G_m(x,y,z) = \frac{V/x^m}{(1+(z/x)^2)^m} + \frac{W/y^m}{(1+(z/y)^2)^m} - \frac{1}{(1+z^2)^m} \geq \Big( \frac{V}{x^m} + \frac{W}{y^m} - 1 \Big) \frac{1}{(1+z^2)^m}.$$

☐

**Lemma 3.3.6.** *The polynomial $P_4(X)$ is admissible. Additionally, if $0 < a \leq b \leq c$, $A > 0$, and $B, C \geq 0$, then*

$$\mathrm{Re}\{C \cdot P_4\big(\frac{a}{c+it}\big) + B \cdot P_4\big(\frac{a}{b+it}\big) - A \cdot P_4\big(\frac{a}{a+it}\big)\} \geq 0, \tag{3.19}$$

*provided*

$$\frac{C}{c^4} + \frac{B}{b^4} \geq \frac{A}{a^4}.$$

*Proof.* The proof that $P_4(X)$ is admissible is given in [HB92, Section 4]. It remains to prove (3.19). By direct computation, one can verify that

$$\mathrm{Re}\{P_4\big(\frac{a}{b+it}\big)\} = \frac{16}{5}\frac{(ab)^4}{(b^2+t^2)^4} + \frac{a(b-a)}{5(b^2+t^2)^3}Q(a,b,t), \tag{3.20}$$

where $Q(a,b,t) = 5t^4 + 2(5b^2 + 5ab - a^2)t^2 + b^2(5b^2 + 10ab + 14a^2)$ is clearly positive for $0 < a \leq b$ and $t \in \mathbb{R}$. Thus, for $0 < a \leq b$ and $t \in \mathbb{R}$, we have

$$\mathrm{Re}\{P_4\big(\frac{a}{b+it}\big)\} \geq \frac{16}{5}\frac{(ab)^4}{(b^2+t^2)^4}. \tag{3.21}$$

Now, consider the LHS of (3.19). Apply (3.21) to the first and second term and (3.20) to the third term. Thus, the LHS of (3.19) is

$$\geq \frac{16a^4}{5} \cdot \Big(C \cdot \frac{c^4}{(c^2+t^2)^4} + B \cdot \frac{b^4}{(b^2+t^2)^4} - A \cdot \frac{a^4}{(a^2+t^2)^4}\Big) \geq \frac{16A}{5} \cdot G_4\big(\tfrac{c}{a}, \tfrac{b}{a}, \tfrac{t}{a}\big), \tag{3.22}$$

where $G_4(x,y,z)$ is defined in Lemma 3.3.5 with $V = C/A$ and $W = B/A$. Applying Lemma 3.3.5 to $G_4\big(\tfrac{c}{a}, \tfrac{b}{a}, \tfrac{t}{a}\big)$ immediately implies (3.19) with the desired condition. □

## 3.4 Smoothed explicit inequality

We further generalize the "classical explicit inequality" (Proposition 3.2.3) to smoothly weighted versions of $-\frac{L'}{L}(s,\chi)$, similar to the well-known Weil's explicit formula. For any Hecke character $\chi \pmod{H}$ and function $f : [0,\infty) \to \mathbb{R}$ with compact support, define

$$\begin{aligned} \mathcal{W}(s,\chi;f) &:= \sum_{\mathfrak{n} \subseteq \mathcal{O}} \Lambda_K(\mathfrak{n})\chi(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{-s} f\Big(\frac{\log \mathrm{N}\mathfrak{n}}{\mathcal{L}}\Big) \qquad \text{for } \sigma > 1, \\ \mathcal{K}(s,\chi;f) &:= \mathrm{Re}\{\mathcal{W}(s,\chi;f)\}. \end{aligned} \tag{3.23}$$

We begin with the same setup as [HB92, Section 5]. Assume $f$ satisfies the following:

**Condition 1** *Let $f$ be a continuous function from $[0,\infty)$ to $\mathbb{R}$, supported in $[0,x_0)$ and bounded absolutely by $M$. Let $f$ be twice differentiable on $(0,x_0)$, with $f''$ being continuous and bounded by $B$.*

Recall that the *Laplace transform of $f$* is given by

$$F(z) := \int_0^\infty e^{-zt} f(t)dt, \qquad z \in \mathbb{C}. \tag{3.24}$$

Note $F(z)$ is entire since $f$ has compact support. For $\mathrm{Re}(z) > 0$, we have

$$F(z) = \frac{1}{z} f(0) + F_0(z), \tag{3.25}$$

where

$$|F_0(z)| \leq |z|^{-2} A(f) \tag{3.26}$$

with

$$A(f) = 3Bx_0 + \frac{2|f(0)|}{x_0}.$$

Define the *content of $f$* to be

$$\mathcal{C} = \mathcal{C}(f) := (x_0, M, B, f(0)). \tag{3.27}$$

For the purposes of generality, estimates in this section will depend only on the content of $f$. For all subsequent sections, we will ignore this distinction and allow dependence on $f$ in general. We first reduce our analysis to primitive characters and then prove the main result.

**Lemma 3.4.1.** *Assume $H$ is a primitive congruence class group. Suppose $\chi \pmod H$ is induced from $\chi^* \pmod{\mathfrak{f}_\chi}$. For $\epsilon > 0$ and $f$ satisfying Condition 1,*

$$\mathcal{W}(s,\chi;f) = \mathcal{W}(s,\chi^*;f) + O_{\mathcal{C}}\big(\epsilon^{-1} n_K + \epsilon \mathcal{L}^*\big)$$

*uniformly in the region $\sigma > 1$.*

*Proof.* Since $H$ is primitive, notice $\chi \pmod{H}$ is a Hecke character modulo $\mathfrak{f}_H$. Thus,

$$\begin{aligned}\Big|\mathcal{W}(s,\chi;f) - \mathcal{W}(s,\chi^*;f)\Big| &\leq \sum_{(\mathfrak{n},\mathfrak{f}_H)\neq 1} \frac{\Lambda_K(\mathfrak{n})}{\mathrm{N}\mathfrak{n}} |f(\mathcal{L}^{-1}\log \mathrm{N}\mathfrak{n})| \\ &\leq M \sum_{(\mathfrak{n},\mathfrak{f}_H)\neq 1} \frac{\Lambda_K(\mathfrak{n})}{\mathrm{N}\mathfrak{n}} = M \sum_{\mathfrak{p}|\mathfrak{f}_H} \sum_{j\geq 1} \frac{\log \mathrm{N}\mathfrak{p}}{(\mathrm{N}\mathfrak{p})^j} \leq 2M \sum_{\mathfrak{p}|\mathfrak{f}_H} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}}.\end{aligned}$$

The desired result then follows from Lemmas 2.4.4 and 2.4.7. □

**Proposition 3.4.2.** *Assume $H$ is a primitive congruence class group. Let $\chi \pmod{H}$ and $\epsilon > 0$ be arbitrary, and suppose $s = \sigma + it$ satisfies*

$$|\sigma - 1| \leq \frac{(\log \mathcal{L})^{1/2}}{\mathcal{L}}, \quad |t| \leq \mathcal{T}.$$

*Suppose $f$ satisfies Condition 1 and that $f(0) \geq 0$. Then there exists $\delta = \delta(\epsilon, \mathcal{C}) \in (0,1)$ depending only on $\epsilon$ and the content of $f$ such that*

$$\begin{aligned}\tfrac{1}{\mathcal{L}} \cdot \mathcal{K}(s,\chi;f) \leq E_0(\chi) \cdot \mathrm{Re}\{F((s-1)\mathcal{L})\} &- \sum_{|1+it-\rho|\leq\delta} \mathrm{Re}\{F((s-\rho)\mathcal{L})\} \\ &+ f(0)\phi\tfrac{\mathcal{L}_\chi}{\mathcal{L}} + \epsilon,\end{aligned} \tag{3.28}$$

*provided $\mathcal{L}$ is sufficiently large depending on $\epsilon$ and the content of $f$.*

*Proof.* The proof will closely follow the arguments of [HB92, Lemma 5.2]. Let $\chi^* \pmod{\mathfrak{f}_\chi}$ be the primitive character inducing $\chi$. From Lemma 3.4.1,

$$\mathcal{K}(s,\chi;f) = \mathcal{K}(s,\chi^*;f) + O_{\mathcal{C}}(\epsilon^{-1}n_K + \epsilon\mathcal{L}^*).$$

Dividing both sides by $\mathcal{L}$ and recalling $n_K = o(\mathcal{L})$, it follows that

$$\mathcal{L}^{-1}\mathcal{K}(s,\chi;f) \leq \mathcal{L}^{-1}\mathcal{K}(s,\chi^*;f) + \epsilon$$

for $\mathcal{L}$ sufficiently large depending on $\epsilon$ and the content of $f$. Thus, we may prove (3.28) with $\mathcal{K}(s,\chi^*;f)$ instead of $\mathcal{K}(s,\chi;f)$.

Let $\sigma \geq 1 + 2\mathcal{L}^{-1}$ and set $\sigma_0 := 1 + \mathcal{L}^{-1}$ so $\sigma_0 < \sigma$. Consider

$$I := \frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0+i\infty} \Big(-\frac{L'}{L}(w,\chi^*)\Big) F_0((s-w)\mathcal{L})dw. \tag{3.29}$$

Since $F_0$ satisfies (3.26) and

$$-\frac{L'}{L}(w,\chi^*) \ll |\frac{\zeta_K'}{\zeta_K}(\sigma_0)| \ll n_K(\sigma_0-1)^{-1}$$

by Lemma 2.3.1, the integral converges absolutely. Hence, we may compute $I$ by interchanging the summation and integration, and calculating the integral against $(\mathrm{N}\mathfrak{n})^{-w}$ term-wise. That is to say,

$$I = \sum_{\mathfrak{n}\subseteq\mathcal{O}} \Lambda(\mathfrak{n})\chi^*(\mathfrak{n})\Big(\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} (\mathrm{N}\mathfrak{n})^{-w}F_0((s-w)\mathcal{L})dw\Big). \tag{3.30}$$

Arguing as in [HB92, Section 5, p.21] and using Lebesgue's Dominated Convergence Theorem, one can verify that

$$\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} (\mathrm{N}\mathfrak{n})^{-w}F_0((s-w)\mathcal{L})dw = \frac{(\mathrm{N}\mathfrak{n})^{-s}}{\mathcal{L}}\cdot\big(f(\mathcal{L}^{-1}\log \mathrm{N}\mathfrak{n}) - f(0)\big),$$

since $f$ satisfies Condition 1. Substituting this result into (3.30), we see that

$$I = \frac{1}{\mathcal{L}}\Big(\mathcal{W}(s,\chi^*;f) + \frac{L'}{L}(s,\chi^*)f(0)\Big). \tag{3.31}$$

Returning to (3.29), we shift the line of integration from $(\sigma_0 \pm \infty)$ to $(-\frac{1}{2}\pm\infty)$ yielding

$$\begin{aligned} I &= E_0(\chi)F_0((s-1)\mathcal{L}) - \sum_{\rho} F_0((s-\rho)\mathcal{L}) \\ &\quad - r(\chi)F_0(s\mathcal{L}) + \frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\Big(-\frac{L'}{L}(w,\chi^*)\Big)F_0((s-w)\mathcal{L})dw, \end{aligned} \tag{3.32}$$

where the sum is over the non-trivial zeros of $L(w,\chi)$ and $r(\chi)\geq 0$ is the order of the trivial zero $w=0$ of $L(w,\chi^*)$. From (2.5) and (2.8), notice $r(\chi)\leq n_K$ so by (3.26),

$$r(\chi)|F_0(s\mathcal{L})| \ll \frac{n_K A(f)}{|s\mathcal{L}|^2} \ll \frac{n_K A(f)}{\mathcal{L}^2} \ll \frac{A(f)}{\mathcal{L}}.$$

To bound the remaining integral in (3.32), we apply the functional equation (2.7) of $L(w,\chi^*)$ and Lemma 2.3.4. Namely, using Lemma 2.3.1, we note for $\mathrm{Re}\{w\} = -1/2$ that

$$-\frac{L'}{L}(w,\chi^*) = \mathcal{L}_\chi^* + \frac{L'}{L}(1-w,\overline{\chi^*}) + O(n_K\log(2+|w|)) = \mathcal{L}_\chi^* + O(n_K\log(2+|w|)).$$

From (3.26), we therefore find that

$$\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\Big(-\frac{L'}{L}(w,\chi^*)\Big)F_0((s-w)\mathcal{L})dw$$
$$=\frac{\mathcal{L}_\chi^*}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}F_0((s-w)\mathcal{L})dw+O\Big(\frac{A(f)}{\mathcal{L}^2}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\frac{n_K\cdot\log(2+|w|)}{|s-w|^2}dw\Big).$$

Since $F_0$ is entire and satisfies (3.26), we may pull the line of integration in the first integral as far left as we desire, concluding that the first integral vanishes. One can readily verify that integral in the error term is

$$\ll\frac{A(f)}{\mathcal{L}^2}\cdot n_K\log(2+|s|)\ll\frac{A(f)n_K\log\mathcal{T}}{\mathcal{L}^2}\ll\frac{A(f)}{\mathcal{L}}$$

by Lemma 3.1.1. Combining these bounds into (3.32) and comparing with (3.31) yields

$$\tfrac{1}{\mathcal{L}}\cdot\mathcal{W}(s,\chi^*;f)=-\frac{L'}{L}(s,\chi^*)f(0)\tfrac{1}{\mathcal{L}}+E_0(\chi)\cdot F_0((s-1)\mathcal{L})-\sum_\rho F_0((s-\rho)\mathcal{L})+O\Big(\frac{A(f)}{\mathcal{L}}\Big).\tag{3.33}$$

We wish to apply Proposition 3.2.3 giving $\delta=\delta(\epsilon)$, but we must discard zeros in the above sum where $|1+it-\rho|\geq\delta$. By (3.26), [LMO79, Lemma 2.1], and Lemma 3.1.1, this discard induces an error

$$\ll\sum_{|1+it-\rho|\geq\delta}\frac{A(f)}{\mathcal{L}^2|s-\rho|^2}\ll_\delta\frac{A(f)}{\mathcal{L}^2}\sum_\rho\frac{1}{1+|t-\gamma|^2}\ll_\delta\frac{A(f)}{\mathcal{L}^2}(\mathcal{L}^*+n_K\log\mathcal{T})\ll_\delta\frac{A(f)}{\mathcal{L}}.$$

Hence, taking real parts of (3.33), applying Proposition 3.2.3, and using (3.25), we find

$$\frac{\mathcal{K}(s,\chi^*;f)}{\mathcal{L}}\leq E_0(\chi)\mathrm{Re}\Big\{F((s-1)\mathcal{L})-\sum_{|1+it-\rho|<\delta}F((s-\rho)\mathcal{L})\Big\}+f(0)\big(\phi+\epsilon\big)\tfrac{\mathcal{L}_\chi}{\mathcal{L}}+O_{\delta,\mathcal{C}}\big(\mathcal{L}^{-1}\big).$$

Taking $\mathcal{L}$ sufficiently large depending on $\epsilon$ and the content of $f$, the error term may be made arbitrarily small. Upon choosing a new $\epsilon$, we have established (3.28) in the range

$$1+2\mathcal{L}^{-1}\leq\sigma\leq1+(\log\mathcal{L})^{1/2}\mathcal{L}^{-1}.$$

Similar to the discussion in [HB92, Section 5, p.22-23], one may show (3.28) holds in the desired extended range by considering $g(t)=e^{\alpha t}f(t)$ for $0\leq\alpha\leq(\log\mathcal{L})/3x_0$. $\square$

In analogy with Proposition 3.3.2 and Lemma 3.3.4, we would like to use Proposition 3.4.2 by restricting the sum over zeros $\rho$ to just a few specified zeros. To do so, we require our weight

$f$ to satisfy an additional condition which was introduced in [HB92, Section 6].

**Condition 2** *The function $f$ is non-negative. Moreover, its Laplace transform $F$ satisfies*

$$\mathrm{Re}\{F(z)\} \geq 0 \quad \textit{for } \mathrm{Re}(z) \geq 0.$$

Condition 2 implies that, viewed as a real-variable function of $t \in \mathbb{R}$, $F(t)$ is a positive decreasing real-valued function. We may now give a more convenient version of Proposition 3.4.2 in the following lemma.

**Lemma 3.4.3.** *Let $\epsilon \in (0,1)$ be arbitrary, and let $s = \sigma + it$ with*

$$|\sigma - 1| \leq \frac{\log\log \mathcal{T}}{C_0 \mathcal{L}}, \qquad |t| \leq 5T_0,$$

*where constants $C_0 > 0$ and $T_0 \geq 1$ come from Lemma 3.1.2. Write $\sigma = 1 - \lambda/\mathcal{L}$ and $t = \mu/\mathcal{L}$.*

*Let $\chi \pmod{H}$ be an arbitrary Hecke character and let $\mathcal{Z} := \{\tilde{\rho}_1, \tilde{\rho}_2, \ldots, \tilde{\rho}_J\}$ be a finite, possibly empty, multiset of zeros of $L(s,\chi)$ (called the extracted zeros) containing the multiset*

$$\{\rho_\chi : \sigma < \beta_\chi \leq 1, \quad |\gamma_\chi| \leq T_0\}.$$

*Write $\tilde{\rho}_j = \tilde{\beta}_j + i\tilde{\gamma}_j = \Big(1 - \dfrac{\tilde{\lambda}_j}{\mathcal{L}}\Big) + i \cdot \dfrac{\tilde{\mu}_j}{\mathcal{L}}$ for $1 \leq j \leq J$ and suppose $f$ satisfies Conditions 1 and 2. Then*

$$\mathcal{L}^{-1} \cdot \mathcal{K}(s,\chi;f) \leq E_0(\chi) \cdot \mathrm{Re}\{F(-\lambda + i\mu)\} - \sum_{j=1}^{J} \mathrm{Re}\{F(\tilde{\lambda}_j - \lambda - i(\tilde{\mu}_j - \mu))\} + f(0)\phi\frac{\mathcal{L}_\chi}{\mathcal{L}} + \epsilon$$

*for $\mathcal{L}$ sufficiently large depending only on $\epsilon$, $J$, and the content of $f$.*

**Remark.** The dependence of "sufficiently large" on $J$ is insignificant for our purposes, as we will employ the lemma with $0 \leq J \leq 10$ in all of our applications.

*Proof.* From Proposition 3.4.2, it follows that

$$\begin{aligned} \frac{\mathcal{K}(s,\chi;f)}{\mathcal{L}} \leq\ & f(0)\big(\phi\tfrac{\mathcal{L}_\chi}{\mathcal{L}} + \epsilon\big) + E_0(\chi) \cdot \mathrm{Re}\{F(-\lambda + i\mu)\} \\ & - \sum_{|1+it-\rho| \leq \delta} \mathrm{Re}\{F((s-\rho)\mathcal{L})\} \end{aligned} \tag{3.34}$$

for some $\delta = \delta(\epsilon, \mathcal{C})$. We consider the sum over zeros depending on whether $\rho \in \mathcal{Z}$ or not. For any $\rho = \tilde{\rho}_j \in \mathcal{Z}$, if $|1 + it - \tilde{\rho}_j| \geq \delta$, then $|\tilde{\mu}_j - \mu| \gg_\delta \mathcal{L}$ or $\tilde{\lambda}_j \gg_\delta \mathcal{L}$. From (3.25) and (3.26), it follows that

$$\mathrm{Re}\{F((s-\tilde{\rho}_j)\mathcal{L})\} \ll_{\epsilon,\mathcal{C}} \mathcal{L}^{-1}.$$

implying

$$\sum_{\substack{|1+it-\rho|\leq\delta \\ \rho\in\mathcal{Z}}} \mathrm{Re}\{F((s-\rho)\mathcal{L})\} = \sum_{j=1}^{J} \mathrm{Re}\{F(\tilde{\lambda}_j - \lambda - i(\tilde{\mu}_j - \mu))\} + O_{\epsilon,\mathcal{C}}(J\mathcal{L}^{-1}). \tag{3.35}$$

Next, for all zeros $\rho = \beta + i\gamma \notin \mathcal{Z}$ satisfying $|1+it-\rho| \leq \delta$, we claim $\beta \leq \sigma$. Assuming the claim, it follows by Condition 2 that

$$\sum_{\substack{|1+it-\rho|\leq\delta \\ \rho\notin\mathcal{Z}}} \mathrm{Re}\{F((s-\rho)\mathcal{L})\} \geq 0. \tag{3.36}$$

To see the claim, assume for a contradiction that $\sigma \leq \beta \leq 1$ for some zero $\rho = \beta + i\gamma$ occurring in (3.36). As $|1+it-\rho| \leq \delta$, it follows that

$$|\gamma| \leq |t| + \delta \leq 5T_0 + 1 \leq 6T_0.$$

Hence, from Lemma 3.1.2, either

$$|\gamma| \leq T_0 \qquad \text{or} \qquad \beta \leq 1 - \frac{\log\log\mathcal{T}}{C_0\mathcal{L}}.$$

In the latter case, it follows $\beta \leq \sigma$ which is a contradiction, so it must be that $|\gamma| \leq T_0$ and $\sigma \leq \beta \leq 1$. By the assumptions of the lemma, it follows $\rho \in \mathcal{Z}$, which is also a contradiction. This proves the claim.

Therefore, combining (3.35) and (3.36), we may conclude that

$$-\sum_{|1+it-\rho|\leq\delta} \mathrm{Re}\{F((s-\rho)\mathcal{L})\} \leq -\sum_{j=1}^{J} \mathrm{Re}\{F(\tilde{\lambda}_j - \lambda - i(\tilde{\mu}_j - \mu))\} + O_{\epsilon,\mathcal{C}}(J\mathcal{L}^{-1}).$$

Using this bound in (3.34) and taking $\mathcal{L}$ sufficiently large depending on $\epsilon, \mathcal{C}$ and $J$, we have the desired result upon choosing a new $\epsilon$. □

We also record a lemma useful for applications of Lemma 3.4.3 in Sections 4.2 and 4.3.

**Lemma 3.4.4.** *Suppose $f$ satisfies Conditions 1 and 2. For $a, b \geq 0$ and $y \in \mathbb{R}$, we have that*

$$\operatorname{Re}\{F(-a+iy) - F(iy) - F(b-a+iy)\} \leq \begin{cases} F(-a) - F(0) & \text{if } b \geq a, \\ F(-a) - F(b-a) & \text{if } b \leq a. \end{cases}$$

*Proof.* If $b \geq a$, then by Condition 2, $\operatorname{Re}\{F(b-a+iy)\} \geq 0$ so the LHS of the desired inequality is

$$\begin{aligned} \leq \operatorname{Re}\{F(-a+iy) - F(iy)\} &= \int_0^\infty f(t)(e^{at}-1)\cos(yt)dt \\ &\leq \int_0^\infty f(t)(e^{at}-1)dt \\ &= F(-a) - F(0) \end{aligned}$$

since $f(t) \geq 0$ and $a \geq 0$. A similar argument holds for $b \leq a$, except we exclude $\operatorname{Re}\{F(iy)\}$ by positivity in this case. $\square$

# Chapter 4

# Zero-free regions and zero repulsion

*"You can't cross the sea merely by standing and staring at the water."*

– Rabindranath Tagore.

Let $H$ be a congruence class group for a number field $K$. We will retain the notation of Chapters 2 and 3 and, in particular, the definitions in Section 3.1. This chapter consists of various results about the distribution of zeros for the product of Hecke $L$-functions given by

$$\prod_{\chi \,(\mathrm{mod}\, H)} L(s, \chi).$$

We will establish explicit zero-free regions and zero repulsion results by exploiting the explicit inequalities of Chapter 3. Our chief inspiration will continue to be Heath-Brown's paper [HB92] on Dirichlet $L$-functions.

We will postpone establishing the full form of two of the three key principles of Linnik: the Deuring–Heilbronn phenomenon and the log-free zero density estimate. These two principles require vastly different approaches than the methods employed in this chapter. While somewhat deviating from the literature, we make a subtle non-technical distinction between "zero repulsion" and "Deuring–Heilbronn phenomenon". For us, the former occurs when a zero (real or complex) close to $\mathrm{Re}\{s\} = 1$ repels other zeros but not far into the critical strip; the latter occurs when a simple real zero very close to $s = 1$ repels other zeros deep into the critical strip. Therefore, in Chapter 4, we will discuss zero repulsion but not Deuring–Heilbronn phenomenon.

## 4.1 Statement of results

Let us now state the main theorems of this chapter. Recall $K$ is an arbitrary number field, $H$ is an arbitrary congruence class group of $K$, and $Q = Q_H$ is given by (2.2). Throughout this chapter, $T_\star \geq 1$ is a *fixed* positive real number and $\nu(x)$ is *any fixed* increasing real-variable function $\geq 4$ such that $\nu(x) \gg \log(x+4)$. Thus, any implied constants (e.g. coming from a "sufficiently large" condition) depend implicitly on $T_\star$ and $\nu$.

**Theorem 4.1.1.** *Let $H$ be a congruence class group of a number field $K$. Suppose $D_K Q n_K^{n_K}$ is sufficiently large and let $r \geq 1$ be an integer. Then the function $\prod_{\substack{\chi \,(\mathrm{mod}\, H) \\ \mathrm{ord}\,\chi \geq r}} L(s,\chi)$ has at most 1 zero, counting with multiplicity, in the rectangle*

$$\sigma \geq 1 - \frac{c_0}{\log D_K + \frac{3}{4}\log Q + n_K \cdot \nu(n_K)}, \qquad |t| \leq T_\star,$$

*where $s = \sigma + it$ and*

$$c_0 = \begin{cases} 0.1764 & \text{if } r \geq 6, \\ 0.1489 & \text{if } r = 5, \\ 0.1227 & \text{if } r = 2,3,4, \\ 0.0875 & \text{if } r = 1. \end{cases}$$

*Further, if this exceptional zero $\rho_1$ exists, then it and its associated character $\chi_1$ are both real.*

**Remark.** Here $\mathrm{ord}\,\chi$ is the multiplicative order of $\chi$.

As mentioned in Chapter 1, some explicit results have been shown by Kadiri [Kad12] and Ahn and Kwon [AK14]. However, these results are for a single Hecke $L$-function $L(s,\chi)$ instead of $\prod_{\chi \,(\mathrm{mod}\, H)} L(s,\chi)$. Further, those zero-free regions are of the form

$$\sigma \geq 1 - \frac{\tilde{c}_0}{\log D_K + \log Q}, \qquad |t| \leq 0.13. \tag{4.1}$$

Note that the dependence on the degree $n_K$ is "absorbed" by $\log D_K$. It has been shown that

$L(s,\chi)$ is zero-free (except possibly for one real zero when $\chi$ is real) in the rectangle (4.1) for

$$\tilde{c}_0 = \begin{cases} 0.1149 & \text{if } \operatorname{ord}\chi \geq 5 \text{ [AK14]}, \\ 0.1004 & \text{if } \operatorname{ord}\chi = 4 \text{ [AK14]}, \\ 0.0662 & \text{if } \operatorname{ord}\chi = 3 \text{ [AK14]}, \\ 0.0392 & \text{if } \operatorname{ord}\chi = 2 \text{ and } D_K \text{ is sufficiently large [Kad12]}^1, \\ 0.0784 & \text{if } \operatorname{ord}\chi = 1 \text{ and } D_K \text{ is sufficiently large [Kad12]}. \end{cases}$$

Note that the results of [Kad12] also allow for $|t| \leq 1$ to be used in (4.1). Comparing the above known values for $\tilde{c}_0$ with $c_0$ in Theorem 4.1.1, if a given family of number fields $K$ satisfies

$$n_K = o\Big(\frac{\log D_K}{\log\log D_K}\Big), \tag{4.2}$$

then, for a suitable choice of $\nu(x)$, Theorem 4.1.1 is superior to all previously known cases, especially in the $Q$-aspect. A classical theorem of Minkowski states, for any number field $K$,

$$n_K = O(\log D_K)$$

so, unless $n_K$ is unusually large, one would expect that (4.2) typically holds.

We also establish a result, similar to those of [Gra81] and [HB92] for Dirichlet $L$-functions, giving a larger zero-free region but allowing more zeros.

**Theorem 4.1.2.** *Let $H$ be a congruence class group of a number field $K$. Suppose $D_K Q n_K^{n_K}$ is sufficiently large. Then $\prod_{\chi \,(\mathrm{mod}\, H)} L(s,\chi)$ has at most 2 zeros, counting with multiplicity, in the rectangle*

$$\sigma \geq 1 - \frac{0.2866}{\log D_K + \frac{3}{4}\log Q + n_K \cdot \nu(n_K)} \qquad |t| \leq T_\star.$$

*Moreover, the Dedekind zeta function $\zeta_K(s)$ has at most 2 zeros, counting with multiplicity, in the rectangle*

$$\sigma \geq 1 - \frac{0.2909}{\log D_K + n_K \cdot \nu(n_K)} \qquad |t| \leq T_\star.$$

When an exceptional zero $\rho_1$ from Theorem 4.1.1 exists, we prove an explicit version of strong zero repulsion.

**Theorem 4.1.3.** *Let $H$ be a congruence class group of a number field $K$. Suppose $\chi_1 \,(\mathrm{mod}\, H)$*

[1] This case is not explicitly written in the cited paper but is directly implied by the case $\operatorname{ord}\chi = 1$.

*is a real character and*

$$\beta_1 = 1 - \frac{\lambda_1}{\log D_K + \frac{3}{4}\log Q + n_K \cdot \nu(n_K)}$$

*is a real zero of* $L(s,\chi_1)$ *with* $\lambda_1 > 0$*. Then, provided* $D_K Q n_K^{n_K}$ *is sufficiently large (depending on* $R \geq 1$ *and possibly* $\epsilon > 0$*), the function* $\prod_{\chi \pmod H} L(s,\chi)$ *has only the one zero* $\beta_1$*, counting with multiplicity, in the rectangle*

$$\sigma \geq 1 - \frac{\min\{c_1 \log(1/\lambda_1), R\}}{\log D_K + \frac{3}{4}\log Q + n_K \cdot \nu(n_K)}, \qquad |t| \leq T_\star,$$

*where* $s = \sigma + it$ *and*

$$c_1 = \begin{cases} \frac{1}{2} - \epsilon & \text{if } \chi_1 \text{ is quadratic and } \lambda_1 \leq 10^{-10}, \\ 0.2103 & \text{if } \chi_1 \text{ is quadratic and } \lambda_1 \leq 0.1227, \\ 1 - \epsilon & \text{if } \chi_1 \text{ is principal and } \lambda_1 \leq 10^{-5}, \\ 0.7399 & \text{if } \chi_1 \text{ is principal and } \lambda_1 \leq 0.0875. \end{cases}$$

Kadiri and Ng [KN12] have established an explicit version of strong zero repulsion for zeros of the Dedekind zeta function $\zeta_K(s)$ with

$$c_1 = \begin{cases} 0.9045 & \text{if } \lambda_1 \leq 10^{-6}, \\ 0.6546 & \text{if } \lambda_1 \leq 0.0784. \end{cases}$$

Hence, Theorem 4.1.3 improves upon their result when (4.2) holds and when the primary term $c_1 \log(1/\lambda_1)$ dominates, as normally is the case in applications.

We also establish some explicit numerical bounds related to the zero density of Hecke $L$-functions. While Chapter 5 is dedicated to the log-free zero density estimate in its complete form, we have chosen to include these numerical bounds in Chapter 4 since the techniques employed are similar to the other theorems herein. For $\lambda > 0$ and $T_\star \geq 1$, define

$$N(\lambda) = N(\lambda; T_\star) = \#\{\chi \pmod H : \chi \neq \chi_0, L(s,\chi) \text{ has a zero in the region } \mathcal{S}(\lambda)\ \},$$

where

$$\mathcal{S}(\lambda) = \mathcal{S}(\lambda; T_\star) = \Big\{ s \in \mathbb{C} : \sigma \geq 1 - \frac{\lambda}{\log D_K + \frac{3}{4}\log Q + n_K \cdot \nu(n_K)},\ |t| \leq T_\star \Big\}. \qquad (4.3)$$

In the classical case $K = \mathbb{Q}$, $H = P_{\mathfrak{q}}$, and $\mathfrak{q} = (q)$, this quantity has been analyzed by [Gra81, HB92] for a slowly growing range ($\lambda \ll \log\log\log q$) and by [HB92] for a bounded range ($\lambda \leq 2$). We establish a result in the same vein as the latter. To do so, we require some technical assumptions.

Let $0 < \lambda \leq 2$ be given. Let $f \in C_c^2([0,\infty))$ have Laplace transform $F(z) = \int_0^\infty f(t)e^{-zt}dt$. Suppose $f$ satisfies all of the following:

$$\begin{aligned} &f(t) \geq 0 \text{ for } t \geq 0; && \mathrm{Re}\{F(z)\} \geq 0 \text{ for } \mathrm{Re}\{z\} \geq 0; \\ &F(\lambda) > \tfrac{1}{3}f(0); && \Big(F(\lambda) - \tfrac{1}{3}f(0)\Big)^2 > \tfrac{1}{3}f(0)\Big(\tfrac{1}{4}f(0) + F(0)\Big). \end{aligned} \tag{4.4}$$

We therefore have the following result.

**Theorem 4.1.4.** *Let $\epsilon > 0$ and $0 < \lambda \leq 2$. Suppose $f \in C_c^2([0,\infty))$ satisfies* (4.4). *Then unconditionally,*

$$N(\lambda) \leq \frac{\Big(\frac{1}{4}f(0) + F(0)\Big)\Big(F(0) - \frac{1}{12}f(0)\Big)}{\Big(F(\lambda) - \frac{1}{3}f(0)\Big)^2 - \frac{1}{3}f(0)\Big(\frac{1}{4}f(0) + F(0)\Big)} + \epsilon$$

*for $D_K Q n_K^{n_K}$ sufficiently large depending on $\epsilon$ and $f$.*

**Remark.** Let $\rho_1$ be a certain zero of a Hecke $L$-function $L(s,\chi_1)$ with the property that $\mathrm{Re}\{\rho_1\} \geq \mathrm{Re}\{\rho_\chi\}$ for any character $\chi \pmod{H}$ with a zero $\rho_\chi$ in the rectangle $\mathcal{S}(\lambda)$ given by (4.3). By introducing dependence on $\rho_1$, the bound on $N(\lambda)$ in Theorem 4.1.4 can be improved. See Section 3.1 for the choice of $\rho_1$ and Theorem 4.5.1 for further details.

Theorem 4.1.4 and its proof are inspired by [HB92, Section 12] and so similarly, the obtained bounds are non-trivial only for a narrow range of $\lambda$. By choosing $f$ roughly optimally, we exhibit a table of bounds derived from Theorem 4.1.4 below.

| $\lambda$ | .100 | .125 | .150 | .175 | .200 | .225 | .250 | .275 | .300 | .325 | .350 | .375 | .400 | .425 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $N(\lambda)$ | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 6 | 7 | 9 | 11 | 15 | 22 | 46 |

One can see that the estimates obtained are comparable to Theorems 4.1.1 and 4.1.2 which respectively imply that $N(0.1227) \leq 1$ and $N(0.2866) \leq 2$.

In the classical case $K = \mathbb{Q}$, Heath-Brown substantially improved upon all preceding work for zeros of Dirichlet $L$-functions. For general number fields $K$, we have taken advantage of the innovations founded in [HB92] to improve on the existing aforementioned results and also to establish new explicit estimates. As such, the general structure of this chapter is reminiscent of his work and is subject to small improvements similar to those suggested in [HB92, Section 16]. Xylouris implemented a number of those suggestions in [Xyl11a] so, in principle, one could refine the results here by the same methods.

Finally, we emphasize that, throughout Chapter 4, we will use the notation established in Chapter 2 and Section 3.1. The identified zeros in Section 3.1 will play an especially key role. The results of Chapter 3 form the technical crux of all proofs of this chapter.

For the reader who wishes to proceed quickly to the proofs of the theorems:

- Theorem 4.1.1 is proved in Section 4.4.
- Theorem 4.1.2 is an immediate corollary of Propositions 4.2.7, 4.2.13, 4.3.4 and 4.3.10.
- Theorem 4.1.3 is an immediate corollary of Propositions 4.2.7 and 4.2.13.
- Theorem 4.1.4 is a special case of Theorem 4.5.1.

## 4.2 Zero repulsion: $\chi_1$ and $\rho_1$ are real

Recall the indexing of zeros from Section 3.1. Throughout this section, we assume $\chi_1$ and $\rho_1$ are real. We wish to quantify the zero repulsion of $\rho_1$ with $\rho'$ and $\rho_2$ using the results of Sections 3.3 and 3.4 along with various trigonometric identities analogous to the classical one: $3 + 4\cos\theta + \cos 2\theta \geq 0$. We emphasize that $\chi_1$ can be quadratic or possibly principal.

We will primarily use the smoothed explicit inequality (Lemma 3.4.3) so we assume that the weight function $f$ continues to satisfy Conditions 1 and 2. For simplicity, henceforth denote $\mathcal{K}(s,\chi) = \mathcal{K}(s,\chi;f)$, which is given by (3.23). Suppose characters $\chi, \chi_*$ have zeros $\rho, \rho_*$ respectively. Our starting point is the trigonometric identity

$$0 \leq \chi_0(\mathfrak{n})\big(1 + \mathrm{Re}\{\chi(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{i\gamma}\}\big)\big(1 + \mathrm{Re}\{\chi_*(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{i\gamma_*}\}\big).$$

Multiplying by $\Lambda(\mathfrak{n})f(\mathcal{L}^{-1}\log \mathrm{N}\mathfrak{n})(\mathrm{N}\mathfrak{n})^{-\sigma}$ and summing over $\mathfrak{n}$, it follows that

$$\begin{aligned} 0 \leq\ & \mathcal{K}(\sigma,\chi_0) + \mathcal{K}(\sigma+i\gamma,\chi) + \mathcal{K}(\sigma+i\gamma_*,\chi_*) \\ & + \frac{1}{2}\mathcal{K}(\sigma+i\gamma+i\gamma_*,\chi\chi_*) + \frac{1}{2}\mathcal{K}(\sigma+i\gamma-i\gamma_*,\chi\overline{\chi_*}) \end{aligned} \qquad \text{for } \sigma > 0. \tag{4.5}$$

In some cases, we will use a simpler trigonometric identity:

$$0 \leq \chi_0(\mathfrak{n}) + \mathrm{Re}\{\chi(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{i\gamma}\},$$

which similarly yields

$$0 \leq \mathcal{K}(\sigma, \chi_0) + \mathcal{K}(\sigma + i\gamma, \chi) \qquad \text{for } \sigma > 0. \tag{4.6}$$

### 4.2.1 Bounds for $\lambda'$

Recall $\rho_1, \lambda_1, \rho'$ and $\lambda'$ are defined in Section 3.1.1. We establish zero repulsion results for $\rho'$ in terms of $\rho_1$, using different methods depending on various ranges of $\lambda_1$. In this subsection, we intentionally include more details to proofs but in later subsections we shall omit these extra explanations as the arguments will be similar to those found here.

**Lemma 4.2.1.** *Assume $\chi_1$ and $\rho_1$ are real. Let $\epsilon > 0$ and suppose $f$ satisfies Conditions 1 and 2. Provided $\mathcal{L}$ is sufficiently large depending on $\epsilon$ and $f$, the following holds:*

*(a) If $\chi_1$ is quadratic and $\lambda' \leq \lambda_2$, then with $\psi = 4\phi$ it follows that*

$$\begin{aligned} 0 \leq \quad & F(-\lambda') - F(0) - F(\lambda_1 - \lambda') + f(0)\psi + \epsilon \\ & + \mathrm{Re}\{F(-\lambda' + i\mu') - F(i\mu') - F(\lambda_1 - \lambda' + i\mu')\}. \end{aligned}$$

*(b) If $\chi_1$ is principal, then with $\psi = 2\phi$ it follows that*

$$\begin{aligned} 0 \leq \quad & F(-\lambda') - F(0) - F(\lambda_1 - \lambda') + f(0)\psi + \epsilon \\ & + \mathrm{Re}\{F(-\lambda' + i\mu') - F(i\mu') - F(\lambda_1 - \lambda' + i\mu')\}. \end{aligned}$$

*Proof.* (a) In (4.5), choose $\chi = \chi_* = \chi_1, \rho = \rho'$ and $\rho_* = \rho_1$ with $\sigma = \beta'$ in (4.5) giving

$$0 \leq \mathcal{K}(\beta', \chi_0) + \mathcal{K}(\beta' + i\gamma', \chi_1) + \mathcal{K}(\beta', \chi_1) + \mathcal{K}(\beta' + i\gamma', \chi_0). \tag{4.7}$$

Apply Lemma 3.4.3 to each $\mathcal{K}(*, *)$ term and extract the relevant zeros as follows:

- For $\mathcal{K}(\beta', \chi_0)$ and $\mathcal{K}(\beta' + i\gamma', \chi_0)$, extract no zeros since by assumption $\lambda' \leq \lambda_2$ yielding

$$\begin{aligned} \mathcal{L}^{-1}\mathcal{K}(\beta', \chi_0) &\leq f(0)\phi\tfrac{\mathcal{L}_0}{\mathcal{L}} + F(-\lambda') + \epsilon, \\ \mathcal{L}^{-1}\mathcal{K}(\beta' + i\gamma', \chi_0) &\leq f(0)\phi\tfrac{\mathcal{L}_0}{\mathcal{L}} + \mathrm{Re}\{F(-\lambda' + i\mu')\} + \epsilon. \end{aligned} \tag{4.8}$$

- For $\mathcal{K}(\beta' + i\gamma', \chi_1)$ and $\mathcal{K}(\beta', \chi_1)$, extract $\{\rho_1, \rho'\}$ implying

$$\begin{aligned} \mathcal{L}^{-1}\mathcal{K}(\beta' + i\gamma', \chi_1) &\leq f(0)\phi\tfrac{\mathcal{L}_{\chi_1}}{\mathcal{L}} - F(0) - \mathrm{Re}\{F(\lambda_1 - \lambda' + i\mu')\} + \epsilon, \\ \mathcal{L}^{-1}\mathcal{K}(\beta', \chi_1) &\leq f(0)\phi\tfrac{\mathcal{L}_{\chi_1}}{\mathcal{L}} - F(\lambda_1 - \lambda') - \mathrm{Re}\{F(i\mu')\} + \epsilon. \end{aligned} \tag{4.9}$$

Using (4.8) and (4.9) in (4.7) and rescaling $\epsilon$, the desired inequality follows except with $\psi = \phi \cdot \frac{2\mathcal{L}_0 + 2\mathcal{L}_{\chi_1}}{\mathcal{L}}$. From Lemma 3.1.1, $\psi \leq 4\phi$ so we may use $\psi = 4\phi$ instead.
(b) Use (4.6) with $\chi = \chi_0, \sigma = \beta'$ and $\rho = \rho'$, from which we deduce

$$0 \leq \mathcal{K}(\beta', \chi_0) + \mathcal{K}(\beta' + i\gamma', \chi_0) \quad \text{for } \sigma > 0.$$

Similar to (a), for both $\mathcal{K}(*, \chi_0)$ terms, apply Lemma 3.4.3 extracting both zeros $\{\rho_1, \rho'\}$ yielding

$$\mathcal{L}^{-1}\mathcal{K}(\beta', \chi_0) \leq f(0)\phi\tfrac{\mathcal{L}_0}{\mathcal{L}} + F(-\lambda') - F(\lambda_1 - \lambda') - \mathrm{Re}\{F(i\mu')\} + \epsilon$$
$$\mathcal{L}^{-1}\mathcal{K}(\beta' + i\gamma', \chi_0) \leq f(0)\phi\tfrac{\mathcal{L}_0}{\mathcal{L}} + \mathrm{Re}\{F(-\lambda' + i\mu') - F(\lambda_1 - \lambda' + i\mu')\} - F(0) + \epsilon$$

Combined with the previous inequality, this yields the desired result with $\psi = 2\phi \cdot \frac{\mathcal{L}_0}{\mathcal{L}}$. By Lemma 3.1.1, we may use $\psi = 2\phi$ instead. □

### $\lambda_1$ very small

We now obtain a preliminary version of the strong zero repulsion for zeros of $L(s, \chi_1)$.

**Lemma 4.2.2.** *Assume $\chi_1$ and $\rho_1$ are real. Let $\epsilon > 0$ and suppose $\mathcal{L}$ is sufficiently large depending on $\epsilon$.*

*(a) If $\chi_1$ is quadratic and $\lambda' \leq \lambda_2$, then either $\lambda' < 4e$ or*

$$\lambda' \geq \Big(\frac{1}{2} - \epsilon\Big) \log(\lambda_1^{-1}),$$

*which is non-trivial for $\lambda_1 \leq 3.5 \times 10^{-10}$.*

*(b) If $\chi_1$ is principal, then either $\lambda' < 4e$ or*

$$\lambda' \geq \big(1 - \epsilon\big) \log(\lambda_1^{-1}),$$

*which is non-trivial for $\lambda_1 \leq 1.8 \times 10^{-5}$.*

*Proof.* The proof is a close adaptation of [HB92, p. 37]. From Lemma 3.4.4 and Lemma 4.2.1, we have that

$$0 \leq 2F(-\lambda') - F(0) - 2F(\lambda_1 - \lambda') + f(0)(\psi + \epsilon).$$

where $\psi$ depends on the cases in Lemma 4.2.1 and we assume $f(0) > 0$. As in [HB92, p. 37], choose

$$f(t) = \begin{cases} x_0 - t & \text{if } 0 \le t \le x_0, \\ 0 & \text{if } t \ge x_0, \end{cases}$$

for which Conditions 1 and 2 hold. Then by the same calculations, we see that

$$2F(-\lambda') - 2F(\lambda_1 - \lambda') \le \frac{2x_0\lambda_1 \exp(x_0\lambda')}{(\lambda')^2}, \qquad F(0) = \tfrac{1}{2}x_0^2, \qquad f(0) = x_0.$$

Hence, from the first inequality, we have that

$$2x_0\lambda_1(\lambda')^{-2}\exp(x_0\lambda') - \frac{1}{2}x_0^2 + x_0(\psi + \epsilon) \ge 0.$$

Choose $x_0 := 2\psi + \frac{1}{\lambda'} + 2\epsilon$ so that the dependence on $f$ is uniform for $\lambda' \ge 1$. With this choice, our inequality above then leads to

$$\lambda_1 \ge \frac{\lambda'}{4}\exp(-x_0\lambda') = \frac{\lambda'}{4e}\exp(-(2\psi + 2\epsilon)\lambda').$$

When $\lambda' \ge 4e$, we conclude that

$$\lambda' \ge (\tfrac{1}{2\psi} - \epsilon)\log(\lambda_1^{-1}).$$

The result in each case follows from the value of $\psi$ given in Lemma 4.2.1 and noting $\phi = 1/4$. $\square$

### $\lambda_1$ small

Here we create a "numerical version" of Lemma 4.2.1.

**Lemma 4.2.3.** *Let $\epsilon > 0$ and for $b \ge 0$, assume $0 < \lambda_1 \le b$ and retain the assumptions of Lemma 4.2.1. Suppose, for some $\lambda_b' > 0$, we have*

$$2F(-\lambda_b') - 2F(b - \lambda_b') - F(0) + f(0)\psi \le 0 \tag{4.10}$$

*where $\psi = 4\phi$ or $2\phi$ if $\chi_1$ is quadratic or principal respectively. Then $\lambda' \ge \lambda_b' - \epsilon$ for $\mathcal{L}$ sufficiently large depending on $\epsilon, b$ and $f$.*

*Proof.* Lemma 4.2.1 and Lemma 3.4.4 imply that

$$0 \le 2F(-\lambda') - 2F(\lambda_1 - \lambda') - F(0) + f(0)\psi + \epsilon.$$

| $\lambda_1 \le$ | $\frac{1}{2}\log\lambda_1^{-1} \ge$ | $\lambda^\star \ge$ | $\lambda$ |
|---|---|---|---|
| $10^{-10}$ | 11.51 | 10.99 | .8010 |
| $10^{-9}$ | 10.36 | 9.920 | .7975 |
| $10^{-8}$ | 9.210 | 8.838 | .7930 |
| $10^{-7}$ | 8.059 | 7.740 | .7873 |
| $10^{-6}$ | 6.908 | 6.623 | .7796 |
| $10^{-5}$ | 5.756 | 5.481 | .7687 |
| $10^{-4}$ | 4.605 | 4.303 | .7521 |
| .001 | 3.454 | 3.075 | .7239 |
| .005 | 2.649 | 2.176 | .6896 |
| .010 | 2.303 | 1.778 | .6679 |
| .015 | 2.100 | 1.542 | .6522 |
| .020 | 1.956 | 1.374 | .6394 |
| .025 | 1.844 | 1.244 | .6283 |

| $\lambda_1 \le$ | $\frac{1}{2}\log\lambda_1^{-1} \ge$ | $\lambda^\star \ge$ | $\lambda$ |
|---|---|---|---|
| .030 | 1.753 | 1.137 | .6183 |
| .035 | 1.676 | 1.048 | .6092 |
| .040 | 1.609 | .9699 | .6007 |
| .045 | 1.551 | .9016 | .5927 |
| .050 | 1.498 | .8407 | .5852 |
| .055 | 1.450 | .7859 | .5780 |
| .060 | 1.407 | .7362 | .5711 |
| .065 | 1.367 | .6906 | .5644 |
| .070 | 1.330 | .6487 | .5580 |
| .075 | 1.295 | .6098 | .5517 |
| .080 | 1.263 | .5738 | .5457 |
| .085 | 1.233 | .5401 | .5397 |

Table 4.1: Bounds for $\lambda^\star = \lambda'$ with $\chi_1$ quadratic, $\rho_1$ real and $\lambda_1$ small;
and for $\lambda^\star = \lambda_2$ with $\chi_1$ quadratic, $\rho_1$ real, $\chi_2$ principal and $\lambda_1$ small.

Now, by Conditions 1 and 2, the function

$$F(-\lambda) - F(b-\lambda) = \int_0^\infty f(t)e^\lambda(1-e^{-b})dt$$

is an increasing function of $\lambda$ and also of $b$. Hence, the previous inequality implies that

$$0 \le 2F(-\lambda') - 2F(b-\lambda') - F(0) + f(0)\psi + \epsilon.$$

On the other hand, from the increasing behaviour of $F(-\lambda) - F(b-\lambda)$, we may deduce that, if (4.10) holds for some $\lambda_b'$, then

$$0 \le 2F(-\lambda) - 2F(b-\lambda) - F(0) + f(0)\psi \qquad \text{only if } \lambda \ge \lambda_b'.$$

Comparing with the previous inequality and choosing a new value of $\epsilon$, we conclude that $\lambda' \ge \lambda_b' - \epsilon$. See [KN12, p.773] for details on this last argument. □

In each case, employing Lemma 4.2.3 for various values of $b$ requires a choice of $f$ depending on $b$ which maximizes the computed value of $\lambda_b'$. Based on numerical experimentation, we choose $f = f_\lambda$ from [HB92, Lemma 7.2] with parameter $\lambda = \lambda(b)$. This produces Tables 4.1 and 4.2. Note that the bounds in Table 4.1 are applicable in a later subsection for bounds on $\lambda_2$.

| $\lambda_1 \leq$ | $\log \lambda_1^{-1} \geq$ | $\lambda' \geq$ | $\lambda$ | $\lambda_1 \leq$ | $\log \lambda_1^{-1} \geq$ | $\lambda' \geq$ | $\lambda$ |
|---|---|---|---|---|---|---|---|
| $10^{-5}$ | 11.51 | 11.66 | 1.545 | .085 | 2.465 | 1.869 | 1.193 |
| $10^{-4}$ | 9.210 | 9.324 | 1.516 | .0875 | 2.436 | 1.836 | 1.189 |
| .001 | 6.908 | 6.902 | 1.468 | .090 | 2.408 | 1.803 | 1.185 |
| .005 | 5.298 | 5.135 | 1.413 | .095 | 2.354 | 1.741 | 1.178 |
| .010 | 4.605 | 4.352 | 1.379 | .100 | 2.303 | 1.681 | 1.170 |
| .015 | 4.200 | 3.887 | 1.355 | .105 | 2.254 | 1.625 | 1.163 |
| .020 | 3.912 | 3.555 | 1.336 | .110 | 2.207 | 1.572 | 1.156 |
| .025 | 3.689 | 3.297 | 1.319 | .115 | 2.163 | 1.521 | 1.149 |
| .030 | 3.507 | 3.084 | 1.304 | .120 | 2.120 | 1.472 | 1.142 |
| .035 | 3.352 | 2.905 | 1.291 | .125 | 2.079 | 1.426 | 1.135 |
| .040 | 3.219 | 2.749 | 1.279 | .130 | 2.040 | 1.381 | 1.129 |
| .045 | 3.101 | 2.611 | 1.267 | .135 | 2.002 | 1.338 | 1.122 |
| .050 | 2.996 | 2.488 | 1.257 | .140 | 1.966 | 1.297 | 1.116 |
| .055 | 2.900 | 2.377 | 1.246 | .145 | 1.931 | 1.258 | 1.110 |
| .060 | 2.813 | 2.275 | 1.237 | .150 | 1.897 | 1.220 | 1.103 |
| .065 | 2.733 | 2.181 | 1.227 | .155 | 1.864 | 1.183 | 1.097 |
| .070 | 2.659 | 2.095 | 1.218 | .160 | 1.833 | 1.148 | 1.091 |
| .075 | 2.590 | 2.015 | 1.210 | .165 | 1.802 | 1.113 | 1.085 |
| .080 | 2.526 | 1.940 | 1.201 | .170 | 1.772 | 1.080 | 1.079 |

Table 4.2: Bounds for $\lambda'$ with $\chi_1$ principal, $\rho_1$ real and $\lambda_1$ small.

### $\lambda_1$ medium

As a first attempt, we use techniques similar to before.

**Lemma 4.2.4.** *Assume $\chi_1$ and $\rho_1$ are real. Provided $\mathcal{L}$ is sufficiently large, it follows that if $\rho'$ is real then*

$$\lambda' \geq \begin{cases} 0.6069 & \text{if } \chi_1 \text{ is quadratic and } \lambda' \leq \lambda_2, \\ 1.2138 & \text{if } \chi_1 \text{ is principal,} \end{cases}$$

*and if $\rho'$ is complex then*

$$\lambda' \geq \begin{cases} 0.1722 & \text{if } \chi_1 \text{ is quadratic and } \lambda' \leq \lambda_2, \\ 0.3444 & \text{if } \chi_1 \text{ is principal.} \end{cases}$$

*Proof.* If $\rho'$ is real, then $\mu' = 0$. From Lemma 4.2.1 it follows that

$$0 \leq F(-\lambda') - F(0) - F(\lambda_1 - \lambda') + \tfrac{1}{2} f(0)\psi + \epsilon,$$

where $\epsilon, f, \psi$ are specified in Lemma 4.2.1. Since $F$ is decreasing by Condition 2,

$$0 \le F(-\lambda') - 2F(0) + \tfrac{1}{2} f(0)\psi + \epsilon.$$

We select the function from [HB92, Lemma 7.5] corresponding to $k = 2$. Hence,

$$\frac{1}{\lambda'} \cos^2\theta \le \tfrac{1}{2}\psi + \epsilon.$$

For $k = 2$, we find $\theta = 0.9873...$ and so $\lambda' \ge \frac{0.6069}{\psi}$ for an appropriate choice of $\epsilon$. If $\rho'$ is complex, then we follow a similar argument selecting $f$ from [HB92, Lemma 7.5] corresponding to $k = \frac{3}{2}$ (i.e. $\theta = 1.2729...$). □

For $\rho'$ complex, a method based on Section 3.3 leads to better bounds than Lemma 4.2.4.

**Lemma 4.2.5.** *Assume $\chi_1$ and $\rho_1$ is real and also suppose $\rho'$ is complex. Let $\lambda > 0$ and $J > 0$. If $\mathcal{L}$ is sufficiently large depending on $\epsilon, \lambda$ and $J$ then*

$$0 \le (J^2 + \tfrac{1}{2})\big(P_4(1) - P_4\big(\frac{\lambda}{\lambda + \lambda_1}\big)\big) - 2J \cdot P_4\big(\frac{\lambda}{\lambda+\lambda'}\big) + \begin{cases} 2\phi(J+1)^2\lambda + \epsilon & \text{if } \chi_1 \text{ is quadratic}, \\ \phi(J+1)^2\lambda + \epsilon & \text{if } \chi_1 \text{ is principal}, \end{cases}$$

*provided*

$$\frac{J_0}{(\lambda+\lambda')^4} + \frac{1}{(\lambda+\lambda_1)^4} > \frac{1}{\lambda^4} \qquad \text{with } J_0 = \min\{\tfrac{J}{2} + \tfrac{1}{2J}, 4J\}. \tag{4.11}$$

**Remark.** Recall $P_4(X)$ is defined by (3.18).

*Proof.* For an admissible polynomial $P(X) = \sum_{k=1}^d a_k X^k$, we begin with the inequality

$$\begin{aligned} 0 &\le \chi_0(\mathfrak{n})(1+\chi_1(\mathfrak{n}))\big(J + \mathrm{Re}\{\chi_1(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{-i\gamma'}\}\big)^2 \\ &= (J^2 + \tfrac{1}{2})\big(\chi_0(\mathfrak{n}) + \chi_1(\mathfrak{n})\big) + 2J \cdot \big(\mathrm{Re}\{\chi_0(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{-i\gamma'}\} + \mathrm{Re}\{\chi_1(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{-i\gamma'}\}\big) \\ &\quad + \tfrac{1}{2} \cdot \big(\mathrm{Re}\{\chi_0(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{-2i\gamma'}\} + \mathrm{Re}\{\chi_1(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{-2i\gamma'}\}\big). \end{aligned}$$

To introduce $\mathcal{P}(s,\chi) = \mathcal{P}(s,\chi;P)$, we multiply the above inequality by

$$\frac{\Lambda(\mathfrak{n})}{(\mathrm{N}\mathfrak{n})^\sigma}\Big(\sum_{k=1}^d a_k \frac{\big((\sigma-1)\log \mathrm{N}\mathfrak{n}\big)^{k-1}}{(k-1)!}\Big)$$

with $\sigma = 1 + \frac{\lambda}{\mathcal{L}}$ and sum over ideals $\mathfrak{n}$ yielding

$$\begin{aligned} 0 &\le (J^2 + \tfrac{1}{2})\big(\mathcal{P}(\sigma,\chi_0) + \mathcal{P}(\sigma,\chi_1)\big) + 2J \cdot \big(\mathcal{P}(\sigma + i\gamma',\chi_0) + \mathcal{P}(\sigma+i\gamma',\chi_1)\big) \\ &\quad + \tfrac{1}{2} \cdot \big(\mathcal{P}(\sigma + 2i\gamma',\chi_0) + \mathcal{P}(\sigma+2i\gamma',\chi_1)\big). \end{aligned} \tag{4.12}$$

Taking $P(X) = P_4(X)$ so $a_1 = 1$, we consider the two cases depending on $\chi_1$.

*(a) $\chi_1$ is quadratic:* Apply Lemma 3.3.4 to each $\mathcal{P}(*,*)$ term in (4.12) extracting the pole from $\chi_0$-terms and the zeros $\rho_1, \rho'$ (and possibly $\overline{\rho'}$) from the $\chi_1$-terms. Each of these applications yields the following:

$$\begin{aligned}
\mathcal{L}^{-1}\cdot\mathcal{P}(\sigma,\chi_0) &\leq \phi\tfrac{\mathcal{L}_0}{\mathcal{L}} + \epsilon + \frac{P_4(1)}{\lambda},\\
\mathcal{L}^{-1}\cdot\mathcal{P}(\sigma,\chi_1) &\leq \phi\tfrac{\mathcal{L}_{\chi_1}}{\mathcal{L}} + \epsilon - \frac{1}{\lambda}\cdot\Big(P_4(\frac{\lambda}{\lambda+\lambda_1}) + \mathrm{Re}\{P_4(\frac{\lambda}{\lambda+\lambda'+i\mu'})\}\Big),\\
\mathcal{L}^{-1}\cdot\mathcal{P}(\sigma+i\gamma',\chi_0) &\leq \phi\tfrac{\mathcal{L}_0}{\mathcal{L}} + \epsilon + \frac{1}{\lambda}\cdot\mathrm{Re}\Big\{P_4(\frac{\lambda}{\lambda+i\mu'})\Big\},\\
\mathcal{L}^{-1}\cdot\mathcal{P}(\sigma+i\gamma',\chi_1) &\leq \phi\tfrac{\mathcal{L}_{\chi_1}}{\mathcal{L}} + \epsilon - \frac{1}{\lambda}\cdot\Big(P_4(\frac{\lambda}{\lambda+\lambda'}) + \mathrm{Re}\{P_4(\frac{\lambda}{\lambda+\lambda_1+i\mu'}) + P_4(\frac{\lambda}{\lambda+\lambda'+2i\mu'})\}\Big),\\
\mathcal{L}^{-1}\cdot\mathcal{P}(\sigma+2i\gamma',\chi_0) &\leq \phi\tfrac{\mathcal{L}_0}{\mathcal{L}} + \epsilon + \frac{1}{\lambda}\cdot\mathrm{Re}\Big\{P_4(\frac{\lambda}{\lambda+2i\mu'})\Big\},\\
\mathcal{L}^{-1}\cdot\mathcal{P}(\sigma+2i\gamma',\chi_1) &\leq \phi\tfrac{\mathcal{L}_{\chi_1}}{\mathcal{L}} + \epsilon - \frac{1}{\lambda}\cdot\mathrm{Re}\{P_4(\frac{\lambda}{\lambda+\lambda_1+2i\mu'}) + P_4(\frac{\lambda}{\lambda+\lambda'+i\mu'})\},
\end{aligned}$$

provided $\mathcal{L}$ is sufficiently large depending on $\epsilon$ and $\lambda$. For the term $\mathcal{P}(\sigma+i\gamma',\chi_1)$, we extracted all 3 zeros of $\chi_1$ since $\mu' \neq 0$. Substituting these inequalities into (4.12) and noting $\frac{\mathcal{L}_0+\mathcal{L}_{\chi_1}}{\mathcal{L}} \leq 2$ by Lemma 3.1.1, we find that

$$0 \leq (J^2+\tfrac{1}{2})\big(P_4(1) - P_4(\frac{\lambda}{\lambda+\lambda_1})\big) - 2JP_4(\frac{\lambda}{\lambda+\lambda'}) - A - B + 2\phi(J+1)^2\lambda + \epsilon, \tag{4.13}$$

where

$$\begin{aligned}
A &= \mathrm{Re}\Big\{(J^2+1)\cdot P_4(\frac{\lambda}{\lambda+\lambda'+i\mu'})\} + 2J\cdot P_4(\frac{\lambda}{\lambda+\lambda_1+i\mu'}) - 2J\cdot P_4(\frac{\lambda}{\lambda+i\mu'})\Big\},\\
B &= \mathrm{Re}\Big\{2J\cdot P_4(\frac{\lambda}{\lambda+\lambda'+2i\mu'}) + \frac{1}{2}\cdot P_4(\frac{\lambda}{\lambda+\lambda_1+2i\mu'}) - \frac{1}{2}\cdot P_4(\frac{\lambda}{\lambda+2i\mu'})\Big\}.
\end{aligned}$$

From Lemma 3.3.6, we see that $A, B \geq 0$, provided

$$\frac{J^2+1}{(\lambda+\lambda')^4} + \frac{2J}{(\lambda+\lambda_1)^4} > \frac{2J}{\lambda^4} \qquad \text{and} \qquad \frac{2J}{(\lambda+\lambda')^4} + \frac{1/2}{(\lambda+\lambda_1)^4} > \frac{1/2}{\lambda^4}.$$

Assumption (4.11) implies both of these inequalities.

*(b) $\chi_1$ is principal:* Then (4.12) becomes

$$0 \leq (2J^2+1)\mathcal{P}(\sigma,\chi_0) + 4J\cdot\mathcal{P}(\sigma+i\gamma',\chi_0) + \mathcal{P}(\sigma+2i\gamma',\chi_0).$$

We similarly apply Lemma 3.3.4 to each term above extracting the pole and zeros $\rho_1, \rho'$ (and possibly $\overline{\rho'}$). Each of these applications yields the following:

$$\begin{aligned}
\mathcal{L}^{-1}\cdot\mathcal{P}(\sigma,\chi_0) &\leq \phi\tfrac{\mathcal{L}_0}{\mathcal{L}} + \epsilon + \frac{1}{\lambda}\Big(P_4(1) - P_4\big(\frac{\lambda}{\lambda+\lambda_1}\big) + \mathrm{Re}\Big\{P_4\big(\frac{\lambda}{\lambda+\lambda'+i\mu'}\big)\Big\}\Big),\\
\mathcal{L}^{-1}\cdot\mathcal{P}(\sigma+i\gamma',\chi_0) &\leq \phi\tfrac{\mathcal{L}_0}{\mathcal{L}} + \epsilon + \frac{1}{\lambda}\Big(-P_4\big(\frac{\lambda}{\lambda+\lambda'}\big) + \mathrm{Re}\Big\{P_4\big(\frac{\lambda}{\lambda+i\mu'}\big) - P_4\big(\frac{\lambda}{\lambda+\lambda_1+i\mu'}\big)\\
&\qquad\qquad - P_4\big(\frac{\lambda}{\lambda+\lambda'+2i\mu'}\big)\Big\}\Big),\\
\mathcal{L}^{-1}\cdot\mathcal{P}(\sigma+2i\gamma',\chi_0) &\leq \phi\tfrac{\mathcal{L}_0}{\mathcal{L}} + \epsilon + \frac{1}{\lambda}\cdot\mathrm{Re}\Big\{P_4\big(\frac{\lambda}{\lambda+2i\mu'}\big) - P_4\big(\frac{\lambda}{\lambda+\lambda_1+2i\mu'}\big) - P_4\big(\frac{\lambda}{\lambda+\lambda'+i\mu'}\big)\Big\}.
\end{aligned}$$

Substituting these into the previous inequality, noting $\frac{\mathcal{L}_0}{\mathcal{L}} \leq 1$, and dividing by 2, we obtain (4.13) except with $2\phi$ replaced by $\phi$. Following the same argument, we obtain the desired result. □

Again, we exhibit a "numerical version" of Lemma 4.2.5.

**Corollary 4.2.6.** *Assume $\chi_1$ and $\rho_1$ is real and suppose $\rho'$ is complex. Let $\epsilon > 0$. Suppose $0 < \lambda_1 \leq b, \lambda > 0, J > 0$ and that there exists $\lambda_b' \in [0,\infty)$ satisfying*

$$(J^2+\tfrac{1}{2})\big(P_4(1) - P_4\big(\frac{\lambda}{\lambda+b}\big)\big) - 2J\cdot P_4\big(\frac{\lambda}{\lambda+\lambda_b'}\big) + \psi(J+1)^2\lambda \leq 0$$

*where $\psi = 2\phi$ or $\phi$ if $\chi_1$ is quadratic or principal respectively. If*

$$\frac{J_0}{(\lambda+\lambda_b')^4} + \frac{1}{(\lambda+b)^4} > \frac{1}{\lambda^4}, \qquad \textit{where } J_0 = \min\{\tfrac{J}{2}+\tfrac{1}{2J}, 4J\},$$

*then $\lambda' \geq \lambda_b' - \epsilon$ provided $\mathcal{L}$ is sufficiently large depending on $\epsilon, \lambda$ and $J$.*

*Proof.* From Lemma 4.2.5,

$$0 \leq (J^2+\tfrac{1}{2})\big(P_4(1) - P_4\big(\frac{\lambda}{\lambda+\lambda_1}\big)\big) - 2J\cdot P_4\big(\frac{\lambda}{\lambda+\lambda'}\big) + (2-E_0(\chi_1))\cdot\phi(J+1)^2\lambda + \epsilon.$$

Since $P_4$ has non-negative coefficients and $P_4(0) = 0$, the above expression is *increasing* with $\lambda_1$ and $\lambda'$. From this observation, the desired result follows. □

| $\lambda_1 \le$ | $\frac{1}{2}\log(1/\lambda_1)$ | $\lambda^\star \ge$ | $\lambda$ | $J$ |
|---|---|---|---|---|
| .09 | 1.204 | .5261 | 1.239 | .8837 |
| .10 | 1.151 | .5063 | 1.265 | .8793 |
| .11 | 1.104 | .4880 | 1.289 | .8752 |
| .12 | 1.060 | .4709 | 1.310 | .8714 |
| .1227 | 1.049 | .4665 | 1.316 | .8704 |
| .13 | 1.020 | .4549 | 1.330 | .8677 |
| .14 | .9831 | .4398 | 1.348 | .8642 |
| .15 | .9486 | .4257 | 1.364 | .8608 |
| .16 | .9163 | .4122 | 1.379 | .8575 |
| .17 | .8860 | .3995 | 1.393 | .8544 |
| .18 | .8574 | .3874 | 1.405 | .8513 |
| .19 | .8304 | .3759 | 1.417 | .8483 |
| .20 | .8047 | .3649 | 1.428 | .8454 |
| .21 | .7803 | .3544 | 1.438 | .8426 |
| .22 | .7571 | .3443 | 1.447 | .8398 |
| .23 | .7348 | .3347 | 1.455 | .8370 |
| .24 | .7136 | .3254 | 1.463 | .8343 |
| .25 | .6931 | .3165 | 1.471 | .8316 |
| .26 | .6735 | .3080 | 1.477 | .8289 |
| .27 | .6547 | .2998 | 1.483 | .8263 |
| .28 | .6365 | .2918 | 1.489 | .8237 |
| .2866 | .6248 | .2868 | 1.493 | .8220 |

Table 4.3: Bounds for $\lambda^\star = \lambda'$ with $\chi_1$ quadratic, $\rho_1$ real and $\lambda_1$ medium; and for $\lambda^\star = \lambda_2$ with $\chi_1$ quadratic, $\rho_1$ real, $\chi_2$ principal, and $\rho_2$ complex.

Corollary 4.2.6 gives lower bounds for $\lambda'$ for certain ranges of $\lambda_1$. For each range $0 < \lambda_1 \le b$, we choose $\lambda = \lambda(b) > 0$, $J = J(b) > 0$ to produce an optimal lower bound $\lambda'_b$ for $\lambda'$. This produces Tables 4.3 and 4.4.

### Summary of bounds on $\lambda'$

We collect the results for each range of $\lambda_1$ into a single result for ease of use.

**Proposition 4.2.7.** *Assume $\chi_1$ and $\rho_1$ are real. Suppose $\mathcal{L}$ is sufficiently large depending on $\epsilon > 0$. Then:*

*(a) Suppose $\chi_1$ is quadratic and $\lambda' \le \lambda_2$. Then*

$$\lambda' \ge \begin{cases} (\frac{1}{2} - \epsilon)\log\lambda_1^{-1} & \text{if } \lambda_1 \le 10^{-10} \\ 0.2103\log\lambda_1^{-1} & \text{if } \lambda_1 \le 0.1227 \end{cases}$$

*and if $\lambda_1 > 0.1227$ then the bounds in Table 4.3 apply and $\lambda' \ge 0.2866$.*

| $\lambda_1 \leq$ | $\log(1/\lambda_1) \geq$ | $\lambda' \geq$ | $\lambda$ | $J$ |
|---|---|---|---|---|
| .18 | 1.715 | 1.052 | 2.478 | .8837 |
| .19 | 1.661 | 1.032 | 2.505 | .8815 |
| .20 | 1.609 | 1.013 | 2.530 | .8793 |
| .21 | 1.561 | .9939 | 2.555 | .8772 |
| .22 | 1.514 | .9759 | 2.578 | .8752 |
| .23 | 1.470 | .9586 | 2.600 | .8733 |
| .24 | 1.427 | .9418 | 2.621 | .8714 |
| .25 | 1.386 | .9255 | 2.641 | .8695 |
| .26 | 1.347 | .9098 | 2.660 | .8677 |
| .27 | 1.309 | .8945 | 2.678 | .8659 |
| .28 | 1.273 | .8797 | 2.695 | .8642 |
| .29 | 1.238 | .8653 | 2.712 | .8625 |
| .30 | 1.204 | .8513 | 2.728 | .8608 |
| .31 | 1.171 | .8377 | 2.743 | .8592 |
| .32 | 1.139 | .8245 | 2.758 | .8575 |
| .33 | 1.109 | .8116 | 2.772 | .8560 |
| .34 | 1.079 | .7990 | 2.785 | .8544 |
| .35 | 1.050 | .7867 | 2.798 | .8528 |
| .36 | 1.022 | .7748 | 2.811 | .8513 |
| .37 | .9943 | .7631 | 2.822 | .8498 |
| .38 | .9676 | .7517 | 2.834 | .8483 |
| .39 | .9416 | .7406 | 2.845 | .8469 |
| .40 | .9163 | .7297 | 2.855 | .8454 |
| .41 | .8916 | .7191 | 2.866 | .8440 |
| .42 | .8675 | .7087 | 2.875 | .8426 |
| .43 | .8440 | .6985 | 2.885 | .8412 |
| .44 | .8210 | .6886 | 2.894 | .8398 |
| .45 | .7985 | .6788 | 2.903 | .8384 |
| .46 | .7765 | .6693 | 2.911 | .8370 |
| .47 | .7550 | .6600 | 2.919 | .8356 |
| .48 | .7340 | .6508 | 2.927 | .8343 |
| .49 | .7133 | .6418 | 2.934 | .8329 |
| .50 | .6931 | .6330 | 2.941 | .8316 |
| .51 | .6733 | .6244 | 2.948 | .8303 |
| .52 | .6539 | .6159 | 2.955 | .8289 |
| .53 | .6349 | .6076 | 2.961 | .8276 |
| .54 | .6162 | .5995 | 2.967 | .8263 |
| .55 | .5978 | .5915 | 2.973 | .8250 |
| .56 | .5798 | .5837 | 2.978 | .8237 |
| .57 | .5621 | .5760 | 2.984 | .8224 |
| .5733 | .5563 | .5735 | 2.985 | .8220 |

Table 4.4: Bounds for $\lambda'$ with $\chi_1$ principal, $\rho_1$ real and $\lambda_1$ medium.

*(b) Suppose $\chi_1$ is principal. If $\lambda_1 \leq 0.0875$, then*

$$\lambda' \geq \begin{cases} (1-\epsilon)\log \lambda_1^{-1} & \text{if } \lambda_1 \leq 10^{-5} \\ 0.7399 \log \lambda_1^{-1} & \text{if } \lambda_1 \leq 0.0875 \end{cases}$$

*and if $\lambda_1 > 0.0875$ then the bounds in Tables 4.2 and 4.4 apply and $\lambda' \geq 0.5733$.*

**Remark.** The constants $0.1227$ and $0.0875$ come a posteriori from the corresponding zero-free regions established in Section 4.4.

*Proof.* (a) Suppose $\lambda_1 \leq 10^{-10}$. From Table 4.1, we see that $\lambda' \geq 10.99 > 4e$ and so the desired bound follows from Lemma 4.2.2. Suppose $\lambda_1 \leq 0.1227$. One compares Lemma 4.2.4 and Table 4.3 and finds that the latter gives weaker bounds. Thus, we only consider Tables 4.1 and 4.3 for this range of $\lambda_1$. For the subinterval $\lambda_1 \in [0.12, 0.1227]$, it follows that

$$\lambda' \geq 0.4663 \geq \frac{0.4663}{\log 1/0.12} \log \lambda_1^{-1} \geq 0.2200 \log \lambda_1^{-1}.$$

Repeat this process for each subinterval $[10^{-10}, 10^{-9}], [10^{-9}, 10^{-8}], \ldots, [0.85, 0.9], \ldots, [0.12, 0.1227]$ to obtain the desired bound. For $\lambda_1 > 0.1227$, one again compares Lemma 4.2.4 and Table 4.3 and finds that the latter gives weaker bounds. For (b), we argue analogous to (a) except we only use Table 4.2 for $\lambda_1 \leq 0.0875$. □

### 4.2.2 Bounds for $\lambda_2$

Recall $\rho_1, \lambda_1, \rho_2$ and $\lambda_2$ are defined in Section 3.1.1. We follow the same general approach as $\lambda'$ with natural modifications. Throughout, we shall assume $\lambda_2 \leq \lambda'$; otherwise, we may use the bounds from Section 4.2.1 on $\lambda'$.

**Lemma 4.2.8.** *Assume $\chi_1$ and $\rho_1$ are real and also that $\lambda_2 \leq \lambda'$. Suppose $f$ satisfies Conditions 1 and 2. For $\epsilon > 0$, provided $\mathcal{L}$ is sufficiently large depending on $\epsilon$ and $f$, the following holds:*

*(a) If $\chi_1, \chi_2$ are non-principal, then, with $\psi = 4\phi$, it follows that*

$$0 \leq F(-\lambda_2) - F(0) - F(\lambda_1 - \lambda_2) + f(0)\psi + \epsilon.$$

*(b) If $\chi_1$ is principal, then $\chi_2$ is necessarily non-principal and, with $\psi = 2\phi$, it follows that*

$$0 \leq \quad F(-\lambda_2) - F(0) - F(\lambda_1 - \lambda_2) + f(0)\psi + \epsilon.$$

*(c) If $\chi_2$ is principal, then $\chi_1$ is necessarily non-principal and, with $\psi = 4\phi$, it follows that*

$$\begin{aligned} 0 \leq \quad & F(-\lambda_2) - F(0) - F(\lambda_1 - \lambda_2) + f(0)\psi + \epsilon \\ & + \mathrm{Re}\{F(-\lambda_2 + i\mu_2) - F(i\mu_2) - F(\lambda_1 - \lambda_2 + i\mu_2)\}. \end{aligned}$$

*Proof.* In (4.5), set $(\chi, \rho) = (\chi_1, \rho_1)$ and $(\chi_*, \rho_*) = (\chi_2, \rho_2)$ and $\sigma = \beta_2$, which gives

$$0 \leq \mathcal{K}(\beta_2, \chi_0) + \mathcal{K}(\beta_2, \chi_1) + \mathcal{K}(\beta_2 + i\gamma_2, \chi_2) + \tfrac{1}{2}\mathcal{K}(\beta_2 + i\gamma_2, \chi_1\chi_2) + \tfrac{1}{2}\mathcal{K}(\beta_2 - i\gamma_2, \chi_1\overline{\chi_2}). \tag{4.14}$$

The arguments involved are entirely analogous to Lemma 4.2.1 so we omit the details here. For all cases, one applies Lemma 3.4.3 to each $\mathcal{K}(*, *)$ term, extracting $\rho_1$ or $\rho_2$ whenever possible. We remark that $\chi_1\chi_2$ and $\chi_1\overline{\chi_2}$ are always non-principal by construction (see Section 3.1). □

### $\lambda_1$ very small

We include the final result here without proof for the sake of brevity.

**Lemma 4.2.9.** *Assume $\chi_1$ and $\rho_1$ are real and $\lambda_2 \leq \lambda'$. Suppose $\mathcal{L}$ is sufficiently large depending on $\epsilon > 0$.*

*(a) If $\chi_1, \chi_2$ are non-principal, then either $\lambda_2 < 2e$ or*

$$\lambda_2 \geq \Big(\frac{1}{2} - \epsilon\Big) \log(\lambda_1^{-1}),$$

*which is non-trivial for $\lambda_1 \leq 1.8 \times 10^{-5}$.*

*(b) If $\chi_1$ is principal, then $\chi_2$ is necessarily non-principal and either $\lambda_2 < 2e$ or*

$$\lambda_2 \geq \big(1 - \epsilon\big) \log(\lambda_1^{-1}),$$

*which is non-trivial for $\lambda_1 \leq 4.3 \times 10^{-3}$.*

*(c) If $\chi_2$ is principal, then $\chi_1$ is necessarily non-principal and either $\lambda_2 < 4e$ or*

$$\lambda_2 \geq \Big(\frac{1}{2} - \epsilon\Big) \log(\lambda_1^{-1}),$$

*which is non-trivial for $\lambda_1 \leq 3.5 \times 10^{-10}$.*

*Proof.* Analogous to Lemma 4.2.2 using Lemma 4.2.8 in place of Lemma 4.2.1. We omit the details for brevity. □

### $\lambda_1$ small

**Lemma 4.2.10.** *Assume $\chi_1$ and $\rho_1$ are real and also that $\lambda_2 \leq \lambda'$. Suppose $f$ satisfies Conditions 1 and 2. Let $\epsilon > 0$ and assume $0 < \lambda_1 \leq b$ for some $b > 0$. Suppose, for some $\tilde{\lambda}_b > 0$, we have*

$$F(-\tilde{\lambda}_b) - F(b - \tilde{\lambda}_b) - F(0) + 4\phi f(0) \leq 0 \qquad \text{if } \chi_1, \chi_2 \text{ are non-principal,}$$

$$F(-\tilde{\lambda}_b) - F(b - \tilde{\lambda}_b) - F(0) + 2\phi f(0) \leq 0 \qquad \text{if } \chi_1 \text{ is principal,}$$

$$2F(-\tilde{\lambda}_b) - 2F(b - \tilde{\lambda}_b) - F(0) + 4\phi f(0) \leq 0 \quad \text{if } \chi_2 \text{ is principal.}$$

*Then, according to the above cases, $\lambda_2 \geq \tilde{\lambda}_b - \epsilon$ provided $\mathcal{L}$ is sufficiently large depending on $\epsilon, b$ and $f$.*

*Proof.* Analogous to Lemma 4.2.3 using Lemma 4.2.8 in place of Lemma 4.2.1. Hence, we omit the proof. □

As before, Lemma 4.2.10 requires a choice of $f$ depending on $b$ which maximizes the computed value of $\tilde{\lambda}_b$. Based on numerical experimentation, we choose $f = f_\lambda$ from [HB92, Lemma 7.2] with parameter $\lambda = \lambda(b)$ for all cases. This produces Tables 4.1, 4.5 and 4.6.

### $\lambda_1$ medium

We first deal with the case when $\rho_2$ is real and $\chi_2$ is principal, i.e. $\mu_2 = 0$.

**Lemma 4.2.11.** *Assume $\chi_1$ and $\rho_1$ are real. Suppose $\mathcal{L}$ is sufficiently large. If $\rho_2$ is real, then*

$$\lambda_2 \geq \begin{cases} 0.3034 & \textit{if } \chi_1, \chi_2 \textit{ are non-principal}, \\ 0.6069 & \textit{otherwise}. \end{cases}$$

*If $\rho_2$ is complex, then*

$$\lambda_2 \geq \begin{cases} 0.3034 & \textit{if } \chi_1, \chi_2 \textit{ are non-principal}, \\ 0.6069 & \textit{if } \chi_1 \textit{ is principal}, \\ 0.1722 & \textit{if } \chi_2 \textit{ is principal}. \end{cases}$$

*Proof.* Analogous to Lemma 4.2.4 using Lemma 4.2.8 in place of Lemma 4.2.1. The arguments lead to selecting $f$ from [HB92, Lemma 7.5] corresponding to $k = 2$ (i.e. $\theta = 0.9873...$)

| $\lambda_1 \leq$ | $\frac{1}{2}\log\lambda_1^{-1} \geq$ | $\lambda_2 \geq$ | $\lambda$ |
|---|---|---|---|
| $10^{-5}$ | 5.756 | 5.828 | .7725 |
| $10^{-4}$ | 4.605 | 4.662 | .7579 |
| .001 | 3.454 | 3.451 | .7342 |
| .005 | 2.649 | 2.569 | .7065 |
| .010 | 2.303 | 2.178 | .6896 |
| .015 | 2.100 | 1.947 | .6776 |
| .020 | 1.956 | 1.783 | .6679 |
| .025 | 1.844 | 1.654 | .6596 |
| .030 | 1.753 | 1.550 | .6522 |
| .035 | 1.676 | 1.461 | .6455 |
| .040 | 1.609 | 1.384 | .6394 |
| .045 | 1.551 | 1.317 | .6337 |
| .050 | 1.498 | 1.256 | .6283 |
| .055 | 1.450 | 1.202 | .6232 |
| .060 | 1.407 | 1.152 | .6183 |
| .065 | 1.367 | 1.107 | .6137 |
| .070 | 1.330 | 1.065 | .6092 |
| .075 | 1.295 | 1.026 | .6049 |
| .080 | 1.263 | .9895 | .6007 |
| .085 | 1.233 | .9555 | .5967 |
| .090 | 1.204 | .9236 | .5928 |
| .095 | 1.177 | .8935 | .5890 |
| .100 | 1.151 | .8652 | .5853 |
| .105 | 1.127 | .8383 | .5816 |
| .110 | 1.104 | .8127 | .5781 |
| .115 | 1.081 | .7884 | .5746 |
| .120 | 1.060 | .7653 | .5712 |
| .125 | 1.040 | .7432 | .5679 |
| .130 | 1.020 | .7221 | .5646 |
| .135 | 1.001 | .7019 | .5613 |
| .140 | .9831 | .6825 | .5582 |
| .145 | .9655 | .6639 | .5550 |
| .150 | .9486 | .6460 | .5520 |

| $\lambda_1 \leq$ | $\frac{1}{2}\log\lambda_1^{-1} \geq$ | $\lambda_2 \geq$ | $\lambda$ |
|---|---|---|---|
| .155 | .9322 | .6288 | .5489 |
| .160 | .9163 | .6122 | .5459 |
| .165 | .9009 | .5962 | .5429 |
| .170 | .8860 | .5808 | .5400 |
| .175 | .8715 | .5659 | .5371 |
| .180 | .8574 | .5515 | .5342 |
| .185 | .8437 | .5376 | .5314 |
| .190 | .8304 | .5242 | .5286 |
| .195 | .8174 | .5111 | .5258 |
| .200 | .8047 | .4985 | .5231 |
| .205 | .7924 | .4863 | .5203 |
| .210 | .7803 | .4744 | .5176 |
| .215 | .7686 | .4629 | .5150 |
| .220 | .7571 | .4517 | .5123 |
| .225 | .7458 | .4408 | .5097 |
| .230 | .7348 | .4302 | .5070 |
| .235 | .7241 | .4200 | .5044 |
| .240 | .7136 | .4100 | .5018 |
| .245 | .7032 | .4002 | .4993 |
| .250 | .6931 | .3908 | .4967 |
| .255 | .6832 | .3816 | .4942 |
| .260 | .6735 | .3726 | .4916 |
| .265 | .6640 | .3638 | .4891 |
| .270 | .6547 | .3553 | .4866 |
| .275 | .6455 | .3470 | .4841 |
| .280 | .6365 | .3389 | .4817 |
| .285 | .6276 | .3310 | .4792 |
| .290 | .6189 | .3233 | .4768 |
| .295 | .6104 | .3158 | .4743 |
| .300 | .6020 | .3084 | .4719 |

Table 4.5: Bounds for $\lambda_2$ with $\chi_1$ quadratic, $\rho_1$ real, $\chi_2$ non-principal and $\lambda_1$ small.

| $\lambda_1 \le$ | $\log \lambda_1^{-1} \ge$ | $\lambda_2 \ge$ | $\lambda$ |
|---|---|---|---|
| .004 | 5.521 | 6.150 | 1.448 |
| .006 | 5.116 | 5.705 | 1.434 |
| .008 | 4.828 | 5.386 | 1.422 |
| .010 | 4.605 | 5.137 | 1.413 |
| .015 | 4.200 | 4.682 | 1.394 |
| .020 | 3.912 | 4.357 | 1.379 |
| .025 | 3.689 | 4.103 | 1.366 |
| .03 | 3.507 | 3.895 | 1.355 |
| .04 | 3.219 | 3.565 | 1.336 |
| .05 | 2.996 | 3.309 | 1.319 |
| .06 | 2.813 | 3.099 | 1.304 |
| .07 | 2.659 | 2.922 | 1.291 |
| .08 | 2.526 | 2.769 | 1.279 |
| .0875 | 2.436 | 2.666 | 1.270 |
| .10 | 2.303 | 2.513 | 1.257 |
| .12 | 2.120 | 2.304 | 1.237 |
| .14 | 1.966 | 2.130 | 1.218 |
| .16 | 1.833 | 1.979 | 1.201 |
| .18 | 1.715 | 1.847 | 1.186 |
| .20 | 1.609 | 1.730 | 1.171 |
| .22 | 1.514 | 1.625 | 1.156 |
| .24 | 1.427 | 1.531 | 1.142 |
| .26 | 1.347 | 1.444 | 1.129 |
| .28 | 1.273 | 1.365 | 1.116 |
| .30 | 1.204 | 1.292 | 1.104 |
| .32 | 1.139 | 1.224 | 1.092 |
| .34 | 1.079 | 1.162 | 1.080 |
| .36 | 1.022 | 1.103 | 1.068 |
| .38 | .9676 | 1.048 | 1.057 |
| .40 | .9163 | .9970 | 1.046 |
| .42 | .8675 | .9488 | 1.035 |
| .44 | .8210 | .9033 | 1.025 |
| .46 | .7765 | .8605 | 1.014 |
| .48 | .7340 | .8199 | 1.004 |
| .50 | .6931 | .7816 | .9934 |
| .52 | .6539 | .7452 | .9833 |
| .54 | .6162 | .7106 | .9733 |
| .56 | .5798 | .6778 | .9633 |
| .58 | .5447 | .6466 | .9535 |
| .60 | .5108 | .6168 | .9438 |
| .6068 | .4996 | .6070 | .9405 |

Table 4.6: Bounds for $\lambda_2$ with $\chi_1$ principal, $\rho_1$ real and $\lambda_1$ small.

when $\rho_2$ is real or $\chi_2$ is non-principal, and to $k = 3/2$ (i.e. $\theta = 1.2729...$) when $\rho_2$ is complex and $\chi_2$ is principal. $\square$

For $\chi_2$ principal and $\rho_2$ complex, the "polynomial method" of Section 3.3 yields better bounds.

**Lemma 4.2.12.** *Assume $\chi_1$ is quadratic and $\rho_1$ is real. Further suppose $\chi_2$ is principal and $\rho_2$ is complex. Let $\lambda > 0$ and $J > 0$. If $\mathcal{L}$ is sufficiently large depending on $\epsilon$, $\lambda$ and $J$, then*

$$0 \le (J^2 + \tfrac{1}{2})\big(P_4(1) - P_4\big(\frac{\lambda}{\lambda+\lambda_1}\big)\big) - 2JP_4\big(\frac{\lambda}{\lambda+\lambda_2}\big) + 2\phi(J+1)^2\lambda + \epsilon,$$

*provided*

$$\frac{J_0}{(\lambda+\lambda_2)^4} + \frac{1}{(\lambda+\lambda_1)^4} > \frac{1}{\lambda^4}, \tag{4.15}$$

*where $J_0 = \min\{\frac{J}{2} + \frac{1}{2J}, 4J\}$.*

*Proof.* This is analogous to Lemma 4.2.5 so we give a brief outline here. We begin with the inequality

$$\begin{aligned} 0 &\le \chi_0(\mathfrak{n})(1+\chi_1(\mathfrak{n}))\big(J + \mathrm{Re}\{(\mathrm{N}\mathfrak{n})^{-i\gamma_2}\}\big)^2 \\ &= (J^2+\tfrac{1}{2})\big(\chi_0(\mathfrak{n}) + \chi_1(\mathfrak{n})\big) + 2J\cdot\big(\mathrm{Re}\{(\mathrm{N}\mathfrak{n})^{-i\gamma_2}\} + \mathrm{Re}\{\chi_1(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{-i\gamma_2}\}\big) \\ &\quad + \tfrac{1}{2}\cdot\big(\mathrm{Re}\{(\mathrm{N}\mathfrak{n})^{-2i\gamma_2}\} + \mathrm{Re}\{\chi_1(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{-2i\gamma_2}\}\big). \end{aligned}$$

We introduce $\mathcal{P}(s,\chi) = \mathcal{P}(s,\chi;P_4)$ in the usual way with $\sigma = 1 + \frac{\lambda}{\mathcal{L}}$, yielding

$$\begin{aligned} 0 &\le (J^2+\tfrac{1}{2})\big(\mathcal{P}(\sigma,\chi_0) + \mathcal{P}(\sigma,\chi_1)\big) + 2J\cdot\big(\mathcal{P}(\sigma+i\gamma_2,\chi_0) + \mathcal{P}(\sigma+i\gamma_2,\chi_1)\big) \\ &\quad + \tfrac{1}{2}\cdot\big(\mathcal{P}(\sigma+2i\gamma_2,\chi_0) + \mathcal{P}(\sigma+2i\gamma_2,\chi_1)\big). \end{aligned} \tag{4.16}$$

Next, apply Lemma 3.3.4 to each $\mathcal{P}(*,*)$ term in (4.16) extracting the zero $\rho_2$ from $\chi_0$-terms and the zero $\rho_1$ from the $\chi_1$-terms. One also extracts both zeros $\{\rho_2, \overline{\rho_2}\}$ from $\mathcal{P}(\sigma+i\gamma_2,\chi_0)$. Noting $\frac{\mathcal{L}_0+\mathcal{L}_{\chi_1}}{\mathcal{L}} \le 2$ by Lemma 3.1.1 and choosing a new $\epsilon$, these applications yield the following:

$$0 \le (J^2 + \tfrac{1}{2})\big(P_4(1) - P_4\big(\frac{\lambda}{\lambda+\lambda_1}\big)\big) - 2JP_4\big(\frac{\lambda}{\lambda+\lambda_2}\big) - A - B + 2\phi(J+1)^2\lambda + \epsilon, \tag{4.17}$$

provided $\mathcal{L}$ is sufficiently large depending on $\epsilon$ and $\lambda$ and where

$$A = (J^2+1)\mathrm{Re}\{P_4(\frac{\lambda}{\lambda+\lambda_2+i\mu_2})\} + 2J\cdot\mathrm{Re}\{P_4(\frac{\lambda}{\lambda+\lambda_1+i\mu_2})\} - 2J\cdot\mathrm{Re}\Big\{P_4(\frac{\lambda}{\lambda+i\mu_2})\Big\},$$
$$B = 2J\cdot\mathrm{Re}\{P_4(\frac{\lambda}{\lambda+\lambda_2+2i\mu_2})\} + \frac{1}{2}\cdot\mathrm{Re}\{P_4(\frac{\lambda}{\lambda+\lambda_1+2i\mu_2})\} - \frac{1}{2}\cdot\mathrm{Re}\Big\{P_4(\frac{\lambda}{\lambda+2i\mu_2})\Big\}.$$

Assumption (4.15) implies $A, B \geq 0$ by Lemma 3.3.6 yielding the desired result from (4.17). □

With an appropriate numerical version of Lemma 4.2.12, analogous to Corollary 4.2.6, we obtain lower bounds for $\lambda_2$ for $\lambda_1 \in [0,b]$ and fixed $b > 0$. Optimally choosing $\lambda = \lambda(b) > 0, J = J(b) > 0$ produces Table 4.3 again.

**Summary of bounds on $\lambda_2$**

We collect the estimates of the previous subsections for each range of $\lambda_1$ into a one result for ease of use.

**Proposition 4.2.13.** *Assume $\chi_1$ and $\rho_1$ are real. Suppose $\mathcal{L}$ is sufficiently large depending on $\epsilon > 0$:*

*(a) Suppose $\chi_1$ is quadratic and $\lambda_2 \leq \lambda'$. Then*

$$\lambda_2 \geq \begin{cases} (\frac{1}{2}-\epsilon)\log\lambda_1^{-1} & \text{if } \lambda_1 \leq 10^{-10} \\ 0.2103\log\lambda_1^{-1} & \text{if } \lambda_1 \leq 0.1227 \end{cases}$$

*and if $\lambda_1 > 0.1227$ then the bounds in Table 4.3 apply and $\lambda_2 \geq 0.2866$.*

*(b) Suppose $\chi_1$ is principal. Then*

$$\lambda_2 \geq (1-\epsilon)\log\lambda_1^{-1} \quad \text{if } \lambda_1 \leq 0.0875.$$

*and if $\lambda_1 > 0.0875$ then the bounds in Table 4.6 apply and $\lambda_2 \geq 0.6069$.*

**Remark.** After comparing Propositions 4.2.7 and 4.2.13 in the case when $\chi_1$ is quadratic, we realize that the additional assumptions $\lambda' \leq \lambda_2$ or $\lambda_2 \leq \lambda'$ are superfluous.

*Proof.* (a) First, suppose $\chi_2$ is non-principal. For $\lambda_1 \leq 10^{-5}$, we see from Table 4.5 that $\lambda_2 \geq 5.828 > 2e$ so the desired bound follows form Lemma 4.2.9. For $10^{-5} \leq \lambda_1 \leq$

$0.1227$, consider Table 4.5. Apply the same process as in Proposition 4.2.7 to each subinterval $[10^{-5}, 10^{-4}], \ldots, [0.12, 0.125]$ to obtain

$$\lambda_2 \geq 0.3506 \log \lambda_1^{-1}.$$

Now, suppose $\chi_2$ is principal. For $\lambda_1 \leq 10^{-10}$, we see from Table 4.1 that $\lambda_2 \geq 10.99 > 4e$ so the desired bound follows from Lemma 4.2.9. For $10^{-10} \leq \lambda_1 \leq 0.1227$, consider Tables 4.1 and 4.3. Apply the same process as in Proposition 4.2.7 to each subinterval

$$[10^{-10}, 10^{-9}], \ldots, [0.85, 0.9], \ldots [0.12, 0.1227]$$

and obtain

$$\lambda_2 \geq 0.2103 \log \lambda_1^{-1}.$$

Upon comparing the two cases, the latter gives weaker results in the range $\lambda_1 \leq 0.1227$. For $\lambda_1 > 0.1227$, we compare Lemma 4.2.11 and Tables 4.3 and 4.5 to see that Table 4.3 gives the weakest bounds.

(b) Similar to (a) except we use Table 4.6 in conjunction with Lemma 4.2.9. The range $\lambda_1 \leq 0.004$ gives the bound $\lambda_2 \geq (1-\epsilon) \log \lambda_1^{-1}$. The range $0.004 \leq \lambda_1 \leq 0.0875$ turns out to give a better bound but we opt to write a bound uniform for $\lambda_1 \leq 0.0875$. For $\lambda_1 > 0.0875$, we use Lemma 4.2.11 and Table 4.6. $\square$

## 4.3 Zero repulsion: $\chi_1$ or $\rho_1$ is complex

When $\chi_1$ or $\rho_1$ is complex, the effect of zero repulsion is lesser than when $\chi_1$ and $\rho_1$ are real. Nonetheless, we will follow the same general outline as the previous section, but with modified trigonometric identities and more frequently using the "polynomial method" of Section 3.3. Also, whether $\chi_1$ is principal naturally affects our arguments in a significant manner so, for clarity, we further subdivide our results on this condition. Recall the definitions of zeros $\rho_1, \rho'$, and $\rho_2$ in Section 3.1.1.

### 4.3.1 Bounds for $\lambda'$

**$\chi_1$ non-principal**

**Lemma 4.3.1.** *Assume $\chi_1$ or $\rho_1$ is complex with $\chi_1$ non-principal. Let $\lambda > 0, J \geq \frac{1}{4}$. If $\mathcal{L}$ is sufficiently large depending on $\epsilon, \lambda$ and $J$ then*

$$0 \leq (J^2 + \tfrac{1}{2})P_4(1) - (J^2 + \tfrac{1}{2}) \cdot P_4\big(\frac{\lambda}{\lambda+\lambda'}\big) - 2J \cdot P_4\big(\frac{\lambda}{\lambda+\lambda_1}\big) + 2(J+1)^2\phi\lambda + \epsilon,$$

*provided*

$$\frac{J_0}{(\lambda+\lambda_1)^4} + \frac{1}{(\lambda+\lambda')^4} > \frac{1}{\lambda^4} \qquad \text{with } J_0 = \min\{J + \tfrac{3}{4J}, 4J\}. \tag{4.18}$$

*Proof.* For simplicity, denote $\mathcal{P}(s,\chi) = \mathcal{P}(s,\chi;P_4)$. Our starting point is the inequality

$$0 \leq \chi_0(\mathfrak{n})\big(1 + \mathrm{Re}\{\chi_1(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{i\gamma'}\}\big)\big(J + \mathrm{Re}\{\chi_1(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{i\gamma_1}\}\big)^2.$$

In the usual way, it follows that

$$\begin{aligned} 0 \leq\; & (J^2 + \tfrac{1}{2})\big\{\mathcal{P}(\sigma,\chi_0) + \mathcal{P}(\sigma + i\gamma',\chi_1)\big\} + \\ & + J\mathcal{P}(\sigma + i(\gamma_1+\gamma'),\chi_1^2) + 2J\mathcal{P}(\sigma + i\gamma_1,\chi_1) + J\mathcal{P}(\sigma + i(\gamma_1 - \gamma'),\chi_0) \\ & + \tfrac{1}{4}\mathcal{P}(\sigma + i(2\gamma_1+\gamma'),\chi_1^3) + \tfrac{1}{2}\mathcal{P}(\sigma + 2i\gamma_1,\chi_1^2) + \tfrac{1}{4}\mathcal{P}(\sigma + i(2\gamma_1 - \gamma'),\chi_1), \end{aligned} \tag{4.19}$$

where $\sigma = 1 + \frac{\lambda}{\mathcal{L}}$. To each term $\mathcal{P}(\,\cdot\,,\chi_1^r)$ above, we apply Lemma 3.3.4 extracting zeros depending on the order of $\chi_1$ and the value of $r$. We divide our argument into cases.

(i) $(\mathrm{ord}\,\chi_1 \geq 4)$ *Extract $\{\rho_1,\rho'\}$ from $\mathcal{P}(\,\cdot\,,\chi_1^r)$ when $r = 1$.* From (4.19), we deduce

$$0 \leq (J^2 + \tfrac{1}{2})P_4(1) - (J^2 + \tfrac{1}{2})P_4\big(\frac{\lambda}{\lambda+\lambda'}\big) - 2JP_4\big(\frac{\lambda}{\lambda+\lambda_1}\big) - A + \lambda\psi + \epsilon, \tag{4.20}$$

where $\psi = (J^2 + 3J + \frac{3}{2})\phi\frac{\mathcal{L}_{\chi_1}}{\mathcal{L}} + (J^2 + J + \frac{1}{2})\phi\frac{\mathcal{L}_0}{\mathcal{L}}$, and

$$A = \mathrm{Re}\Big\{(J^2 + \tfrac{3}{4})P_4\big(\frac{\lambda}{\lambda+\lambda_1+it_1}\big) + 2J \cdot P_4\big(\frac{\lambda}{\lambda+\lambda'+it_1}\big) - J \cdot P_4\big(\frac{\lambda}{\lambda+it_1}\big)\Big\}$$

with $t_1 = \mu' - \mu_1$. One can easily verify that $J^2 + 3J + \frac{3}{2} \leq 3 \cdot (J^2 + J + \frac{1}{2})$ and so by Lemma 3.1.1, we may more simply take $\psi = 2(J+1)^2\phi$ in (4.20). By Lemma 3.3.6, assumption (4.18) implies $A \geq 0$, completing the proof of case (i).

(ii) $(\mathrm{ord}\,\chi_1 = 3)$ *Extract $\{\rho_1,\rho'\}$ or $\{\overline{\rho_1},\overline{\rho'}\}$ from $\mathcal{P}(\,\cdot\,,\chi_1^r)$ when $r = 1$ or $2$ respectively.*

Then by (4.19),

$$0 \le (J^2+\tfrac{1}{2})P_4(1) - (J^2+\tfrac{1}{2})P_4\big(\frac{\lambda}{\lambda+\lambda'}\big) - 2JP_4\big(\frac{\lambda}{\lambda+\lambda_1}\big) - A - B + \lambda\psi + \epsilon, \tag{4.21}$$

where $\psi = (J^2+3J+\frac{5}{4})\phi\frac{\mathcal{L}_{\chi_1}}{\mathcal{L}} + (J^2+J+\frac{3}{4})\phi\frac{\mathcal{L}_0}{\mathcal{L}}$, the quantity $A$ is as defined in case (i), and

$$B = \mathrm{Re}\Big\{J\cdot P_4\big(\frac{\lambda}{\lambda+\lambda_1+it_2}\big) + \tfrac{1}{2}\cdot P_4\big(\frac{\lambda}{\lambda+\lambda'+it_2}\big) - \tfrac{1}{4}\cdot P_4\big(\frac{\lambda}{\lambda+it_2}\big)\Big\}$$

with $t_2 = \mu' + 2\mu_1$. Again, one can check that $J^2+3J+\frac{5}{4} \le 3\cdot(J^2+J+\frac{3}{4})$ and so by Lemma 3.1.1, we may take $\psi = 2(J+1)^2\phi$ in (4.21). Similar to (i), Lemma 3.3.6 and assumption (4.18) imply $A, B \ge 0$.

(iii) $(\mathrm{ord}\,\chi_1 = 2)$ *Extract* $\{\rho_1, \overline{\rho_1}, \rho'\}$ *from* $\mathcal{P}(\,\cdot\,,\chi_1^r)$ *when* $r=1$ *or* $3$. Again, apply Lemma 3.3.4 to the terms in (4.19) except with a slightly more careful analysis. We outline these modifications here.

- Write $2J\cdot\mathcal{P}(\sigma+i\gamma_1,\chi_1) = J\cdot\mathcal{P}(\sigma+i\gamma_1,\chi_1)+J\cdot\mathcal{P}(\sigma+i\gamma_1,\chi_1)$. Extract $\{\rho_1,\overline{\rho_1},\rho'\}$ from the first term and extract $\{\rho_1,\overline{\rho_1},\overline{\rho'}\}$ from the second term.
- For $\frac{1}{4}\mathcal{P}(\sigma+i(2\gamma_1+\gamma'),\chi_1)$ and $\frac{1}{4}\mathcal{P}(\sigma+i(2\gamma_1-\gamma'),\chi_1)$, extract $\{\rho_1,\rho'\}$ and $\{\rho_1,\overline{\rho'}\}$ respectively.

With these modifications, (4.19) overall yields

$$\begin{aligned}0 \le &(J^2+\tfrac{1}{2})P_4(1) - (J^2+\tfrac{1}{2})P_4\big(\frac{\lambda}{\lambda+\lambda'}\big) - 2JP_4\big(\frac{\lambda}{\lambda+\lambda_1}\big) + \lambda(\psi+\epsilon)\\
&- \mathrm{Re}\Big\{(J^2+\tfrac{3}{4})P_4\big(\frac{\lambda}{\lambda+\lambda_1+it_1}\big) + J\cdot P_4\big(\frac{\lambda}{\lambda+\lambda'+it_1}\big) - J\cdot P_4\big(\frac{\lambda}{\lambda+it_1}\big)\Big\}\\
&- \mathrm{Re}\Big\{(J^2+\tfrac{3}{4})\cdot P_4\big(\frac{\lambda}{\lambda+\lambda_1+it_3}\big) + J\cdot P_4\big(\frac{\lambda}{\lambda+\lambda'+it_3}\big) - J\cdot P_4\big(\frac{\lambda}{\lambda+it_3}\big)\Big\}\\
&- \mathrm{Re}\Big\{2J\cdot P_4\big(\frac{\lambda}{\lambda+\lambda_1+it_4}\big) + \tfrac{1}{2}\cdot P_4\big(\frac{\lambda}{\lambda+\lambda'+it_4}\big) - \tfrac{1}{2}\cdot P_4\big(\frac{\lambda}{\lambda+it_4}\big)\Big\}\end{aligned} \tag{4.22}$$

where $t_1 = \mu'-\mu_1$; $t_3 = \mu'+\mu_1$; $t_4 = 2\mu_1$; and $\psi = (J^2+2J+1)\phi\frac{\mathcal{L}_{\chi_1}}{\mathcal{L}} + (J^2+2J+1)\phi\frac{\mathcal{L}_0}{\mathcal{L}}$. Trivially $J^2+2J+1 \le 3\cdot(J^2+2J+1)$ and so by Lemma 3.1.1, we may more simply take $\psi = 2(J+1)^2\phi$. The three terms $\mathrm{Re}\{\dots\}$ in (4.22) are all $\ge 0$ by Lemma 3.3.6 and (4.18) and hence can be ignored.

This completes the proof in all cases. □

A suitable numerical version of Lemma 4.3.1 produces Table 4.7.

| $\lambda_1 \leq$ | $\lambda' \geq$ | $\lambda$ | $J$ |
|---|---|---|---|
| .1227 | .7391 | 1.097 | .7788 |
| .125 | .7266 | 1.104 | .7797 |
| .130 | .7007 | 1.120 | .7817 |
| .135 | .6766 | 1.135 | .7836 |
| .140 | .6540 | 1.149 | .7854 |
| .145 | .6328 | 1.162 | .7872 |
| .150 | .6128 | 1.174 | .7889 |
| .155 | .5939 | 1.185 | .7906 |
| .160 | .5759 | 1.195 | .7923 |
| .165 | .5589 | 1.204 | .7939 |
| .170 | .5427 | 1.213 | .7955 |
| .175 | .5272 | 1.221 | .7971 |
| .180 | .5124 | 1.229 | .7986 |
| .185 | .4982 | 1.236 | .8001 |
| .190 | .4846 | 1.242 | .8016 |
| .195 | .4715 | 1.249 | .8030 |
| .200 | .4590 | 1.254 | .8045 |
| .205 | .4469 | 1.259 | .8059 |

| $\lambda_1 \leq$ | $\lambda' \geq$ | $\lambda$ | $J$ |
|---|---|---|---|
| .210 | .4353 | 1.264 | .8073 |
| .215 | .4241 | 1.269 | .8087 |
| .220 | .4132 | 1.273 | .8100 |
| .225 | .4027 | 1.276 | .8114 |
| .230 | .3926 | 1.280 | .8127 |
| .235 | .3828 | 1.283 | .8140 |
| .240 | .3733 | 1.285 | .8153 |
| .245 | .3641 | 1.288 | .8166 |
| .250 | .3552 | 1.290 | .8179 |
| .255 | .3465 | 1.292 | .8191 |
| .260 | .3381 | 1.294 | .8204 |
| .265 | .3300 | 1.295 | .8216 |
| .270 | .3220 | 1.296 | .8229 |
| .275 | .3143 | 1.297 | .8241 |
| .280 | .3068 | 1.298 | .8253 |
| .285 | .2995 | 1.299 | .8265 |
| .290 | .2924 | 1.299 | .8277 |
| .2909 | .2911 | 1.299 | .8279 |

Table 4.7: Bounds for $\lambda'$ with $\chi_1$ or $\rho_1$ complex and $\chi_1$ non-principal

### $\chi_1$ principal

**Lemma 4.3.2.** *Assume $\chi_1$ is principal, $\rho_1$ is complex, and $\rho'$ is real. Suppose $f$ satisfies Conditions 1 and 2. For $\epsilon > 0$, provided $\mathcal{L}$ is sufficiently large depending on $\epsilon$ and $f$, the following holds:*

$$0 \leq 2F(-\lambda') - 2F(\lambda_1 - \lambda') - F(0) + 2\phi f(0) + \epsilon.$$

*Proof.* This is analogous to Lemma 4.2.1. To be brief, use (4.6) with $(\chi, \gamma) = (\chi_0, \gamma_1)$ and $\sigma = \beta'$ and apply Lemma 3.4.3 extracting $\{\rho', \rho_1, \overline{\rho_1}\}$ from $\mathcal{K}(\beta', \chi_0)$ and $\{\rho', \rho_1\}$ from $\mathcal{K}(\beta' + i\gamma_1, \chi_0)$. □

A numerical version of Lemma 4.3.2 yields bounds for $\lambda'$ with $f = f_\lambda$ taken from [HB92, Lemma 7.2], producing Table 4.8. The remaining case consists of $\chi_1$ principal with both $\rho_1$ and $\rho'$ complex.

**Lemma 4.3.3.** *Assume $\chi_1$ is principal, $\rho_1$ is complex and $\rho'$ is complex. Let $\lambda > 0$ and $J > 0$. If $\mathcal{L}$ is sufficiently large depending on $\epsilon, \lambda$ and $J$ then*

$$0 \leq (J^2 + \tfrac{1}{2})P_4(1) - (J^2 + \tfrac{1}{2}) \cdot P_4\Big(\frac{\lambda}{\lambda + \lambda_1}\Big) - 2J \cdot P_4\Big(\frac{\lambda}{\lambda + \lambda'}\Big) + 2(J+1)^2\phi\lambda + \epsilon$$

| $\lambda_1 \leq$ | $\lambda' \geq$ | $\lambda$ |
|---|---|---|
| .0875 | 1.836 | 1.189 |
| .09 | 1.803 | 1.185 |
| .10 | 1.681 | 1.170 |
| .11 | 1.572 | 1.156 |
| .12 | 1.472 | 1.142 |
| .13 | 1.381 | 1.129 |
| .14 | 1.297 | 1.116 |
| .15 | 1.220 | 1.103 |
| .16 | 1.148 | 1.091 |
| .17 | 1.080 | 1.079 |
| .18 | 1.017 | 1.068 |
| .19 | .9578 | 1.056 |
| .20 | .9020 | 1.045 |
| .21 | .8493 | 1.034 |

| $\lambda_1 \leq$ | $\lambda' \geq$ | $\lambda$ |
|---|---|---|
| .22 | .7994 | 1.023 |
| .23 | .7522 | 1.013 |
| .24 | .7073 | 1.002 |
| .25 | .6646 | .9917 |
| .26 | .6239 | .9813 |
| .27 | .5851 | .9711 |
| .28 | .5480 | .9609 |
| .29 | .5126 | .9508 |
| .30 | .4787 | .9407 |
| .31 | .4462 | .9307 |
| .32 | .4150 | .9208 |
| .33 | .3851 | .9108 |
| .34 | .3565 | .9009 |
| .3443 | .3445 | .8966 |

Table 4.8: Bounds for $\lambda'$ with $\chi_1$ principal, $\rho_1$ complex and $\rho'$ real

*provided both of the following hold:*

$$\frac{1}{(\lambda+\lambda_1)^4} + \frac{J_0}{(\lambda+\lambda')^4} > \frac{1}{\lambda^4} \qquad \textit{and} \qquad \frac{2}{(\lambda+\lambda_1)^4} + \frac{J_1}{(\lambda+\lambda')^4} > \frac{1}{\lambda^4}, \tag{4.23}$$

*where* $J_0 = \min\{J + \frac{3}{4J}, 4J\}$ *and* $J_1 = 4J/(J^2+1)$.

*Proof.* Analogous to Lemma 4.3.1 but we exchange the roles of $\rho_1$ and $\rho'$ using that

$$0 \leq \chi_0(\mathfrak{n})\big(1 + \mathrm{Re}\{(\mathrm{N}\mathfrak{n})^{i\gamma_1}\}\big)\big(J + \mathrm{Re}\{(\mathrm{N}\mathfrak{n})^{i\gamma'}\}\big)^2.$$

Writing $\mathcal{P}(s) = \mathcal{P}(s, \chi_0; P_4)$, it follows in the usual way that

$$\begin{aligned} 0 \leq (J^2 + \tfrac{1}{2})\big\{\mathcal{P}(\sigma) + \mathcal{P}(\sigma + i\gamma_1)\big\} + J\mathcal{P}(\sigma + i(\gamma' + \gamma_1)) + 2J\mathcal{P}(\sigma + i\gamma') + J\mathcal{P}(\sigma + i(\gamma' - \gamma_1)) \\ + \tfrac{1}{4}\mathcal{P}(\sigma + i(2\gamma' + \gamma_1)) + \tfrac{1}{2}\mathcal{P}(\sigma + 2i\gamma') + \tfrac{1}{4}\mathcal{P}(\sigma + i(2\gamma' - \gamma_1)), \end{aligned} \tag{4.24}$$

where $\sigma = 1 + \frac{\lambda}{\mathcal{L}}$. Next, apply Lemma 3.3.4 to each term according to the following outline:

- $\mathcal{P}(\sigma)$ and $\mathcal{P}(\sigma + i\gamma')$ extract all 4 zeros $\{\rho_1, \overline{\rho_1}, \rho', \overline{\rho'}\}$.
- $\mathcal{P}(\sigma + i\gamma_1)$ and $\mathcal{P}(\sigma + i(\gamma' + \gamma_1))$ extract only $\{\rho_1, \rho', \overline{\rho'}\}$.
- $\mathcal{P}(\sigma + i(\gamma' - \gamma_1))$ extract only $\{\overline{\rho_1}, \rho', \overline{\rho'}\}$.
- $\mathcal{P}(\sigma + i(2\gamma' + \gamma_1))$ and $\mathcal{P}(\sigma + i(2\gamma' - \gamma_1))$ extract $\{\rho_1, \rho'\}$ and $\{\overline{\rho_1}, \rho'\}$ respectively.

- $\mathcal{P}(\sigma + 2i\gamma')$ extract only $\{\rho_1, \overline{\rho_1}, \rho'\}$.

When necessary, we utilize that $P_4(\overline{X}) = \overline{P_4(X)}$. Then overall we obtain:

$$\begin{aligned} 0 \leq &(J^2 + \tfrac{1}{2})P_4(1) - (J^2 + \tfrac{1}{2})P_4\big(\frac{\lambda}{\lambda + \lambda_1}\big) - 2J \cdot P_4\big(\frac{\lambda}{\lambda + \lambda'}\big) + 2\phi(J+1)^2\lambda + \epsilon \\ &- \sum_{r=1}^{7} \mathrm{Re}\Big\{A_r \cdot P_4\big(\frac{\lambda}{\lambda + \lambda_1 + it_r}\big) + B_r \cdot P_4\big(\frac{\lambda}{\lambda + \lambda' + it_r}\big) - C_r \cdot P_4\big(\frac{\lambda}{\lambda + it_r}\big)\Big\}, \end{aligned} \tag{4.25}$$

where $t_r, A_r, B_r$, and $C_r$ are given by the following table.

| $r$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $t_r$ | $\mu_1$ | $\mu'$ | $\mu' + \mu_1$ | $\mu' - \mu_1$ | $2\mu'$ | $2\mu' + \mu_1$ | $2\mu' - \mu_1$ |
| $A_r$ | $2J^2 + 1$ | $2J$ | $2J$ | $2J$ | $1/2$ | $1/2$ | $1/2$ |
| $B_r$ | $2J$ | $2J^2 + 3/2$ | $J^2 + 3/4$ | $J^2 + 3/4$ | $2J$ | $J$ | $J$ |
| $C_r$ | $J^2 + 1/2$ | $2J$ | $J$ | $J$ | $1/2$ | $1/4$ | $1/4$ |

It suffices to show the sum over $r$ in (4.25) is non-negative. By Lemma 3.3.6, the sum is $\geq 0$ provided

$$\frac{A_r}{(\lambda + \lambda_1)^4} + \frac{B_r}{(\lambda + \lambda')^4} > \frac{C_r}{\lambda^4} \qquad \text{for } r = 1, 2, \ldots, 7.$$

After inspection, the most stringent conditions are $r = 1, 2$ and $5$, which are implied by assumption (4.23). □

This produces Table 4.9 in the usual fashion.

### Summary of bounds

We collect the results in the subsection into a single proposition for the reader's convenience.

**Proposition 4.3.4.** *Assume $\chi_1$ or $\rho_1$ is complex. Provided $\mathcal{L}$ is sufficiently large, we have the following:*

*(a) If $\chi_1$ is non-principal then $\lambda' \geq 0.2909$ and the bounds for $\lambda'$ in Table 4.7 apply.*

*(b) If $\chi_1$ is principal then $\lambda' \geq 0.2909$ and the bounds for $\lambda'$ in Table 4.9 apply.*

*Proof.* If $\chi_1$ is non-principal, then the only bounds available come from Table 4.7. If $\chi_1$ is principal, then upon comparing Tables 4.8 and 4.9, one finds that the latter gives weaker bounds. □

| $\lambda_1 \leq$ | $\lambda' \geq$ | $\lambda$ | $J$ |
|---|---|---|---|
| .0875 | .5330 | 1.155 | .8815 |
| .090 | .5278 | 1.161 | .8804 |
| .095 | .5179 | 1.171 | .8782 |
| .100 | .5083 | 1.181 | .8762 |
| .105 | .4991 | 1.190 | .8742 |
| .110 | .4902 | 1.198 | .8723 |
| .115 | .4817 | 1.206 | .8704 |
| .120 | .4734 | 1.213 | .8686 |
| .125 | .4654 | 1.220 | .8669 |
| .130 | .4577 | 1.226 | .8652 |
| .135 | .4502 | 1.232 | .8636 |
| .140 | .4429 | 1.238 | .8620 |
| .145 | .4359 | 1.243 | .8605 |
| .150 | .4290 | 1.248 | .8590 |
| .155 | .4223 | 1.252 | .8576 |
| .160 | .4159 | 1.257 | .8562 |
| .165 | .4096 | 1.261 | .8548 |
| .170 | .4034 | 1.265 | .8534 |
| .175 | .3974 | 1.268 | .8521 |
| .180 | .3916 | 1.271 | .8509 |
| .185 | .3859 | 1.274 | .8496 |
| .190 | .3804 | 1.277 | .8484 |
| .195 | .3749 | 1.280 | .8472 |
| .200 | .3696 | 1.282 | .8460 |
| .205 | .3645 | 1.284 | .8449 |
| .210 | .3594 | 1.286 | .8437 |
| .215 | .3545 | 1.288 | .8426 |
| .220 | .3497 | 1.290 | .8415 |
| .225 | .3449 | 1.291 | .8405 |
| .230 | .3403 | 1.293 | .8394 |
| .235 | .3358 | 1.294 | .8384 |
| .240 | .3314 | 1.295 | .8374 |
| .245 | .3270 | 1.296 | .8364 |
| .250 | .3228 | 1.297 | .8354 |
| .255 | .3186 | 1.297 | .8344 |
| .260 | .3145 | 1.298 | .8335 |
| .265 | .3106 | 1.298 | .8326 |
| .270 | .3066 | 1.299 | .8317 |
| .275 | .3028 | 1.299 | .8308 |
| .280 | .2990 | 1.299 | .8299 |
| .285 | .2953 | 1.299 | .8290 |
| .290 | .2917 | 1.299 | .8281 |
| .2909 | .2911 | 1.299 | .8280 |

Table 4.9: Bounds for $\lambda'$ with $\chi_1$ principal, $\rho_1$ complex and $\rho'$ complex

### 4.3.2 Bounds for $\lambda_2$

Before dividing into cases, we begin with the following lemma analogous to Lemma 4.3.1.

**Lemma 4.3.5.** *Assume $\chi_1$ or $\rho_1$ is complex. Suppose $f$ satisfies Conditions 1 and 2. For $\epsilon > 0$, provided $\mathcal{L}$ is sufficiently large depending on $\epsilon$ and $f$, the following holds:*

*(a) If $\chi_1, \chi_2$ are non-principal, then*

$$0 \leq F(-\lambda_1) - F(0) - F(\lambda_2 - \lambda_1) + 4\phi f(0) + \epsilon.$$

*(b) If $\chi_1$ is principal, then $\rho_1$ is complex, $\chi_2$ is non-principal and*

$$\begin{aligned} 0 \leq \quad & F(-\lambda_1) - F(0) - F(\lambda_2 - \lambda_1) + 4\phi f(0) + \epsilon \\ & + \operatorname{Re}\{F(-\lambda_1 + i\mu_1) - F(i\mu_1) - F(\lambda_2 - \lambda_1 + i\mu_1)\}. \end{aligned}$$

*(c) If $\chi_2$ is principal, then $\chi_1$ is non-principal and*

$$\begin{aligned} 0 \leq \quad & F(-\lambda_1) - F(0) - F(\lambda_2 - \lambda_1) + 4\phi f(0) + \epsilon \\ & + \operatorname{Re}\{F(-\lambda_1 + i\mu_2) - F(i\mu_2) - F(\lambda_2 - \lambda_1 + i\mu_2)\}. \end{aligned}$$

*Proof.* The arguments involved are very similar to Lemma 4.2.1 and Lemma 4.2.8 so we omit most of the details. Briefly, use (4.5) by setting $(\chi, \rho) = (\chi_1, \rho_1)$ and $(\chi_*, \rho_*) = (\chi_2, \rho_2)$ and $\sigma = \beta_1$, which gives

$$\begin{aligned} 0 \leq\ & \mathcal{K}(\beta_1, \chi_0) + \mathcal{K}(\beta_1 + i\gamma_1, \chi_1) + \mathcal{K}(\beta_1 + i\gamma_2, \chi_2) \\ & + \tfrac{1}{2}\mathcal{K}(\beta_1 + i(\gamma_1 + \gamma_2), \chi_1\chi_2) + \tfrac{1}{2}\mathcal{K}(\beta_1 + i(\gamma_1 - \gamma_2), \chi_1\overline{\chi_2}). \end{aligned} \tag{4.26}$$

Apply Lemma 3.4.3 to each $\mathcal{K}(*, *)$ term, extracting zeros $\rho_1$ or $\rho_2$ whenever possible, depending on the cases. Recall $\chi_1\chi_2$ and $\chi_1\overline{\chi_2}$ are always non-principal by construction (see Section 3.1). □

#### $\chi_1$ and $\chi_2$ non-principal

A numerical version of Lemma 4.3.5 suffices here.

**Lemma 4.3.6.** *Assume $\chi_1$ or $\rho_1$ is complex with $\chi_1, \chi_2$ non-principal. Let $\epsilon > 0$ and for $b > 0$, assume $0 < \lambda_1 \leq b$. Suppose, for some $\tilde{\lambda}_b > 0$, we have*

$$F(-b) - F(0) - F(\tilde{\lambda}_b - b) + 4\phi f(0) \leq 0.$$

| $\lambda_1 \leq$ | $\lambda_2 \geq$ | $\lambda$ |
|---|---|---|
| .1227 | .4890 | .3837 |
| .13 | .4779 | .3888 |
| .135 | .4706 | .3922 |
| .140 | .4635 | .3955 |
| .145 | .4566 | .3986 |
| .150 | .4499 | .4017 |
| .155 | .4433 | .4047 |
| .160 | .4370 | .4077 |
| .165 | .4308 | .4105 |
| .170 | .4247 | .4133 |
| .175 | .4188 | .4160 |
| .180 | .4131 | .4187 |
| .185 | .4075 | .4213 |
| .190 | .4020 | .4238 |
| .195 | .3966 | .4263 |
| .200 | .3914 | .4287 |
| .205 | .3862 | .4311 |
| .210 | .3812 | .4334 |
| .215 | .3763 | .4357 |

| $\lambda_1 \leq$ | $\lambda_2 \geq$ | $\lambda$ |
|---|---|---|
| .220 | .3715 | .4380 |
| .225 | .3668 | .4402 |
| .230 | .3622 | .4423 |
| .235 | .3576 | .4444 |
| .240 | .3532 | .4465 |
| .245 | .3488 | .4486 |
| .250 | .3446 | .4506 |
| .255 | .3404 | .4526 |
| .260 | .3363 | .4545 |
| .265 | .3322 | .4564 |
| .270 | .3283 | .4583 |
| .275 | .3244 | .4602 |
| .280 | .3205 | .4620 |
| .285 | .3168 | .4638 |
| .290 | .3131 | .4656 |
| .295 | .3094 | .4673 |
| .300 | .3059 | .4690 |
| .3034 | .3035 | .4702 |

Table 4.10: Bounds for $\lambda_2$ with $\chi_1$ or $\rho_1$ complex and $\chi_1, \chi_2$ non-principal

*Then $\lambda_2 \geq \tilde{\lambda}_b - \epsilon$ provided $\mathcal{L}$ is sufficiently large depending on $\epsilon$ and $f$.*

*Proof.* Analogous to Lemma 4.2.10 using Lemma 4.3.5 in place of Lemma 4.2.8. Hence, we omit the proof. □

This produces Table 4.10 by taking $f = f_\lambda$ from [HB92, Lemma 7.2] with parameter $\lambda = \lambda(b)$.

### $\chi_1$ principal or $\chi_2$ is principal

When $\chi_2$ is principal and $\rho_2$ is real, a numerical version of Lemma 4.3.5 suffices.

**Lemma 4.3.7.** *Assume $\chi_1$ or $\rho_1$ is complex. Further assume $\chi_2$ is principal and $\rho_2$ is real. Let $\epsilon > 0$ and for $b > 0$, assume $0 < \lambda_1 \leq b$. Suppose, for some $\tilde{\lambda}_b > 0$, we have*

$$F(-b) - F(0) - F(\tilde{\lambda}_b - b) + 2\phi f(0) \leq 0.$$

*Then $\lambda_2 \geq \tilde{\lambda}_b - \epsilon$ provided $\mathcal{L}$ is sufficiently large depending on $\epsilon$ and $f$.*

This produces Table 4.11 by taking $f$ from [HB92, Lemma 7.2] with parameter $\lambda = \lambda(b)$.

| $\lambda_1 \le$ | $\lambda_2 \ge$ | $\lambda$ | $\lambda_1 \le$ | $\lambda_2 \ge$ | $\lambda$ |
|---|---|---|---|---|---|
| .1227 | 1.221 | .6530 | .37 | .8149 | .8425 |
| .13 | 1.203 | .6620 | .39 | .7932 | .8526 |
| .15 | 1.155 | .6846 | .41 | .7725 | .8622 |
| .17 | 1.112 | .7049 | .43 | .7526 | .8714 |
| .19 | 1.073 | .7234 | .45 | .7335 | .8803 |
| .21 | 1.037 | .7403 | .47 | .7152 | .8889 |
| .23 | 1.003 | .7560 | .49 | .6977 | .8971 |
| .25 | .9710 | .7707 | .51 | .6807 | .9051 |
| .27 | .9412 | .7844 | .53 | .6644 | .9128 |
| .29 | .9132 | .7973 | .55 | .6487 | .9203 |
| .31 | .8867 | .8095 | .57 | .6336 | .9276 |
| .33 | .8615 | .8210 | .59 | .6189 | .9346 |
| .35 | .8377 | .8320 | .6068 | .6070 | .9404 |

Table 4.11: Bounds for $\lambda_2$ with $\chi_1$ or $\rho_1$ complex and $\chi_2$ principal and $\rho_2$ real

Now, when $\chi_1$ is principal or when $\chi_2$ is principal and $\rho_2$ is complex, we employ the "polynomial method".

**Lemma 4.3.8.** *Suppose $\chi_j$ is principal and $\rho_j$ is complex, and let $\chi_k \neq \chi_j$. Let $\epsilon, \lambda, J > 0$. If $\mathcal{L}$ is sufficiently large depending on $\epsilon, \lambda$ and $J$, then*

$$0 \le (J^2 + \tfrac{1}{2})\big\{P_4(1) - P_4\big(\frac{\lambda}{\lambda+\lambda_k}\big)\big\} - 2JP_4\big(\frac{\lambda}{\lambda+\lambda_j}\big) + 2\phi(J+1)^2\lambda + \epsilon$$

*provided*

$$\frac{J_0}{(\lambda+\lambda_j)^4} + \frac{1}{(\lambda+\lambda_k)^4} > \frac{1}{\lambda^4}, \qquad \textit{where } J_0 = \min\{J + \tfrac{3}{4J}, 4J\}. \tag{4.27}$$

*Proof.* Write $\mathcal{P}(s,\chi) = \mathcal{P}(s,\chi;P_4)$. We begin with the inequality

$$0 \le \chi_0(\mathfrak{n})\big(1 + \mathrm{Re}\{\chi_k(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{-i\gamma_k}\}\big)\big(J + \mathrm{Re}\{(\mathrm{N}\mathfrak{n})^{-i\gamma_j}\}\big)^2.$$

It follows in the usual fashion that

$$\begin{aligned} 0 \le (J^2 + \tfrac{1}{2})&\big\{\mathcal{P}(\sigma,\chi_0) + \mathcal{P}(\sigma + i\gamma_k,\chi_k)\big\} + \\ &+ J\mathcal{P}(\sigma + i(\gamma_j+\gamma_k),\chi_k) + 2J\mathcal{P}(\sigma + i\gamma_j,\chi_0) + J\mathcal{P}(\sigma + i(\gamma_j-\gamma_k),\overline{\chi_k}) \\ &+ \tfrac{1}{4}\mathcal{P}(\sigma + i(2\gamma_j+\gamma_k),\chi_k) + \tfrac{1}{2}\mathcal{P}(\sigma + 2i\gamma_j,\chi_0) + \tfrac{1}{4}\mathcal{P}(\sigma + i(2\gamma_j-\gamma_k),\overline{\chi_k}), \end{aligned} \tag{4.28}$$

where $\sigma = 1 + \frac{\lambda}{\mathcal{L}}$. Next, apply Lemma 3.3.4 to each $\mathcal{P}(*,*)$ term in (4.28) extracting $\{\rho_j,\overline{\rho_j}\}$ from $\chi_0$-terms, $\rho_k$ from the $\chi_k$-terms, and $\overline{\rho_k}$ from $\overline{\chi_k}$-terms. When necessary, we also use that

$P_4(\overline{X}) = \overline{P_4(X)}$. Then overall

$$0 \le (J^2 + \tfrac{1}{2})P_4(1) - (J^2 + \tfrac{1}{2})P_4\big(\frac{\lambda}{\lambda + \lambda_k}\big) - 2JP_4\big(\frac{\lambda}{\lambda + \lambda_j}\big) + \psi\lambda + \epsilon - A - B, \tag{4.29}$$

where

$$A = \mathrm{Re}\Big\{(2J^2 + \tfrac{3}{2})P_4\big(\frac{\lambda}{\lambda + \lambda_j + i\mu_j}\big) + 2J \cdot P_4\big(\frac{\lambda}{\lambda + \lambda_k + i\mu_j}\big) - 2J \cdot P_4\big(\frac{\lambda}{\lambda + i\mu_j}\big)\Big\},$$
$$B = \mathrm{Re}\Big\{2J \cdot P_4\big(\frac{\lambda}{\lambda + \lambda_j + 2i\mu_j}\big) + \tfrac{1}{2} \cdot P_4\big(\frac{\lambda}{\lambda + \lambda_k + 2i\mu_j}\big) - \tfrac{1}{2} \cdot P_4\big(\frac{\lambda}{\lambda + 2i\mu_j}\big)\Big\},$$

and $\psi = (J^2 + 2J + 1)\phi\frac{\mathcal{L}_{\chi_k}}{\mathcal{L}} + (J^2 + 2J + 1)\phi\frac{\mathcal{L}_0}{\mathcal{L}}$. Trivially $J^2 + 2J + 1 \le 3 \cdot (J^2 + 2J + 1)$ so, by Lemma 3.1.1, we may more simply take $\psi = 2(J + 1)^2\phi$ in (4.29). From Lemma 3.3.6 and (4.27), it follows that $A, B \ge 0$. □

We record a numerical version of Lemma 4.3.8 without proof.

**Corollary 4.3.9.** *Suppose $\chi_1$ or $\rho_1$ is complex. For $b > 0$, assume $0 < \lambda_1 \le b$ and let $\lambda, J > 0$. Denote $J_0 := \min\{J + \frac{3}{4J}, 4J\}$. Assume one of the following holds:*

*(a) $\chi_1$ is principal, $\rho_1$ is complex. Further there exists $\tilde{\lambda_b} \in [0, \infty)$ satisfying*

$$0 = (J^2 + \tfrac{1}{2})\big(P_4(1) - P_4\big(\frac{\lambda}{\lambda + \tilde{\lambda_b}}\big)\big) - 2J \cdot P_4\big(\frac{\lambda}{\lambda + b}\big) + 2\phi(J + 1)^2\lambda + \epsilon$$

*and*

$$\frac{J_0}{(\lambda + b)^4} + \frac{1}{(\lambda + \tilde{\lambda_b})^4} > \frac{1}{\lambda^4}.$$

*(b) $\chi_2$ is principal, $\rho_2$ is complex. Further there exists $\tilde{\lambda_b} \in [0, \infty)$ satisfying*

$$0 = (J^2 + \tfrac{1}{2})\big(P_4(1) - P_4\big(\frac{\lambda}{\lambda + b}\big)\big) - 2J \cdot P_4\big(\frac{\lambda}{\lambda + \tilde{\lambda_b}}\big) + 2\phi(J + 1)^2\lambda + \epsilon$$

*and*

$$\frac{1}{(\lambda + b)^4} + \frac{J_0}{(\lambda + \tilde{\lambda_b})^4} > \frac{1}{\lambda^4}.$$

*Then, in either case, $\lambda_2 \ge \tilde{\lambda_b} - \epsilon$ for $\mathcal{L}$ sufficiently large depending on $\epsilon, b, \lambda$ and $J$.*

This produces Tables 4.12 and 4.13.

| $\lambda_1 \leq$ | $\lambda_2 \geq$ | $\lambda$ | $J$ |
|---|---|---|---|
| .0875 | 1.017 | .9321 | .7627 |
| .090 | .9892 | .9474 | .7640 |
| .095 | .9385 | .9760 | .7666 |
| .100 | .8937 | 1.002 | .7690 |
| .105 | .8537 | 1.026 | .7713 |
| .110 | .8175 | 1.048 | .7735 |
| .115 | .7846 | 1.069 | .7757 |
| .120 | .7544 | 1.087 | .7777 |
| .125 | .7266 | 1.104 | .7797 |
| .130 | .7007 | 1.120 | .7817 |
| .135 | .6766 | 1.135 | .7836 |
| .140 | .6540 | 1.149 | .7854 |
| .145 | .6328 | 1.162 | .7872 |
| .150 | .6128 | 1.174 | .7889 |
| .155 | .5939 | 1.185 | .7906 |
| .160 | .5759 | 1.195 | .7923 |
| .165 | .5589 | 1.204 | .7939 |
| .170 | .5427 | 1.213 | .7955 |
| .175 | .5272 | 1.221 | .7971 |
| .180 | .5124 | 1.229 | .7986 |
| .185 | .4982 | 1.236 | .8001 |
| .190 | .4846 | 1.242 | .8016 |

| $\lambda_1 \leq$ | $\lambda_2 \geq$ | $\lambda$ | $J$ |
|---|---|---|---|
| .195 | .4715 | 1.249 | .8030 |
| .200 | .4590 | 1.254 | .8045 |
| .205 | .4469 | 1.259 | .8059 |
| .210 | .4353 | 1.264 | .8073 |
| .215 | .4241 | 1.269 | .8087 |
| .220 | .4132 | 1.273 | .8100 |
| .225 | .4027 | 1.276 | .8114 |
| .230 | .3926 | 1.280 | .8127 |
| .235 | .3828 | 1.283 | .8140 |
| .240 | .3733 | 1.285 | .8153 |
| .245 | .3641 | 1.288 | .8166 |
| .250 | .3552 | 1.290 | .8179 |
| .255 | .3465 | 1.292 | .8191 |
| .260 | .3381 | 1.294 | .8204 |
| .265 | .3300 | 1.295 | .8216 |
| .270 | .3220 | 1.296 | .8229 |
| .275 | .3143 | 1.297 | .8241 |
| .280 | .3068 | 1.298 | .8253 |
| .285 | .2995 | 1.299 | .8265 |
| .290 | .2924 | 1.299 | .8277 |
| .2909 | .2911 | 1.299 | .8279 |

Table 4.12: Bounds for $\lambda_2$ with $\chi_1$ principal and $\rho_1$ complex

| $\lambda_1 \leq$ | $\lambda_2 \geq$ | $\lambda$ | $J$ |
|---|---|---|---|
| .1227 | .4691 | 1.217 | .8677 |
| .125 | .4654 | 1.220 | .8669 |
| .130 | .4577 | 1.226 | .8652 |
| .135 | .4502 | 1.232 | .8636 |
| .140 | .4429 | 1.238 | .8620 |
| .145 | .4359 | 1.243 | .8605 |
| .150 | .4290 | 1.248 | .8590 |
| .155 | .4223 | 1.252 | .8576 |
| .160 | .4159 | 1.257 | .8562 |
| .165 | .4096 | 1.261 | .8548 |
| .170 | .4034 | 1.265 | .8534 |
| .175 | .3974 | 1.268 | .8521 |
| .180 | .3916 | 1.271 | .8509 |
| .185 | .3859 | 1.274 | .8496 |
| .190 | .3804 | 1.277 | .8484 |
| .195 | .3749 | 1.280 | .8472 |
| .200 | .3696 | 1.282 | .8460 |
| .205 | .3645 | 1.284 | .8449 |
| .210 | .3594 | 1.286 | .8437 |
| .215 | .3545 | 1.288 | .8426 |
| .220 | .3497 | 1.290 | .8415 |
| .225 | .3449 | 1.291 | .8405 |
| .230 | .3403 | 1.293 | .8394 |
| .235 | .3358 | 1.294 | .8384 |
| .240 | .3314 | 1.295 | .8374 |
| .245 | .3270 | 1.296 | .8364 |
| .250 | .3228 | 1.297 | .8354 |
| .255 | .3186 | 1.297 | .8344 |
| .260 | .3145 | 1.298 | .8335 |
| .265 | .3106 | 1.298 | .8326 |
| .270 | .3066 | 1.299 | .8317 |
| .275 | .3028 | 1.299 | .8308 |
| .280 | .2990 | 1.299 | .8299 |
| .285 | .2953 | 1.299 | .8290 |
| .290 | .2917 | 1.299 | .8281 |
| .2909 | .2911 | 1.299 | .8280 |

Table 4.13: Bounds for $\lambda_2$ with $\chi_1$ or $\rho_1$ complex and $\chi_2$ principal and $\rho_2$ complex

**Summary of bounds**

We collect the results in the subsection into a single proposition for the reader's convenience.

**Proposition 4.3.10.** *Assume $\chi_1$ or $\rho_1$ is complex. Provided $\mathcal{L}$ is sufficiently large, the following holds:*

*(a) If $\chi_1$ is non-principal, then $\lambda_2 \geq 0.2909$ and the bounds for $\lambda_2$ in Table 4.13 apply.*

*(b) If $\chi_1$ is principal, then $\lambda_2 \geq 0.2909$ and the bounds for $\lambda_2$ in Table 4.12 apply.*

*Proof.* If $\chi_1$ is non-principal then one compares Table 4.10, Table 4.11 and Table 4.13 and finds that the last one gives the weakest bounds. If $\chi_1$ is principal, then the only bounds available come from Table 4.12. $\square$

## 4.4 Zero-free region

*Proof of Theorem 4.1.1*: If $\chi_1$ and $\rho_1$ are both real, then Theorem 4.1.1 is implied by Propositions 4.2.7 and 4.2.13. Thus, it remains to consider when $\chi_1$ or $\rho_1$ is complex, dividing our cases according to the order of $\chi_1$.

### $\chi_1$ has order $\geq 5$

We begin with the inequality

$$0 \leq \chi_0(\mathfrak{n})\big(3 + 10 \cdot \mathrm{Re}\{\chi_1(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{-i\gamma_1}\}\big)^2\big(9 + 10 \cdot \mathrm{Re}\{\chi_1(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{-i\gamma_1}\}\big)^2, \tag{4.30}$$

which was also used in [HB92, Section 9]. This will also be roughly optimal for our purposes. We shall use the smoothed explicit inequality with a weight $f$ satisfying Conditions 1 and 2. By the usual arguments, we expand out the above identity, multiply by the appropriate factor and sum over $\mathfrak{n}$. Overall this yields

$$\begin{aligned} 0 \leq 14379 \cdot \mathcal{K}(\sigma, \chi_0) &+ 24480 \cdot \mathcal{K}(\sigma + i\gamma_1, \chi_1) + 14900 \cdot \mathcal{K}(\sigma + 2i\gamma_1, \chi_1^2) \\ &+ 6000 \cdot \mathcal{K}(\sigma + 3i\gamma_1, \chi_1^3) + 1250 \cdot \mathcal{K}(\sigma + 4i\gamma_1, \chi_1^4), \end{aligned} \tag{4.31}$$

where $\mathcal{K}(s, \chi) = \mathcal{K}(s, \chi; f)$ and $\sigma = 1 - \frac{\lambda^\star}{\mathcal{L}}$ with constant $\lambda^\star$ satisfying

$$\lambda_1 \leq \lambda^\star \leq \min\{\lambda', \lambda_2\}.$$

Now, apply Lemma 3.4.3 to each term in (4.31) and consider cases depending on $\mathrm{ord}\,\chi_1$. For $\mathcal{K}(\sigma + ni\gamma_1, \chi_1^n)$:

- $(\mathrm{ord}\,\chi_1 \geq 6)$ Extract $\{\rho_1\}$ if $n = 1$ only.
- $(\mathrm{ord}\,\chi_1 = 5)$ Set $\lambda^\star = \lambda_1$ and extract $\{\rho_1\}$ if $n = 1$ only.

It follows that

$$0 \leq 14379 \cdot F(-\lambda^\star) - 24480 \cdot F\big(\lambda_1 - \lambda^\star\big) + Bf(0)\phi + \epsilon, \tag{4.32}$$

where $B = 14379 \cdot \frac{\mathcal{L}_0}{\mathcal{L}} + 46630 \cdot \frac{\mathcal{L}_{\chi_1}}{\mathcal{L}}$. From Lemma 3.1.1, $B \leq 57516 + 3493\frac{\mathcal{L}_{\chi_1}}{\mathcal{L}} \leq 62174$ so (4.32) reduces to

$$0 \leq 14379 \cdot F(-\lambda^\star) - 24480 \cdot F\big(\lambda_1 - \lambda^\star\big) + 62174\phi f(0) + \epsilon. \tag{4.33}$$

We now consider cases.

- $(\mathrm{ord}\,\chi_1 \geq 6)$ Without loss, we may assume $\lambda_1 \leq 0.180$. From Propositions 4.3.4 and 4.3.10, we may take $\lambda^\star = 0.3916$. Choose $f$ according to [HB92, Lemma 7.1] with parameters $\theta = 1$ and $\lambda = 0.243$. Then (4.33) implies $\lambda_1 \geq 0.1764$.

- $(\operatorname{ord}\chi_1 = 5)$ Since $\lambda^\star = \lambda_1$ in this case, (4.33) becomes

$$0 \le 14379 \cdot F(-\lambda_1) - 24480 \cdot F(0) + 62174 f(0)\phi + \epsilon.$$

We choose $f$ according to [HB92, Lemma 7.5] with $k = 24480/14379$ giving $\theta = 1.1580...$ and

$$\lambda_1^{-1}\cos^2\theta \le \frac{1}{4}\cdot\frac{62174}{14379} + \epsilon,$$

whence $\lambda_1 \ge 0.1489$.

## $\chi_1$ has order $2, 3$ or $4$

We use the same identity (4.30), but instead we will apply the "polynomial method" with $P_4(X)$. In the usual way, it follows from (4.30) that

$$\begin{aligned} 0 \le 14379 \cdot \mathcal{P}(\sigma,\chi_0) &+ 24480 \cdot \mathcal{P}(\sigma+i\gamma_1,\chi_1) + 14900 \cdot \mathcal{P}(\sigma+2i\gamma_1,\chi_1^2) \\ &+ 6000 \cdot \mathcal{P}(\sigma+3i\gamma_1,\chi_1^3) + 1250 \cdot \mathcal{P}(\sigma+4i\gamma_1,\chi_1^4), \end{aligned} \tag{4.34}$$

where $\sigma = 1 + \frac{\lambda}{\mathcal{L}}$ with $\lambda > 0$. The above identity will be roughly optimal for our purposes. Now, we apply Lemma 3.3.4 to each term above and consider cases depending on $\operatorname{ord}\chi_1$. For each term $\mathcal{P}(\sigma + ni\gamma_1, \chi_1^n)$:

- $(\operatorname{ord}\chi_1 = 4)$ Extract $\{\rho_1\}$ if $n = 1$ and $\{\overline{\rho_1}\}$ if $n = 3$.

- $(\operatorname{ord}\chi_1 = 3)$ Extract $\{\rho_1\}$ if $n = 1$ or $4$ and $\{\overline{\rho_1}\}$ if $n = 2$.

- $(\operatorname{ord}\chi_1 = 2)$ Extract $\{\rho_1, \overline{\rho_1}\}$ if $n = 1$ or $3$ since $\rho_1$ is necessarily complex.

It follows that

$$0 \le 14379 \cdot P_4(1) - 24480 \cdot P_4\Big(\frac{\lambda}{\lambda+\lambda_1}\Big) + A_{\chi_1} + B_{\chi_1}\phi\lambda + \epsilon, \tag{4.35}$$

where

$$A_{\chi_1} = \begin{cases} \mathrm{Re}\{1250 \cdot P_4\big(\frac{\lambda}{\lambda+4i\mu_1}\big) - 6000 \cdot P_4\big(\frac{\lambda}{\lambda+\lambda_1+4i\mu_1}\big)\}, & \mathrm{ord}\,\chi_1 = 4, \\ \mathrm{Re}\{6000 \cdot P_4\big(\frac{\lambda}{\lambda+3i\mu_1}\big) - 16150 \cdot P_4\big(\frac{\lambda}{\lambda+\lambda_1+3i\mu_1}\big)\}, & \mathrm{ord}\,\chi_1 = 3, \\ \mathrm{Re}\{14900 \cdot P_4\big(\frac{\lambda}{\lambda+2i\mu_1}\big) - 30480 \cdot P_4\big(\frac{\lambda}{\lambda+\lambda_1+2i\mu_1}\big)\} & \mathrm{ord}\,\chi_1 = 2, \\ \quad +\mathrm{Re}\{1250 \cdot P_4\big(\frac{\lambda}{\lambda+4i\mu_1}\big) - 6000 \cdot P_4\big(\frac{\lambda}{\lambda+\lambda_1+4i\mu_1}\big)\}, & \end{cases}$$

and

$$B_{\chi_1} = \begin{cases} 15629 \cdot \frac{\mathcal{L}_0}{\mathcal{L}} + 45380 \cdot \frac{\mathcal{L}_{\chi_1}}{\mathcal{L}} & \text{if } \mathrm{ord}\,\chi_1 = 4, \\ 20379 \cdot \frac{\mathcal{L}_0}{\mathcal{L}} + 40630 \cdot \frac{\mathcal{L}_{\chi_1}}{\mathcal{L}} & \text{if } \mathrm{ord}\,\chi_1 = 3, \\ 30529 \cdot \frac{\mathcal{L}_0}{\mathcal{L}} + 30480 \cdot \frac{\mathcal{L}_{\chi_1}}{\mathcal{L}} & \text{if } \mathrm{ord}\,\chi_1 = 2. \end{cases}$$

By Lemma 3.1.1, we observe $B_{\chi_1} \leq 61009$. Furthermore, applying Lemma 3.3.6 to $A_{\chi_1}$, it follows that $A_{\chi_1} \leq 0$ in all cases provided

$$\frac{14900}{\lambda^4} - \frac{30480}{(\lambda+\lambda_1)^4} \leq 0. \tag{4.36}$$

Thus, (4.35) implies

$$0 \leq 14379 \cdot P_4(1) - 24480 \cdot P_4\big(\frac{\lambda}{\lambda+\lambda_1}\big) + 61009\phi\lambda + \epsilon$$

provided (4.36) holds. Taking $\lambda = 0.9421$ yields $\lambda_1 \geq 0.1227$.

## $\chi_1$ is principal

Recall in this case we assume $\rho_1$ is complex. We begin with a slightly different inequality:

$$0 \leq \chi_0(\mathfrak{n})\big(0 + 10 \cdot \mathrm{Re}\{(\mathrm{N}\mathfrak{n})^{-i\gamma_1}\}\big)^2\big(7 + 10 \cdot \mathrm{Re}\{(\mathrm{N}\mathfrak{n})^{-i\gamma_1}\}\big)^2.$$

Again using the "polynomial method" with $P_4(X)$, it similarly follows that

$$\begin{aligned} 0 \leq 620 \cdot \mathcal{P}(\sigma, \chi_0) + 1050 \cdot \mathcal{P}(\sigma + i\gamma_1, \chi_0) + 745 \cdot \mathcal{P}(\sigma + 2i\gamma_1, \chi_0) \\ + 350 \cdot \mathcal{P}(\sigma + 3i\gamma_1, \chi_0) + 125 \cdot \mathcal{P}(\sigma + 4i\gamma_1, \chi_0), \end{aligned} \tag{4.37}$$

where $\sigma = 1 + \frac{\lambda}{\mathcal{L}}$ with $\lambda > 0$. Apply Lemma 3.3.4 to each term above, extracting $\{\rho_1, \overline{\rho_1}\}$ since $\rho_1$ is necessarily complex. Since $\mathcal{L}_0 \le \mathcal{L}$, we have that

$$0 \le 620 \cdot P_4(1) - 1050 \cdot P_4\Big(\frac{\lambda}{\lambda + \lambda_1}\Big) + A_0 + 2890\phi\lambda + \epsilon, \tag{4.38}$$

where

$$\begin{aligned} A_0 = \mathrm{Re}\{1050 \cdot P_4\Big(\frac{\lambda}{\lambda + i\mu_1}\Big) - 1365 \cdot P_4\Big(\frac{\lambda}{\lambda + \lambda_1 + i\mu_1}\Big)\} \\ + \mathrm{Re}\{745 \cdot P_4\Big(\frac{\lambda}{\lambda + 2i\mu_1}\Big) - 1400 \cdot P_4\Big(\frac{\lambda}{\lambda + \lambda_1 + 2i\mu_1}\Big)\} \\ + \mathrm{Re}\{350 \cdot P_4\Big(\frac{\lambda}{\lambda + 3i\mu_1}\Big) - 870 \cdot P_4\Big(\frac{\lambda}{\lambda + \lambda_1 + 3i\mu_1}\Big)\} \\ + \mathrm{Re}\{125 \cdot P_4\Big(\frac{\lambda}{\lambda + 4i\mu_1}\Big) - 350 \cdot P_4\Big(\frac{\lambda}{\lambda + \lambda_1 + 4i\mu_1}\Big)\}. \end{aligned}$$

Applying Lemma 3.3.6 to each term of $A_0$, it follows that $A_0 \le 0$ provided

$$\frac{1050}{\lambda^4} - \frac{1365}{(\lambda + \lambda_1)^4} \le 0. \tag{4.39}$$

Thus, (4.38) implies

$$0 \le 620 \cdot P_4(1) - 1050 \cdot P_4\Big(\frac{\lambda}{\lambda + \lambda_1}\Big) + 2890\phi\lambda + \epsilon$$

provided (4.39) is satisfied. Taking $\lambda = 1.291$ yields $\lambda_1 \ge 0.0875$. This completes the proof of Theorem 4.1.1. □

## 4.5 Numerical zero density estimate

Recall $T_\star \ge 1$ is fixed and $H$ is an arbitrary congruence class group of $K$. Let us first introduce some notation intended only for this section.

#### Worst low-lying zeros of each character

Consider the rectangle

$$\{s \in \mathbb{C} : 0 \le \sigma \le 1, \quad |t| \le T_\star\}.$$

For each character $\pmod H$ with a zero in this rectangle, index it $\chi^{(k)}$ for $k = 1, 2, \dots$ with a zero $\rho^{(k)}$ in this rectangle defined by:

$$\mathrm{Re}(\rho^{(k)}) = \max\{\mathrm{Re}(\rho) : L(\rho, \chi^{(k)}) = 0, |\gamma| \leq T_\star\},$$

so $\chi^{(j)} \neq \chi^{(k)}$ for $j \neq k$. Write

$$\rho^{(k)} := \beta^{(k)} + i\gamma^{(k)}, \quad \beta^{(k)} = 1 - \frac{\lambda^{(k)}}{\mathcal{L}}, \quad \gamma^{(k)} = \frac{\mu^{(k)}}{\mathcal{L}}.$$

Without loss, we may assume $\lambda^{(1)} \leq \lambda^{(2)} \leq \dots$ and so on.

**Remark.** Upon comparing with the indexing given in Section 3.1, we always have the bound $\lambda_k \geq \lambda^{(k)}$ for all $k$ where both quantities exist. Further, $\lambda_1 = \lambda^{(1)}$.

### 4.5.1 Low-lying zero density

For $\lambda \geq 0$, consider the rectangle

$$\mathcal{S} = \mathcal{S}(\lambda) := \{s \in \mathbb{C} : 1 - \frac{\lambda}{\mathcal{L}} \leq \sigma \leq 1, \quad |t| \leq T_\star\}.$$

Define

$$N = N(\lambda) := \#\{\chi \pmod H : \chi \neq \chi_0, L(s, \chi) \text{ has a zero in } \mathcal{S}(\lambda)\} = \sum_{\substack{\lambda^{(k)} \leq \lambda \\ \chi^{(k)} \neq \chi_0}} 1.$$

Below is the main result of this section which gives bounds on $N(\lambda)$ using the smoothed explicit inequality.

**Theorem 4.5.1.** *Suppose $f$ satisfies Conditions 1 and 2 and let $\epsilon > 0$. Assume $\lambda_1 \geq b$ for some $b \geq 0$. For $\lambda \geq 0$, if*

$$F(\lambda - b) > \tfrac{4}{3} f(0)\phi,$$

*and*

$$\Big(F(\lambda - b) - \tfrac{4}{3} f(0)\phi\Big)^2 > \tfrac{4}{3} f(0)\phi\Big(f(0)\phi + F(-b)\Big)$$

*then unconditionally,*

$$N(\lambda) \leq \frac{\Big(f(0)\phi + F(-b)\Big)\Big(F(-b) - \frac{1}{3} f(0)\phi\Big)}{\Big(F(\lambda - b) - \frac{4}{3} f(0)\phi\Big)^2 - \frac{4}{3} f(0)\phi\Big(f(0)\phi + F(-b)\Big)} + \epsilon \tag{4.40}$$

*for $\mathcal{L}$ sufficiently large depending on $\epsilon$, $T_\star$, and $f$.*

**Remark.** If $\zeta_K(s)$ has a real zero in $\mathcal{S}(\lambda)$, then one can extract this zero from $\mathcal{K}(\sigma, \chi_0; f)$ in the argument below and hence improve (4.40) to

$$N(\lambda) \leq \frac{\Big(f(0)\phi + F(-b) - F(\lambda - b)\Big)\Big(F(-b) - F(\lambda - b) - \frac{1}{3}f(0)\phi\Big)}{\Big(F(\lambda - b) - \frac{4}{3}f(0)\phi\Big)^2 - \frac{4}{3}f(0)\phi\Big(f(0)\phi + F(-b) - F(\lambda - b)\Big)} + \epsilon$$

with naturally modified assumptions. The utility of such a bound is not entirely clear. If the real zero is exceptional, then the zero repulsion from Section 4.2 would likely be a better substitute.

*Proof.* We closely follow the arguments in [HB92, Section 12]. Let $\chi \pmod H$ denote a non-principal character with a zero $\tilde{\rho} = \tilde{\beta} + i\tilde{\gamma}$ in $\mathcal{S}(\lambda)$; that is, $b \leq \lambda_1 \leq \tilde{\lambda} \leq \lambda$. Applying Lemma 3.4.3 with $s = \sigma + i\tilde{\gamma}$ where $\sigma = 1 - \frac{b}{\mathcal{L}}$ and $\mathcal{Z} = \{\tilde{\rho}\}$ we find that

$$\mathcal{L}^{-1} \cdot \mathcal{K}(\sigma + i\tilde{\gamma}, \chi; f) \leq f(0)\phi\tfrac{\mathcal{L}_\chi}{\mathcal{L}} - F(\tilde{\lambda} - b) + \epsilon \tag{4.41}$$

for $\mathcal{L}$ sufficiently large depending on $\epsilon$ and the content of $f$. Since $F$ is decreasing by Condition 2, it follows that $F(\tilde{\lambda} - b) \geq F(\lambda - b)$. Also recalling that $\frac{\mathcal{L}_\chi}{\mathcal{L}} \leq \frac{4}{3}$ by (3.3) and (3.4), we see that (4.41) implies:

$$\mathcal{L}^{-1} \cdot \mathcal{K}(\sigma + i\tilde{\gamma}, \chi; f) \leq f(0)\tfrac{4}{3}\phi - F(\lambda - b) + \epsilon. \tag{4.42}$$

Summing (4.42) over $\chi = \chi^{(j)}$ (which are non-principal by construction) and $\tilde{\gamma} = \gamma^{(j)}$ for $j = 1, \dots, N$ where $N = N(\lambda)$, we deduce that

$$\begin{aligned}\Big(F(\lambda - b) - f(0)\tfrac{4}{3}\phi - \epsilon\Big)N\mathcal{L} &\leq -\sum_{j\leq N}\mathcal{K}(\sigma + i\gamma^{(j)}, \chi^{(j)}; f)\\ &= -\sum_{(\mathfrak{n},\mathfrak{q})=1}\Lambda(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{-\sigma}f\Big(\frac{\log \mathrm{N}\mathfrak{n}}{\mathcal{L}}\Big)\mathrm{Re}\Big\{\sum_{j\leq N}\chi^{(j)}(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{-i\gamma^{(j)}}\Big\}\\ &\leq \sum_{(\mathfrak{n},\mathfrak{q})=1}\Lambda(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{-\sigma}f\Big(\frac{\log \mathrm{N}\mathfrak{n}}{\mathcal{L}}\Big)\Big|\sum_{j\leq N}\chi^{(j)}(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{-i\gamma^{(j)}}\Big|.\end{aligned} \tag{4.43}$$

The LHS of (4.43) is positive by assumption so after squaring both sides of (4.43), we apply Cauchy-Schwarz to the last expression on the RHS implying

$$\big(\text{LHS of (4.43)}\big)^2 \leq S_1 S_2,$$

where

$$S_1 = \sum_{(\mathfrak{n},\mathfrak{q})=1} \Lambda(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{-\sigma} f\Big(\frac{\log \mathrm{N}\mathfrak{n}}{\mathcal{L}}\Big) = \mathcal{K}(\beta, \chi_0; f),$$

and

$$\begin{aligned} S_2 &= \sum_{(\mathfrak{n},\mathfrak{q})=1} \Lambda(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{-\sigma} f\Big(\frac{\log \mathrm{N}\mathfrak{n}}{\mathcal{L}}\Big)\Big| \sum_{j\leq N} \chi^{(j)}(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{-i\gamma^{(j)}}\Big|^2 \\ &= \sum_{j,k\leq N} \mathcal{K}(\sigma + i(\gamma^{(j)} - \gamma^{(k)}), \chi^{(j)}\overline{\chi}^{(k)}; f). \end{aligned}$$

The $1$ term from $S_1$ and the $N$ terms in $S_2$ with $j = k$ give

$$\mathcal{K}(\sigma, \chi_0; f) \leq \mathcal{L}\big(f(0)\phi + F(-b) + \epsilon\big)$$

by Lemma 3.4.3. For the $N^2 - N$ terms in $S_2$ with $j \neq k$, apply Lemma 3.4.3 extracting no zeros to see that

$$\mathcal{K}(\sigma + i(\gamma^{(j)} - \gamma^{(k)}), \chi^{(j)}\overline{\chi}^{(k)}; f) \leq \mathcal{L}\big(f(0)\tfrac{4}{3}\phi + \epsilon\big).$$

Therefore, from (4.43), we conclude that

$$\begin{aligned} &\Big(F(\lambda - b) - f(0)\tfrac{4}{3}\phi - \epsilon\Big)^2 N^2\mathcal{L}^2 \\ &\quad\leq \mathcal{L}\Big[f(0)\phi + \epsilon + F(-b)\Big] \times \mathcal{L}\Big[\Big(f(0)\phi + \epsilon + F(-b)\Big)N + \big(f(0)\tfrac{4}{3}\phi + \epsilon\big)(N^2 - N)\Big]. \end{aligned}$$

Dividing both sides by $N\mathcal{L}^2$, solving the inequality, and choosing a new $\epsilon > 0$ depending on $f$, we find

$$N \leq \frac{\Big(f(0)\phi + F(-b)\Big)\Big(F(-b) - \frac{1}{3}f(0)\phi\Big)}{\Big(F(\lambda - b) - \frac{4}{3}f(0)\phi\Big)^2 - \frac{4}{3}f(0)\phi\Big(f(0)\phi + F(-b)\Big)} + \epsilon$$

provided the denominator is positive, which is one of our hypotheses. □

To demonstrate the utility of Theorem 4.5.1, we produce a table of numerical bounds for $N(\lambda)$. Just as in Heath-Brown's case [HB92, Table 13], it turns out that the acquired bounds only hold for certain bounded ranges of $\lambda \in [0, \lambda_b]$ depending on $\lambda_1 \geq b$. However, for small values of $\lambda$, the resulting bounds are better than similar ones obtained in Chapter 5 (cf. Table 5.1).

We apply Theorem 4.5.1 using the weight $f = f_{\hat{\theta},\hat{\lambda}}$ from [HB92, Lemma 7.1] with param-

| | $\lambda_1 \geq 0$ | $\lambda_1 \geq .0875$ | $\lambda_1 \geq .1$ | $\lambda_1 \geq .1227$ | $\lambda_1 \geq .15$ | $\lambda_1 \geq .20$ | $\lambda_1 \geq .25$ | $\lambda_1 \geq .30$ | $\lambda_1 \geq .35$ |
|---|---|---|---|---|---|---|---|---|---|
| $\lambda$ | $N(\lambda)$ | $N(\lambda)$ | $N(\lambda)$ | $N(\lambda)$ | $N(\lambda)$ | $N(\lambda)$ | $N(\lambda)$ | $N(\lambda)$ | $N(\lambda)$ |
| .1 | 2 | 2 | | | | | | | |
| .125 | 2 | 2 | 2 | 2 | | | | | |
| .150 | 3 | 3 | 3 | 3 | | | | | |
| .175 | 3 | 3 | 3 | 3 | 3 | | | | |
| .200 | 4 | 4 | 4 | 3 | 3 | | | | |
| .225 | 4 | 4 | 4 | 4 | 4 | 4 | | | |
| .250 | 5 | 5 | 5 | 5 | 4 | 4 | | | |
| .275 | 6 | 6 | 5 | 5 | 5 | 5 | 5 | | |
| .300 | 7 | 6 | 6 | 6 | 6 | 6 | 5 | | |
| .325 | 9 | 8 | 7 | 7 | 7 | 7 | 6 | 6 | |
| .350 | 11 | 9 | 9 | 9 | 8 | 8 | 7 | 7 | |
| .375 | 15 | 11 | 11 | 10 | 10 | 9 | 8 | 8 | 7 |
| .400 | 22 | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 |
| .425 | 46 | 22 | 20 | 18 | 16 | 14 | 12 | 11 | 10 |
| .450 | $\infty$ | 41 | 36 | 29 | 24 | 19 | 16 | 13 | 12 |
| .475 | | 1087 | 207 | 85 | 51 | 30 | 22 | 18 | 15 |
| .500 | | $\infty$ | $\infty$ | $\infty$ | $\infty$ | 90 | 40 | 27 | 21 |
| .525 | | | | | | $\infty$ | 413 | 61 | 34 |
| .550 | | | | | | | $\infty$ | $\infty$ | 127 |
| .575 | | | | | | | | | $\infty$ |
| .600 | | | | | | | | | |

Table 4.14: Bounds for $N(\lambda)$ in Theorem 4.5.1

eters $\hat{\theta}$ and $\hat{\lambda}$, say, taking

$$\hat{\theta} = 1.63 + 1.28b - 4.35\lambda, \qquad \hat{\lambda} = \lambda.$$

This is roughly optimal based on numerical experimentation and produces Table 4.14. Only non-trivial bounds are displayed since trivially $N(\lambda) \leq 1$ for $\lambda < \lambda_1$.

### 4.5.2 Extending the low-lying zero density estimate

To extend the valid range of $\lambda$ in Table 4.14, we introduce a variant inspired by suggestion 8 of [HB92, Section 12]. For $\lambda \geq \lambda_\star > 0$ fixed, define

$$\mathcal{S}(\lambda, \lambda_\star) := \{s \in \mathbb{C} : 1 - \frac{\lambda}{\mathcal{L}} \leq \sigma < 1 - \frac{\lambda_\star}{\mathcal{L}}\,,\ |t| \leq T_\star\}$$

and

$$N(\lambda,\lambda_\star) = \#\{\chi \,(\mathrm{mod}\, H) : \chi \neq \chi_0, L(s,\chi) \text{ has a zero in } \mathcal{S}(\lambda,\lambda_\star)\ \} = \sum_{\substack{\lambda_\star < \lambda^{(k)} \leq \lambda \\ \chi^{(k)} \neq \chi_0}} 1.$$

Trivially, $\mathcal{S}(\lambda, 0) = \mathcal{S}(\lambda)$ and

$$N(\lambda) = N(\lambda_\star) + N(\lambda,\lambda_\star).$$

To bound the latter quantity, construct a subset $\mathcal{M}(\lambda,\lambda_\star)$ of $\mathcal{M} := \{\chi^{(k)} : k \geq 1\}$ as follows:

1. Remove the trivial character from $\mathcal{M}$.
2. Delete every character $\chi = \chi^{(k)}$ from $\mathcal{M}$ such that $L(s,\chi)$ has a zero in $\mathcal{S}(\lambda_\star)$.
3. Select a character $\psi \in \mathcal{M}$ such that $\psi$ has a zero in $\mathcal{S}(\lambda,\lambda_\star)$. Put $\psi$ in $\mathcal{M}(\lambda,\lambda_\star)$.
4. Delete[2] $\psi$ and $\psi\chi$ from $\mathcal{M}$ for every character $\chi$ with a zero in $\mathcal{S}(\lambda_\star)$.
5. Repeat Steps 3 and 4 until there are no more characters to choose from $\mathcal{M}$.

Denote $M(\lambda,\lambda_\star) = \#\mathcal{M}(\lambda,\lambda_\star)$. By construction, if $\psi_1$ and $\psi_2$ are distinct characters of $\mathcal{M}(\lambda,\lambda_\star)$ then $\psi_1\overline{\psi}_2 \neq \chi$ for any $\chi$ with a zero in $\mathcal{S}(\lambda_\star)$. Moreover, it follows that

$$N(\lambda,\lambda_\star) \leq \{N(\lambda_\star)+1\}M(\lambda,\lambda_\star).$$

since, for each $\psi \in \mathcal{M}(\lambda,\lambda_\star)$, we deleted at most $N(\lambda_\star)$ characters (as well as $\psi$ itself) which could have a zero in $\mathcal{S}(\lambda,\lambda_\star)$. Combining this with our previous bound for $N(\lambda)$, we deduce

$$N(\lambda) \leq \{N(\lambda_\star)+1\}M(\lambda,\lambda_\star) + N(\lambda_\star) \tag{4.44}$$

for $\lambda \geq \lambda_\star > 0$. Bounding $M(\lambda,\lambda_\star)$ for values of $\lambda$ exceeding Table 4.14 will therefore allow us to extend the range for $N(\lambda)$ as well. Unfortunately, the possible existence of a complex zero in $\mathcal{S}(\lambda_\star)$ for the Dedekind zeta function $\zeta_K(s)$ (i.e. the trivial character $\chi_0 \,(\mathrm{mod}\, H)$) limits the potential of this argument.

**Proposition 4.5.2.** *Let $\lambda \geq b > 0$ be fixed and set $M = M(\lambda, b)$. Assume the Dedekind zeta function $\zeta_K(s)$ does not have a complex zero in the region*

$$\mathrm{Re}\{s\} > 1 - \frac{b}{\mathcal{L}}, \qquad |\mathrm{Im}\{s\}| \leq T_\star.$$

[2] Note that Step 4 automatically deletes $\psi\overline{\chi}$ as well since if $L(s,\chi)$ has a zero in $\mathcal{S}(\lambda_\star)$ then so does $L(s,\overline{\chi})$.

*If*

$$F(\lambda - b) > \tfrac{4}{3} f(0)\phi,$$

*and*

$$\Big(F(\lambda - b) - \tfrac{4}{3} f(0)\phi\Big)^2 > \tfrac{4}{3} f(0)\phi\Big(f(0)\phi + F(-b)\Big)$$

*then unconditionally,*

$$M(\lambda, b) \leq \frac{\Big(f(0)\phi + F(-b)\Big)\Big(F(-b) - \frac{1}{3} f(0)\phi\Big)}{\Big(F(\lambda - b) - \frac{4}{3} f(0)\phi\Big)^2 - \frac{4}{3} f(0)\phi\Big(f(0)\phi + F(-b)\Big)} + \epsilon \tag{4.45}$$

*for $\mathcal{L}$ sufficiently large depending on $\epsilon$, $T_\star$, and $f$.*

*Proof.* We sketch the proof since it is a straightforward adaptation of the proof of Theorem 4.5.1 using the characters of $\mathcal{M}(\lambda, b)$. One first deduces, by Lemma 3.4.3, that

$$\mathcal{L}^{-1} \cdot \mathcal{K}(\sigma + i\gamma^{(j)}, \chi^{(j)}; f) \leq f(0)\tfrac{4}{3}\phi - F(\lambda - b) + \epsilon \tag{4.46}$$

for all $\chi^{(j)}$ ranging over $\mathcal{M}(\lambda, b)$, so $1 \leq j \leq M$. The above holds since no character $\chi \in \mathcal{M}(\lambda, b)$ has a zero in $\mathcal{S}(b)$. Rearranging and applying Cauchy-Schwarz, one must bound the analogous $S_1$ and $S_2$. To do so, we again see that

$$\mathcal{K}(\sigma, \chi_0; f) \leq \mathcal{L}\big(f(0)\phi + F(-b) + \epsilon\big)$$

by Lemma 3.4.3. It is here we use that $L(s, \chi_0)$ has no *complex* zero in $\mathcal{S}(b)$; if any real zero $\beta = 1 - \frac{\lambda}{\mathcal{L}}$ of $L(s, \chi)$ arises in our application of Lemma 3.4.3, we may discard it by Condition 2 as $F(\lambda - b) \geq 0$. For the $M^2 - M$ terms in $S_2$ with $j \neq k$, we again apply Lemma 3.4.3 extracting no zeros to see that

$$\mathcal{K}(\sigma + i(\gamma^{(j)} - \gamma^{(k)}), \chi^{(j)}\overline{\chi}^{(k)}; f) \leq \mathcal{L}\big(f(0)\tfrac{4}{3}\phi + \epsilon\big).$$

Note that no zeros are extracted due to the construction of $\mathcal{M}(\lambda, b)$. In particular, $\chi^{(j)}\overline{\chi}^{(k)} \neq \chi$ for any $\chi$ with a zero in $\mathcal{S}(b)$. Continuing with the same arguments as in Theorem 4.5.1, we conclude the desired result. □

We could similarly produce a table like Table 4.14 using Proposition 4.5.2. However, we will not require such precision in its application in Section 7.4. We will content ourselves with the following corollary.

**Corollary 4.5.3.** *If $\mathcal{L}$ is sufficiently large then $N(0.569) \leq 3365$ unconditionally.*

*Proof.* We wish to apply Proposition 4.5.2 using the weight $f = f_{\hat{\lambda}}$ from [HB92, Lemma 7.2] with parameter $\hat{\lambda}$, say. By Theorem 4.1.1, $b = 0.0875$ is a valid choice and $N(0.0875) \leq 1$. Taking $\hat{\lambda} = 0.2784$, we deduce, by Proposition 4.5.2, that

$$M(0.569, 0.0875) \leq 1682.$$

Thus, by (4.44), we conclude $N(0.569) \leq 2 \cdot 1682 + 1 = 3365$, as desired. $\square$

We emphasize that the goal of Corollary 4.5.3 is to maximize the value of $\lambda$ for which we can obtain a reasonable bound for $N(\lambda)$. The precise quality of the bound is not of serious concern since it will still be far better than those obtained in Chapter 5 (cf. Table 5.1). With this purpose in mind, one can see that $\lambda = 0.569$ in Corollary 4.5.3 exceeds $\lambda = 0.425$ in the first column of Table 4.14. In fact, it exceeds the range of $\lambda$ in every column of Table 4.14.

# Chapter 5

# Log-free zero density estimates

*"So me put in work, work, work, work, work, work..."*

– Rihanna.

In this chapter, we use the power sum method to prove explicit versions of the log-free zero density estimates for Hecke $L$-functions due to Weiss. These results serve as generalizations of the classical log-free zero density estimate (1.9) for Dirichlet characters. We will retain the notation of Chapter 2 but we will abandon the notation introduced in Chapters 3 and 4.

## 5.1 Statement of results

Let $H$ be an arbitrary congruence class group of a number field $K$. For a Hecke character $\chi \,(\mathrm{mod}\, H)$, $0 < \sigma < 1$, and $T \geq 1$ arbitrary, define

$$N(\sigma, T, \chi) := \#\{\rho = \beta + i\gamma : L(\rho, \chi) = 0, \sigma < \beta < 1, |\gamma| \leq T\},$$

where the nontrivial zeros $\rho$ of $L(s, \chi)$ are counted with multiplicity. Weiss [Wei83, Corollary 4.4] proved that if $\frac{1}{2} \leq \sigma < 1$ and $T \geq n_K^2 h_H^{1/n_K}$, then

$$\sum_{\chi \,(\mathrm{mod}\, H)} N(\sigma, T, \chi) \ll (e^{O(n_K)} D_K^2 Q T^{n_K})^{C(1-\sigma)}, \tag{5.1}$$

where $C > 0$ is some absolute constant. We will prove the following.

**Theorem 5.1.1.** *Let $H$ be a congruence class group of a number field $K$. If $\frac{1}{2} \leq \sigma < 1$ and $T \geq \max\{n_K^{5/6}(D_K^{4/3} Q^{4/9})^{-1/n_K}, 1\}$, then*

$$\sum_{\chi \,(\mathrm{mod}\, H)} N(\sigma, T, \chi) \ll \{e^{O(n_K)} D_K^2 Q T^{n_K+2}\}^{81(1-\sigma)}. \tag{5.2}$$

*If* $1 - 10^{-3} \leq \sigma < 1$*, then one may replace 81 with 73.5.*

**Remark.**

- Theorem 5.1.1 noticeably improves Weiss' density estimate (5.1) in the range of $T$. If $n_K \leq 2(\log D_K)/\log\log D_K$, then Theorem 5.1.1 holds for $T \geq 1$. Thus we may take $T \geq 1$ for most choices of $K$.

- One can verify from Minkowski's lower bound for $D_K$ and the valid range of $T$ that the $e^{O(n_K)}$ factor is always negligible, regardless of how $n_K$ compares to $(\log D_K)/\log\log D_K$.

To obtain the precise numerical exponents in Theorems 1.3.2 and 1.3.4, we will require a more explicit version of Theorem 5.1.1 when $\sigma$ is very close to 1 and $T$ is fixed. This desired precision necessitates the introduction of a few important quantities.

Let $\delta_0 > 0$ be fixed and sufficiently small. For this chapter only[1], define

$$\mathscr{L} := \begin{cases} (\frac{1}{3} + \delta_0) \log D_K + (\frac{19}{36} + \delta_0) \log Q + (\frac{5}{12} + \delta_0) n_K \log n_K & \text{if } n_K^{5n_K/6} \geq D_K^{4/3} Q^{4/9}, \\ (1 + \delta_0) \log D_K + (\frac{3}{4} + \delta_0) \log Q + \delta_0 n_K \log n_K & \text{otherwise.} \end{cases} \tag{5.3}$$

Notice that

$$\mathscr{L} \geq (1+\delta_0) \log D_K + (\tfrac{3}{4}+\delta_0) \log Q + \delta_0 n_K \log n_K \quad \text{and} \quad \mathscr{L} \geq (\tfrac{5}{12}+\delta_0) n_K \log n_K \tag{5.4}$$

unconditionally. First, we restate a slightly weaker form of Theorem 5.1.1 using $\mathscr{L}$.

**Theorem 5.1.2.** *Let $H$ be a congruence class group of a number field $K$. Let $T \geq 1$ be arbitrary. If $0 < \lambda < \mathscr{L}$ then*

$$\sum_{\chi \pmod{H}} N(1 - \tfrac{\lambda}{\mathscr{L}}, T, \chi) \ll e^{162\lambda}$$

*provided $\mathscr{L}$ is sufficiently large depending only on $T$.*

*Proof.* One can verify this in a straightforward manner from (5.3) and Theorem 5.1.1. □

In addition to Theorem 5.1.2, we will require a more explicit zero density estimate for

[1] Actually, we will return to this quantity in later chapters but emphasize this point here to avoid confusion.

"low-lying" zeros. For $T \geq 1$, define

$$\begin{aligned}\mathcal{N}(\lambda) = \mathcal{N}_H(\lambda, T) &:= \sum_{\chi \,(\mathrm{mod}\, H)} N(1 - \tfrac{\lambda}{\mathscr{L}}, T, \chi) \\ &= \sum_{\chi \,(\mathrm{mod}\, H)} \#\{\rho : L(\rho, \chi) = 0, 1 - \tfrac{\lambda}{\mathscr{L}} < \mathrm{Re}\{\rho\} < 1, |\mathrm{Im}\{\rho\}| \leq T\}.\end{aligned} \tag{5.5}$$

Notice $\mathcal{N}(\lambda)$ defined here is *not* the same as $N(\lambda)$ as defined by (4.3). Instead, one has $N(\lambda) \leq \mathcal{N}(\lambda)$. By Theorems 4.1.1 and 4.1.2, observe that $\mathcal{N}(0.0875) \leq 1$ and $\mathcal{N}(0.2866) \leq 2$. In light of these bounds, we search for explicit numerical estimates for $\mathcal{N}(\lambda)$ with $0.287 \leq \lambda \leq 1$. These are given by Table 5.1 and help establish the following explicit bound on $\mathcal{N}(\lambda)$.

**Theorem 5.1.3.** *Let $H$ be a congruence class group of a number field $K$. Let $\epsilon_0 > 0$ be fixed and sufficiently small. If $0 < \lambda < \epsilon_0 \mathscr{L}$ and $T \geq 1$ then*

$$\mathcal{N}(\lambda) = \mathcal{N}_H(\lambda, T) \leq e^{162\lambda + 188}$$

*for $\mathscr{L}$ sufficiently large depending only on $T$. If $0 < \lambda \leq 1$, then the bounds for $\mathcal{N}(\lambda)$ in Table 5.1 are superior.*

The proof of Theorem 5.1.3 is given in Section 5.4.2 and essentially relies on Theorem 5.3.3 with a careful choice of parameters for each fixed value of $\lambda$.

### 5.1.1 Comparing the mollifier method with power sums

It is instructive to compare the two primary methods for proving log-free zero density estimates. The basic idea behind the proof of (1.9) (the so-called mollifier method) is to construct a Dirichlet polynomial which detects zeros by assuming large values at the zeros of a Dirichlet $L$-function. The optimal Dirichlet polynomial for this task will look like a version of $\mu(n)$, where

$$\mu(n) = \begin{cases} (-1)^r & \text{if } n \text{ is squarefree with } r \text{ prime factors,} \\ 0 & \text{otherwise} \end{cases}$$

is the usual Möbius function. In order to efficiently sum the large values contributed by each of the detected zeros, one relies on the fact that the partial sums of $\mu(n)$ exhibit significant cancellation. To see why this is true, observe that the Prime Number Theorem (with the error term of Hadamard and de la Vallée-Poussin) is equivalent to the statement that there exists an

| $\lambda$ | $\log \mathcal{N}(\lambda) \leq$ | $\alpha$ | $\eta$ | $\omega$ | $\xi$ | $J(\xi\lambda)$ | $Y_{\xi\lambda}$ | $X_{\xi\lambda}$ |
|---|---|---|---|---|---|---|---|---|
| .287 | 198.1 | .3448 | .09955 | .03466 | 1.0082 | .46 | 5.8 | 993 |
| .288 | 198.3 | .3444 | .09943 | .03462 | 1.0082 | .46 | 5.8 | 991 |
| .289 | 198.5 | .3441 | .09931 | .03458 | 1.0082 | .46 | 5.8 | 988 |
| .290 | 198.7 | .3437 | .09918 | .03454 | 1.0082 | .46 | 5.8 | 986 |
| .291 | 198.9 | .3433 | .09906 | .03450 | 1.0082 | .46 | 5.8 | 984 |
| .292 | 199.1 | .3429 | .09894 | .03446 | 1.0081 | .46 | 5.8 | 982 |
| .293 | 199.3 | .3426 | .09882 | .03442 | 1.0081 | .46 | 5.8 | 979 |
| .294 | 199.5 | .3422 | .09870 | .03439 | 1.0081 | .46 | 5.8 | 977 |
| .295 | 199.8 | .3418 | .09859 | .03435 | 1.0081 | .46 | 5.8 | 975 |
| .296 | 200.0 | .3415 | .09847 | .03431 | 1.0081 | .46 | 5.8 | 973 |
| .297 | 200.2 | .3411 | .09835 | .03427 | 1.0080 | .46 | 5.8 | 970 |
| .298 | 200.4 | .3408 | .09823 | .03423 | 1.0080 | .46 | 5.8 | 968 |
| .299 | 200.6 | .3404 | .09811 | .03420 | 1.0080 | .46 | 5.8 | 966 |
| .300 | 200.8 | .3400 | .09800 | .03416 | 1.0080 | .46 | 5.8 | 964 |
| .325 | 205.9 | .3316 | .09518 | .03326 | 1.0075 | .47 | 5.8 | 914 |
| .350 | 211.0 | .3240 | .09257 | .03242 | 1.0071 | .47 | 5.7 | 871 |
| .375 | 216.0 | .3171 | .09014 | .03163 | 1.0067 | .47 | 5.7 | 833 |
| .400 | 220.9 | .3108 | .08787 | .03090 | 1.0064 | .48 | 5.7 | 800 |
| .425 | 225.7 | .3054 | .08678 | .02878 | 1.0061 | .46 | 5.6 | 769 |
| .450 | 230.4 | .2998 | .08373 | .02956 | 1.0059 | .48 | 5.6 | 744 |
| .475 | 235.1 | .2948 | .08184 | .02895 | 1.0056 | .48 | 5.6 | 720 |
| .500 | 239.8 | .2903 | .08006 | .02837 | 1.0054 | .49 | 5.6 | 699 |
| .550 | 249.0 | .2821 | .07677 | .02729 | 1.0050 | .49 | 5.5 | 661 |
| .600 | 258.0 | .2748 | .07379 | .02631 | 1.0046 | .50 | 5.5 | 629 |
| .650 | 266.9 | .2684 | .07109 | .02542 | 1.0043 | .50 | 5.4 | 602 |
| .700 | 275.6 | .2627 | .06862 | .02460 | 1.0041 | .50 | 5.4 | 579 |
| .750 | 284.3 | .2576 | .06634 | .02383 | 1.0039 | .51 | 5.4 | 559 |
| .800 | 292.9 | .2529 | .06424 | .02313 | 1.0037 | .51 | 5.4 | 541 |
| .850 | 301.4 | .2486 | .06230 | .02247 | 1.0035 | .51 | 5.3 | 525 |
| .900 | 309.8 | .2447 | .06049 | .02186 | 1.0033 | .51 | 5.3 | 510 |
| .950 | 318.2 | .2412 | .05880 | .02128 | 1.0032 | .52 | 5.3 | 497 |
| 1.00 | 326.5 | .2378 | .05722 | .02074 | 1.0030 | .52 | 5.3 | 486 |

Table 5.1: Bounds for $\mathcal{N}(\lambda)$ in Theorem 5.1.3

absolute constant $c_3 > 0$ such that if $x$ is sufficiently large, then

$$\sum_{n\leq x}\mu(n) \ll x\exp(-c_3(\log x)^{1/2}). \tag{5.6}$$

The fact that (5.6) is a part of the proofs of the log-free zero density estimates in [Gra77, HB92, IK04, Jut77] may not be immediately obvious. After summing the mollified Dirichlet polynomials over all characters $\chi \pmod q$ and applying duality, one must ultimately minimize the quadratic form

$$S(x) = \sum_{n\leq x}\Big(\sum_{d|n}\lambda_d\Big)^2$$

subject to the constraint

$$\lambda_d = \begin{cases} \mu(d) & \text{if } 1 \leq d < z_1, \\ 0 & \text{if } d > z_2, \end{cases}$$

where $1 < z_1 < z_2$ are given real numbers. See, for example, [IK04, Pages 430–431]. For the purpose of proving a log-free zero density estimate, it is convenient to define

$$\lambda_d = \begin{cases} \mu(d)\min\Big(1, \dfrac{\log(z_2/d)}{\log(z_2/z_1)}\Big) & \text{if } 1 \leq d \leq z_2, \\ 0 & \text{if } d > z_2. \end{cases}$$

Each of [Gra77, HB92, IK04, Jut77] uses the beautiful work of Graham [Gra78] to estimate $S(x)$ with this choice of $\lambda_d$; Graham proves that

$$S(x) = \frac{x}{\log(z_2/z_1)}\Big(1 + O\Big(\frac{1}{\log(z_2/z_1)}\Big)\Big). \tag{5.7}$$

At several points in the proof of (5.7), Graham uses the asymptotic Prime Number Theorem in the form (5.6).

For a number field $K$, let $\mu_K(\mathfrak{n})$ be the extension of the Möbius function to the prime ideals of $K$. For the sake of simplicity, suppose that the Dedekind zeta function $\zeta_K(s)$ has no exceptional zero. The effective form of the Prime Ideal Theorem proven in [LO77] is equivalent to the statement that there exists an absolute constant $c_4 > 0$ such that if $\log x \gg n_K(\log D_K)^2$, then

$$\sum_{\mathrm{N}\mathfrak{n}\leq x}\mu_K(\mathfrak{n}) \ll x\exp\Big(-c_4\Big(\frac{\log x}{n_K}\Big)^{1/2}\Big).$$

Therefore, to generalize (5.7) to the Möbius function of $K$, $x$ needs to be larger than any polynomial in $D_K$ before the partial sums of $\mu_K(\mathfrak{n})$ up to $x$ begin to exhibit cancellation. Thus if one extends the preceding arguments to prove an analogue of (1.9) for the Hecke characters of $K$, then the ensuing log-free zero density estimate will not have the $K$-uniformity which is necessary to prove Theorem 1.3.2.

Turán developed an alternate formulation of log-free zero density estimates. The idea is to take high derivatives of $-\frac{L'}{L}(s,\chi)$. This produces a large sum of complex numbers involving zeros of $L(s,\chi)$, which can be bounded below by the Turán power sum method (see Proposition 5.3.1). The integral of a certain zero-detecting polynomial (which is not defined in terms of the Möbius function) gives an upper bound for these high derivatives. Therefore, when a certain zero-detecting polynomial (which is not defined in terms of the Möbius function) encounters a zero of $L(s,\chi)$, its integral will be bounded away from zero because of the lower bound given by the power sum method. The contributions from the detected zeros up to height $T$ are summed efficiently using a particular large sieve inequality (see Section 5.2).

The advantage of using the power sum method in our proofs lies in the fact that the method is a purely Diophantine result, independent of the number fields in our proofs; this allows for noticeably better field uniformity than the mollifier method. The disadvantage is that the lower bound in the power sum method is quite small, which, for example, would inflate the constant 12/5 in (1.9). To our knowledge, the power sum method is the only tool available that will produce a $K$-uniform log-free zero density estimate of the form (1.9) which is strong enough to deduce a conclusion as strong as Theorem 1.3.2. Limitations to the power sum method indicate a genuine obstacle to any substantive improvements in the constants in Theorem 1.3.2 when using these methods.

To prove the large sieve inequality (5.11) used in the proof of Theorem 5.1.1, we bounded certain sums over integral ideals, which required smoothing the sums using a kernel that is $n_K$-times differentiable. Unfortunately, the smoothing introduces the powers of ${n_K}^{n_K}$ (see the comments immediately preceding [Wei83, Section 1]). As mentioned after Theorem 1.3.2, the factor of ${n_K}^{n_K}$ is negligible if $n_K$ is small compared to $(\log D_K)/\log\log D_K$, which is expected to be the case in most applications.

## 5.2 Mean values of Dirichlet polynomials

In [Gal70], Gallagher proves the following mean value results for Dirichlet polynomials.

**Theorem.** *Let $\{a_n\}$ be a sequence of complex numbers such that $\sum_{n\geq 1} n|a_n|^2 < \infty$.*

*(i) If $T \geq 1$, then*

$$\sum_{\chi \bmod q} \int_{-T}^{T} \Big| \sum_{n=1}^{\infty} a_n \chi(n) n^{it} \Big|^2 dt \ll \sum_{n=1}^{\infty} (qT+n)|a_n|^2, \tag{5.8}$$

*where the sum is over Dirichlet characters $\chi \bmod q$.*

*(ii) Let $R \geq 2$, and assume $a_n = 0$ if $n$ has any prime factor less than $R$. If $T \geq 1$, then*

$$\sum_{q \leq R} \log \frac{R}{q} \sideset{}{^*}\sum_{\chi \bmod q} \int_{-T}^{T} \Big| \sum_{n=1}^{\infty} a_n \chi(n) n^{it} \Big|^2 dt \ll \sum_{n=1}^{\infty} (R^2 T + n)|a_n|^2. \tag{5.9}$$

*Here, $\sum^*$ denotes the restriction to primitive Dirichlet characters $\chi \bmod q$.*

In (5.9), the $\log(R/q)$ weighting on the left hand side (which arises from the support of $a_n$) turns out to be decisive in some applications, such as the proof of (1.6). To prove Theorem 5.1.1, we need a $K$-uniform analogue of (5.8) when $a_n$ is supported as in (5.9). Weiss used the Selberg sieve to prove such a result in his Ph.D. thesis [Wei80, Theorem $3'$, p. 98].

**Theorem** (Weiss)**.** *Let $b(\,\cdot\,)$ be a complex-valued function on the integral ideals $\mathfrak{n}$ of $K$, and suppose that $\sum_{\mathfrak{n}} (\mathrm{N}\mathfrak{n})|b(\mathfrak{n})|^2 < \infty$. Let $T \gg 1$. Suppose that $b(\mathfrak{n}) = 0$ when $\mathfrak{n}$ has a prime ideal factor $\mathfrak{p}$ with $\mathrm{N}\mathfrak{p} \leq z$, and define $V(z) = \sum_{\mathrm{N}\mathfrak{n} \leq z} \mathrm{N}\mathfrak{n}^{-1}$. If $0 < \epsilon < 1/2$, then*

$$\sum_{\chi \,(\mathrm{mod}\, H)} \int_{-T}^{T} \Big| \sum_{\mathfrak{n}} b(\mathfrak{n}) \chi(\mathfrak{n}) \mathrm{N}\mathfrak{n}^{-it} \Big|^2 dt \ll \sum_{\mathfrak{n}} |b(\mathfrak{n})|^2 \Big( \frac{\kappa_K}{V(z)} \mathrm{N}\mathfrak{n} + c(\epsilon)(n_K^{n_K} D_K Q T^{n_K} z^4)^{1/2+\epsilon} h_H T \Big)$$

*for some constant $c(\epsilon) > 0$ depending only on $\epsilon$.*

**Remark.** Assuming the Lindelöf Hypothesis for Hecke $L$-functions, the upper bound becomes

$$\ll \sum_{\mathfrak{n}} |b(\mathfrak{n})|^2 \Big( \frac{\kappa_K}{V(z)} \mathrm{N}\mathfrak{n} + c(\epsilon)(D_K Q)^{\epsilon} h_H T^{1+\epsilon n_K} z^{2+\epsilon} \Big).$$

This appears to be optimal when using the Selberg sieve, considering that when $K = \mathbb{Q}$, the second term is roughly $(qTz^2)^{1+\epsilon}$. For related unconditional results, see Duke [Duk89, Section 1].

This result is interesting in its own right, but to make the result more practical for the applications at hand, Weiss chooses $b(\mathfrak{n})$ to be supported on the prime ideals $\mathfrak{p}$ such that $y < \mathrm{N}\mathfrak{p} \leq y^{c_1}$. Then, Weiss sets $z = y^{1/3}$ and chooses $\log y \geq c_2 \log(D_K Q T^{n_K})$ and $\epsilon = 1/3$. By Corollary 2.4.2 and taking $c_1$ and $c_2$ to be sufficiently large, Weiss' result reduces to

$$\sum_{\chi(H)=1} \int_{-T}^{T} \Big| \sum_{y < \mathrm{N}\mathfrak{p} \leq y^{c_1}} b(\mathfrak{p}) \chi(\mathfrak{n}) \mathrm{N}\mathfrak{n}^{-it} \Big|^2 dt \ll \frac{1}{\log y} \sum_{y < \mathfrak{p} \leq y^{c_1}} |b(\mathfrak{p})|^2 \mathrm{N}\mathfrak{p}.$$

In [Wei83, Corollary 3.8], Weiss recasts this estimate with more generally.

**Corollary 5.2.1** (Weiss). *Let $b(\,\cdot\,)$ be a complex-valued function on the prime ideals $\mathfrak{p}$ of $K$ such that $\sum_{\mathfrak{p}}(\mathrm{N}\mathfrak{p})|b(\mathfrak{p})|^2 < \infty$ and $b(\mathfrak{p}) = 0$ whenever $\mathrm{N}\mathfrak{p} \leq y$. Let $H$ be a primitive congruence class group of $K$. If $y \geq (h_H n_K^{2n_K} D_K Q T^{2n_K})^8$, then*

$$\sum_{\chi(H)=1} \int_{-T}^{T} \Big| \sum_{\mathfrak{p}} b(\mathfrak{p})\chi(\mathfrak{n})\mathrm{N}\mathfrak{n}^{-it} \Big|^2 dt \ll \frac{1}{\log y} \sum_{\mathfrak{p}} |b(\mathfrak{p})|^2 \mathrm{N}\mathfrak{p}.$$

The exponent 8 in the range of $y$ in Corollary 5.2.1 is large enough to influence the value of $C$ in (5.1), which affects $B_2$ in (1.24). In this section, we improve Corollary 5.2.1 so that it does not influence the exponents in Theorem 5.1.1.

**Theorem 5.2.2.** *Let $\upsilon \geq \epsilon > 0$ be arbitrary. Let $b(\,\cdot\,)$ be a complex-valued function on the prime ideals $\mathfrak{p}$ of $K$ such that $\sum_{\mathfrak{p}}(\mathrm{N}\mathfrak{p})|b(\mathfrak{p})|^2 < \infty$ and $b(\mathfrak{p}) = 0$ whenever $\mathrm{N}\mathfrak{p} \leq y$. Let $H$ be a primitive congruence class group of $K$. If $T \geq 1$ and*

$$y \geq C_\epsilon \big\{ h_H n_K^{(5/4+\upsilon)n_K} D_K^{3/2+\upsilon} Q^{1/2} T^{n_K/2+1} \big\}^{1+\epsilon} \tag{5.10}$$

*for some sufficiently large $C_\epsilon > 0$ then*

$$\sum_{\chi \,(\mathrm{mod}\, H)} \int_{-T}^{T} \Big| \sum_{\mathfrak{p}} b(\mathfrak{p})\chi(\mathfrak{p})\mathrm{N}\mathfrak{p}^{-it} \Big|^2 dt \leq \Big( \frac{5\pi\{1 - \frac{1}{1+\upsilon}\}^{-1}}{\frac{1}{1+\epsilon}\log(\frac{y}{h_H}) - \mathcal{L}'} + O_\epsilon(y^{-\frac{\epsilon}{2}}) \Big) \sum_{\mathfrak{p}} \mathrm{N}\mathfrak{p}|b(\mathfrak{p})|^2, \tag{5.11}$$

*where $\mathcal{L}' = \frac{1}{2}\log D_K + \frac{1}{2}\log Q + \frac{1}{4} n_K \log n_K + (\frac{n_K}{2} + 1)\log T + O_\epsilon(1)$.*

**Remark.** Taking $\upsilon = \epsilon$ and using Lemma 2.4.6, we improve the range of $y$ in Corollary 5.2.1 to

$$y \gg e^{O_\epsilon(n_K)} \big\{ n_K^{5/4 n_K} D_K^2 Q^{3/2} T^{n_K/2+1} \big\}^{1+\epsilon}.$$

## 5.2.1 Preparing for the Selberg sieve

To apply the Selberg sieve, we will require several weighted estimates involving Hecke characters. Before we begin, we highlight the necessary properties of our weight $\Psi$.

**Lemma 5.2.3.** *For $T \geq 1$, let $A = T\sqrt{2n_K}$. Define*

$$\widehat{\Psi}(s) = \Big[ \frac{\sinh(s/A)}{s/A} \Big]^{2n_K}$$

*and let*

$$\Psi(x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \widehat{\Psi}(s) x^{-s} ds$$

*be the inverse Mellin transform of $\widehat{\Psi}(s)$. Then:*

*(i)* $0 \le \Psi(x) \le A/2$ *and* $\Psi(x)$ *is a compactly supported function vanishing outside the interval* $e^{-2n_K/A} \le x \le e^{2n_K/A}$.

*(ii)* $\widehat{\Psi}(s)$ *is an entire function.*

*(iii) For all complex* $s = \sigma + it$, $|\widehat{\Psi}(s)| \le (A/|s|)^{2n_K} e^{|\sigma|/A}$.

*(iv) For* $|s| \le A$, $|\widehat{\Psi}(s)| \le (1 + |s|^2/(5A^2))^{2n_K}$.

*(v) Uniformly for* $|\sigma| \le A/\sqrt{2n_K}$, $|\widehat{\Psi}(s)| \ll 1$.

*(vi) Let* $\{b_m\}_{m\ge 1}$ *be a sequence of complex numbers with* $\sum_m |b_m| < \infty$. *Then*

$$\int_{-T}^{T} \Big|\sum_m b_m m^{-it}\Big|^2 dt \le \frac{5\pi}{2}\int_0^\infty \Big|\sum_m b_m \Psi\Big(\frac{x}{m}\Big)\Big|^2 \frac{dx}{x}$$

*Proof.* For (i)–(v), see [Wei83, Lemma 3.2]; in his notation, $\Psi(x) = H_{2n_K}(x)$ with parameter $A = T\sqrt{2n_K}$. Statement (vi) follows easily from the proof of [Wei83, Corollary 3.3]. $\square$

For the remainder of this section, assume:

- $H \pmod{\mathfrak{q}}$ is an arbitrary *primitive* congruence class group of $K$.
- $0 < \epsilon < 1/2$ and $T \ge 1$ is arbitrary.
- $\Psi$ is the weight function of Lemma 5.2.3.

Next, we establish improved analogues of [Wei83, Lemmas 3.4 and 3.6 and Corollary 3.5].

**Lemma 5.2.4.** *Let* $\chi \pmod{H}$ *be a Hecke character. For* $x > 0$,

$$\Big|\sum_{\mathfrak{n}} \frac{\chi(\mathfrak{n})}{\mathrm{N}\mathfrak{n}} \cdot \Psi\Big(\frac{x}{\mathrm{N}\mathfrak{n}}\Big) - E_0(\chi)\frac{\varphi(\mathfrak{q})}{\mathrm{N}\mathfrak{q}}\kappa_K\Big| \ll_\epsilon \Big\{n_K^{n_K/4} D_K^{1/2} Q^{1/2} T^{n_K/2+1}\Big\}^{1+\epsilon}.$$

*Proof.* The quantity we wish to bound equals

$$\frac{1}{2\pi i}\int_{-1-i\infty}^{-1+i\infty} L(s+1,\chi)\widehat{\Psi}(s)x^s ds. \tag{5.12}$$

If $\chi \pmod{\mathfrak{q}}$ is induced by the primitive character $\chi^* \pmod{\mathfrak{f}_\chi}$, then

$$L(s,\chi) = L(s,\chi^*)\prod_{\substack{\mathfrak{p}\mid\mathfrak{q}\\ \mathfrak{p}\nmid\mathfrak{f}_\chi}}(1 - \chi^*(\mathfrak{p})\mathrm{N}\mathfrak{p}^{-s}).$$

Thus $|L(it,\chi)| \leq 2^{\omega(\mathfrak{q})}|L(it,\chi^*)|$ where $\omega(\mathfrak{q})$ is the number of distinct prime ideal divisors of $\mathfrak{q}$. Since $H \pmod{\mathfrak{q}}$ is primitive, $\mathfrak{q} = \mathfrak{f}_H$ so $\omega(\mathfrak{q}) \leq 6e^{4/\epsilon}n_K + \frac{\epsilon}{2}\log(D_K Q)$, by [Wei83, Lemma 1.13]. Thus, for $\mathrm{Re}\{s\} = -1$, $|L(s+1,\chi)| \ll e^{O_\epsilon(n_K)}(D_K Q)^{\epsilon/2}|L(s+1,\chi^*)|$. Thus, by the convexity bound (Lemma 2.3.2), the expression in (5.12) is

$$\ll e^{O_\epsilon(n_K)}(D_K Q)^{\frac{1}{2}+\epsilon}x^{-1}\int_0^\infty (1+|t|)^{(\frac{1}{2}+\epsilon)n_K}|\widehat{\Psi}(-1+it)|dt$$

as $D_\chi \leq D_K Q$. By Lemma 5.2.3(iii) and (iv), this integral is

$$\ll \int_0^{\frac{A}{2}} (1+|t|)^{(\frac{1}{2}+\epsilon)n_K}|\widehat{\Psi}(-1+it)|dt + \int_{\frac{A}{2}}^\infty (1+|t|)^{(\frac{1}{2}+\epsilon)n_K}|\widehat{\Psi}(-1+it)|dt,$$

which is $\ll e^{O(n_K)}A^{(\frac{1}{2}+\epsilon)n_K+1}$. Collecting the above estimates, the claimed bound, up to a factor of $\epsilon$, follows upon recalling $A = T\sqrt{2n_K}$ and noting $e^{O(n_K)} \ll_\epsilon (n_K^{n_K})^\epsilon$. □

**Corollary 5.2.5.** *Let $C$ be a coset of $H$, and let $\mathfrak{d}$ be an integral ideal coprime to $\mathfrak{q}$. For all $x > 0$, we have*

$$\Big|\sum_{\substack{\mathfrak{n}\in C\\ \mathfrak{d}|\mathfrak{n}}} \frac{1}{\mathrm{N}\mathfrak{n}}\Psi\Big(\frac{x}{\mathrm{N}\mathfrak{n}}\Big) - \frac{\varphi(\mathfrak{q})}{\mathrm{N}\mathfrak{q}}\frac{\kappa_K}{h_H}\cdot\frac{1}{\mathrm{N}\mathfrak{d}}\Big| \ll_\epsilon \{n_K^{n_K/4}D_K^{1/2}Q^{1/2}T^{n_K/2+1}\}^{1+\epsilon}\cdot\frac{1}{x}.$$

*Proof.* The proof is essentially the same as that of [Wei83, Corollary 3.5], except for the fact that we have an improved bound in Lemma 5.2.4. □

We now apply the Selberg sieve. For $z \geq 1$, define

$$S_z = \{\mathfrak{n} \colon \mathfrak{p} \mid \mathfrak{n} \implies \mathrm{N}\mathfrak{p} > z\} \qquad \text{and} \qquad V(z) = \sum_{\mathrm{N}\mathfrak{n}\leq z}\frac{1}{\mathrm{N}\mathfrak{n}}. \tag{5.13}$$

**Lemma 5.2.6.** *Let $C$ be a coset of $H$. For $x > 0$ and $z \geq 1$,*

$$\sum_{\mathfrak{n}\in C\cap S_z}\frac{1}{\mathrm{N}\mathfrak{n}}\Psi\Big(\frac{x}{\mathrm{N}\mathfrak{n}}\Big) \leq \frac{\kappa_K}{h_H V(z)} + O_\epsilon\Big(\frac{\{n_K^{n_K/4}D_K^{1/2}Q^{1/2}T^{n_K/2+1}\}^{1+\epsilon}z^{2+2\epsilon}}{x}\Big).$$

*Proof.* The proof is essentially the same as that of [Wei83, Lemma 3.6], except for the fact that we have an improved bound in Lemma 5.2.4. □

### 5.2.2 Proof of Theorem 5.2.2

Let $z$ be a parameter satisfying $1 \leq z \leq y$, which we will specify later. Extend $b(\mathfrak{n})$ to all integral ideals $\mathfrak{n}$ of $K$ by zero. Applying Lemma 5.2.3 and writing $b_m = \sum_{\mathrm{N}\mathfrak{n}=m} b(\mathfrak{n})\chi(\mathfrak{n})$, for each Hecke character $\chi \pmod{H}$, it follows that

$$\sum_{\chi \,(\mathrm{mod}\, H)} \int_{-T}^{T} \Big| \sum_{\mathfrak{n}} b(\mathfrak{n})\chi(\mathfrak{n})\mathrm{N}\mathfrak{n}^{-it} \Big|^2 dt \leq \frac{5\pi}{2} \int_0^\infty \sum_{\chi \,(\mathrm{mod}\, H)} \Big| \sum_{\mathfrak{n}} b(\mathfrak{n})\chi(\mathfrak{n})\Psi\Big(\frac{x}{\mathrm{N}\mathfrak{n}}\Big) \Big|^2 \frac{dx}{x}. \tag{5.14}$$

By the orthogonality of characters and the Cauchy-Schwarz inequality,

$$\sum_{\chi \,(\mathrm{mod}\, H)} \Big| \sum_{\mathfrak{n}} b(\mathfrak{n})\chi(\mathfrak{n})\Psi\Big(\frac{x}{\mathrm{N}\mathfrak{n}}\Big) \Big|^2 \leq h_H \sum_{C \in I(\mathfrak{q})/H} \Big( \sum_{\mathfrak{n} \in C} \mathrm{N}\mathfrak{n}|b(\mathfrak{n})|^2 \Psi\Big(\frac{x}{\mathrm{N}\mathfrak{n}}\Big) \Big) \sum_{\mathfrak{n} \in C \cap S_z} \frac{\Psi(\frac{x}{\mathrm{N}\mathfrak{n}})}{\mathrm{N}\mathfrak{n}},$$

since $z \leq y$ and $b(\mathfrak{n})$ is supported on prime ideals with norm greater than $y$. For $\delta = \delta(\epsilon) > 0$ sufficiently small and $B_\delta > 0$ sufficiently large, denote

$$M_\delta' = M_\delta z^{2+2\delta} \qquad \text{and} \qquad M_\delta = B_\delta \big\{ n_K^{n_K/4} D_K^{1/2} Q^{1/2} T^{n_K/2+1} \big\}^{1+\delta}.$$

By Lemma 5.2.6, the right hand side of the preceding inequality is therefore at most

$$\sum_{C \in I(\mathfrak{q})/H} \sum_{\mathfrak{n} \in C} \mathrm{N}\mathfrak{n}|b(\mathfrak{n})|^2 \Psi\Big(\frac{x}{\mathrm{N}\mathfrak{n}}\Big)\Big(\frac{\kappa_K}{V(z)} + \frac{h_H M_\delta'}{x}\Big) \leq \sum_{\mathfrak{n}} \mathrm{N}\mathfrak{n}|b(\mathfrak{n})|^2 \Psi\Big(\frac{x}{\mathrm{N}\mathfrak{n}}\Big)\Big(\frac{\kappa_K}{V(z)} + \frac{h_H M_\delta'}{x}\Big).$$

Combining the above estimates into (5.14) yields

$$\begin{aligned}
&\sum_{\chi \,(\mathrm{mod}\, H)} \int_{-T}^{T} \Big| \sum_{\mathfrak{p}} b(\mathfrak{p})\chi(\mathfrak{p})\mathrm{N}\mathfrak{p}^{-it} \Big|^2 dt \\
&\leq \frac{5\pi}{2} \sum_{\mathfrak{n}} \mathrm{N}\mathfrak{n}|b(\mathfrak{n})|^2 \Big( \frac{\kappa_K}{V(z)} \int_0^\infty \Psi\Big(\frac{x}{\mathrm{N}\mathfrak{n}}\Big)\frac{dx}{x} + h_H M_\delta' \int_0^\infty \frac{1}{x}\Psi\Big(\frac{x}{\mathrm{N}\mathfrak{n}}\Big)\frac{dx}{x} \Big) \\
&\leq \frac{5\pi}{2} \sum_{\mathfrak{n}} \mathrm{N}\mathfrak{n}|b(\mathfrak{n})|^2 \Big( \frac{\kappa_K}{V(z)} |\widehat{\Psi}(0)| + \frac{h_H M_\delta'}{\mathrm{N}\mathfrak{n}} |\widehat{\Psi}(1)| \Big),
\end{aligned}$$

by Lemma 5.2.3(v). Since $b(\mathfrak{n})$ is supported on prime ideals whose norm is greater than $y$, the above is $\leq \frac{5\pi}{2}\big(\frac{\kappa_K}{V(z)} + O(h_H M_\delta z^{2+2\delta} y^{-1})\big) \sum_{\mathfrak{p}} \mathrm{N}\mathfrak{p}|b(\mathfrak{p})|^2$. Now, select $z$ satisfying

$$z = \Big( \frac{y^{(1+\delta)/(1+\epsilon)}}{h_H M_\delta} \Big)^{1/(2+2\delta)}, \tag{5.15}$$

so $1 \leq z \leq y$. Hence,

$$\sum_{\chi \,(\mathrm{mod}\, H)} \int_{-T}^{T} \Big| \sum_{\mathfrak{p}} b(\mathfrak{p})\chi(\mathfrak{p})\mathrm{N}\mathfrak{p}^{-it} \Big|^2 dt \leq \frac{5\pi}{2}\Big( \frac{\kappa_K}{V(z)} + O_\epsilon(y^{-\epsilon/2}) \Big) \sum_{\mathfrak{p}} \mathrm{N}\mathfrak{p}|b(\mathfrak{p})|^2 \tag{5.16}$$

for $\delta = \delta(\epsilon) > 0$ sufficiently small. From (5.10) and (5.15), it follows that $z \geq 3(n_K^{n_K} D_K)^{1/2+\upsilon/2}$ provided $C_\epsilon$ in (5.10) is sufficiently large. Applying Corollary 2.4.2 to (5.16), it follows that

$$\sum_{\chi \,(\mathrm{mod}\, H)} \int_{-T}^{T} \Big| \sum_{\mathfrak{p}} b(\mathfrak{p})\chi(\mathfrak{p})\mathrm{N}\mathfrak{p}^{-it} \Big|^2 dt \leq \Big( \frac{5\pi\upsilon}{2\{1+\upsilon\}\log z + O_\epsilon(1)} + O_\epsilon(y^{-\epsilon/2}) \Big) \sum_{\mathfrak{p}} \mathrm{N}\mathfrak{p}|b(\mathfrak{p})|^2,$$

since $\upsilon \geq \epsilon > 0$. Finally, by (5.10) and (5.15),

$$2\log z \geq \tfrac{1}{1+\epsilon}\log(\tfrac{y}{h_H}) - \tfrac{1}{2}\{\log D_K + \log Q + \tfrac{1}{2} n_K \log n_K + (n_K+2)\log T + O_\epsilon(1)\}.$$

Substituting this estimate into the previous inequality, we obtain the desired conclusion. $\square$

## 5.3 Detecting the zeros of Hecke $L$-functions

### 5.3.1 Notation

We first specify some additional notation to be used throughout this section.

#### Arbitrary Quantities

- Let $H \,(\mathrm{mod}\, \mathfrak{q})$ be a *primitive* congruence class group.
- Let $\epsilon \in (0, 1/8)$ and $\phi = 1 + \frac{4}{\pi}\epsilon + 16\epsilon^2 + 340\epsilon^{10}$.
- Let $T \geq 1$. Recall $Q = Q_H$ as in (2.2) and define

$$\mathcal{L} = \mathcal{L}_{T,\epsilon} := \log D_K + \tfrac{1}{2}\log Q + (\tfrac{n_K}{2} + 1)\log(T+3) + \Theta n_K, \tag{5.17}$$

  where $\Theta = \Theta(\epsilon) \geq 1$ is sufficiently large depending on $\epsilon$.

- Let $\lambda_0 > \frac{1}{20}$. Suppose $\tau \in \mathbb{R}$ and $\lambda > 0$ satisfy

$$\lambda_0 \leq \lambda \leq \tfrac{1}{16}\mathcal{L} \quad \text{and} \quad |\tau| \leq T. \tag{5.18}$$

  Furthermore, denote $r = \frac{\lambda}{\mathcal{L}}$.

**Fixed Quantities**

- Let $\alpha, \eta \in (0, \infty)$ and $\omega \in (0,1)$ be fixed.

- Define $A \geq 1$ so that $A_1 = \sqrt{A^2+1}$ satisfies

$$A_1 = 2(4e(1+1/\alpha))^{\alpha}(1+\eta). \tag{5.19}$$

- Let $x = e^{X\mathcal{L}}$ and $y = e^{Y\mathcal{L}}$ with $X, Y > 0$ given by

$$\begin{aligned} Y = Y_\lambda &= \frac{1}{eA_1} \cdot \frac{1}{\alpha}\Big\{2\phi A + \frac{8}{\lambda}\Big\}, \\ X = X_\lambda &= \frac{2\log\left(\frac{2A_1}{1-\omega}\right)}{(1-\omega)} \cdot \frac{1+\alpha}{\alpha}\Big\{2\phi A + \frac{8}{\lambda}\Big\}, \end{aligned} \tag{5.20}$$

  and $\alpha, \eta, \omega$ are chosen so that $2 < Y < X$. Notice $X = X_\lambda$ and $Y = Y_\lambda$ depend on the arbitrary quantities $\epsilon$ and $\lambda$, but they are uniformly bounded above and below in terms of $\alpha, \eta$, and $\omega$, i.e. $X \asymp 1$ and $Y \asymp 1$. For this reason, while $X$ and $Y$ are technically not fixed quantities, they may be treated as such.

### 5.3.2 Key ingredients

**Detecting Zeros**

The first goal of this section is to prove the following proposition.

**Proposition 5.3.1.** *Let $\chi \pmod H$ be a Hecke character. Suppose $L(s,\chi)$ has a non-trivial zero $\rho$ satisfying*

$$|1 + i\tau - \rho| \leq r = \tfrac{\lambda}{\mathcal{L}}. \tag{5.21}$$

*Further assume*

$$J(\lambda) := \frac{W_1\lambda + W_2}{A_1(1+\eta)^{k_0}} < 1, \tag{5.22}$$

*where $X = X_\lambda, Y = Y_\lambda$,*

$$\begin{aligned} k_0 = k_0(\lambda) &= \alpha^{-1}\big(2\phi A\lambda + 8\big), \\ W_1 = W_1(\lambda) &= 8A_1\big(1 + \tfrac{1}{k_0}\big) + 2eA_1\big(Y + \tfrac{1}{2} + \{2X+1\}e^{-\omega\lambda X}\big) + O(\epsilon), \\ W_2 = W_2(\lambda) &= 2e\omega^{-1}A_1 e^{-\omega\lambda X} + 18 + O(\epsilon). \end{aligned}$$

*If* $\lambda < \frac{\epsilon}{A_1}\mathcal{L}$ *and* $2 < Y < X$ *then*

$$r^4 \log\Big(\frac{x}{y}\Big) \int_y^x \Big| \sum_{y \leq \mathrm{N}\mathfrak{p} < u} \frac{\chi(\mathfrak{p}) \log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{1+i\tau}} \Big|^2 \frac{du}{u} + E_0(\chi) \mathbf{1}_{\{|\tau| < Ar\}}(\tau)$$
$$\geq \Big( \frac{\alpha/(1+\alpha)}{8e2^{1/\alpha}} \Big)^{4\phi A\lambda + 16} \frac{(1 - J(\lambda))^2}{4}.$$

**Remark.** Note that $W_j(\lambda) \ll 1$ for $j = 1, 2$.

The proof of Proposition 5.3.1 is divided into two main steps, with the final arguments culminating in Section 5.3.5. The method critically hinges on the following power sum estimate due to Kolesnik and Straus [KS83].

**Theorem 5.3.2** (Kolesnik–Straus)**.** *For any integer* $M \geq 0$ *and complex numbers* $z_1, \dots, z_N$*, there is an integer* $k$ *with* $M+1 \leq k \leq M+N$ *such that* $|z_1^k + \cdots + z_N^k| \geq 1.007(\frac{N}{4e(M+N)})^N |z_1|^k$.

**Remark.** For any $M \geq 1$, one can verify that the expression $\big(\frac{N}{4e(M+N)}\big)^N$ is a decreasing function of $N$. We will use this fact without mention.

Makai [Mak64] showed that the constant $4e$ is essentially optimal.

### Explicit Zero Density Estimate

Using Theorem 5.2.2 and Proposition 5.3.1, the second and primary goal of this section is to establish an explicit log-free zero density estimate. Recall, for a Hecke character $\chi$,

$$N(\sigma, T, \chi) = \#\{\rho : L(\rho, \chi) = 0, \sigma < \mathrm{Re}\{\rho\} < 1, |\mathrm{Im}(\rho)| \leq T\}. \tag{5.23}$$

where $\sigma \in (0, 1)$ and $T \geq 1$.

**Theorem 5.3.3.** *Let* $\xi \in (1, \infty)$ *and* $\upsilon \in (0, \frac{1}{10}]$ *be fixed and denote* $\sigma = 1 - \frac{\lambda}{\mathcal{L}}$*. Suppose*

$$\lambda_0 \leq \lambda < \tfrac{\epsilon}{\xi A_1}\mathcal{L}, \quad X > Y > 4.6, \quad \textit{and} \quad T \geq \max\{n_K^{5/6}(D_K^{4/3}Q^{4/9})^{-1/n_K}, 1\}, \tag{5.24}$$

*where* $X = X_{\xi\lambda}$ *and* $Y = Y_{\xi\lambda}$*. Then*

$$\sum_{\chi \pmod{H}} N(\sigma, T, \chi) \leq \frac{4\xi}{\sqrt{\xi^2 - 1}} \cdot (C_4\lambda^4 + C_3\lambda^3 + C_1\lambda + C_0) e^{B_1\lambda + B_2} \cdot \{1 - J(\xi\lambda)\}^{-2},$$

*where $J(\cdot)$ is defined by* (5.22) *satisfying $J(\xi\lambda) < 1$, and*

$$B_1 = 4\phi A\xi \log(4e\alpha^{-1}(1+\alpha)2^{(1+\alpha)/\alpha}), \quad B_2 = 16\log(4e\alpha^{-1}(1+\alpha)2^{(1+\alpha)/\alpha}),$$
$$C_4 = \frac{5\pi e\phi X(X-Y)^2(X+Y+1+\epsilon)\xi^4}{\left(1-\frac{1}{1+\upsilon}\right)\left(\frac{1}{1+\epsilon}Y-4\right)}, \; C_3 = \frac{4}{\phi\xi}C_4, \; C_1 = 4\phi A\xi, \; C_0 = 16A+\epsilon. \tag{5.25}$$

**Remark.**

- In Section 5.4, we will employ Theorem 5.3.3 with various choices of parameters $\alpha, \eta, \upsilon, \epsilon, \omega$, and $\xi$ depending on the range of $\sigma$. Consequently, this result is written without any explicit choice of the fixed or arbitrary quantities found in Section 5.3.1.
- The quantities $C_4$ and $C_3$ are technically not constants with respect to $\lambda$ or $\epsilon$ but one can see that both are bounded absolutely according to the definitions in Section 5.3.1.

Sections 5.3.3 and 5.3.4 are dedicated to preparing for the proof of Proposition 5.3.1 which is contained in Section 5.3.5. The proof of Theorem 5.3.3 is finalized in Section 5.3.6.

### 5.3.3 A large derivative

Suppose $\chi \pmod H$ is induced from the primitive character $\chi^*$. Denote $F(s) := \frac{L'}{L}(s,\chi^*)$ and $z := 1 + r + i\tau$. Using Theorem 5.3.2, the goal of this subsection is to show $F(s)$ has a large high order derivative, which we establish in the following lemma.

**Lemma 5.3.4.** *Keep the above notation and suppose $L(s,\chi)$ has a zero $\rho$ satisfying* (5.21). *If $\lambda < \frac{\epsilon}{A_1}\mathcal{L}$ and $\mathbf{1}_S$ is the indicator function of a set $S$, then*

$$E_0(\chi)\mathbf{1}_{\{|\tau|<Ar\}}(\tau) + \left|\frac{r^{k+1}}{k!}F^{(k)}(z)\right| \geq \frac{\left(\frac{\alpha}{4e(1+\alpha)}\right)^{2\phi A\lambda+8}}{2^{k+1}}\left\{1 - \frac{\left\{8(1+\frac{1}{k})A_1 + O(\epsilon)\right\}\lambda + 18}{A_1(1+\eta)^k}\right\}$$

*for some integer $k$ in the range $\frac{1}{\alpha}\cdot(2\phi A\lambda+8) \leq k \leq \frac{1+\alpha}{\alpha}\cdot(2\phi A\lambda+8)$.*

*Proof.* By [Wei83, Lemma 1.10],

$$F(s) + \frac{E_0(\chi)}{s-1} = \sum_{|1+i\tau-\rho|<1/2} \frac{1}{s-\rho} + G(s)$$

uniformly in the region $|1+i\tau-s| < 1/2$, where $G(s)$ is analytic and $|G(s)| \ll \mathcal{L}$ in this region. Differentiating the above formula $k$ times and evaluating at $z = 1+r+i\tau$, we deduce

$$\frac{(-1)^k}{k!}\cdot F^{(k)}(z) + \frac{E_0(\chi)}{(z-1)^{k+1}} = \sum_{|1+i\tau-\rho|<1/2} \frac{1}{(z-\rho)^{k+1}} + O(4^k\mathcal{L}),$$

since $r = \frac{\lambda}{\mathcal{L}} < \frac{1}{16}$ by assumption (5.18). The error term arises from bounding $G^{(k)}(z)$ using Cauchy's integral formula with a circle of radius $1/4$. For zeros $\rho$ that satisfy $Ar < |1 + i\tau - \rho| < 1/2$, notice that

$$(A^2 + 1)r^2 < r^2 + |1 + i\tau - \rho|^2 \le |z - \rho|^2 \le (r + |1 + i\tau - \rho|)^2 \le (r + 1/2)^2 < 1.$$

Recalling $A_1 = \sqrt{A^2 + 1}$, it follows by partial summation that

$$\sum_{Ar < |1+i\tau-\rho| < 1/2} \frac{1}{|z - \rho|^{k+1}} \le \int_{A_1 r}^{1} u^{-k-1} dN_\chi(u; z) = (k + 1) \int_{A_1 r}^{1} \frac{N_\chi(u; z)}{u^{k+2}} du + O(\mathcal{L})$$

where we bounded $N_\chi(1; z) \ll \mathcal{L}$ using [LMO79, Lemma 2.2]. By Lemma 2.3.7, the above is therefore

$$\le (k + 1) \int_{A_1 r}^{\infty} \frac{4u\mathcal{L} + 8}{u^{k+2}} du + O(\mathcal{L}) \le \frac{4\{1 + \frac{1}{k}\}A_1 r\mathcal{L} + 8}{(A_1 r)^{k+1}} + O(\mathcal{L}).$$

By considering cases, one may bound the $E_0(\chi)$-term as follows:

$$r^{k+1} \cdot \Big|\frac{E_0(\chi)}{(z - 1)^{k+1}}\Big| \le E_0(\chi) \cdot \mathbf{1}_{\{|\tau| < Ar\}}(\tau) + \frac{1}{A_1^{k+1}}. \tag{5.26}$$

The above results now yield

$$\begin{aligned} E_0(\chi)\mathbf{1}_{\{|\tau|<Ar\}}(\tau) + \Big|\frac{r^{k+1}F^{(k)}(z)}{k!}\Big| \\ \ge \Big|\sum_{|1+i\tau-\rho|\le Ar} \frac{r^{k+1}}{(z - \rho)^{k+1}}\Big| - \Big[\frac{4\{1 + \frac{1}{k}\}A_1 r\mathcal{L} + 9}{A_1^{k+1}} + O\big((4r)^{k+1}\mathcal{L}\big)\Big]. \end{aligned} \tag{5.27}$$

To lower bound the remaining sum over zeros, we wish to apply Theorem 5.3.2. Denote

$$N = N_\chi(Ar; 1 + i\tau) = \#\{\rho : L(\rho, \chi) = 0, |1 + i\tau - \rho| \le Ar\}.$$

Since $\lambda < \frac{\epsilon}{A_1}\mathcal{L} < \frac{\epsilon}{A}\mathcal{L}$ and $\epsilon < \frac{1}{8}$, it follows by Lemma 3.2.4 and (5.17) that $N \le 2\phi A\lambda + 8$. Define $M := \lfloor \frac{2\phi A\lambda+8}{\alpha} \rfloor$. Thus, from Theorem 5.3.2 and assumption (5.21),

$$\Big|\sum_{|1+i\tau-\rho|\le Ar} \frac{1}{(z - \rho)^{k+1}}\Big| \ge \Big(\frac{\alpha}{4e(1 + \alpha)}\Big)^{2\phi A\lambda+8} \frac{1}{(2r)^{k+1}} \tag{5.28}$$

for some $M+1 \le k \le M+N$. To simplify the right hand side of (5.27), observe that

$$(4r)^{k+1}\mathcal{L} \le 4\lambda(4r)^k \ll \lambda(4\epsilon)^k A_1^{-k} \ll \epsilon\lambda A_1^{-k}, \tag{5.29}$$

since $r = \frac{\lambda}{\mathcal{L}} < \frac{\epsilon}{A_1} < \frac{1}{4A_1}$ by assumption. Moreover, our choice of $A_1$ in (5.19) implies

$$A_1^{-(k+1)} = \Big(\frac{\alpha}{4e(1+\alpha)}\Big)^{\alpha k}\frac{1}{2^k}\cdot\frac{1}{A_1(1+\eta)^k} \le \Big(\frac{\alpha}{4e(1+\alpha)}\Big)^{2\phi A\lambda+8}\frac{1}{2^{k+1}}\cdot\frac{2}{A_1(1+\eta)^k} \tag{5.30}$$

since $\alpha k \ge \alpha(M+1) \ge 2\phi A\lambda + 8$. Incorporating (5.28)-(5.30) into (5.27) yields the desired result. The range of $k$ in Lemma 5.3.4 is determined by the above choice of $M$ and $N$. □

### 5.3.4 Short sum over prime ideals

Continuing with the discussion and notation of Section 5.3.3, from the Euler product for $L(s,\chi^*)$, we have that

$$F(s) = \frac{L'}{L}(s,\chi^*) = -\sum_{\mathfrak{n}} \chi^*(\mathfrak{n})\Lambda_K(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{-s}$$

for $\mathrm{Re}\{s\} > 1$, where $\Lambda_K(\,\cdot\,)$ is given by (2.5). Differentiating the above formula $k$ times, we deduce that

$$\frac{(-1)^{k+1}r^{k+1}}{k!}\cdot F^{(k)}(z) = \sum_{\mathfrak{n}} \frac{\Lambda_K(\mathfrak{n})\chi^*(\mathfrak{n})}{\mathrm{N}\mathfrak{n}^{1+r+i\tau}}\cdot rJ_k(r\log \mathrm{N}\mathfrak{n}) \tag{5.31}$$

for any integer $k \ge 1$, where $z = 1+r+i\tau$ and $J_k(u) = u^k/k!$. From Stirling's bound (see [oST]) in the form $k^k e^{-k}\sqrt{2\pi k} \le k! \le k^k e^{-k}\sqrt{2\pi k}e^{1/12k}$, one can verify that

$$J_k(u) \le \begin{cases} A_1^{-k}e^u & \text{if } u \le \dfrac{k}{eA_1}, \\ A_1^{-k}e^{(1-\omega)u} & \text{if } u \ge \frac{2}{1-\omega}\log\big(\frac{2A_1}{1-\omega}\big)k, \end{cases} \tag{5.32}$$

for any $k \ge 1$ and $A_1 > 1, \omega \in (0,1)$ defined in Section 5.3.1. The goal of this subsection is to bound the infinite sum in (5.31) by an integral average of short sums over prime ideals.

**Lemma 5.3.5.** *Suppose the integer $k$ is in the range given in Lemma 5.3.4. If $\lambda < \frac{\epsilon}{A_1}\mathcal{L}$ then*

$$\Big|\sum_{\mathfrak{n}} \frac{\chi^*(\mathfrak{n})\Lambda_K(\mathfrak{n})}{\mathrm{N}\mathfrak{n}^{1+r+i\tau}}\cdot rJ_k(r\log\mathrm{N}\mathfrak{n})\Big| \le r^2\int_y^x\Big|\sum_{y\le\mathrm{N}\mathfrak{p}<u}\frac{\chi^*(\mathfrak{p})\log\mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{1+i\tau}}\Big|\frac{du}{u}$$
$$+ (e[Y+\tfrac{1}{2}+\{2X+1\}e^{-\omega\lambda X}+O(\epsilon)]\lambda + e^{1-\omega\lambda X}/\omega)A_1^{-k},$$

*where $x = e^{X\mathcal{L}}$ and $y = e^{Y\mathcal{L}}$ with $X = X_\lambda, Y = Y_\lambda$ defined by* (5.20).

*Proof.* First, divide the sum on the left hand side into four sums:

$$\sum_{\mathfrak{n}} = \sum_{\mathrm{N}\mathfrak{p}<y} + \sum_{y \le \mathrm{N}\mathfrak{p}<x} + \sum_{\mathrm{N}\mathfrak{p}\ge x} + \sum_{\mathfrak{n}\text{ not prime}} = S_1 + S_2 + S_3 + S_4.$$

Observe that (5.20) and (5.32), along with the range of $k$ in Lemma 5.3.4, imply that

$$J_k(r\log \mathrm{N}\mathfrak{n}) \le \begin{cases} A_1^{-k}(\mathrm{N}\mathfrak{n})^r & \text{if } \mathrm{N}\mathfrak{n} \le y, \\ A_1^{-k}(\mathrm{N}\mathfrak{n})^{(1-\omega)r} & \text{if } \mathrm{N}\mathfrak{n} \ge x. \end{cases} \tag{5.33}$$

Hence, for $S_1$, it follows by Lemma 2.4.3 that

$$|S_1| \le rA_1^{-k}\sum_{\mathrm{N}\mathfrak{p}<y}\frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}} \le rA_1^{-k}\cdot e\log(eD_K^{1/2}y) \le e\Big(\lambda Y + \frac{\lambda}{2} + \epsilon\Big)A_1^{-k},$$

since $r = \frac{\lambda}{\mathcal{L}} < \epsilon, \log D_K \le \mathcal{L}$, and $y = e^{Y\mathcal{L}}$. Similarly, for $S_3$, apply partial summation using Lemma 2.4.3 to deduce that

$$|S_3| \le rA_1^{-k}\sum_{\mathrm{N}\mathfrak{p}\ge x}\frac{\log \mathrm{N}\mathfrak{p}}{(\mathrm{N}\mathfrak{p})^{1+\omega r}} \le rA_1^{-k}\int_x^\infty \frac{\omega r e\log(eD_K^{1/2}t)}{t^{1+\omega r}}dt \le (\{X+\tfrac{1}{2}\}\lambda+\omega^{-1}+\epsilon)\frac{e^{1-\omega\lambda X}}{A_1^k}.$$

For $S_4$, since $\frac{u^k}{k!} \le e^u$ for $u > 0$, observe that

$$J_k(r\log \mathrm{N}\mathfrak{n}) = (2r)^k(\tfrac{1}{2}\log \mathrm{N}\mathfrak{n})^k/k! \le (2r)^k(\mathrm{N}\mathfrak{n})^{\frac{1}{2}}.$$

Thus, by Lemma 2.4.3,

$$\begin{aligned}|S_4| \le r\sum_{\mathfrak{p}}\sum_{m\ge 2}\frac{\log \mathrm{N}\mathfrak{p}}{(\mathrm{N}\mathfrak{p}^m)^{1+r}}J_k(r\log \mathrm{N}\mathfrak{p}^m) \le (2r)^k r\sum_{\mathfrak{p}}\sum_{m\ge 2}\frac{\log \mathrm{N}\mathfrak{p}}{(\mathrm{N}\mathfrak{p}^m)^{1/2+r}} &\ll (2r)^k r\sum_{\mathfrak{p}}\frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{1+2r}} \\ &\ll \lambda\epsilon A_1^{-k},\end{aligned}$$

since $\log D_K \le \mathcal{L}$ and $\mathcal{L}^{-1} \ll r = \frac{\lambda}{\mathcal{L}} < \frac{\epsilon}{A_1}$. Also note that $\epsilon \in (0, \frac{1}{8})$ implies $(2\epsilon)^k \ll \epsilon$. Finally, for the main term $S_2$, define

$$W(u) = W_\chi(u;\tau) := \sum_{y\le \mathrm{N}\mathfrak{p}<u}\frac{\chi(\mathfrak{p})\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{1+i\tau}},$$

so by partial summation,

$$S_2 = rW(x)x^{-r}J_k(r\log x) - r^2\int_y^x W(u)\frac{d}{dt}[e^{-t}J_k(t)]\Big|_{t=r\log u}\frac{du}{u} \tag{5.34}$$

as $W(y) = 0$. Similar to $S_1, S_3$, and $S_4$, it follows from (5.33) and Lemma 2.4.3 that

$$|rW(x)x^{-r}J_k(r\log x)| \leq rA_1^{-k}x^{-\omega r}\sum_{y\leq \mathrm{N}\mathfrak{p}<x}\frac{\Lambda_K(\mathfrak{n})}{\mathrm{N}\mathfrak{n}} \leq e\big(\{X+\tfrac{1}{2}\}\lambda+\epsilon\big)e^{-\omega\lambda X}A_1^{-k}.$$

Observe $|\frac{d}{dt}(e^{-t}J_k(t))| = |e^{-t}J_{k-1}(t) - e^{-t}J_k(t)| \leq e^{-t}[J_{k-1}(t)+J_k(t)] \leq 1$ from the definition of $J_k(t)$ and since $\sum_{k=0}^{\infty}J_k(t) = e^t$. Hence,

$$|S_2| \leq r^2\int_y^x |W(u)|\frac{du}{u} + e\big(\{X+\tfrac{1}{2}\}\lambda+\epsilon\big)e^{-\omega\lambda X}A_1^{-k}.$$

Collecting all of our estimates, we conclude the desired result as $\lambda \geq \lambda_0 \gg 1$. □

### 5.3.5 Proof of Proposition 5.3.1

If $E_0(\chi)\mathbf{1}_{\{|\tau|<Ar\}}(\tau) = 1$ then the inequality in Proposition 5.3.1 holds trivially, as the right hand side is certainly less than 1. Thus, we may assume otherwise.

Combining Lemmas 5.3.4 and 5.3.5 via (5.31), it follows that

$$r^2\int_y^x\Big|\sum_{y\leq \mathrm{N}\mathfrak{p}<u}\frac{\chi^*(\mathfrak{p})\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{1+i\tau}}\Big|\frac{du}{u} \geq \Big(\frac{\alpha}{4e(1+\alpha)}\Big)^{2\phi A\lambda+8}\cdot\frac{1}{2^{k+1}}\big\{1-J(\lambda)\big\}, \tag{5.35}$$

after bounding $A_1^{-k}$ as in (5.30) and noting $k \geq k_0$ in the range of Lemma 5.3.4. By assumption, $J(\lambda) < 1$ and hence the right hand side of (5.35) is positive. Therefore, squaring both sides and applying Cauchy-Schwarz to the left hand side gives

$$r^4\log(x/y)\int_y^x\Big|\sum_{y\leq \mathrm{N}\mathfrak{p}<u}\frac{\chi^*(\mathfrak{p})\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{1+i\tau}}\Big|^2\frac{du}{u} \geq \Big(\frac{\alpha}{4e(1+\alpha)}\Big)^{4\phi A\lambda+16}\cdot\frac{1}{2^{2k+2}}\big\{1-J(\lambda)\big\}^2.$$

By assumption, $y = e^{Y\mathcal{L}} > e^{2\mathcal{L}} \geq \mathrm{N}\mathfrak{f}_\chi$, so it follows $\chi^*(\mathfrak{p}) = \chi(\mathfrak{p})$ for $y \leq \mathrm{N}\mathfrak{p} < x$ so we may replace $\chi^*$ with $\chi$ in the above sum over prime ideals. Finally, we note $k \leq \frac{1+\alpha}{\alpha}(2\phi A\lambda+8)$ since $k$ is in the range of Lemma 5.3.4, yielding the desired result. □

### 5.3.6 Proof of Theorem 5.3.3

For $\chi \pmod H$, consider zeros $\rho = \beta + i\gamma$ of $L(s,\chi)$ such that

$$1 - \lambda/\mathcal{L} \leq \beta < 1, \qquad |\gamma| \leq T. \tag{5.36}$$

Denote $\lambda^\star = \xi\lambda$ and $r^\star = \lambda^\star/\mathcal{L} = \xi(1-\sigma)$, so by (5.24) we have $r^\star < \frac{\epsilon}{A_1}$. For any zero $\rho = \beta + i\gamma$ of $L(s,\chi)$, define $\Phi_{\rho,\chi}(\tau) := \mathbf{1}_{\{|1+i\tau-\rho|\leq r^\star\}}(\tau)$. If $\rho$ satisfies (5.36) then one can verify by elementary arguments that

$$\frac{1}{r^\star}\int_{-T}^{T} \Phi_{\rho,\chi}(\tau)d\tau \geq \frac{\sqrt{\xi^2-1}}{\xi}.$$

Applying Proposition 5.3.1 to such zeros $\rho$, it follows that

$$\begin{aligned}\int_{-T}^{T} \frac{1}{r^\star}\Phi_{\rho,\chi}(\tau)\Big[(r^\star)^4\log(x/y)\int_y^x \Big|\sum_{y\leq \mathrm{N}\mathfrak{p}<u} \frac{\chi(\mathfrak{p})\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{1+i\tau}}\Big|^2\frac{du}{u} + E_0(\chi)\mathbf{1}_{\{|\tau|<Ar^\star\}}(\tau)\Big]d\tau \\ \geq \frac{\sqrt{\xi^2-1}}{4\xi}\Big(\frac{\alpha}{4e(1+\alpha)2^{(1+\alpha)/\alpha}}\Big)^{2\phi A\xi\lambda+16} \times \big\{1-J(\xi\lambda)\big\} =: w(\lambda),\end{aligned}$$

say. Note $x = e^{X\mathcal{L}}$ and $y = e^{Y\mathcal{L}}$ where $X = X_{\lambda^\star}$ and $Y = Y_{\lambda^\star}$. Summing over all zeros $\rho$ of $L(s,\chi)$ satisfying (5.36), we have that

$$\begin{aligned}w(\lambda)N(\sigma,T,\chi) \leq (X-Y)(2\phi r^\star\mathcal{L}+8)(r^\star)^3\mathcal{L}\int_y^x \Big(\int_{-T}^{T}\Big|\sum_{y\leq \mathrm{N}\mathfrak{p}<u} \frac{\chi(\mathfrak{p})\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{1+i\tau}}\Big|^2 d\tau\Big)\frac{du}{u} \\ + E_0(\chi)(4\phi Ar^\star\mathcal{L}+16A),\end{aligned} \tag{5.37}$$

because, by Lemma 3.2.4,

$$\sum_{\substack{\rho \\ L(\rho,\chi)=0}} \Phi_{\rho,\chi}(\tau) = N_\chi(r^\star; 1+i\tau) \leq 2\phi r^\star\mathcal{L} + 8$$

for $|\tau| \leq T$ and $r^\star < \epsilon$. From the conditions on $Y$ and $T$ in (5.24) and the definition of $\mathcal{L}$ in (5.17), observe that, for $\nu = \nu(\epsilon) > 0$ sufficiently small, Lemma 2.4.6 implies that

$$y = e^{Y\mathcal{L}} \geq C_\nu\{h_H n_K^{(5/4+2\upsilon)n_K} D_K^{3/2+2\upsilon} Q^{1/2} T^{n_K/2+1}\}^{1+\nu},$$

since $\upsilon \leq \frac{1}{10}$ and $\Theta = \Theta(\epsilon) \geq 1$ is sufficiently large. Therefore, we may sum (5.37) over $\chi \pmod{H}$ and apply Theorem 5.2.2 with $b(\mathfrak{p}) = \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}}$ for $y \leq \mathrm{N}\mathfrak{p} < u$ to deduce

$$w(\lambda) \sum_{\chi \pmod{H}} N(\sigma, T, \chi) \leq \Big( C'(2\phi r^\star \mathcal{L} + 8)(r^\star)^3 + O_\epsilon\big(\frac{(r^\star)^4 \mathcal{L}^2}{e^{\epsilon Y \mathcal{L}/2}}\big)\Big) \int_y^x \sum_{y \leq \mathrm{N}\mathfrak{p} < u} \frac{(\log \mathrm{N}\mathfrak{p})^2}{\mathrm{N}\mathfrak{p}} \frac{du}{u} + 4A\phi r^\star \mathcal{L} + 16A, \tag{5.38}$$

where $C' = 5\pi(X - Y)(1 - \frac{1}{1+\upsilon})^{-1}(\frac{1}{1+\epsilon}Y - 4)^{-1}$. To calculate $C'$, we replaced $\mathcal{L}'$ (found in Theorem 5.2.2) by observing from Lemma 2.4.6 that $\mathcal{L}' + \frac{1}{1+\epsilon}\log h_H \leq 4\mathcal{L}$ since $T \geq \max\{n_K^{5/6} D_K^{-4/3n_K} Q^{-4/9n_K}, 1\}$ and $\Theta = \Theta(\epsilon)$ is sufficiently large. For the remaining integral in (5.38), notice by Lemma 2.4.3 that

$$\int_y^x \sum_{y \leq \mathrm{N}\mathfrak{p} < u} \frac{(\log \mathrm{N}\mathfrak{p})^2}{\mathrm{N}\mathfrak{p}} \frac{du}{u} \leq \log x \int_y^x e \log(e D_K^{1/2} u) \frac{du}{u} \leq \tfrac{e}{2} X(X - Y)(X + Y + 1 + \tfrac{2}{\mathcal{L}})\mathcal{L}^3.$$

Substituting this estimate in (5.38) and recalling $r^\star = \lambda^\star/\mathcal{L} = \xi\lambda/\mathcal{L}$, we have shown

$$w(\lambda) \sum_{\chi \pmod{H}} N(\sigma, T, \chi) \leq 2\phi C'' \xi^4 \cdot \lambda^4 + 8C''\xi^3 \cdot \lambda^3 + 4\phi A \xi \cdot \lambda + 16A + O_\epsilon(\lambda^3 \mathcal{L} e^{-\epsilon \mathcal{L}}),$$

where $C'' = \frac{e}{2} X(X - Y)(X + Y + 1 + \frac{2}{\mathcal{L}})C'$. Since $\mathcal{L} \geq \Theta$ and $\Theta$ is sufficiently large depending on $\epsilon$, the big-O error term above and the quantity $\frac{2}{\mathcal{L}}$ in $C''$ may both be bounded by $\epsilon$. This completes the proof of Theorem 5.3.3. □

## 5.4 Proofs of log-free zero density estimates

Having established Theorem 5.3.3, we may deduce Theorems 5.1.1 and 5.1.3.

### 5.4.1 Proof of Theorem 5.1.1

Without loss, we may assume $H \pmod{\mathfrak{q}}$ is primitive because $Q = Q_H = Q_{H'}, h_H = h_{H'}$ and

$$\sum_{\chi \pmod{H}} N(\sigma, T, \chi) = \sum_{\chi \pmod{H'}} N(\sigma, T, \chi)$$

if $H'$ induces $H$. Suppose $\frac{1}{2} \leq \sigma \leq 1 - \frac{0.05}{4}$. By a naive application of [LMO79, Lemma 2.1], one can verify that for $T \geq 1$,

$$\sum_{\chi \,(\mathrm{mod}\, H)} N(\sigma, T, \chi) \ll h_H T \log(D_K Q T^{n_K}) \ll (e^{O(n_K)} D_K^2 Q T^{n_K+2})^{81(1-\sigma)} \tag{5.39}$$

after bounding $h_H$ with Lemma 2.4.6.

Now, let $\epsilon \in (0, 1/8)$ be fixed and define $\mathcal{L}$ as in (5.17). Suppose $1 - \frac{\epsilon}{4} < \sigma < 1$. Let $R \geq 1$ be fixed and sufficiently large. By applying the bound in Lemma 2.4.6 to [Wei83, Theorem 4.3], we deduce that for $T \geq 1$,

$$\sum_{\chi \,(\mathrm{mod}\, H)} N(1 - \tfrac{R}{\mathcal{L}}, T, \chi) \ll 1, \tag{5.40}$$

so it suffices to bound $\sum_{\chi \,(\mathrm{mod}\, H)} N(\sigma, T, \chi)$ in the range

$$1 - \frac{\epsilon}{4} < \sigma < 1 - \frac{R}{\mathcal{L}}. \tag{5.41}$$

Equivalently, if $\sigma = 1 - \frac{\lambda}{\mathcal{L}}$ then we consider the range $R < \lambda < \frac{\epsilon}{4}\mathcal{L}$. According to Theorem 5.3.3 and the notation defined in Section 5.3.1, select

$$\xi = 1 + 10^{-5}, \qquad \upsilon = 10^{-5}, \qquad \eta = 10^{-5}, \qquad \omega = 10^{-5}, \quad \text{and} \quad \alpha = 0.15.$$

It follows that the constants $B_2, C_0, C_1, C_3, C_4$ in Theorem 5.3.3 are bounded absolutely,

$$X > Y > 4.6, \quad B_1 \leq 146.15\phi, \quad \text{and} \quad \xi A_1 < 4$$

where $\phi = 1 + \frac{4}{\pi}\epsilon + 16\epsilon^2 + 340\epsilon^{10}$. Moreover, since $\lambda > R$, $J(\xi\lambda) \ll \frac{\lambda}{(1+10^{-5})^{\lambda}} \ll \frac{R}{(1+10^{-5})^{R}}$ and therefore $J(\xi\lambda) < \frac{1}{2}$ for $R$ sufficiently large. Thus, by Theorem 5.3.3,

$$\sum_{\chi \,(\mathrm{mod}\, H)} N(\sigma, T, \chi) \ll \lambda^4 e^{146.15\phi\lambda} \ll e^{146.2\phi\lambda} = e^{146.2\phi(1-\sigma)\mathcal{L}} \tag{5.42}$$

for $\sigma$ satisfying (5.41) and $T \geq \max\{n_K^{5/6} D_K^{-4/3n_K} Q^{-4/9n_K}, 1\}$. To complete the proof of Theorem 5.1.1, it remains to choose $\epsilon$ in (5.42). If $\epsilon = 0.05$ then $146.2\phi < 162 = 2 \cdot 81$ yielding the desired result when combined with (5.39). If $\epsilon = 10^{-3}$ then $146.2\phi < 147 = 2 \cdot 73.5$ as claimed. $\square$

### 5.4.2 Proof of Theorem 5.1.3

For $T \geq 1$, set $T_0 := \max\{n_K^{5/6} D_K^{-4/3n_K} Q^{-4/9n_K}, T\}$. Comparing $\mathcal{L} = \mathcal{L}_{T_0,\delta_0}$ given by (5.17) with $\mathscr{L}$ given by (5.3), one may deduce $\mathcal{L} \leq \mathscr{L}$ for $\mathscr{L}$ sufficiently large depending only on $T$. Hence, for $\lambda > 0$ and $\mathscr{L}$ sufficiently large depending only on $T$, we have that

$$\mathcal{N}(\lambda) = \mathcal{N}_H(\lambda) = \sum_{\chi \,(\mathrm{mod}\, H)} N(1 - \tfrac{\lambda}{\mathscr{L}}, T, \chi) \leq \sum_{\chi \,(\mathrm{mod}\, H)} N(1 - \tfrac{\lambda}{\mathcal{L}}, T, \chi), \tag{5.43}$$

where $N(\sigma, T, \chi)$ defined in (5.23). For $\lambda \leq 0.2866$, the result follows as $\mathcal{N}(0.2866) \leq 2$ by Theorem 4.1.2. For each fixed value of $0.2866 \leq \lambda \leq 1$ appearing in Table 5.1, we apply Theorem 5.3.3 with $\upsilon = 0.1$ and $\epsilon \in (0, 10^{-5})$ assumed to be fixed and sufficiently small; this yields a bound for $\mathcal{N}(\lambda\mathscr{L}/\mathcal{L})$. By (5.43), the same bound holds for $\mathcal{N}(\lambda)$. Using MATLAB, we roughly optimize the bound in Theorem 5.3.3 by numerical experimentation over the remaining parameters $(\alpha, \eta, \omega, \xi)$ which produces Table 5.1. Note that we have verified $J(\xi\lambda) < 1$ and $X_{\xi\lambda} > Y_{\xi\lambda} > 4.6$ in each case. It remains to consider $\lambda \geq 1$. Apply Theorem 5.3.3 with

$$\begin{aligned} &T = 1, && \lambda_0 = 1, && \alpha = 0.1549, && \eta = 0.05722, \\ &\epsilon = 10^{-5}, && \upsilon = 0.1, && \xi = 1.0030, && \omega = 0.02074. \end{aligned}$$

This choice of values is motivated by the last row of Table 5.1, but with a more suitable choice for $\alpha$. With this selection, one can check that for any $\lambda \geq 1$,

$$4.61 \leq Y_{\xi\lambda} \leq 9.2, \qquad 264 \leq X_{\xi\lambda} \leq 526, \qquad J(\xi\lambda) \leq 0.272.$$

These inequalities can be verified by elementary arguments and the definitions in Section 5.3.1 and (5.22). In particular, for any $\lambda \geq 1$, the assumptions of Theorem 5.3.3 are satisfied for all $1 \leq \lambda < \epsilon_0\mathscr{L}$. Denoting $C_4, C_3, C_1, C_0, B_2, B_1$ as in Theorem 5.3.3, it follows that:

$$\begin{aligned} &C_4 = C_4(\lambda) \leq 6.0 \times 10^{13}, && C_1 \leq 17, && B_2 \leq 154, \\ &C_3 = C_3(\lambda) \leq 2.4 \times 10^{14}, && C_0 \leq 65, && B_1 \leq 156, \end{aligned}$$

for $\lambda \geq 1$. Thus, by Theorem 5.3.3, for $1 \leq \lambda \leq \epsilon_0\mathscr{L}$,

$$\begin{aligned} \mathcal{N}(\lambda) &\leq 52\big(6.0 \times 10^{13} \cdot \lambda^4 + 2.4 \times 10^{14} \cdot \lambda^3 + 17 \cdot \lambda + 65\big)e^{156\lambda+154} \\ &\leq 52 \cdot 6.7 \times 10^{12} \cdot \Big(\frac{(6\lambda)^4}{4!} + \frac{(6\lambda)^3}{3!} + 6\lambda + 1\Big)e^{156\lambda+154} \\ &\leq 52 \cdot 6.7 \times 10^{12} \cdot e^{162\lambda+154} \leq e^{162\lambda+188}. \qquad \square \end{aligned}$$

# Chapter 6

# Deuring–Heilbronn phenomenon

*"Everybody, try laughing. Then whatever scares you will go away!"*

– Tatsuo Kusakabe, My Neighbor Totoro.

The Deuring–Heilbronn phenomenon for Hecke $L$-functions quantifies the zero repulsion effect of a simple real zero attached to a real (possibly trivial) Hecke character all the way to the critical line. In this chapter, we establish explicit variants of the Deuring–Heilbronn phenomenon for the Dedekind zeta function of a number field $K$ and for the Hecke $L$-functions of characters $\chi \,(\mathrm{mod}\, H)$ where $H$ is an arbitrary congruence class group of $K$. As usual, we retain the notation of Chapter 2 only.

The only known proof method which retains the appropriate field uniformity utilizes power sums. This technique originates from the work of Lagarias–Montgomery–Odlyzko [LMO79, Theorem 5.1] and appears again in a paper of Weiss [Wei83, Theorem 4.3]. In all cases, our approach follows the general structure of [LMO79, Theorem 5.1] with a more careful analysis.

## 6.1 Statement of results

We begin by stating a variant for Hecke $L$-functions.

**Theorem 6.1.1.** *Let $H$ be a congruence class group of a number field $K$ with $Q = Q_H$ given by* (2.2). *Let $\psi \,(\mathrm{mod}\, H)$ be a real Hecke character and suppose $L(s,\psi)$ has a real zero $\beta_1$. Let $T \geq 1$ be arbitrary, and $\chi \,(\mathrm{mod}\, H)$ be an arbitrary Hecke character. Let $\rho' = \beta' + i\gamma'$ be a zero of $L(s,\chi)$ satisfying $\frac{1}{2} \leq \beta' < 1$ and $|\gamma'| \leq T$. Then, for $\epsilon > 0$ arbitrary,*

$$\beta' \leq 1 - \frac{\log\left(\dfrac{c_\epsilon}{(1-\beta_1)\log(D_K \cdot Q \cdot T^{n_K} e^{O_\epsilon(n_K)})}\right)}{b_1 \log D_K + b_2 \log Q + b_3 n_K \log T + O_\epsilon(n_K)}$$

*for some absolute effective constant $c_\epsilon > 0$, where*

$$(b_1, b_2, b_3) = \begin{cases} (48+\epsilon, 60+\epsilon, 24+\epsilon) & \text{if } \psi \text{ is quadratic}, \\ (24+\epsilon, 12+\epsilon, 12+\epsilon) & \text{if } \psi \text{ is trivial.} \end{cases}$$

The above result is the first explicit variant of its kind. To prove Theorem 1.3.1, we need to quantify the Deuring–Heilbronn phenomenon for only the Dedekind zeta function of $K$, which is the special case when $\psi$ and $\chi$ are both trivial in Theorem 6.1.1. However, we will require the following more precise version.

**Theorem 6.1.2.** *Let $K$ be an arbitrary number field and $T \geq 1$ be fixed. Suppose $\zeta_K(s)$ has a real zero $\beta_1$ and let $\rho' = \beta' + i\gamma'$ be another zero of $\zeta_K(s)$ satisfying*

$$\tfrac{1}{2} \leq \beta' < 1 \quad \text{and} \quad |\gamma'| \leq T. \tag{6.1}$$

*Then, for $D_K$ sufficiently large,*

$$\beta' \leq 1 - \frac{\log\left(\dfrac{c}{(1-\beta_1)\log D_K}\right)}{C \log D_K},$$

*where $c = c(T) > 0$ and $C = C(T) > 0$ are absolute effective constants. In particular, one may take $T$ and $C = C(T)$ according to the table below.*

| $T$ | 1 | 3.5 | 8.7 | 22 | 54 | 134 | 332 | 825 | 2048 | 5089 | 12646 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $C$ | 31.4 | 32.7 | 35.0 | 38.4 | 42.0 | 45.9 | 49.7 | 53.6 | 57.4 | 61.2 | 65.0 |

**Remark.**

(i) This result for general $T \geq 1$ follows from [LMO79, Theorem 5.1] but our primary concern is verifying the table of values for $T$ and $C$. The choices of $T$ in the given table are obviously not special; one can compute $C$ for any fixed $T$ by a simple modification to our argument below. We made these selections primarily for their application in the proof of Theorem 1.3.1.

(ii) If $n_K = o(\log D_K)$ then one can take $C = 24.01$ for any fixed $T$.

(iii) Kadiri and Ng [KN12] alternatively show that if

$$1 - \frac{\log\log D_K}{13.84 \log D_K} \leq \beta' < 1 \quad \text{and} \quad |\gamma'| \leq 1 \tag{6.2}$$

and $D_K$ is sufficiently large then

$$\beta' \leq 1 - \frac{\log\left(\frac{1}{(1-\beta_1)\log D_K}\right)}{1.53 \log D_L}.$$

While the repulsion constant $1.53$ is much better than $31.4$ given by Theorem 6.1.1, the permitted range of $\beta'$ in (6.1) is much larger than that of (6.2) therefore allowing Theorem 6.1.2 to deal with Siegel zeros which are extremely close to $1$. Thus, to distinguish this feature, we refer to Kadiri and Ng's result as "zero repulsion" whereas Theorem 6.1.2 is "Deuring–Heilbronn" phenomenon. The same type of comment holds true when comparing Theorem 4.1.3 (zero repulsion) with Theorem 6.1.1 (Deuring–Heilbronn phenomenon).

If $\rho'$ is a real zero in Theorem 6.1.2, then one can improve upon the above theorem.

**Theorem 6.1.3.** *Suppose $\zeta_K(s)$ has a real zero $\beta_1$ and let $\beta'$ be another real zero of $\zeta_K(s)$ satisfying $0 < \beta' < 1$. Then, for $D_K$ sufficiently large,*

$$\beta' \leq 1 - \frac{\log\left(\frac{c}{(1-\beta_1)\log D_K}\right)}{16.6 \log D_K},$$

*where $c > 0$ is an absolute effective constant.*

**Remark.** If $n_K = o(\log D_K)$ then $16.6$ can be replaced by $12.01$.

Applying the above theorem to the zero $\beta' = 1 - \beta_1$ of $\zeta_K(s)$ immediately yields the following corollary which will play a key role in our proof of Theorem 1.3.1.

**Corollary 6.1.4.** *Suppose $\zeta_K(s)$ has a real zero $\beta_1$. Then, for $D_K$ sufficiently large,*

$$1 - \beta_1 \gg D_K^{-16.6},$$

*where the implicit constant is absolute and effective.*

**Remark.** Corollary 6.1.4 makes explicit [LMO79, Corollary 5.2] and so, as remarked therein, Stark [Sta74] gives a better lower bound for $1 - \beta_1$ when $K$ has a tower of normal extensions with base $\mathbb{Q}$. However, if $\log D_K = o(n_K \log n_K)$ then the above bound is superior to [Sta74]. This condition on $\log D_K$ holds, for example, when $K$ runs through an infinite $\ell$-class field tower above some fixed number field $F \neq \mathbb{Q}$ and for some fixed prime $\ell$.

# 6.2 Preliminaries

## 6.2.1 Power sum inequality

We record a power sum inequality and its proof from [LMO79, Theorem 4.2] specialized to our intended application.

**Lemma 6.2.1.** *Define*

$$P(r,\theta) := \sum_{j=1}^{J} \Big(1 - \frac{j}{J+1}\Big) r^j \cos(j\theta).$$

*Then*

*(i)* $P(r,\theta) \geq -\frac{1}{2}$ *for* $0 \leq r \leq 1$ *and all* $\theta$.

*(ii)* $P(1,0) = J/2$.

*(iii)* $|P(r,\theta)| \leq \frac{3}{2} r$ *for* $0 \leq r \leq 1/3$.

*Proof.* See [LMO79, Lemma 4.1] for details. □

**Theorem 6.2.2** (Lagarias–Montgomery–Odlyzko)**.** *Let* $\epsilon > 0$ *and a sequence of complex numbers* $\{z_n\}_n$ *be given. Let* $s_m = \sum_{n=1}^{\infty} z_n^m$ *and suppose that* $|z_n| \leq |z_1|$ *for all* $n \geq 1$*. Define*

$$M := \frac{1}{|z_1|} \sum_n |z_n|. \tag{6.3}$$

*Then there exists* $m_0$ *with* $1 \leq m_0 \leq (12+\epsilon)M$ *such that*

$$\mathrm{Re}\{s_{m_0}\} \geq \frac{\epsilon}{48+5\epsilon} |z_1|^{m_0}.$$

*Proof.* This is a simplified version of [LMO79, Theorem 4.2]; our focus was to reduce their constant $24$ to $12+\epsilon$ by some minor modifications. We reiterate the proof here for clarity. Rescaling we may suppose $|z_1| = 1$. Write $z_n = r_n \exp(i\theta_n)$ so $r_n \in [0,1]$. Then

$$\begin{aligned}
S_J &:= \sum_{j=1}^{J} \Big(1 - \frac{j}{J+1}\Big) \mathrm{Re}\{s_j\}(1 + \cos j\theta_1) \\
&= \sum_{n=1}^{\infty} \sum_{j=1}^{J} \Big(1 - \frac{j}{J+1}\Big) (\cos j\theta_n)(1 + \cos j\theta_1) r_n^j \\
&= \sum_{n=1}^{\infty} \big\{ P(r_n,\theta_n) + \tfrac{1}{2} P(r_n, \theta_n - \theta_1) + \tfrac{1}{2} P(r_n, \theta_n + \theta_1) \big\}.
\end{aligned}$$

Using Lemma 6.2.1, we lower bound the contribution of each term. For $n = 1$, we obtain a contribution $\geq \left(\frac{J+1}{4} - r_1\right)$. Terms $n > 1$ satisfying $r_n \geq 1/3$ contribute $\geq -1 \geq -3r_n$. Each of the remaining terms satisfying $r_n < 1/3$ are bounded using Lemma 6.2.1(iii) and therefore contribute $\geq -3r_n$. Choosing $J = \lfloor (12+\epsilon)M \rfloor$, we deduce that

$$S_J \geq \frac{J+1}{4} - 3M \geq \frac{\epsilon M}{4}, \tag{6.4}$$

as $J+1 \geq (12+\epsilon)M$. Now, suppose for a contradiction that $\mathrm{Re}\{s_j\} < \frac{\epsilon}{48+5\epsilon}$ for all $1 \leq j \leq J$. Then, as $(1 - \frac{j}{J+1})(1 + \cos j\theta_1)$ is non-negative for all $1 \leq j \leq J$,

$$S_J \leq \frac{\epsilon}{48+5\epsilon} \sum_{j=1}^{J} \Big(1 - \frac{j}{J+1}\Big)(1 + \cos j\theta_1) < \frac{\epsilon}{48+5\epsilon} \cdot 2P(1,0) = \frac{\epsilon J}{48+5\epsilon}.$$

Comparing with (6.4) and noting $J \leq (12+\epsilon)M$, we obtain a contradiction. □

### 6.2.2 Technical estimates for Hecke $L$-functions

In this subsection, we consider Hecke $L$-functions and certain sums over their zeros, both trivial and non-trivial.

**Lemma 6.2.3.** *Let $\chi \pmod{\mathfrak{q}}$ be a Hecke character. For $\sigma \geq 2$ and $t \in \mathbb{R}$,*

$$-\mathrm{Re}\Big\{\frac{L'}{L}(\sigma+it,\chi)\Big\} \leq -\mathrm{Re}\Big\{\frac{L'}{L}(\sigma+it,\chi^*)\Big\} + \frac{1}{2^\sigma - 1}\big(n_K + \log \mathrm{N}\mathfrak{q}\big),$$

*where $\chi^* \pmod{\mathfrak{f}_\chi}$ is the primitive character inducing $\chi$.*

*Proof.* By definition,

$$L(s,\chi) = P(s,\chi)L(s,\chi^*), \quad \text{where} \quad P(s,\chi) = \prod_{\substack{\mathfrak{p}\mid\mathfrak{q} \\ \mathfrak{p}\nmid\mathfrak{f}_\chi}} \Big(1 - \frac{\chi^*(\mathfrak{p})}{\mathrm{N}\mathfrak{p}^s}\Big).$$

Hence, it suffices to show $|\frac{P'}{P}(s,\chi)| \leq \frac{1}{2^\sigma-1}(n_K + \log \mathrm{N}\mathfrak{q})$. Observe that

$$\Big|\frac{P'}{P}(s,\chi)\Big| = \Big|\sum_{\substack{\mathfrak{p}\mid\mathfrak{q} \\ \mathfrak{p}\nmid\mathfrak{f}_\chi}} \sum_{k=1}^{\infty} \frac{\chi^*(\mathfrak{p}^k)\log \mathrm{N}\mathfrak{p}^k}{k(\mathrm{N}\mathfrak{p}^k)^s}\Big| = \sum_{\mathfrak{p}\mid\mathfrak{q}} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^\sigma - 1} \leq \frac{1}{1-2^{-\sigma}} \cdot \frac{1}{2^{\sigma-1}} \sum_{\mathfrak{p}\mid\mathfrak{q}} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}}.$$

We bound the remaining sum by taking $\epsilon = 1$ in Lemma 2.4.4. This yields the desired estimate. □

**Lemma 6.2.4.** *Let $\chi \pmod{\mathfrak{q}}$ be a Hecke character. For $\sigma > 1$ and $t \in \mathbb{R}$,*

$$\sum_{\omega \text{ trivial}} \frac{1}{|\sigma + it - \omega|^2} \leq \begin{cases} \left(\frac{1}{2\sigma} + \frac{1}{\sigma^2}\right) \cdot n_K & \textit{if } \chi \textit{ is primitive,} \\ \left(\frac{1}{2\sigma} + \frac{1}{\sigma^2}\right) \cdot n_K + \left(\frac{1}{2\sigma} + \frac{2}{\sigma^2 \log 2}\right) \cdot \log \mathrm{N}\mathfrak{q} & \textit{unconditionally,} \end{cases}$$

*where the sum is over all trivial zeros $\omega$ of $L(s, \chi)$, counted with multiplicity.*

*Proof.* Suppose $\chi \pmod{\mathfrak{q}}$ is induced by the primitive character $\chi^* \pmod{\mathfrak{f}_\chi}$. Then

$$L(s, \chi) = P(s, \chi) L(s, \chi^*), \quad \text{where} \quad P(s, \chi) = \prod_{\mathfrak{p} | \mathfrak{q}, \ \mathfrak{p} \nmid \mathfrak{f}_\chi} \Big(1 - \frac{\chi^*(\mathfrak{p})}{\mathrm{N}\mathfrak{p}^s}\Big)$$

for all $s \in \mathbb{C}$. Thus, the trivial zeros of $L(s, \chi)$ are zeros of the finite Euler product $P(s, \chi)$ or trivial zeros of $L(s, \chi^*)$. We consider each separately. From (2.8) and (2.5), observe that

$$\begin{aligned} \sum_{\substack{\omega \text{ trivial} \\ L(\omega, \chi^*) = 0}} \frac{1}{|\sigma + it - \omega|^2} &\leq a(\chi) \sum_{k=0}^{\infty} \frac{1}{(\sigma + 2k)^2 + t^2} + b(\chi) \sum_{k=0}^{\infty} \frac{1}{(\sigma + 2k + 1)^2 + t^2} \\ &\leq n_K \sum_{k=0}^{\infty} \frac{1}{(\sigma + 2k)^2} \leq \Big(\frac{1}{2\sigma} + \frac{1}{\sigma^2}\Big) n_K. \end{aligned}$$

Now, if $\chi$ is primitive then $P(s, \chi) \equiv 1$ and hence never vanishes. Otherwise, notice the zeros of each $\mathfrak{p}$-factor in the Euler product of $P(s, \chi)$ are totally imaginary and are given by $a_\chi(\mathfrak{p}) i + \frac{2\pi i \mathbb{Z}}{\log \mathrm{N}\mathfrak{p}}$ for some $0 \leq a_\chi(\mathfrak{p}) < 2\pi / \log \mathrm{N}\mathfrak{p}$. Translating these zeros $\omega \mapsto \omega + it$ amounts to choosing another representative $0 \leq b_\chi(\mathfrak{p}; t) < 2\pi / \log \mathrm{N}\mathfrak{p}$. Therefore,

$$\sum_{\substack{\omega \text{ trivial} \\ P(\omega, \chi) = 0}} \frac{1}{|\sigma + it - \omega|^2} \leq 2 \sum_{\substack{\mathfrak{p} | \mathfrak{q} \\ \mathfrak{p} \nmid \mathfrak{f}_\chi}} \sum_{k=0}^{\infty} \frac{1}{\sigma^2 + (2\pi k / \log \mathrm{N}\mathfrak{p})^2} \leq \Big(\frac{1}{2\sigma} + \frac{2}{\sigma^2 \log 2}\Big) \log \mathrm{N}\mathfrak{q},$$

as required. $\square$

**Lemma 6.2.5.** *Let $H$ be a congruence class group of the number field $K$. Suppose $\psi \pmod{H}$ is real and $\chi \pmod{H}$ is arbitrary. For $\sigma = \alpha + 1$ with $\alpha \geq 1$ and $t \in \mathbb{R}$,*

$$\begin{aligned} &\sum_{\substack{\rho \\ \zeta_K(\rho) = 0}} \frac{1}{|\sigma - \rho|^2} + \sum_{\substack{\rho \\ L(\rho, \psi) = 0}} \frac{1}{|\sigma - \rho|^2} + \sum_{\substack{\rho \\ L(\rho, \chi) = 0}} \frac{1}{|\sigma + it - \rho|^2} + \sum_{\substack{\rho \\ L(\rho, \psi\chi) = 0}} \frac{1}{|\sigma + it - \rho|^2} \\ &\quad \leq \frac{1}{\alpha} \cdot \Big[\frac{1}{2} \log(D_K^3 Q^2 D_\psi) + \Big(\log(\alpha + 2) + \frac{2}{\alpha + 1} + \frac{1}{2^{\alpha+1} - 1} - 2 \log \pi\Big) n_K \\ &\qquad + n_K \log(\alpha + 2 + |t|) + \frac{2}{2^{\alpha+1} - 1} \log Q + \frac{4}{\alpha} + \frac{4}{\alpha + 1}\Big], \end{aligned}$$

*where the sums are over all non-trivial zeros of the corresponding $L$-functions.*

**Remark.** If $\psi$ is trivial, notice that the left hand side equals

$$2\Big( \sum_{\substack{\rho \\ \zeta_K(\rho)=0}} \frac{1}{|\sigma-\rho|^2} + \sum_{\substack{\rho \\ L(\rho,\chi)=0}} \frac{1}{|\sigma+it-\rho|^2}\Big).$$

This additional factor of 2 will be useful to us later.

*Proof.* Suppose $\psi$ and $\chi$ are induced from the primitive characters $\psi^*$ and $\chi^*$ respectively. From the identity $0 \le (1+\psi^*(\mathfrak{n}))(1+\mathrm{Re}\{\chi^*(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{-it}\})$, it follows that

$$0 \le -\mathrm{Re}\Big\{ \frac{\zeta_K'}{\zeta_K}(\sigma) + \frac{L'}{L}(\sigma,\psi^*) + \frac{L'}{L}(\sigma+it,\chi^*) + \frac{L'}{L}(\sigma+it,\psi^*\chi^*)\Big\}. \tag{6.5}$$

The first three $L$-functions are primitive, but $\xi := \psi^*\chi^*$ is not necessarily primitive. Note $\xi$ is a character modulo $[\mathfrak{f}_\chi,\mathfrak{f}_\psi]$, the least common multiple of $\mathfrak{f}_\psi$ and $\mathfrak{f}_\chi$. Hence, by Lemma 6.2.3, we deduce that

$$0 \le -\mathrm{Re}\Big\{ \frac{\zeta_K'}{\zeta_K}(\sigma) + \frac{L'}{L}(\sigma,\psi^*) + \frac{L'}{L}(\sigma+it,\chi^*) + \frac{L'}{L}(\sigma+it,\xi^*)\Big\} + \frac{n_K + \log \mathrm{N}[\mathfrak{f}_\chi,\mathfrak{f}_\psi]}{2^\sigma-1}.$$

Note $\mathrm{N}[\mathfrak{f}_\chi,\mathfrak{f}_\psi] \le Q^2$ since $\psi$ and $\chi$ are both characters trivial on the congruence subgroup $H$ and therefore the norms of their respective conductors are bounded by $Q$. Substituting this bound into the above, we apply Lemmas 2.2.1 and 2.3.3 to each term. We deduce that

$$\begin{aligned} 0 \le\ & \tfrac{1}{2}\log(D_K D_\psi D_\chi D_\xi) + \frac{2}{2^\sigma-1}\log Q + n_K\log(\sigma+1+|t|) + A_\sigma n_K \\ & - \mathrm{Re}\Big\{ \sum_{\substack{\rho \\ \zeta_K(\rho)=0}} \frac{1}{\sigma-\rho} + \sum_{\substack{\rho \\ L(\rho,\psi)=0}} \frac{1}{\sigma-\rho} + \sum_{\substack{\rho \\ L(\rho,\chi)=0}} \frac{1}{\sigma+it-\rho} + \sum_{\substack{\rho \\ L(\rho,\psi\chi)=0}} \frac{1}{\sigma+it-\rho}\Big\} \\ & + \frac{1+E_0(\psi)}{\alpha} + \frac{1+E_0(\psi)}{\alpha+1} + \mathrm{Re}\Big\{\frac{E_0(\chi)+E_0(\chi\psi)}{\alpha+it} + \frac{E_0(\chi)+E_0(\chi\psi)}{\alpha+1+it}\Big\}, \end{aligned} \tag{6.6}$$

where $A_\sigma = \log(\sigma+1) + \frac{2}{\sigma} + \frac{1}{2^\sigma-1} - 2\log\pi$. Since $0<\beta<1$, notice $\mathrm{Re}\{\frac{1}{\sigma+it-\rho}\} \ge \frac{\alpha}{|\sigma+it-\rho|^2}$ and $\mathrm{Re}\{\frac{1}{\alpha+it} + \frac{1}{\alpha+1+it}\} \le \frac{1}{\alpha} + \frac{1}{\alpha+1}$. Further, $D_\chi$ and $D_\xi$ are both $\le D_K Q$ as $\xi = \psi^*\chi^*$ induces the character $\psi\chi \pmod{\mathfrak{q}}$ which is trivial on $H$. Rearranging (6.6) and employing all of the subsequent observations gives the desired conclusion. □

### 6.2.3 Technical estimates for the Dedekind zeta function

In this subsection, we consider the Dedekind zeta function of a number field $K$ and certain sums over its zeros, both trivial and non-trivial. The estimates are similar to the previous subsection on Hecke $L$-functions but, for some applications, we will desire precise numerical estimates for the special case of the Dedekind zeta function. Recall the notation defined in Chapter 2, especially Section 2.2.

**Lemma 6.2.6.** *For $\alpha > 0$ and $t \geq 0$,*

$$\mathrm{Re}\Big\{\frac{\gamma_K'}{\gamma_K}(\alpha+1) + \frac{\gamma_K'}{\gamma_K}(\alpha+1 \pm it)\Big\} = G_1(\alpha;t)\cdot r_1 + G_2(\alpha;t)\cdot 2r_2,$$

*where*

$$\begin{aligned} G_1(\alpha;t) &:= \frac{\Delta(\alpha+1,0)+\Delta(\alpha+1,t)}{2} - \log\pi, \\ G_2(\alpha;t) &:= \frac{\Delta(\alpha+1,0)+\Delta(\alpha+2,0)+\Delta(\alpha+1,t)+\Delta(\alpha+2,t)}{4} - \log\pi, \end{aligned} \tag{6.7}$$

*and* $\Delta(x,y) = \mathrm{Re}\{\frac{\Gamma'}{\Gamma}(\frac{x+iy}{2})\}$.

**Remark.** For fixed $\alpha > 0$ and $j = 1$ or 2, observe that $G_j(\alpha;t)$ is increasing as a function of $t \geq 0$ by [AK14, Lemma 2].

*Proof.* Denote $\sigma = \alpha+1$. As $\Delta(x,y) = \Delta(x,-y)$, we may assume $t \geq 0$. From (2.11), it follows that

$$\begin{aligned} \mathrm{Re}\Big\{\frac{\gamma_K'}{\gamma_K}(\sigma+it)\Big\} &= \frac{1}{2}\Big[(r_1+r_2)\Delta(\sigma,t) + r_2\Delta(\sigma+1,t) - (r_1+2r_2)\log\pi\Big] \\ &= \frac{1}{2}\Big[r_1(\Delta(\sigma,t)-\log\pi) + 2r_2\cdot(\tfrac{\Delta(\sigma,t)+\Delta(\sigma+1,t)}{2} - \log\pi)\Big]. \end{aligned}$$

Using the same identity for $t = 0$ gives the desired result. □

**Lemma 6.2.7.** *For $\alpha \geq 1$ and $t \in \mathbb{R}$,*

$$\sum_{\omega \text{ trivial}} \frac{1}{|\alpha+1+it-\omega|^2} \leq W_1(\alpha)\cdot r_1 + W_2(\alpha)\cdot r_2,$$

*where the sum is over all trivial zeros $\omega$ of $\zeta_K(s)$,*

$$W_1(\alpha) = \sum_{k=0}^{\infty}\frac{1}{(\alpha+1+2k)^2}, \quad \textit{and} \quad W_2(\alpha) = \sum_{k=0}^{\infty}\frac{1}{(\alpha+1+k)^2}.$$

*Proof.* This follows immediately from (2.13). □

**Lemma 6.2.8.** *For $\alpha \geq 1$ and $t \in \mathbb{R}$,*

$$\begin{aligned}\sum_{\rho}\Big(\frac{1}{|\alpha+1-\rho|^2}+\frac{1}{|\alpha+1+it-\rho|^2}\Big)\\ \leq \frac{1}{\alpha}\big(\log D_K + G_1(\alpha;|t|)\cdot r_1 + G_2(\alpha;|t|)\cdot 2r_2\big)+\frac{2}{\alpha^2}+\frac{2}{\alpha+\alpha^2}.\end{aligned} \tag{6.8}$$

*where the sum is over all non-trivial zeros $\rho$ of $\zeta_K(s)$ and $G_j(\alpha,|t|)$ are defined by* (6.7).

*Proof.* We combine the inequality

$$0 \leq -\mathrm{Re}\Big\{\frac{\zeta_K'}{\zeta_K}(\alpha+1)+\frac{\zeta_K'}{\zeta_K}(\alpha+1+it)\Big\}$$

with Lemmas 2.2.1 and 6.2.6 to deduce that

$$\begin{aligned}0 \leq \log D_K + G_1(\alpha;|t|)\cdot r_1 + G_2(\alpha;|t|)\cdot 2r_2 + \mathrm{Re}\Big\{\frac{1}{\alpha+it}+\frac{1}{\alpha+1+it}\Big\}\\ -\sum_{\rho}\mathrm{Re}\Big\{\frac{1}{\alpha+1-\rho}+\frac{1}{\alpha+1+it-\rho}\Big\}+\frac{1}{\alpha}+\frac{1}{\alpha+1}.\end{aligned} \tag{6.9}$$

Observe, as $\beta \in (0,1)$,

$$\mathrm{Re}\Big\{\frac{1}{\alpha+1+it-\rho}\Big\}=\frac{\alpha+1-\beta}{|\alpha+1+it-\rho|^2}\geq\frac{\alpha}{|\alpha+1+it-\rho|^2}$$

and

$$\mathrm{Re}\Big\{\frac{1}{\alpha+it}+\frac{1}{\alpha+1+it}\Big\}\leq\frac{1}{\alpha}+\frac{1}{\alpha+1}.$$

We rearrange (6.9) and employ these observations to deduce (6.8). □

# 6.3 Proofs of Deuring–Heilbronn phenomenon

## 6.3.1 Proof of Theorem 6.1.1

Recall $H \pmod{\mathfrak{q}}$ is an arbitrary congruence class group of a number field $K$. If $\tilde{H} \pmod{\mathfrak{m}}$ induces $H \pmod{\mathfrak{q}}$, then a character $\chi \pmod{H}$ is induced by a character $\tilde{\chi} \pmod{\tilde{H}}$. It follows that

$$L(s,\chi)=L(s,\tilde{\chi})\prod_{\substack{\mathfrak{p}\mid\mathfrak{q}\\ \mathfrak{p}\nmid\mathfrak{m}}}\Big(1-\frac{\tilde{\chi}(\mathfrak{p})}{\mathrm{N}\mathfrak{p}^s}\Big)$$

for all $s \in \mathbb{C}$. This implies that the non-trivial zeros of $L(s,\chi)$ are the same non-trivial zeros of $L(s,\tilde{\chi})$. Therefore, without loss of generality, we may assume $H \pmod{\mathfrak{q}}$ is primitive.

We divide the proof according to whether $\psi$ is quadratic or trivial. The arguments in each case are similar but require some minor modifications.

**$\psi$ is quadratic.**

Let $m$ be a positive integer, $\alpha \geq 1$ and $\sigma = \alpha + 1$. From the identity $0 \leq (1 + \psi^*(\mathfrak{n}))(1 + \mathrm{Re}\{\chi^*(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{-i\gamma'}\})$ and Lemma 2.3.6 with $s = \sigma + i\gamma'$, it follows that

$$\mathrm{Re}\Big\{\sum_{n=1}^{\infty} z_n^m\Big\} \leq \frac{1}{\alpha^m} - \frac{1}{(\alpha+1-\beta_1)^{2m}} + \mathrm{Re}\Big\{\frac{\delta(\chi)+\delta(\psi\chi)}{(\alpha+i\gamma')^{2m}} - \frac{\delta(\chi)+\delta(\psi\chi)}{(\alpha+1+i\gamma'-\beta_1)^{2m}}\Big\}, \tag{6.10}$$

where $z_n = z_n(\gamma')$ satisfies $|z_1| \geq |z_2| \geq \dots$ and runs over the multisets

$$\begin{aligned}
&\{(\sigma-\omega)^{-2} : \omega \text{ is any zero of } \zeta_K(s)\},\\
&\{(\sigma-\omega)^{-2} : \omega \neq \beta_1 \text{ is any zero of } L(s,\psi^*)\},\\
&\{(\sigma+i\gamma'-\omega)^{-2} : \omega \neq \beta_1 \text{ is any zero of } L(s,\chi^*)\},\\
&\{(\sigma+i\gamma'-\omega)^{-2} : \omega \neq \beta_1 \text{ is any zero of } L(s,\psi^*\chi^*)\}.
\end{aligned} \tag{6.11}$$

Note that the multisets includes trivial zeros of the corresponding $L$-functions and $\psi^*\chi^*$ is a Hecke character (not necessarily primitive) modulo the least common multiple of $\mathfrak{f}_\chi$ and $\mathfrak{f}_\psi$. With this choice, it follows that

$$(\alpha+1/2)^{-2} \leq (\alpha+1-\beta')^{-2} \leq |z_1| \leq \alpha^{-2}. \tag{6.12}$$

The right hand side of (6.10) may be bounded via the observation

$$\Big|\frac{1}{(\alpha+it)^{2m}} - \frac{1}{(\alpha+it+1-\beta_1)^{2m}}\Big| \leq \alpha^{-2m}\Big|1 - \frac{1}{(1+\frac{1-\beta_1}{\alpha+it})^{2m}}\Big| \ll \alpha^{-2m-1}m(1-\beta_1),$$

whence

$$\mathrm{Re}\Big\{\sum_{n=1}^{\infty} z_n^m\Big\} \ll \alpha^{-2m-1}m(1-\beta_1). \tag{6.13}$$

On the other hand, by Theorem 6.2.2, for $\epsilon > 0$, there exists some $m_0 = m_0(\epsilon)$ with $1 \leq m_0 \leq (12+\epsilon)M$ such that

$$\mathrm{Re}\Big\{\sum_{n=1}^{\infty} z_n^{m_0}\Big\} \geq \tfrac{\epsilon}{50}|z_1|^{m_0} \geq \tfrac{\epsilon}{50}(\alpha+1-\beta')^{-2m_0} \geq \tfrac{\epsilon}{50}\alpha^{-2m_0}\exp(-\tfrac{2m_0}{\alpha}(1-\beta')),$$

where $M = |z_1|^{-1}\sum_{n=1}^{\infty}|z_n|$. Comparing with (6.23) for $m = m_0$, it follows that

$$\exp(-(24+2\epsilon)\tfrac{M}{\alpha}(1-\beta')) \ll_\epsilon \tfrac{M}{\alpha}(1-\beta_1). \tag{6.14}$$

Therefore, it suffices to bound $M/\alpha$ and optimize over $\alpha \geq 1$.

By (6.11), the quantity $M$ is a sum involving non-trivial and trivial zeros of certain $L$-functions. For the non-trivial zeros, we employ Lemma 6.2.5 with $D_\psi = D_K \mathrm{N}\mathfrak{f}_\psi \leq D_K Q$ since $\psi$ is quadratic. For the trivial zeros, apply Lemma 6.2.4 in the "primitive" case for $\zeta_K(s), L(s,\psi^*), L(s,\chi^*)$ and in the "unconditional" case for $L(s,\psi^*\chi^*)$. In the latter case, we additionally observe that, as $H \,(\mathrm{mod}\, \mathfrak{q})$ is primitive, $\log \mathrm{N}\mathfrak{q} \leq 2\log Q$ by Lemma 2.4.7. Combining these steps along with (6.22), it follows that

$$\begin{aligned}\frac{M}{\alpha} \leq \frac{(\alpha+1/2)^2}{\alpha^2}\cdot\Big[2\log D_K &+ \Big(\frac{3}{2}+\frac{2\alpha}{2\alpha+2}+\frac{4\alpha}{(\alpha+1)^2\log 2}+\frac{2}{2^{\alpha+1}-1}\Big)\log Q \\ &+ \Big(\log(\alpha+2)+\log(\alpha+3)+2-2\log\pi+\frac{4\alpha}{(\alpha+1)^2}+\frac{1}{2^{\alpha+1}-1}\Big)n_K \\ &+ n_K\log T+\frac{4}{\alpha}+\frac{4}{\alpha+1}\Big],\end{aligned} \tag{6.15}$$

for $\alpha \geq 1$. Note, in applying Lemma 6.2.5, we used that $\log(\alpha+2+T) \leq \log(\alpha+3)+\log T$ for $T \geq 1$. Finally, select $\alpha$ sufficiently large, depending on $\epsilon > 0$, so the right hand side of (6.15) is

$$\leq (2+\tfrac{\epsilon}{100})\log D_K + (2.5+\tfrac{\epsilon}{100})\log Q + (1+\tfrac{\epsilon}{100})n_K\log T + O_\epsilon(n_K).$$

Substituting the resulting bounds in (6.14) completes the proof of Theorem 6.1.1 for $\psi$ quadratic.

**$\psi$ is trivial.**

Begin with the identity $0 \leq 1 + \mathrm{Re}\{\chi^*(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{-i\gamma'}\}$. This similarly implies

$$\mathrm{Re}\Big\{\sum_{n=1}^{\infty} z_n^m\Big\} \leq \frac{1}{\alpha^m} - \frac{1}{(\alpha+1-\beta_1)^{2m}} + \mathrm{Re}\Big\{\frac{\delta(\chi)}{(\alpha+i\gamma')^{2m}} - \frac{\delta(\chi)}{(\alpha+1+i\gamma'-\beta_1)^{2m}}\Big\} \tag{6.16}$$

for a new choice $z_n = z_n(\gamma')$ satisfying $|z_1| \geq |z_2| \geq \dots$ and which runs over the multisets

$$\begin{aligned}&\{(\sigma-\omega)^{-2} : \omega \neq \beta_1 \text{ is any zero of } \zeta_K(s)\},\\ &\{(\sigma+i\gamma'-\omega)^{-2} : \omega \neq \beta_1 \text{ is any zero of } L(s,\chi^*)\}.\end{aligned} \tag{6.17}$$

Following the same arguments as before, we may arrive at (6.24) for the new quantity $M = |z_1|^{-1}\sum_{n=1}^{\infty}|z_n|$. To bound the non-trivial zeros arising in $M$, apply Lemma 6.2.5 with $D_\psi = D_K$ since $\psi$ is trivial. For the trivial zeros, apply Lemma 6.2.4 in the "primitive" case for both $\zeta_K(s)$ and $L(s,\chi^*)$. It follows from (6.22) that, for $\alpha \geq 1$,

$$\begin{aligned}\frac{M}{\alpha} \leq \frac{(\alpha+1/2)^2}{\alpha^2}\cdot\Big[\log D_K + \Big(\frac{1}{2}+\frac{1}{2^{\alpha+1}-1}\Big)\log Q + \frac{1}{2}n_K\log T + \frac{2}{\alpha}+\frac{2}{\alpha+1}\\ + \Big(\frac{1}{2}\log(\alpha+2)+\frac{1}{2}\log(\alpha+3)+1-\log\pi+\frac{2\alpha}{(\alpha+1)^2}+\frac{1/2}{2^{\alpha+1}-1}\Big)n_K\Big].\end{aligned} \tag{6.18}$$

Again, we select $\alpha$ sufficiently large, depending on $\epsilon > 0$, so the right hand side of (6.18) is

$$\leq (1+\tfrac{\epsilon}{50})\log D_K + (0.5+\tfrac{\epsilon}{50})\log Q + (0.5+\tfrac{\epsilon}{50})n_K\log T + O_\epsilon(n_K).$$

Substituting the resulting bound into (6.14) completes the proof of Theorem 6.1.1. □

**Remark.** To obtain a more explicit version of Theorem 6.1.1, the only difference in the proof is selecting an explicit value of $\alpha$, say $\alpha = 18$, in the final step of each case. The possible choice of $\alpha$ is somewhat arbitrary because the coefficients of $\log D_K, \log Q$ and $n_K$ in (6.15) and (6.18) cannot be simultaneously minimized. Hence, in the interest of having relatively small coefficients of comparable size for all quantities, one could choose the value $\alpha = 18$.

### 6.3.2 Proof of Theorem 6.1.2

Let $m$ be a positive integer and $\alpha \geq 1$. From [LMO79, Equation (5.4)] with $s = \alpha+1+i\gamma'$, it follows that

$$\mathrm{Re}\Big\{\sum_{n=1}^{\infty} z_n^m\Big\} \leq \frac{1}{\alpha^m} - \frac{1}{(\alpha+1-\beta_1)^{2m}} + \mathrm{Re}\Big\{\frac{1}{(\alpha+i\gamma')^{2m}} - \frac{1}{(\alpha+i\gamma'+1-\beta_1)^{2m}}\Big\}, \tag{6.19}$$

where $z_n$ satisfies $|z_1| \geq |z_2| \geq \dots$ and runs over the multisets

$$\begin{aligned}&\{(\alpha+1-\omega)^{-2} : \omega \neq \beta_1 \text{ is any zero of } \zeta_L(s)\},\\ &\{(\alpha+1+i\gamma'-\omega)^{-2} : \omega \neq \beta_1 \text{ is any zero of } \zeta_L(s)\}.\end{aligned}$$

If $\omega$ is a trivial zero (and hence a non-positive integer by (2.13)) then $(\alpha+1-\omega)^{-2} \geq 0$. Thus, for any $z_n = (\alpha+1-\omega)^{-2}$ in (6.19) corresponding to a trivial zero $\omega$, we have $z_n^m \geq 0$ so we may discard such $z_n$. It follows that

$$\mathrm{Re}\Big\{ \sum_{n=1}^{\infty} \tilde{z}_n^m \Big\} \leq \frac{1}{\alpha^m} - \frac{1}{(\alpha+1-\beta_1)^{2m}}, \tag{6.20}$$

where $\tilde{z}_n$ satisfies $|\tilde{z}_1| \geq |\tilde{z}_2| \geq \dots$ and runs over the new multisets

$$\begin{aligned} &\{(\alpha+1-\omega)^{-2} : \omega \neq \beta_1 \text{ is any non-trivial zero of } \zeta_L(s)\}, \\ &\{(\alpha+1+i\gamma'-\omega)^{-2} : \omega \neq \beta_1 \text{ is any zero of } \zeta_L(s)\}. \end{aligned} \tag{6.21}$$

As $\rho' = \beta' + i\gamma'$ belongs to the latter multiset, it follows that

$$(\alpha+1-\beta')^{-2} \leq |\tilde{z}_1| \leq \alpha^{-2}. \tag{6.22}$$

Since

$$\Big| \frac{1}{(\alpha+it)^{2m}} - \frac{1}{(\alpha+it+1-\beta_1)^{2m}} \Big| \leq \alpha^{-2m} \Big| 1 - \frac{1}{(1+\frac{1-\beta_1}{\alpha+it})^{2m}} \Big| \ll \alpha^{-2m-1} m(1-\beta_1),$$

equation (6.20) becomes

$$\mathrm{Re}\Big\{ \sum_{n=1}^{\infty} \tilde{z}_n^m \Big\} \ll \alpha^{-2m-1} m(1-\beta_1). \tag{6.23}$$

On the other hand, by Theorem 6.2.2, for $\epsilon > 0$, there exists some $m_0 = m_0(\epsilon)$ with $1 \leq m_0 \leq (12+\epsilon)M$ such that

$$\mathrm{Re}\Big\{ \sum_{n=1}^{\infty} \tilde{z}_n^{m_0} \Big\} \geq \tfrac{\epsilon}{50} |\tilde{z}_1|^{m_0} \geq \tfrac{\epsilon}{50} (\alpha+1-\beta')^{-2m_0} \geq \tfrac{\epsilon}{50} \alpha^{-2m_0} \exp(-\tfrac{2m_0}{\alpha}(1-\beta')),$$

where $M = |\tilde{z}_1|^{-1} \sum_{n=1}^{\infty} |\tilde{z}_n|$ according to our parameters $\tilde{z}_n$ in (6.21). Comparing with (6.23) for $m = m_0$, we have that

$$\exp(-(24+2\epsilon)\tfrac{M}{\alpha}(1-\beta')) \ll_\epsilon \tfrac{M}{\alpha}(1-\beta_1). \tag{6.24}$$

Therefore, it suffices to bound $M/\alpha$ and optimize over $\alpha \geq 1$. By Lemmas 6.2.7 and 6.2.8 and (6.22), notice that

$$\begin{aligned}\frac{M}{\alpha} \leq \frac{(\alpha+1-\beta')^2}{\alpha} \cdot \Big\{ \frac{1}{\alpha}\log D_K + \Big( \frac{G_1(\alpha;|\gamma'|)}{\alpha} + W_1(\alpha) \Big) \cdot r_1 \\ + \Big( \frac{G_2(\alpha;|\gamma'|)}{\alpha} + \frac{1}{2}W_2(\alpha) \Big) \cdot 2r_2 + \frac{2}{\alpha^2} + \frac{2}{\alpha+\alpha^2} \Big\}\end{aligned} \tag{6.25}$$

for $\alpha \geq 1$. To simplify the above, we note $1-\beta' \leq 1/2$ by assumption and $G_j(\alpha;|\gamma'|) \leq G_j(\alpha;T)$ for $j=1,2$ by the remark following Lemma 6.2.6. Also in (6.25), if a coefficient of $r_1$ or $r_2$ is positive, we employ an estimate of Odlyzko [Odl77] which implies

$$(\log 60) \cdot r_1 + (\log 22) \cdot 2r_2 \leq \log D_K \tag{6.26}$$

for $D_K$ sufficiently large. With these observations, it follows that

$$\begin{aligned}\frac{M}{\alpha} \leq \frac{(\alpha+1/2)^2}{\alpha}\Big[ \Big( \frac{1}{\alpha} + \max\Big\{ \frac{G_1(\alpha;T)+\alpha W_1(\alpha)}{\alpha\log 60}, \frac{G_2(\alpha;T)+\frac{1}{2}\alpha W_2(\alpha)}{\alpha \log 22}, 0 \Big\} \Big) \log D_K \\ + \frac{2}{\alpha^2} + \frac{2}{\alpha+\alpha^2} \Big].\end{aligned}$$

Seeking to minimize the coefficient of $\log D_K$, after some numerical calculations, we choose $\alpha = \alpha(T)$ according to the following table:

| $T$ | 1 | 3.5 | 8.7 | 22 | 54 | 134 | 332 | 825 | 2048 | 5089 | 12646 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha$ | 3.50 | 3.77 | 5.39 | 7.30 | 8.92 | 10.17 | 11.21 | 12.26 | 13.22 | 14.17 | 15.23 |

To complete the proof for $T=1$, say, the corresponding choice of $\alpha = 3.50$ implies

$$\frac{M}{\alpha} \leq 1.3067 \log D_K$$

for $D_K$ sufficiently large. Substituting this bound into (6.24) and fixing $\epsilon > 0$ sufficiently small yields the desired result since $24 \times 1.3067 < 31.4$. The other cases follow similarly. □

**Remark.** To clarify remark (ii) following Theorem 6.1.2, notice that if $n_K = o(\log D_K)$ then the coefficients of $r_1$ and $r_2$ in (6.25) can be made arbitrary small for $D_K$ sufficiently large depending on $\alpha \geq 1$. Fixing $\alpha$ sufficiently large (depending on $T$) gives

$$M/\alpha \leq 1.0001 \log D_K$$

for $D_K$ sufficiently large. As $24 \times 1.0001 < 24.01$ the remark follows.

### 6.3.3 Proof of Theorem 6.1.3

The proof is very similar to the above proof for Theorem 6.1.2. Recall $\beta' \neq \beta_1$ is now a real zero of $\zeta_K(s)$, i.e. $\gamma' = 0$. Arguing as in the proof of Theorem 6.1.2, we deduce that

$$\mathrm{Re}\Big\{ \sum_{n=1}^{\infty} \tilde{z}_n^m \Big\} \leq \frac{1}{\alpha^m} - \frac{1}{(\alpha+1-\beta_1)^{2m}}, \tag{6.27}$$

where $\tilde{z}_n$ satisfies $|\tilde{z}_1| \geq |\tilde{z}_2| \geq \dots$ and runs over the multiset

$$\{(\alpha+1-\rho)^{-2} : \rho \neq \beta_1 \text{ is any non-trivial zero of } \zeta_K(s)\}. \tag{6.28}$$

Note we again discarded the trivial zeros by positivity. Equation (6.22) still holds for $\tilde{z}_1$ and we argue similarly to deduce (6.24) holds for $M = |\tilde{z}_1|^{-1} \sum_n |\tilde{z}_n|$. Thus, by Lemma 6.2.8 with $t = 0$, we deduce that

$$\frac{\tilde{M}}{\alpha} \leq \frac{(\alpha+1-\beta')^2}{2\alpha} \cdot \Big\{ \frac{1}{\alpha} \log D_K + \frac{G_1(\alpha;0)}{\alpha} \cdot r_1 + \frac{G_2(\alpha;0)}{\alpha} \cdot 2r_2 + \frac{2}{\alpha^2} + \frac{2}{\alpha+\alpha^2} \Big\} \tag{6.29}$$

for $\alpha \geq 1$. Notice, in particular, the additional factor of 2 in the denominator and the lack of $W_1(\alpha)$ and $W_2(\alpha)$ terms as compared to (6.25). Continuing to argue analogously, we simplify the above by noting $1 - \beta' < 1$ and apply Odlyzko's bound (6.26) to conclude that

$$\frac{\tilde{M}}{\alpha} \leq \frac{(\alpha+1)^2}{2\alpha} \Big[ \Big( \frac{1}{\alpha} + \max\Big\{ \frac{G_1(\alpha;0)}{\alpha \log 60}, \frac{G_2(\alpha;0)}{\alpha \log 22}, 0 \Big\} \Big) \log D_K + \frac{2}{\alpha^2} + \frac{2}{\alpha+\alpha^2} \Big]$$

for $D_K$ sufficiently large. Selecting $\alpha = 5.8$ gives

$$\frac{\tilde{M}}{\alpha} \leq 0.6881 \log D_K$$

for $D_K$ sufficiently large. As $24 \times 0.6881 < 16.6$, we similarly conclude the desired result. $\square$

# Chapter 7

# Least prime ideal

*"Fell deeds awake: fire and slaughter! Spear shall be shaken, shield be splintered, a sword-day, a red day, ere the sun rises!"*

– Théoden, The Lord of the Rings.

Throughout this chapter, let $L/F$ be a Galois extension of number fields with Galois group $G := \mathrm{Gal}(L/F)$ and let $C$ be a conjugacy class of $G$. Our aim is to estimate

$$P(C, L/F) = \min\{\mathrm{N}\mathfrak{p} : \mathfrak{p} \text{ degree 1 prime ideal of } F \text{ unramified in } L \text{ such that } \left[\frac{L/F}{\mathfrak{p}}\right] = C\},$$

where $\mathrm{N} = \mathrm{N}^F_{\mathbb{Q}}$ is the absolute norm of $F$. Informally speaking, we are bounding the least prime ideal which occurs in the Chebotarev Density Theorem. This chapter contains the proofs of Theorems 1.3.1 and 1.3.2, which are two of the main results of this thesis. We will use notation from Section 2.5.

## 7.1 Setup

### 7.1.1 Choice of weight

We will need to select a suitable weight function for counting the prime ideals of the base field $F$ so we describe our choice and its properties here.

**Lemma 7.1.1.** *For real numbers $A, B > 0$ and positive integer $\ell \geq 1$ satisfying $B > 2\ell A$, there exists a real-variable function $f(t) = f_\ell(t; B, A)$ such that:*

*(i) $0 \leq f(t) \leq A^{-1}$ for all $t \in \mathbb{R}$.*

*(ii) The support of $f$ is contained in $[B - 2\ell A, B]$.*

*(iii) Its Laplace transform $F(z) = \int_{\mathbb{R}} f(t)e^{-zt}dt$ is given by*

$$F(z) = e^{-(B-2\ell A)z}\Big(\frac{1-e^{-Az}}{Az}\Big)^{2\ell}. \tag{7.1}$$

*(iv) Let $\mathscr{L} \geq 1$ be arbitrary. Suppose $s = \sigma + it \in \mathbb{C}$ satisfies $\sigma < 1$ and $t \in \mathbb{R}$. Write $\sigma = 1 - \frac{x}{\mathscr{L}}$ and $t = \frac{y}{\mathscr{L}}$. If $0 \leq \alpha \leq 2\ell$ and $t \neq 0$ then*

$$|F((1-s)\mathscr{L})| \leq e^{-(B-2\ell A)x}\Big(\frac{2}{A\sqrt{x^2+y^2}}\Big)^{\alpha} \leq e^{-(B-2\ell A)x}\Big(\frac{2}{A|y|}\Big)^{\alpha}.$$

*Furthermore, for all $t \in \mathbb{R}$,*

$$|F((1-s)\mathscr{L})| \leq e^{-(B-2\ell A)x} \quad \textit{and} \quad F(0) = 1.$$

**Remark.** Heath-Brown [HB92] used the weight $f$ with $\ell = 1$ for his computation of Linnik's constant for the least rational prime in an arithmetic progression. Our choice is also motivated by the work of Weiss [Wei83, Lemma 3.2]. Namely, the weight function $f$ depends on a parameter $\ell$ which will be chosen to be at least of size $O(n_K)$. This forces $f$ to be $O(n_K)$-times differentiable and hence $F(a+ib)$ will decay like $|b|^{-O(n_K)}$ for fixed $a > 0$ and $|b| \to \infty$. This decay rate will be necessary when applying log-free zero density estimates such as Theorem 7.3.6 to bound the contribution of zeros which are high in the critical strip.

*Proof.*

- For parts (i)–(iii), let $\mathbf{1}_S(\,\cdot\,)$ be an indicator function for the set $S \subseteq \mathbb{R}$. For $j \geq 1$, define

$$w_0(t) := \frac{1}{A}\mathbf{1}_{[-A/2,A/2]}(t), \quad \text{and} \quad w_j(t) := (w * w_{j-1})(t).$$

Since $\int_{\mathbb{R}} w_0(t)dt = 1$, it is straightforward verify that $0 \leq w_{2\ell}(t) \leq A^{-1}$ and $w_{2\ell}(t)$ is supported in $[-\ell A, \ell A]$. Observe the Laplace transform $W(z)$ of $w_0$ is given by

$$W(z) = \frac{e^{Az/2} - e^{-Az/2}}{Az} = e^{Az/2}\cdot\Big(\frac{1-e^{-Az}}{Az}\Big),$$

so the Laplace transform $W_{2\ell}(z)$ of $w_{2\ell}$ is given by

$$W_{2\ell}(z) = \Big(\frac{e^{Az/2} - e^{-Az/2}}{Az}\Big)^{2\ell} = e^{\ell Az}\Big(\frac{1-e^{-Az}}{Az}\Big)^{2\ell}.$$

The desired properties for $f$ follow upon choosing $f(t) = w_{2\ell}(t - B + \ell A)$.

- For part (iv), we see by (iii) that

$$|F((1-s)\mathscr{L})| \leq e^{-(B-2\ell A)x}\Big|\frac{1-e^{-A(x+iy)}}{A(x+iy)}\Big|^{2\ell}. \tag{7.2}$$

To bound the above quantity, we observe that for $w = a+ib$ with $a > 0$ and $b \in \mathbb{R}$,

$$\Big|\frac{1-e^{-w}}{w}\Big|^2 \leq \Big(\frac{1-e^{-a}}{a}\Big)^2 \leq 1.$$

This observation can be checked in a straightforward manner (cf. Lemma 7.1.2). It follows that

$$\Big|\frac{1-e^{-A(x+iy)}}{A(x+iy)}\Big|^{2\ell} = \Big|\frac{1-e^{-A(x+iy)}}{A(x+iy)}\Big|^{\alpha} \cdot \Big|\frac{1-e^{-A(x+iy)}}{A(x+iy)}\Big|^{2\ell-\alpha} \leq \Big(\frac{2}{A\sqrt{x^2+y^2}}\Big)^{\alpha}.$$

In the last step, we noted $|1-e^{-A(x+iy)}| \leq 2$ since $x > 0$ by assumption. Combining this with (7.2) yields the desired bound. The additional estimate for $|F((1-s)\mathscr{L})|$ follows similarly. One can also verify $F(0) = 1$ by straightforward calculus arguments.

□

**Lemma 7.1.2.** *For $z = x+iy$ with $x > 0$ and $y \in \mathbb{R}$,*

$$\Big|\frac{1-e^{-z}}{z}\Big|^2 \leq \Big(\frac{1-e^{-x}}{x}\Big)^2.$$

*Proof.* We need only consider $y \geq 0$ by conjugate symmetry. Define

$$\Phi_x(y) := \Big|\frac{1-e^{-z}}{z}\Big|^2 = \frac{1+e^{-2x}-2e^{-x}\cos y}{x^2+y^2} \qquad \text{for } y \geq 0,$$

which is a non-negative smooth function of $y$. Since $\Phi_x(y) \to 0$ as $y \to \infty$, we may choose $y_0 \geq 0$ such that $\Phi_x(y)$ has a global maximum at $y = y_0$. Suppose, for a contradiction, that

$$\Phi_x(y_0) > \Big(\frac{1-e^{-x}}{x}\Big)^2. \tag{7.3}$$

By calculus, one can show $(1-e^{-x})/x \geq e^{-x/2}$ for $x > 0$. With this observation, notice

$$\begin{aligned}\Phi_x'(y_0) = \frac{2e^{-x}\cdot\sin y_0}{x^2+y_0^2} - \frac{2\Phi_x(y_0)\cdot y_0}{x^2+y_0^2} &< \frac{2e^{-x}\cdot\sin y_0}{x^2+y_0^2} - \frac{2\big(\frac{1-e^{-x}}{x}\big)^2\cdot y_0}{x^2+y_0^2} \qquad \text{by (7.3)}\\ &\leq \frac{2e^{-x}\cdot\sin y_0}{x^2+y_0^2} - \frac{2e^{-x}\cdot y_0}{x^2+y_0^2} \leq 0\end{aligned}$$

since $\sin y \leq y$ for $y \geq 0$. On the other hand, $\Phi_x(y)$ has a global max at $y = y_0$ implying $\Phi'_x(y_0) = 0$, a contradiction. $\square$

### 7.1.2 A weighted sum of prime ideals

Recall $L/F$ is a Galois extension of number fields with Galois group $G$ and $C$ is a conjugacy class of $G$. Furthermore, recall the notation and discussion in Section 2.5.

Suppose the integer $\ell \geq 2$ and real numbers $A, B > 0$ satisfy $B - 2\ell A > 0$. Select the weight function $f = f_\ell(\,\cdot\,; B, A)$ from Lemma 7.1.1 according to these parameters. For $\mathscr{L} \geq 1$ arbitrary, define

$$S = S(f) := \sum_{\substack{\mathfrak{p} \text{ unramified in } L \\ \mathfrak{p} \text{ degree } 1}} \Theta_C(\mathfrak{p}) \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}} f\Big(\frac{\log \mathrm{N}\mathfrak{p}}{\mathscr{L}}\Big), \tag{7.4}$$

where the sum is over degree 1 prime ideals $\mathfrak{p}$ of $F$ which are unramified in $L$ and $\Theta_C(\mathfrak{p})$ is defined by (2.27). The parameter $\mathscr{L}$ is left unspecified because the choice is different in Section 7.2 compared with Sections 7.3 and 7.4. In any case, if $S > 0$ then there exists a degree 1 prime ideal $\mathfrak{p}$ of $F$ unramified in $L$ with $[\frac{L/F}{\mathfrak{p}}] = C$ and $\mathrm{N}\mathfrak{p} \leq e^{B\mathscr{L}}$; that is,

$$S > 0 \implies P(C, L/F) \leq e^{B\mathscr{L}}.$$

Equivalently, $S > 0$ implies $\pi_C(x, L/F) \geq 1$ for $x \geq e^{B\mathscr{L}}$. We may take this observation a bit further to obtain a better lower bound for $\pi_C(x, L/F)$, defined by (1.15).

**Lemma 7.1.3.** *In the above notation,*

$$\pi_C(x, L/F) \geq A e^{-2\ell A \mathscr{L}} S \frac{x}{\log x},$$

*where* $x = e^{B\mathscr{L}}$.

*Proof.* Since $f$ is supported in $[B - 2\ell A, B]$ and $|f| \leq A^{-1}$ by Lemma 7.1.1, it follows by the definition of $S$ that

$$\begin{aligned} S &\leq \frac{A^{-1}\log(B\mathscr{L})}{e^{-(B-2\ell A)\mathscr{L}}} \sum_{\substack{\mathfrak{p} \text{ unramified in } L \\ \mathfrak{p} \text{ degree } 1}} \Theta_C(\mathfrak{p}) \\ &= A^{-1} e^{2\ell A \mathscr{L}} \frac{\log x}{x} \pi_C(x, L/F). \end{aligned} \tag{7.5}$$

The last line follows from (2.28) and the fact that $x = e^{B\mathscr{L}}$. Rearranging the inequality gives the lemma. $\square$

Now, we wish to transform $S$ into a contour integral by using the logarithmic derivatives of certain Artin $L$-functions. Recalling the discussion in Section 2.4, one is naturally led to consider the contour

$$I := \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} Z_C(s)F((1-s)\mathscr{L})ds, \tag{7.6}$$

where $Z_C(s)$ is defined by (2.25) and $F(z) = \int_0^\infty f(t)e^{-zt}dt$ is the Laplace transform of $f$. Comparing (7.6) and (2.26), it follows by Mellin inversion that

$$I = \mathscr{L}^{-1}\sum_{\mathfrak{n}} \Theta_C(\mathfrak{n})\frac{\Lambda_F(\mathfrak{n})}{\mathrm{N}\mathfrak{n}} f\Big(\frac{\log \mathrm{N}\mathfrak{n}}{\mathscr{L}}\Big), \tag{7.7}$$

where the sum is over all integral ideals $\mathfrak{n}$ of $F$ and $\Lambda_F(\mathfrak{n})$ is the von Mangoldt $\Lambda$-function for integral ideals of $F$ given by (2.5). Comparing (7.4) and (7.7), it is apparent that the integral $I$ and quantity $\mathscr{L}^{-1}S$ should be equal up to a neglible contribution from: (i) ramified prime ideals, (ii) prime ideals whose norm is not a rational prime (i.e. not degree 1 over $\mathbb{Q}$), and (iii) prime ideal powers. In the following lemma, we prove exactly this by showing that the collective contribution of (i), (ii), and (iii) in (7.7) is bounded by $O(A^{-1}\mathscr{L}e^{-\frac{1}{2}(B-2\ell A)\mathscr{L}}\log D_L)$.

**Lemma 7.1.4.** *In the above notation,*

$$\mathscr{L}^{-1}S = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} Z_C(s)F((1-s)\mathscr{L})ds + O(A^{-1}\mathscr{L}e^{-\frac{1}{2}(B-2\ell A)\mathscr{L}}\log D_L).$$

*Proof.* Denote $Q_1 = e^{(B-2\ell A)\mathscr{L}}$ and $Q_2 = e^{B\mathscr{L}}$.

*Ramified prime ideals.* Since the product of ramified prime ideals $\mathfrak{p} \subseteq \mathcal{O}_F$ divides the relative different $\mathfrak{D}_{L/F}$, it follows that

$$\sum_{\substack{\mathfrak{p}\subseteq\mathcal{O}_F \\ \text{ramified in } L}} \log \mathrm{N}\mathfrak{p} \leq \log D_L.$$

Therefore, by Lemma 7.1.1 and (2.27),

$$\begin{aligned}\sum_{\substack{\mathfrak{p}\subseteq\mathcal{O}_F \\ \text{ramified in } L}} \sum_{m=1}^{\infty} \Theta_C(\mathfrak{p}^m)\frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^m} f\Big(\frac{\log \mathrm{N}\mathfrak{p}^m}{\mathscr{L}}\Big) &\ll A^{-1}\sum_{\substack{\mathfrak{p}\subseteq\mathcal{O}_F \\ \text{ramified in } L}} \log \mathrm{N}\mathfrak{p} \sum_{\substack{m\geq 1 \\ \mathrm{N}\mathfrak{p}^m > Q_1}} \frac{1}{\mathrm{N}\mathfrak{p}^m} \\ &\ll A^{-1}\sum_{\substack{\mathfrak{p}\subseteq\mathcal{O}_F \\ \text{ramified in } L \\ \mathrm{N}\mathfrak{p} > Q_1}} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}} \\ &\ll A^{-1}e^{-(B-2\ell A)\mathscr{L}}\log D_L.\end{aligned}$$

*Prime ideals with norm not equal to a rational prime.* For a given integer $q$, there are at most

$n_F$ prime ideals $\mathfrak{p} \subseteq \mathcal{O}_F$ satisfying $\mathrm{N}\mathfrak{p} = q$. Thus, by Lemma 7.1.1 and (2.27),

$$\begin{aligned}
\sum_{p \text{ prime}} \sum_{k \geq 2} \sum_{\substack{\mathfrak{p} \subseteq \mathcal{O}_F \\ \mathrm{N}\mathfrak{p} = p^k}} \Theta_C(\mathfrak{p}) \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}} f\Big(\frac{\log \mathrm{N}\mathfrak{p}}{\mathscr{L}}\Big) &\ll A^{-1} n_F \mathscr{L} \sum_{p \text{ prime}} \sum_{\substack{k \geq 2 \\ Q_1 < p^k < Q_2}} \frac{1}{p^k} \\
&\ll A^{-1} n_F \mathscr{L} Q_1^{-1/2} \\
&\ll A^{-1} \mathscr{L} e^{-\frac{1}{2}(B - 2\ell A)\mathscr{L}} \log D_L.
\end{aligned}$$

Note in the last step we used the fact that $n_F \leq n_L \ll \log D_L$ by a theorem of Minkowski. *Prime ideal powers.* Arguing similar to the previous case, one may again see that

$$\sum_{p \text{ prime}} \sum_{\substack{\mathfrak{p} \subseteq \mathcal{O}_K \\ \mathrm{N}\mathfrak{p} = p}} \sum_{m \geq 2} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^m} f\Big(\frac{\log \mathrm{N}\mathfrak{p}^m}{\mathscr{L}}\Big) \cdot \Theta(\mathfrak{p}^m) \ll A^{-1} \mathscr{L} e^{-\frac{1}{2}(B - 2\ell A)\mathscr{L}} \log D_L.$$

The desired result follows after comparing (7.4), (7.6) and (7.7) with the three estimates above.

□

Equipped with Lemma 7.1.4, the natural next step is to move the contour to the left of $\mathrm{Re}\{s\} = 1$. Applying Deuring's reduction as described in Section 2.5 combined with Lemma 7.1.4 and (2.34) yields the following:

**Lemma 7.1.5.** *Let $H$ be any abelian subgroup of $G$ such that $H \cap C$ is non-empty. Let $K = L^H$ be the subfield of $L$ fixed by $H$ and let $g_C \in H \cap C$. If $S = S(f)$ is defined by* (7.4) *and $F$ is the Laplace transform of $f$ in Lemma 7.1.1 then*

$$\begin{aligned}
\mathscr{L}^{-1} S = \frac{|C|}{|G|} \sum_{\chi} \frac{\overline{\chi}(g_C)}{2\pi i} \int_{2 - i\infty}^{2 + i\infty} -\frac{L'}{L}(s, \chi, L/K) F((1 - s)\mathscr{L}) ds \\
+ O(A^{-1} \mathscr{L} e^{-\frac{1}{2}(B - 2\ell A)\mathscr{L}} \log D_L),
\end{aligned}$$

*where the sum is over all Hecke characters $\chi$ attached to the abelian extension $L/K$. Here $L(s, \chi, L/K)$ is the (primitive) Hecke $L$-function attached to $\chi$.*

**Remark.** The number of Hecke characters appearing is precisely $\#\mathrm{Gal}(L/K) = [L : K]$.

Now, after pulling the contour in Lemma 7.1.5 to the left of $\mathrm{Re}\{s\} = 1$, we have two options for estimating the non-trivial zeros of the Hecke $L$-functions. By (2.20), we can estimate their contribution as:

(i) the zeros of the Dedekind zeta function $\zeta_L(s)$;

(ii) or the zeros of Hecke $L$-functions $L(s, \chi, L/K)$ averaged over all Hecke characters $\chi$.

Section 7.2 takes strategy (i) whereas Section 7.3 takes strategy (ii). In each case, the ubiquitous quantity $\mathscr{L}$ will be defined differently.

## 7.2 Proof of Theorem 1.3.1

### 7.2.1 Additional preliminaries

For the entirety of Section 7.2, let

$$\mathscr{L} := \log D_L.$$

By a classical theorem of Minkowski, we have that $n_L \ll \mathscr{L}$. We shall use this fact often and without reference. Recall the definitions and quantities related to the Dedekind zeta function of $L$ described in Section 2.2. We summarize a few key results needed for the main argument.

Let $\mathcal{Z}$ be the multiset consisting of zeros of $\zeta_L(s)$ in the rectangle

$$0 < \mathrm{Re}\{s\} < 1, \quad |\mathrm{Im}\{s\}| \leq 1. \tag{7.8}$$

Choose $\rho_1 \in \mathcal{Z}$ such that $\mathrm{Re}\{\rho_1\} = \beta_1 = 1 - \frac{\lambda_1}{\mathscr{L}} \in (0,1)$ is maximal.

**Theorem 7.2.1** (Kadiri [Kad12])**.** *Assume $D_L$ is sufficiently large. If $\lambda_1 < 0.0784$ then $\rho_1$ is a simple real zero of $\zeta_L(s)$.*

For Section 7.2 only, we refer to the case $\lambda_1 < 0.0784$ as the *exceptional case*. Otherwise, $\lambda_1 \geq 0.0784$ is regarded as non-exceptional. The final arguments will be divided according to these two cases. Now, select another zero $\rho' \in \mathcal{Z}$ of $\zeta_L(s)$ such that $\rho' \neq \rho_1$ (counting with multiplicity in $\mathcal{Z}$) and $\mathrm{Re}\{\rho'\} = \beta' = 1 - \frac{\lambda'}{\mathscr{L}}$ is maximal. In the exceptional case, $\rho_1$ is a simple real zero so $\rho'$ is affected by the zero repulsion emanating from $\rho_1$. This is explicitly quantified in [KN12, Theorem 4]; we state a slightly weaker version here.

**Theorem 7.2.2** (Kadiri–Ng [KN12])**.** *Let $\eta > 0$ be arbitrary. If $\lambda_1 \geq \eta$ then $\lambda' \geq 0.6546 \log(1/\lambda_1)$ for $D_L$ sufficiently large depending on $\eta$.*

When $\lambda_1 \leq \eta$, we will defer to Theorem 6.1.2 for the Deuring–Heilbronn phenomenon and its effect on $\rho'$. Next, we reduce Theorem 1.3.1 to verifying the following lemma.

**Lemma 7.2.3.** *Assume $\mathscr{L}$ is sufficiently large. Suppose for every $B \geq 40$ there exists a choice of $A$ and $\ell$ for* (7.4) *satisfying one of:*

*(i)* $A \geq 10^{-2}$, $\ell A \leq 3$, *and* $\frac{|G|}{|C|}\mathscr{L}^{-1} S \gg 1$.

*(ii)* $A \gg \mathscr{L}^{-1}$, $\ell A \leq 1$, *and* $\frac{|G|}{|C|}\mathscr{L}^{-1}S \gg \lambda_1$.

*Then*

$$\pi_C(x, L/F) \gg \frac{1}{D_L^{19}} \frac{|C|}{|G|} \frac{x}{\log x}$$

*for* $x \geq D_L^{40}$ *and* $D_L$ *sufficiently large.*

*Proof.* Let $B = \frac{\log x}{\mathscr{L}}$ so $B \geq 40$. In case (i), we apply Lemma 7.1.3 to deduce that

$$\pi_C(x, L/F) \gg e^{-6\mathscr{L}} \frac{|C|}{|G|} \frac{x}{\log x} \gg \frac{1}{D_L^6} \frac{|C|}{|G|} \frac{x}{\log x}$$

as desired. In case (ii), we again apply Lemma 7.1.3 to deduce that

$$\pi_C(x, L/F) \gg \lambda_1 e^{-2\mathscr{L}} \cdot \frac{|C|}{|G|} \frac{x}{\log x}.$$

From Corollary 6.1.4, we have that $\lambda_1 \gg D_L^{-17}$. Combining with the above yields the desired result as $e^{2\mathscr{L}} = D_L^2$. □

Thus, it suffices to verify the assumptions of Lemma 7.2.3 hold unconditionally.

### 7.2.2 A sum over low-lying zeros

Now, we begin by shifting the contour in Lemma 7.1.5 and reducing the analysis to a careful consideration of contribution coming from zeros $\rho = \beta + i\gamma$ of $\zeta_L(s)$ which are "low-lying".

**Lemma 7.2.4.** *Let $T^\star \geq 1$ be fixed. Keep the notation of Lemma 7.1.5. Then*

$$\begin{aligned} \Big| \tfrac{|G|}{|C|}\mathscr{L}^{-1}S - F(0) \Big| \leq \sum_{\substack{\rho \\ |\gamma| < T^\star}} |F((1-\rho)\mathscr{L})| + O\Big( \mathscr{L}\Big(\frac{2}{AT^\star\mathscr{L}}\Big)^{2\ell} + \frac{\mathscr{L}^2}{A} e^{-(B-2\ell A)\mathscr{L}/2} \Big) \\ + O\Big( \mathscr{L}\Big(\frac{1}{A\mathscr{L}}\Big)^{2\ell} e^{-(B-2\ell A)\mathscr{L}} + \mathscr{L}\Big(\frac{2}{A\mathscr{L}}\Big)^{2\ell} e^{-3(B-2\ell A)\mathscr{L}/2} \Big), \end{aligned} \tag{7.9}$$

*where the sum is over non-trivial zeros $\rho = \beta + i\gamma$ of $\zeta_L(s)$, counted with multiplicity.*

*Proof.* Consider the contour in Lemma 7.1.5. Since Hecke $L$-functions are meromorphic in the entire complex plane, we shift the line of integration to $\mathrm{Re}\{s\} = -\frac{1}{2}$. From (2.20), this picks up exactly the non-trivial zeros of $\zeta_L(s)$, its simple pole at $s = 1$, and its trivial zero at $s = 0$ of order $r_1 + r_2 - 1$. For $\mathrm{Re}\{s\} = -1/2$, we have by (7.1) that

$$F((1-s)\mathscr{L}) \ll e^{-3(B-2\ell A)\mathscr{L}/2} \cdot \Big( \frac{2}{A\mathscr{L}(|s|+1)} \Big)^{2\ell} \tag{7.10}$$

and, from [LO77, Lemma 6.2] and (2.21),

$$\begin{aligned}\sum_{\chi} \Big|\frac{L'}{L}(s,\chi,L/K)\Big| &\ll \sum_{\chi} \big\{ \log D_\chi + n_K \log(|s|+2) \big\} \\ &\ll \mathscr{L} + [L:K]\cdot n_K \log(|s|+2) \\ &\ll \mathscr{L} + n_L \log(|s|+2).\end{aligned}$$

It follows that

$$\sum_{\chi} \frac{\overline{\chi}(g_C)}{2\pi i} \int_{2-i\infty}^{2+i\infty} -\frac{L'}{L}(s,\chi,L/K) F((1-s)\mathscr{L}) ds \ll \mathscr{L}\Big(\frac{2}{A\mathscr{L}}\Big)^{2\ell} e^{-3(B-2\ell A)\mathscr{L}/2},$$

as $n_L \ll \mathscr{L}$. For the zero at $s=0$ of $Z_C(s)$, we may bound its contribution using (7.1) to deduce that

$$(r_1+r_2-1)F(\mathscr{L}) \ll \mathscr{L}\Big(\frac{1}{A\mathscr{L}}\Big)^{2\ell} e^{-(B-2\ell A)\mathscr{L}},$$

since $r_1 + 2r_2 = n_L \ll \mathscr{L}$. These observations and Lemma 7.1.5 therefore yield

$$\begin{aligned}\Big|\tfrac{|G|}{|C|}\mathscr{L}^{-1}S - F(0)\Big| \le \sum_{\rho} |F((1-\rho)\mathscr{L})| + O\Big(\frac{\mathscr{L}^2}{A} e^{-\frac{1}{2}(B-2\ell A)\mathscr{L}} + \mathscr{L}\Big(\frac{1}{A\mathscr{L}}\Big)^{2\ell} e^{-(B-2\ell A)\mathscr{L}}\Big) \\ + O\Big(\mathscr{L}\Big(\frac{2}{A\mathscr{L}}\Big)^{2\ell} e^{-\frac{3}{2}(B-2\ell A)\mathscr{L}}\Big),\end{aligned} \tag{7.11}$$

where the sum is over all non-trivial zeros $\rho = \beta + i\gamma$ of $\zeta_L(s)$. By [LMO79, Lemma 2.1] and (7.1), we have that

$$\sum_{k=0}^{\infty} \sum_{\substack{\rho \\ T^\star + k \le |\gamma| < T^\star + k + 1}} |F((1-\rho)\mathscr{L})| \ll \Big(\frac{2}{A\mathscr{L}}\Big)^{2\ell} \sum_{k=0}^{\infty} \frac{\mathscr{L} + n_L \log(T^\star + k)}{(T^\star+k)^{2\ell}} \ll \mathscr{L}\Big(\frac{2}{AT^\star\mathscr{L}}\Big)^{2\ell},$$

as $n_L \ll \mathscr{L}$ and $\ell \ge 2$. The result follows from (7.11) and the above estimate. □

For the sum over low-lying zeros in Lemma 7.2.4, we bound zeros far away from the line $\mathrm{Re}\{s\} = 1$ using Lemma 7.2.5 below. In the non-exceptional case, this could have been done in a fairly simple manner but when an exceptional zero exists, we will need to partition the zeros according to their height. This will amount to applying a coarse version of partial summation, allowing us to exploit the Deuring–Heilbronn phenomenon more efficiently.

**Lemma 7.2.5.** *Let $J \ge 1$ be given and $T^\star \ge 1$ be fixed. Suppose*

$$2 \le R_1 \le R_2 \le \cdots \le R_J \le \mathscr{L}, \quad 0 = T_0 < T_1 \le T_2 \le \cdots \le T_J = T^\star.$$

*Then*

$$\sum_{\substack{\rho \\ |\gamma|<T^\star}} |F((1-\rho)\mathscr{L})| = \sideset{}{'}\sum_{\rho} |F((1-\rho)\mathscr{L})| + O\Big( \min\Big\{ \Big(\frac{2}{A}\Big)^{2\ell}, \mathscr{L} \Big\} e^{-(B-2\ell A)R_1} \Big)$$
$$+ \sum_{j=2}^{J} O\Big( \mathscr{L}\Big(\frac{2}{AT_{j-1}\mathscr{L}}\Big)^{2\ell} e^{-(B-2\ell A)R_j} \Big), \tag{7.12}$$

*where the marked sum* $\sum'$ *indicates a restriction to zeros* $\rho = \beta + i\gamma$ *of* $\zeta_L(s)$ *satisfying*

$$\beta > 1 - \frac{R_j}{\mathscr{L}}, \qquad T_{j-1} \le |\gamma| < T_j \quad \textit{for some } 1 \le j \le J.$$

*If* $J = 1$ *then the secondary error term in* (7.12) *vanishes.*

**Remark.** To prove Theorem 1.3.1, we will apply the above lemma with $J = 10$ when an exceptional zero exists. One could use higher values of $J$ or a more refined version of Lemma 7.2.5 to obtain some improvement on the final result.

*Proof.* Recall $\ell \ge 2$ for our choice of weight $f$. Let $1 \le j \le J$ be arbitrary. Define the multiset

$$\mathcal{Z}_j := \{\rho : \zeta_L(\rho) = 0,\ \ \beta \le 1 - \frac{R_j}{\mathscr{L}},\ T_{j-1} \le |\gamma| < T_j\}$$

and denote $S_j := \sum_{\rho \in \mathcal{Z}_j} |F((1-\rho)\mathscr{L})|$. Since

$$\sum_{\substack{\rho \\ |\gamma|<T^\star}} |F((1-\rho)\mathscr{L})| = \sideset{}{'}\sum_{\rho} |F((1-\rho)\mathscr{L})| + \sum_{j=1}^{J} S_j,$$

it suffices to show

$$S_1 \ll \min\Big\{ \Big(\frac{2}{A}\Big)^{2\ell}, \mathscr{L} \Big\} e^{-(B-2\ell A)R_1},$$
$$\text{and} \quad S_j \ll \mathscr{L}\Big(\frac{2}{AT_{j-1}\mathscr{L}}\Big)^{2\ell} e^{-(B-2\ell A)R_j} \quad \text{for } 2 \le j \le J.$$

Assume $2 \le j \le J$. As $T_j \le T^\star$ and $T^\star$ is fixed, it follows that $\#\mathcal{Z}_j \ll \mathscr{L}$ by [LMO79, Lemma 2.1]. Hence, by Lemma 7.1.1 and the definition of $\mathcal{Z}_j$,

$$S_j \ll e^{-(B-2\ell A)R_j} \sum_{\rho \in \mathcal{Z}_j} \Big(\frac{2}{A|\gamma|\mathscr{L}}\Big)^{2\ell} \ll \mathscr{L}\Big(\frac{2}{AT_{j-1}\mathscr{L}}\Big)^{2\ell} e^{-(B-2\ell A)R_j}$$

as desired. It remains to consider $S_1$. On one hand, we similarly have $\#\mathcal{Z}_1 \ll \mathscr{L}$ by [LMO79, Lemma 2.1]. Thus, by Lemma 7.1.1 and the definition of $S_1$,

$$S_1 \ll \mathscr{L} e^{-(B-2\ell A)R_1}. \tag{7.13}$$

On the other hand, we may give an alternate bound for $S_1$. For integers $1 \leq m, n \leq \mathscr{L}$, consider the rectangles

$$\mathcal{R}_{m,n} := \Big\{ s = \sigma + it \in \mathbb{C} : 1 - \frac{m+1}{\mathscr{L}} \leq \sigma \leq 1 - \frac{m}{\mathscr{L}}, \quad \frac{n-1}{\mathscr{L}} \leq |t| \leq \frac{n}{\mathscr{L}} \Big\}.$$

We bound the contribution of zeros $\rho$ lying in $\mathcal{R}_{m,n}$ when $m \geq R_1$. If a zero $\rho \in \mathcal{R}_{m,n}$ then

$$|F((1-\rho)\mathscr{L})| \ll e^{-(B-2\ell A)m} \Big( \frac{2}{A\sqrt{m^2+(n-1)^2}} \Big)^{2\ell},$$

by Lemma 7.1.1 with $\alpha = 2\ell$. Further, by [LMO79, Lemma 2.2],

$$\#\{\rho \in \mathcal{R}_{m,n} : \zeta_L(\rho) = 0\} \ll \sqrt{(m+1)^2 + n^2} \ll \sqrt{m^2 + (n-1)^2}.$$

The latter estimate follows since $m, n \geq 1$. Adding up these contributions and using the conjugate symmetry of zeros, we find that

$$\begin{aligned} S_1 &\ll \sum_{\substack{m \geq R_1 \\ n \geq 1}} \sum_{\substack{\rho \in \mathcal{R}_{m,n} \\ \zeta_L(\rho)=0}} |F((1-\rho)\mathscr{L})| \ll \Big(\frac{2}{A}\Big)^{2\ell} \sum_{\substack{m \geq R_1 \\ n \geq 1}} e^{-(B-2\ell A)m} \big(\sqrt{m^2+(n-1)^2}\big)^{-2\ell+1} \\ &\ll \Big(\frac{2}{A}\Big)^{2\ell} e^{-(B-2\ell A)R_1}, \end{aligned}$$

since $\ell \geq 2$. Taking the minimum of the above and (7.13) gives the desired bound for $S_1$. □

If an exceptional zero exists with $\lambda_1$ sufficiently small then we shall choose the parameters in Lemma 7.2.5 so that the restricted sum over zeros is actually empty. Otherwise, Lemma 7.2.5 will be applied with $J = 1$ and $T_1 = T^\star = 1$ so we must handle the remaining restricted sum over zeros in the final arguments. We prepare for this situation via the following lemma.

**Lemma 7.2.6.** *Let $\eta > 0$ and $R \geq 1$ be arbitrary. For $A > 0$ and $\ell \geq 1$, define*

$$\tilde{F}_\ell(z) := \Big(\frac{1-e^{-Az}}{Az}\Big)^{2\ell}.$$

*Suppose $\zeta_L(s)$ is non-zero in the region*

$$\mathrm{Re}\{s\} \geq 1 - \frac{\lambda}{\mathscr{L}}, \qquad |\mathrm{Im}\{s\}| \leq 1$$

*for some $0 < \lambda \leq 10$. Then, provided $D_L$ is sufficiently large depending on $\eta, R$, and $A$,*

$$\sum_{\rho}{}' |\tilde{F}_\ell((1-\rho)\mathscr{L})| \leq \Big(\frac{1-e^{-A\lambda}}{A\lambda}\Big)^{2(\ell-1)} \cdot \Big\{\phi\Big(\frac{1-e^{-2A\lambda}}{A^2\lambda}\Big) + \frac{2A\lambda - 1 + e^{-2A\lambda}}{2A^2\lambda^2} + \eta\Big\}, \quad (7.14)$$

*where $\phi = \frac{1}{2}(1-\frac{1}{\sqrt{5}})$ and the marked sum $\sum'$ indicates a restriction to zeros $\rho = \beta + i\gamma$ of $\zeta_L(s)$ satisfying*

$$\beta \geq 1 - \frac{R}{\mathscr{L}}, \qquad |\gamma| \leq 1.$$

*In particular, as $\lambda \to 0$, the bound in* (7.14) *becomes $\frac{2\phi}{A} + 1 + \eta$.*

*Proof.* This result is motivated by [HB92, Lemma 13.3]. Define

$$h(t) := \begin{cases} A^{-2} \cdot \sinh\big((A-t)\lambda\big) & \text{if } 0 \leq t \leq A, \\ 0 & \text{if } t \geq A, \end{cases}$$

so

$$H(z) = \int_0^\infty e^{-zt} h(t)dt = \frac{1}{2A^2}\Big\{\frac{e^{A\lambda}}{\lambda+z} + \frac{e^{-A\lambda}}{\lambda - z} - \frac{2\lambda e^{-Az}}{\lambda^2 - z^2}\Big\}.$$

As per the argument in [HB92, Lemma 13.3],

$$|\tilde{F}_1(\lambda+z)| \leq \frac{2e^{-A\lambda}}{\lambda} \cdot \mathrm{Re}\{H(z)\} \quad (7.15)$$

for $\mathrm{Re}\{z\} \geq 0$. Combining the above with Lemma 7.1.2, it follows that

$$|\tilde{F}_\ell(\lambda+z)| \leq \Big(\frac{1-e^{-A\lambda}}{A\lambda}\Big)^{2(\ell-1)} \cdot \frac{2e^{-A\lambda}}{\lambda} \cdot \mathrm{Re}\{H(z)\}$$

for $\mathrm{Re}\{z\} \geq 0$, since $(1-e^{-x})/x$ is decreasing for $x > 0$. Setting $\sigma = 1 - \frac{\lambda}{\mathscr{L}} \in \mathbb{R}$, this implies

$$\sum_{\rho}{}' |\tilde{F}_\ell((1-\rho)\mathscr{L})| \leq \Big(\frac{1-e^{-A\lambda}}{A\lambda}\Big)^{2(\ell-1)} \cdot \frac{2e^{-A\lambda}}{\lambda} \sum_{\rho}{}' \mathrm{Re}\{H((\sigma-\rho)\mathscr{L})\},$$

so it suffices to bound the sum on the RHS. Since $h$ and $H$ satisfy Conditions 1 and 2 of [KN12], we apply [KN12, Theorem 3] to bound the sum $\sum'$ on the RHS yielding

$$\begin{aligned} \sum_{\rho}{}' \mathrm{Re}\{H((\sigma-\rho)\mathscr{L})\} &\leq h(0)(\phi+\eta) + H((\sigma-1)\mathscr{L}) - \mathscr{L}^{-1} \sum_{\mathfrak{N} \subseteq \mathcal{O}_L} \frac{\Lambda_L(\mathfrak{N})}{(\mathrm{N}^L_{\mathbb{Q}}\mathfrak{N})^\sigma} h\Big(\frac{\log \mathrm{N}^L_{\mathbb{Q}}\mathfrak{N}}{\mathscr{L}}\Big) \\ &\leq h(0)(\phi+\eta) + H((\sigma-1)\mathscr{L}), \end{aligned}$$

for $D_L$ sufficiently large depending on $\eta, R$ and $A$. Using the definitions of $h$ and $H$ and rescaling $\eta$ approriately, we obtain the desired result. $\square$

### 7.2.3 Non-exceptional case ($\lambda_1 \geq 0.0784$)

Recall the definition of $\rho_1$ given in Section 7.2.1. Here we assume $\lambda_1 \geq 0.0784$. Choose

$$\ell = 2, \quad B \geq 7.41, \quad \text{and} \quad A = 1.5$$

to give a corresponding $f$ and its Laplace transform $F$ defined by Lemma 7.1.1. Observe that $B - 2\ell A \geq 1.41$ for the above choices.

Let $\epsilon > 0$. Apply Lemma 7.2.4 with $T^\star = 1$. Then employ Lemma 7.2.5 with $J = 1, T_1 = T^\star = 1$ and $R_1 = R = R(\epsilon)$ sufficiently large so that

$$\frac{|G|}{|C|}\mathscr{L}^{-1}S \geq 1 - {\sum_{\rho}}' |F((1-\rho)\mathscr{L})| - \epsilon$$

for $D_L$ sufficiently large depending on $\epsilon$. Here the restricted sum is over zeros $\rho = \beta + i\gamma$ satisfying

$$\beta > 1 - \frac{R}{\mathscr{L}} \qquad |\gamma| < 1.$$

It suffices to prove the sum over zeros $\rho$ is $< 1 - \epsilon/2$ for fixed sufficiently small $\epsilon$. Observe by the definition of $\tilde{F}_2$ in Lemma 7.2.6 and our choice of $\rho_1$ that

$${\sum_{\rho}}' |F((1-\rho)\mathscr{L})| = {\sum_{\rho}}' e^{-1.41\lambda}|\tilde{F}_2((1-\rho)\mathscr{L})| \leq e^{-1.41\lambda_1} {\sum_{\rho}}' |\tilde{F}_2((1-\rho)\mathscr{L})|.$$

Since $\lambda_1 \geq 0.0784$, we may bound the remaining sum using Lemma 7.2.6 with $\lambda = 0.0784$. Hence, the above is

$$\leq e^{-1.41\lambda_1} \times 1.1166 \leq e^{-1.41\times 0.0784} \times 1.1166 = 0.9997\cdots < 1,$$

as desired. Thus, $\frac{|G|}{|C|}\mathscr{L}^{-1}S \gg 1$. By Lemma 7.2.3, this completes the proof of Theorem 1.3.1 in the non-exceptional case.

### 7.2.4 Exceptional case ($\lambda_1 < 0.0784$)

For this subsection, let $0 < \eta < 0.0784$ be an absolute arbitrary parameter which will be specified to be fixed and sufficiently small at the end of each subcase. Recall by Theorem 7.2.1 that $\rho_1 = \beta_1$ is a simple real zero of $\zeta_L(s)$.

$\lambda_1$ **small** $(\eta \le \lambda_1 < 0.0784)$

Again, choose the weight function $f$ from Lemma 7.1.1 with

$$\ell = 2, \quad B \ge 2.63, \quad \text{and} \quad A = 0.1,$$

so $B - 2\ell A \ge 2.23$ and $\ell A = 0.2 \le 3$. The argument is similar to the previous case but we take special care of the real zero $\beta_1$. With the same choices as the non-exceptional case, we deduce

$$\frac{|G|}{|C|}\mathscr{L}^{-1}S \ge 1 - |F((1-\beta_1)\mathscr{L})| - \sum_{\rho \ne \beta_1}{}' |F((1-\rho)\mathscr{L})| - \epsilon \tag{7.16}$$

for $D_L$ sufficiently large depending on $\epsilon$. Observe that, since $\rho_1$ is real and $(1-e^{-t})/t \le 1$ for $t > 0$,

$$|F((1-\rho_1)\mathscr{L})| = e^{-2.23\lambda_1}\Big(\frac{1-e^{-0.1\lambda_1}}{0.1\lambda_1}\Big)^4 \le e^{-2.23\lambda_1}.$$

By our choice of $\rho'$ in Section 7.2.1 and a subsequent application of Lemma 7.2.6 with $\lambda = 0$, we have that

$$\sum_{\rho \ne \rho_1}{}' |F((1-\rho)\mathscr{L})| \le e^{-2.23\lambda'} \sum_{\rho \ne \rho_1}{}' |\tilde{F}_2((1-\rho)\mathscr{L})| \le e^{-2.23\lambda'} \times 6.5279.$$

As $\lambda_1 \ge \eta$, we apply Theorem 7.2.2 to see that $\lambda' \ge 0.6546 \log(1/\lambda_1)$ for $D_L$ is sufficiently large depending on $\eta$. Hence, the above is

$$\le 6.5279 \times \lambda_1^{2.23 \times 0.6546} \le 6.5279 \times \lambda_1^{1.4597}.$$

Thus, (7.16) becomes

$$\begin{aligned}\frac{|G|}{|C|}\mathscr{L}^{-1}S &\ge 1 - e^{-2.23\lambda_1} - 6.5279 \times \lambda_1^{1.4597} - \epsilon \\ &\ge \big(2.23 - 6.5279 \times \lambda_1^{0.4597} - 2.4865\lambda_1\big)\lambda_1 - \epsilon,\end{aligned}$$

since $1 - e^{-t} \ge t - t^2/2$ for $t > 0$. The quantity in the brackets is clearly decreasing with $\lambda_1$ so since $\lambda_1 < 0.0784$, we conclude that the above is

$$\begin{aligned}&\ge \big(2.23 - 6.5279 \times 0.0784^{0.4597} - 2.4865 \times 0.0784\big)\lambda_1 - \epsilon \\ &\ge 0.0097\lambda_1 - \epsilon\end{aligned}$$

By taking $\epsilon = 10^{-6}\eta$ and noting $\lambda_1 \geq \eta$, we have $\frac{|G|}{|C|}\mathscr{L}^{-1}S \gg \lambda_1$ for $D_L$ sufficiently large depending on $\eta$. By Lemma 7.2.3, this completes the proof of Theorem 1.3.1 for $\eta \leq \lambda_1 < 0.0784$.

$\lambda_1$ **very small** ($\mathscr{L}^{-200} \leq \lambda_1 < \eta$)

Choose the weight function $f$ from Lemma 7.1.1 with

$$\ell = 101, \quad B \geq 32, \quad \text{and} \quad A = \tfrac{1}{404},$$

so $B - 2\ell A \geq 31.5$ and $\ell A \leq 1$ for $D_L$ sufficiently large. Applying Lemma 7.2.4 with $T^\star = 1$, it follows that

$$\frac{|G|}{|C|}\mathscr{L}^{-1}S \geq 1 - |F((1-\beta_1)\mathscr{L})| - \sum_{\substack{\rho\neq\beta_1 \\ |\gamma|<1}} |F((1-\rho)\mathscr{L})| + O(\mathscr{L}^{-201}).$$

Similar to the previous subcase, we have that $|F((1-\beta_1)\mathscr{L})| \leq e^{-31.5\lambda_1}$. For the remaining sum over zeros, we apply Lemma 7.2.5 with $J = 1, T_\star = T_1 = 1$, and $R_1 = \frac{1}{31.4}\log(c_1/\lambda_1)$ with $c_1 > 0$ absolute and sufficiently small. As $\lambda_1 \geq \mathscr{L}^{-200}$, we may assume without loss that $R_1 < \frac{1}{4}\mathscr{L}$ for $\mathscr{L}$ sufficiently large[1]. Therefore,

$$\frac{|G|}{|C|}\mathscr{L}^{-1}S \geq 1 - e^{-31.5\lambda_1} - \sideset{}{'}\sum_{\rho\neq\beta_1} |F((1-\rho)\mathscr{L})| + O\big(\mathscr{L}^{-201} + \lambda_1^{31.5/31.4}\big), \tag{7.17}$$

where the sum $\sum'$ is defined as per Lemma 7.2.5. By our choice of parameters $T_1$ and $R_1$, it follows from Theorem 6.1.2 that the restricted sum over zeros in (7.17) is actually empty. As $1 - e^{-t} \geq t - t^2/2$ for $t > 0$, we conclude that

$$\frac{|G|}{|C|}\mathscr{L}^{-1}S \geq 31.5\lambda_1 + O(\mathscr{L}^{-201} + \lambda_1^{31.5/31.4}).$$

Since $\mathscr{L}^{-200} \leq \lambda_1 < \eta$ by assumption and $\eta$ is sufficiently small, we conclude that the RHS is $\gg \lambda_1$ after fixing $\eta$. By Lemma 7.2.3, this completes the proof of Theorem 1.3.1 in this case.

[1]This implies the zero $1 - \beta_1$ is already discarded in the error term arising from Lemma 7.2.5. This minor point will be relevant when $\lambda_1$ is extremely small.

$\lambda_1$ **extremely small** $(\lambda_1 < \mathscr{L}^{-200})$

Choose the weight function $f$ from Lemma 7.1.1 with

$$\ell = \lceil \mathscr{L} \rceil, \quad B \geq 35, \quad \text{and} \quad A = \frac{0.88}{\mathscr{L}},$$

so $B - 2\ell A \geq 33.24$ and $\ell A \leq 1$ for $D_L$ sufficiently large. Applying Lemma 7.2.4 with $T^\star = 12646$, it follows that

$$\begin{aligned}\left|\tfrac{|G|}{|C|}\mathscr{L}^{-1}S - F(0)\right| &\leq \sum_{\substack{\rho \\ |\gamma|<12646}} |F((1-\rho)\mathscr{L})| + O\Big(\mathscr{L}e^{2\log\left(\frac{2}{0.88\times 12646}\right)\mathscr{L}} + \mathscr{L}^3 e^{-\frac{33.24}{2}\mathscr{L}}\Big) \\ &\quad + O\Big(\mathscr{L}e^{-33.24\mathscr{L}+2\log\left(\frac{1}{0.88}\right)\mathscr{L}} + \mathscr{L}e^{-\frac{3}{2}\times 33.24\mathscr{L}+2\log\left(\frac{2}{0.88}\right)\mathscr{L}}\Big) \\ &\leq \sum_{\substack{\rho \\ |\gamma|<12646}} |F((1-\rho)\mathscr{L})| + O(\mathscr{L}^3 e^{-16.62\mathscr{L}}).\end{aligned} \tag{7.18}$$

For the remaining sum, we use Lemma 7.2.5 with $J = 10$ selecting $T_j$ and $R_j = \frac{\log(c_j/\lambda_1)}{C_j}$ according to the table below. Note $C_j = C(T_j) > 0$ and $c_j = c(T_j)$ are the absolute constants in Theorem 6.1.2.

| $j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $T_j$ | 3.5 | 8.7 | 22 | 54 | 134 | 332 | 825 | 2048 | 5089 | 12646 |
| $C_j$ | 32.7 | 35.0 | 38.4 | 42.0 | 45.9 | 49.7 | 53.6 | 57.4 | 61.2 | 65.0 |

Therefore,

$$\begin{aligned}\frac{|G|}{|C|}\mathscr{L}^{-1}S &\geq 1 - |F((1-\beta_1)\mathscr{L})| - \sum_{\rho\neq\beta_1,1-\beta_1}{}' |F((1-\rho)\mathscr{L})| - |F(\beta_1\mathscr{L})| + O(\mathscr{L}^3 e^{-16.62\mathscr{L}}) \\ &\quad + O\big(\mathscr{L}\lambda_1^{33.24/32.7}\big) + \sum_{j=2}^{10} O\Big(\mathscr{L}e^{2\log\left(\frac{2}{0.88T_{j-1}}\right)\mathscr{L}}\lambda_1^{33.24/C_j}\Big),\end{aligned} \tag{7.19}$$

where the sum $\sum'$ is defined as per Lemma 7.2.5. Since the zeros of $\zeta_L(s)$ are permuted under the map $\rho \mapsto 1-\rho$, it follows from Theorem 6.1.2 and our choice of parameters $T_j$ and $C_j$ that the restricted sum over zeros in (7.19) is actually empty[2]. For the zeros $1-\beta_1$ and $\beta_1$, notice

$$|F((1-\beta_1)\mathscr{L})| \leq e^{-33.24\lambda_1} \leq e^{-33\lambda_1} \quad \text{and} \quad F(\beta_1\mathscr{L}) \leq e^{-33.24(\mathscr{L}-\lambda_1)} = O(e^{-33\mathscr{L}}),$$

[2]The zero $1-\beta_1$ cannot be discarded via symmetry or Theorem 6.1.2 which is why we must consider its contribution separately.

as $\lambda_1 < 0.0784$. Moreover, as $\lambda_1 < \mathscr{L}^{-200}$ and $\frac{33.24}{32.7} > 1.016$, we observe that

$$\mathscr{L} \cdot \lambda_1^{33.24/32.7} \ll \lambda_1^{-1/200} \cdot \lambda_1^{1.016} \ll \lambda_1^{1.01}.$$

To bound the sum over error terms in the (7.19), notice $\lambda_1 \gg \mathscr{L}e^{-16.6\mathscr{L}}$ by Corollary 6.1.4, which implies that

$$\mathscr{L}e^{2\log\left(\frac{2}{0.88T_j}\right)\mathscr{L}} \lambda_1^{33.24/C_j} \ll \lambda_1 \cdot \mathscr{L}^2 e^{2\log\left(\frac{2}{0.88T_{j-1}}\right)\mathscr{L}+16.6(1-33.24/C_j)\mathscr{L}}.$$

Substituting the prescribed values for $C_j$ and $T_{j-1}$, the above is $\ll \lambda_1 e^{-0.02\mathscr{L}}$ for all $2 \le j \le 10$. Incorporating all of these observations into (7.19) yields

$$\begin{aligned}\frac{|G|}{|C|}\mathscr{L}^{-1}S &\ge 1 - e^{-33\lambda_1} + O\Big(\lambda_1^{1.01} + \lambda_1 e^{-0.02\mathscr{L}} + \mathscr{L}^3 e^{-16.62\mathscr{L}}\Big) \\ &\ge 33\lambda_1 + O\Big(\lambda_1^{1.01} + \lambda_1 e^{-0.02\mathscr{L}} + \mathscr{L}^3 e^{-16.62\mathscr{L}}\Big),\end{aligned}$$

since $1 - e^{-t} \ge t - t^2/2$ for $t > 0$. Again noting that $\lambda_1 \gg \mathscr{L}e^{-16.6\mathscr{L}}$ by Corollary 6.1.4 and $\lambda_1 < \mathscr{L}^{-200}$ by assumption, we finally conclude that the RHS is $\gg \lambda_1$ for $D_L$ sufficiently large. By Lemma 7.2.3, this completes the proof of Theorem 1.3.1 in all cases. □

**Remark.** We outline the minor modifications required to justify the remark following Theorem 1.3.1.

- If there is a sequence of fields $\mathbb{Q} = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_r = L$ such that $L_j$ is normal over $L_{j-1}$ for $1 \le j \le r$ then by [Sta74, Lemmas 10, 11], it follows that $\lambda_1 \gg \mathscr{L}e^{-0.5\mathscr{L}}$. When $\lambda_1$ is extremely small ($\lambda_1 < \mathscr{L}^{-200}$), one may therefore select

$$\ell = \lceil 0.05\mathscr{L} \rceil, \quad B = 32, \quad \text{and} \quad A = \frac{3}{\mathscr{L}},$$

  and apply Lemma 7.2.4 with $T^\star = 12646$. Afterwards, employ Lemma 7.2.5 with $T_j$ and $R_j = \frac{\log(c_j/\lambda_1)}{C_j}$ chosen according to the table below.

| $j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_j$ | 1 | 3.5 | 8.7 | 22 | 54 | 134 | 332 | 825 | 2048 | 5089 | 12646 |
| $C_j$ | 31.4 | 32.7 | 35.0 | 38.4 | 42.0 | 45.9 | 49.7 | 53.6 | 57.4 | 61.2 | 65.0 |

  Following the same arguments yields the desired result.

- If $n_L = o(\log D_L)$ then by remark (ii) following Theorem 6.1.2, applied to Corollary 6.1.4, it follows that $\lambda_1 \gg \mathscr{L}e^{-12.01\mathscr{L}}$. Moreover, by remark (ii) following Theorem 6.1.2, one can use

$$J = 1, \quad T_1 = T^\star = e^{64}, \quad \text{and} \quad R_1 = \frac{\log(c/\lambda_1)}{24.01}$$

in the application of Lemmas 7.2.4 and 7.2.5. One may then modify the case when $\lambda_1$ is very small to consider $\mathscr{L}^{-1000} \leq \lambda_1 < \eta$ and take

$$\ell = 1000, \quad B = 24.1, \quad A = 1/10^6.$$

Similarly, one may modify the case when $\lambda_1$ is extremely small to consider $\lambda_1 < \mathscr{L}^{-1000}$ and take

$$\ell = \lceil 0.1\mathscr{L} \rceil, \quad B = 24.1, \quad \text{and} \quad A = \frac{0.2}{\mathscr{L}}.$$

Following the same arguments yields the claimed result.

- If $\zeta_L(s)$ does not have a Siegel zero then $\lambda_1 \gg 1$ so the cases when $\lambda_1 < \eta$ are unnecessary.

**Remark.** When $\lambda_1$ is extremely small ($\lambda_1 < \mathscr{L}^{-200}$), the selection of parameters $A, B, \ell$, and $T_j$ was primarily based on numerical experimentation but for the previous cases, one can choose them roughly optimally.

## 7.3 Proof of Theorem 1.3.2

### 7.3.1 Additional preliminaries

From Lemma 7.1.5, recall that we are given an arbitrary abelian subgroup $H$ of $G = \mathrm{Gal}(L/F)$ satisfying $H \cap C \neq \emptyset$ and $K = L^H$ is the fixed field of $L$ by $H$. Define the *max conductor* of $L/K$ by

$$\mathcal{Q} = \mathcal{Q}(L/K) := \max\{\mathrm{N}^K_{\mathbb{Q}}\mathfrak{f}_\chi : \chi \in \widehat{\mathrm{Gal}(L/K)}\}, \tag{7.20}$$

where the $K$-integral ideal $\mathfrak{f}_\chi \subseteq \mathcal{O}_K$ is the conductor of the Hecke character $\chi$ attached to the abelian extension $L/K$. For the entirety of Section 7.3, let[3]

$$\mathscr{L} := \begin{cases} (\frac{1}{3} + \delta_0)\log D_K + (\frac{19}{36} + \delta_0)\log \mathcal{Q} + (\frac{5}{12} + \delta_0)n_K \log n_K & \text{if } n_K^{5n_K/6} \geq D_K^{4/3}\mathcal{Q}^{4/9}, \\ (1 + \delta_0)\log D_K + (\frac{3}{4} + \delta_0)\log \mathcal{Q} + \delta_0 n_K \log n_K & \text{otherwise,} \end{cases} \tag{7.21}$$

[3]This is the same quantity as defined in (5.3).

where $\delta_0 > 0$ is fixed and sufficiently small. Notice that

$$\mathscr{L} \geq (1+\delta_0)\log D_K + (\tfrac{3}{4}+\delta_0)\log \mathcal{Q} + \delta_0 n_K \log n_K \quad \text{and} \quad \mathscr{L} \geq (\tfrac{5}{12}+\delta_0) n_K \log n_K \tag{7.22}$$

unconditionally. We exhibit a bound on the degree of the extension $L/K$ in terms of $\mathscr{L}$.

**Lemma 7.3.1.** *$[L:K] \ll e^{4\mathscr{L}/3}$ and $n_L \ll \mathscr{L} e^{4\mathscr{L}/3}$.*

*Proof.* Let $\mathfrak{f} = \mathfrak{f}_{L/K}$ be the Artin conductor attached to $L/K$ by class field theory. Let $I(\mathfrak{f})$ be the group of fractional ideals of $K$ relatively prime to $\mathfrak{f}$. By class field theory, there exists a homomorphism $\phi : I(\mathfrak{f}) \to \mathrm{Gal}(L/K)$. Thus $I(\mathfrak{f})/\ker\phi$ is isomorphic to $\mathrm{Gal}(L/K)$. This induces an isomorphism between their respective character groups and therefore,

$$\mathcal{Q}(L/K) = \max\{\mathrm{N}\mathfrak{f}_\chi : \chi \in \widehat{\mathrm{Gal}}(L/K)\} = \max\{\mathrm{N}\mathfrak{f}_\chi : \chi \in \widehat{I(\mathfrak{f})/\ker\phi}\}.$$

By our previous observations, $|I(\mathfrak{f})/\ker\phi| = |\mathrm{Gal}(L/K)| = [L:K]$. For $\epsilon_0 > 0$ fixed and sufficiently small, we have by Lemma 2.4.6 that $h_{\ker\phi} = |I(\mathfrak{f})/\ker\phi| \ll e^{O_{\epsilon_0}(n_K)} D_K^{1/2+\epsilon_0} \mathcal{Q}^{1+\epsilon_0} \ll e^{4\mathscr{L}/3}$ as desired. To bound $n_L$, observe that $n_L = [L:K]n_K$ and $n_K \ll \mathscr{L}$. $\square$

We will need to carefully analyze the zeros of

$$\prod_\chi L(s,\chi,L/K), \tag{7.23}$$

where the product is over all (necessarily primitive) Hecke characters attached to $L/K$. From the discussion in Section 2.5, the non-trivial zeros of (7.23) are, counting with multiplicity, exactly the non-trivial zeros of (3.1) for some congruence class group $H$ of $K$. In fact, this correspondence occurs for each $L$-function appearing in both (7.23) and (3.1). Thus, all the results of Chapters 3 to 6 regarding the non-trivial zeros of (3.1) can be directly translated to results about (7.23). The remainder of this subsection is dedicated to recording these translated results using the quantity $\mathscr{L}$. The differences are primarily notational.

First, we specify some important zeros of (7.23). These zeros will be used for the remainder of this section. For $T_\star \geq 1$ arbitrary, consider the multiset given by

$$\mathcal{Z} := \Big\{\rho \in \mathbb{C} : \prod_\chi L(\rho,\chi,L/K) = 0, 0 < \mathrm{Re}\{\rho\} < 1, |\mathrm{Im}(\rho)| \leq T_\star\Big\}. \tag{7.24}$$

We select three important zeros in $\mathcal{Z}$ as follows:

- Choose $\rho_1 \in \mathcal{Z}$ such that $\mathrm{Re}\{\rho_1\}$ is maximal. Let $\chi_1$ be its associated Hecke character so $L(\rho_1,\chi_1,L/K) = 0$. Denote $\rho_1 = \beta_1 + i\gamma_1 = (1 - \frac{\lambda_1}{\mathscr{L}}) + i\frac{\mu_1}{\mathscr{L}}$, where $\beta_1 = \mathrm{Re}\{\rho_1\}, \gamma_1 =$

$\mathrm{Im}\{\rho_1\}$, $\lambda_1 > 0$, and $\mu_1 \in \mathbb{R}$.

- Choose[4] $\rho' \in \mathcal{Z} \setminus \{\rho_1, \overline{\rho_1}\}$ satisfying $L(\rho', \chi_1, L/K) = 0$ such that $\mathrm{Re}\{\rho'\}$ is maximal with respect to these conditions. Similarly denote $\rho' = \beta' + i\gamma' = (1 - \frac{\lambda'}{\mathscr{L}}) + i\frac{\mu'}{\mathscr{L}}$.

- Choose $\rho_2 \in \mathcal{Z} \setminus \mathcal{Z}_1$ such that $\mathrm{Re}\{\rho_2\}$ is maximal and where $\mathcal{Z}_1$ is the multiset of zeros of $L(s, \chi_1, L/K)$ contained in $\mathcal{Z}$. Let $\chi_2$ be its associated Hecke character so $L(\rho_2, \chi_2, L/K) = 0$. Similarly, denote $\rho_2 = \beta_2 + i\gamma_2 = (1 - \frac{\lambda_2}{\mathscr{L}}) + i\frac{\mu_2}{\mathscr{L}}$.

If $\mathcal{L}$ is defined by (3.3) with a suitably chosen function $\nu$ then by (7.22) it follows that $\mathscr{L} \geq \mathcal{L}$ for $\mathscr{L}$ sufficiently large. Thus, the results on the distribution of zeros of Hecke $L$-functions, including those from Chapter 4, may be rewritten in the current notation.

**Theorem 7.3.2.** *Assume $\mathscr{L}$ is sufficiently large depending on $T_\star$. If $\lambda_1 < 0.0875$ then $\rho_1$ is a simple real zero of $\prod_\chi L(s, \chi, L/K)$ and is associated with a real character $\chi_1$. Furthermore,* $\min\{\lambda', \lambda_2\} > 0.2866$.

*Proof.* This is the contents of Theorems 4.1.1 and 4.1.2. □

**Theorem 7.3.3.** *Let $\mathscr{L}$ be sufficiently large depending on $T_\star$. If $\lambda_1 < 0.0875$, then* $\min\{\lambda', \lambda_2\} > 0.51$. *If $\eta \leq \lambda_1 < 0.0875$, then* $\min\{\lambda', \lambda_2\} > 0.2103 \log(1/\lambda_1)$.

*Proof.* Follows from Theorems 4.1.3 and 7.3.2. Note $0.2103 \log(1/0.0875) > 0.51$. □

**Theorem 7.3.4.** *Let $T \geq 1$ be arbitrary. Suppose $\chi_1$ is a real character and $\rho_1$ is a real zero. For any character $\chi$ of $L/K$, let $\rho = \beta + i\gamma \neq \rho_1$ be a non-trivial zero of $L(s, \chi, L/K)$ satisfying $1/2 \leq \beta < 1$ and $|\gamma| \leq T$. For $\mathscr{L}$ sufficiently large, there exists an absolute effectively computable constant $c_1 > 0$ such that*

$$\beta < 1 - \frac{\log\left(\dfrac{c_1}{(1-\beta_1)(\mathscr{L} + n_K \log T)}\right)}{81\mathscr{L} + 25 n_K \log T}.$$

*Proof.* This follows immediately from Theorem 6.1.1, since

$$(48 + \epsilon)\log D_K + (60 + \epsilon)\log \mathcal{Q} + O_\epsilon(n_K) \leq (80 + 2\epsilon)\mathscr{L}$$

for $\mathscr{L}$ sufficiently large depending on $\epsilon$. □

**Theorem 7.3.5** (Stark)**.** *Unconditionally,* $\lambda_1 \gg e^{-24\mathscr{L}/5}$.

[4]If $\rho_1$ is real then $\rho' \in \mathcal{Z} \setminus \{\rho_1\}$ instead with the other conditions remaining the same.

*Proof.* By Theorem 7.3.2 or its non-explicit predecessor [Wei83, Theorem 1.9], we may assume $\rho_1$ is real and its associated character $\chi_1$ is real. By considering cases depending on whether $\chi_1$ is quadratic or principal, the result follows from (7.21), (7.22), and the proof of [Sta74, Theorem 1′, p.148]. □

Let $\chi \in \widehat{\mathrm{Gal}}(L/K)$ be a Hecke character. Define

$$N(\sigma, T, \chi) := \#\{\rho = \beta + i\gamma : L(\rho, \chi, L/K) = 0, \sigma < \beta < 1, |\gamma| \leq T\}$$

for $0 < \sigma < 1$ and $T \geq 1$. Further denote

$$N(\sigma, T) := \sum_{\chi} N(\sigma, T, \chi). \tag{7.25}$$

We emphasize that the following estimate does *not* assume $\mathscr{L}$ is sufficiently large.

**Theorem 7.3.6.** *For $0 < \sigma < 1$ and $T \geq 1$, $N(\sigma, T) \ll (e^{162\mathscr{L}} T^{81n_K+162})^{1-\sigma}$.*

*Proof.* This follows from Theorem 5.1.1. To remove the condition on $T$ in Theorem 5.1.1, we used the definition of $\mathscr{L}$ in (7.21). □

We will also require a more explicit zero density estimate for "low-lying" zeros. Set

$$T_0 := \max\{n_K^{5/6} D_K^{-4/3n_K} \mathcal{Q}^{-4/9n_K}, T_\star\}.$$

Comparing $\mathcal{L} = \mathcal{L}_{T_0,\delta_0}$ given by (5.17) with $\mathscr{L}$, we deduce $\mathcal{L} \leq \mathscr{L}$ for $\mathscr{L}$ sufficiently large depending on $T_\star$. This observation implies that, for $\lambda > 0$,

$$N(1 - \tfrac{\lambda}{\mathscr{L}}, T, \chi) \leq N(1 - \tfrac{\lambda}{\mathcal{L}}, T, \chi). \tag{7.26}$$

Hence, the results of Chapter 5 can be transferred into the current notation with $\mathscr{L}$. Abusing notation, define for $0 < \lambda < \mathscr{L}$,

$$\mathcal{N}(\lambda) = \mathcal{N}(\lambda; T_\star) := \sum_{\chi} N(1 - \tfrac{\lambda}{\mathscr{L}}, T_\star, \chi). \tag{7.27}$$

Theorem 7.3.2 states that $\mathcal{N}(0.0875) \leq 1$ and $\mathcal{N}(0.2866) \leq 2$ for $\mathscr{L}$ sufficiently large depending on $T_\star$. For larger values of $\lambda$, we use the following:

**Theorem 7.3.7.** *Assume $\mathscr{L}$ is sufficiently large depending on $T_\star$. Let $\epsilon_0 > 0$ be fixed and sufficiently small. If $0 < \lambda < \epsilon_0 \mathscr{L}$ then*

$$\mathcal{N}(\lambda) \leq e^{162\lambda + 188}.$$

*The bounds for $\mathcal{N}(\lambda)$ in Table 5.1 are superior when* $0 < \lambda \leq 1$.

*Proof.* This is the same as Theorem 5.1.3. $\square$

Finally, we reduce the proof of Theorem 1.3.2 to verifying the following lemma.

**Lemma 7.3.8.** *Let $\eta > 0$ be sufficiently small and arbitrary. Assume $\mathscr{L}$ is sufficiently large depending only on $\eta > 0$. Let $A = 4/\mathscr{L}$ and $\ell = \lfloor \eta\mathscr{L} \rfloor$. If every $B \geq 693.5$ defining* (7.4) *implies $\frac{|G|}{|C|}\mathscr{L}^{-1}S \gg_\eta \min\{1, \lambda_1\}$ then*

$$\pi_C(x, L/F) \gg \frac{1}{D_K^5 \mathcal{Q}^4 n_K^{3n_K}} \frac{|C|}{|G|} \frac{x}{\log x}$$

*for $x \geq D_K^{694}\mathcal{Q}^{521} + D_K^{232}\mathcal{Q}^{367}n_K^{290n_K}$ and $D_K\mathcal{Q}n_K^{n_K}$ sufficiently large.*

*Proof.* Let $B = \frac{\log x}{\mathscr{L}}$ so $B \geq 693.5$ by assumption and the definition of $\mathscr{L}$ in (7.21). Fix $\eta > 0$ sufficiently small. By Lemma 7.1.3, our assumption on $S$, and Theorem 7.3.5, it follows that

$$\pi_C(x, L/F) \gg \min\{1, \lambda_1\} e^{-10\eta\mathscr{L}} \frac{|C|}{|G|} \frac{x}{\log x} \gg e^{-(\frac{24}{5}+10\eta)\mathscr{L}} \frac{|C|}{|G|} \frac{x}{\log x}$$

as desired. By (7.21), we somewhat crudely bound $\mathscr{L}$ to note that $e^{(\frac{24}{5}+10\eta)\mathscr{L}} \ll D_K^5\mathcal{Q}^4 n_K^{3n_K}$. Combining with the above yields the desired result. $\square$

**Remark.** The quality of exponents for $D_K^5\mathcal{Q}^4n_K^{3n_K}$ in the lower bound for $\pi_C(x, L/F)$ can be easily improved by simple modifications to our bounds in Theorem 7.3.5 and our bound of $\mathscr{L}$ in the above arguments. For simplicity, we did not pursue the optimal exponents.

### 7.3.2 A sum over low-lying zeros

We again begin by shifting the contour in Lemma 7.1.5 and reducing the analysis to a careful consideration of contribution coming from zeros of Hecke $L$-functions which are "low-lying". Recall that $T_\star \geq 1$ is arbitrary throughout this section, though we will emphasize it in the statements of some lemmas.

**Lemma 7.3.9.** *Let $T_\star \geq 1$ be arbitrary, and let $\rho_1$ and $\chi_1$ be as in Section 7.3.1. If*

$$B - 2\ell A > 162, \qquad \ell > \frac{81n_K + 162}{4}, \qquad A > \frac{1}{\mathscr{L}},$$

*and $\mathscr{L}$ is sufficiently large then*

$$\left|\tfrac{|G|}{|C|}\mathscr{L}^{-1}S - F(0) + \overline{\chi_1}(g_C)F((1-\rho_1)\mathscr{L})\right| \leq \sum_{\chi}\sum_{\rho}{}' |F((1-\rho)\mathscr{L})| + O\Big(\Big(\frac{2}{AT_\star\mathscr{L}}\Big)^{2\ell} T_\star^{40.5n_K+81} + e^{-78\mathscr{L}}\Big)$$

*where the sum $\sum'$ indicates a restriction to non-trivial zeros $\rho \neq \rho_1$ of $L(s,\chi,L/K)$, counted with multiplicity, satisfying $0 < \mathrm{Re}\{\rho\} < 1$ and $|\mathrm{Im}\{\rho\}| \leq T_\star$.*

*Proof.* Shift the contour in Lemma 7.1.5 to the line $\mathrm{Re}\{s\} = -\frac{1}{2}$. For each Hecke character $\chi$ (which is necessarily primitive), this picks up the non-trivial zeros of $L(s,\chi,L/K)$, the simple pole at $s=1$ when $\chi$ is trivial, and the trivial zero at $s=0$ of $L(s,\chi,L/K)$ of order $r(\chi)$. To bound the remaining contour, by [LMO79, Lemma 2.2] and Lemma 7.1.1(iv) with $\alpha = 2$, for $\mathrm{Re}\{s\} = -1/2$ we have that

$$-\frac{L'}{L}(s,\chi,L/K) \ll \mathscr{L} + n_K\log(|s|+2), \quad \text{and} \quad |F((1-s)\mathscr{L})| \ll \mathscr{L}^2 e^{-\frac{3}{2}(B-2\ell A)\mathscr{L}}\cdot|s|^{-2},$$

since $A > 1/\mathscr{L}$. It follows that

$$\frac{1}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty} -\frac{L'}{L}(s,\chi,L/K)F((1-s)\mathscr{L})ds \ll \mathscr{L}^3 e^{-\frac{3}{2}(B-2\ell A)\mathscr{L}}.$$

Moreover, by the conductor-discriminant formula (2.21), our condition on $A$, (7.21), and Lemma 7.3.1, we have that

$$A^{-1}\mathscr{L}e^{-(B-2\ell A)\mathscr{L}/2}\log D_L \ll \mathscr{L}^2 e^{-(B-2\ell A)\mathscr{L}/2}[L:K]\log(D_K\mathcal{Q}) \ll e^{-(B-2\ell A-4)\mathscr{L}/2}.$$

Substituting all of these calculations in Lemma 7.1.5 implies

$$\left|\tfrac{|G|}{|C|}\mathscr{L}^{-1}S - F(0) + \sum_{\chi}\overline{\chi}(g_C)\sum_{\rho}F((1-\rho)\mathscr{L})\right| \ll \sum_{\chi} r(\chi)F(\mathscr{L}) + e^{-(B-2\ell A-4)\mathscr{L}/2}, \tag{7.28}$$

where the inner sum over $\rho = \rho_\chi$ is over all non-trivial zeros of $L(s,\chi,L/K)$. From (2.5) and (2.8), notice $r(\chi) \leq n_K$. Thus, by Lemmas 7.1.1 and 7.3.1,

$$\sum_{\chi} r(\chi)F(\mathscr{L}) \ll [L:K]n_K e^{-(B-2\ell A)\mathscr{L}} \ll e^{-(B-2\ell A-2)\mathscr{L}}.$$

It follows from (7.28) that

$$\tfrac{|G|}{|C|}\mathscr{L}^{-1}S = F(0) - \sum_{\chi}\overline{\chi}(g_C)\sum_{\rho}F((1-\rho)\mathscr{L}) + O\Big(e^{-(B-2\ell A-4)\mathscr{L}/2}\Big).$$

The error term is bounded by $O(e^{-78\mathscr{L}})$ as $B-2\ell A > 162$. Therefore, it suffices to show

$$Z := \sum_{\chi}\sum_{k=0}^{\infty}\sum_{\substack{\rho \\ 2^k T_\star \le \mathrm{Im}\{\rho\} < 2^{k+1}T_\star}} |F((1-\rho)\mathscr{L})| \ll \Big(\frac{2}{AT_\star\mathscr{L}}\Big)^{2\ell} T_\star^{40.5n_K+81}.$$

From Lemma 7.1.1, writing $\rho = \beta + i\gamma$ with $\beta \ge 1/2$, observe that

$$|F(\rho\mathscr{L})| + |F((1-\rho)\mathscr{L})| \le 2e^{-(B-2\ell A)(1-\beta)\mathscr{L}}\Big(\frac{2}{A|\gamma|\mathscr{L}}\Big)^{2\ell}.$$

Moreover, from Theorem 7.3.6,

$$\tilde{N}(\sigma) = \tilde{N}(\sigma, T) := \sum_{\chi} N(\sigma, 2T, \chi) \ll \big(e^{162\mathscr{L}}T^{81n_K+162}\big)^{(1-\sigma)}$$

for $\frac{1}{2} \le \sigma \le 1, T \ge 1$, and $\mathscr{L}$ sufficiently large. Thus, denoting $B' = B - 2\ell A$, it follows by partial summation that

$$\begin{aligned}
&\sum_{\chi}\sum_{\substack{\rho \\ T \le |\mathrm{Im}\{\rho\}| \le 2T}} |F((1-\rho)\mathscr{L})| \\
&\quad \ll \Big(\frac{2}{AT\mathscr{L}}\Big)^{2\ell}\int_1^{1/2} e^{-B'(1-\sigma)\mathscr{L}}\, d\tilde{N}(\sigma) \\
&\quad \ll \Big(\frac{2}{AT\mathscr{L}}\Big)^{2\ell}\Big[\tilde{N}(1/2)e^{-B'\mathscr{L}/2} + B'\mathscr{L}\int_1^{1/2} e^{-(B'-162)(1-\sigma)\mathscr{L}}T^{(81n_K+162)(1-\sigma)}d\sigma\Big] \\
&\quad \ll \Big(\frac{2}{AT\mathscr{L}}\Big)^{2\ell}\Big[e^{-(B'-162)\mathscr{L}/2}T^{40.5n_K+81} + 1\Big] \\
&\quad \ll \Big(\frac{2}{A\mathscr{L}}\Big)^{2\ell}T^{40.5n_K+81-2\ell}
\end{aligned}$$

since $B' > 162$. Note we have used that the zeros of $\prod_{\chi} L(s,\chi,L/K)$ are symmetric across the critical line $\mathrm{Re}\{s\} = 1/2$. Overall, since $\ell > \frac{81n_K+162}{4}$, we deduce that

$$Z \ll \Big(\frac{2}{A\mathscr{L}}\Big)^{2\ell}T_\star^{40.5n_K+81-2\ell}\sum_{k=0}^{\infty}(2^k)^{40.5n_K+81-2\ell} \ll \Big(\frac{2}{AT_\star\mathscr{L}}\Big)^{2\ell}T_\star^{40.5n_K+81}.$$

□

Next, we further restrict the sum over zeros in Lemma 7.3.9 to zeros $\rho$ close to the line $\mathrm{Re}\{s\} = 1$. To simplify the statement, we also select parameters $\ell$ and $A$ for the weight function.

**Lemma 7.3.10.** *Let $T_\star \geq 1, \eta \in (0,1)$ and $1 \leq R \leq \mathscr{L}$ be arbitrary. Suppose*

$$B - 2\ell A > 162, \qquad A = \frac{4}{\mathscr{L}}, \qquad \ell = \lfloor \eta\mathscr{L} \rfloor. \tag{7.29}$$

*If $\mathscr{L}$ is sufficiently large depending only on $T_\star$ and $\eta$ then*

$$\begin{aligned}\Big|\tfrac{|G|}{|C|}\mathscr{L}^{-1}S - F(0) + \overline{\chi_1}(g_C)F((1-\rho_1)\mathscr{L})\Big| &\leq \sum_{\chi}\sideset{}{^\star}\sum_{\rho}|F((1-\rho)\mathscr{L})| \\ &\quad + O(e^{-(B-2\ell A-162)R} + (2T_\star)^{-2\eta\mathscr{L}}e^{\eta\mathscr{L}} + e^{-78\mathscr{L}}),\end{aligned}$$

*where the marked sum $\sum^\star$ runs over zeros $\rho \neq \rho_1$ of $L(s,\chi,L/K)$, counting with multiplicity, satisfying $1 - \frac{R}{\mathscr{L}} < \mathrm{Re}\{\rho\} < 1$ and $|\mathrm{Im}\{\rho\}| \leq T_\star$.*

*Proof.* For $\mathscr{L}$ sufficiently large depending on $\eta$, the quantities $B, A$ and $\ell$ satisfy the assumptions of Lemma 7.3.9. Denote $B' = B - 2\ell A$. We claim it suffices to show

$$\sum_{\chi}\sideset{}{'}\sum_{\mathrm{Re}\{\rho\}\leq 1-R/\mathscr{L}}|F((1-\rho)\mathscr{L})| \ll e^{-(B'-162)R}, \tag{7.30}$$

where $\sum'$ is defined in Lemma 7.3.9. To see the claim, we need only show that the error term in Lemma 7.3.9 is absorbed by that of Lemma 7.3.10. For $\mathscr{L}$ sufficiently large, notice $T_\star^{40.5n_K} \leq e^{\eta\mathscr{L}}$ as $n_K \log T_\star = o(\mathscr{L})$. Hence, for our choices of $A$ and $\ell$, we have that

$$\Big(\frac{2}{AT_\star\mathscr{L}}\Big)^{2\ell}T_\star^{40.5n_K+81} \leq \Big(\frac{1}{2T_\star}\Big)^{2\eta\mathscr{L}}e^{\eta\mathscr{L}}.$$

This proves the claim. Now, to establish (7.30), define the multiset of zeros

$$\mathcal{R}_m(\chi) := \big\{\rho : L(\rho,\chi) = 0, \quad 1 - \tfrac{m+1}{\mathscr{L}} \leq \mathrm{Re}\{\rho\} \leq 1 - \tfrac{m}{\mathscr{L}}, \quad |\mathrm{Im}(\rho)| \leq T_\star\big\}$$

for $1 \leq m \leq \mathscr{L}$. By Theorem 7.3.6 and Lemma 7.1.1, it follows that

$$\sum_{\chi}\sum_{\rho\in R_m(\chi)}|F((1-\rho)\mathscr{L})| \leq e^{-B'm}\sum_{\chi}\#\mathcal{R}_m(\chi) \ll e^{-(B'-162)m}$$

for $\mathscr{L}$ sufficiently large depending on $T_\star$. Summing over $m \geq R$ yields the desired conclusion. $\square$

Next, we proceed to the final arguments for the proof of Theorem 1.3.2 by dividing into cases depending on whether $\lambda_1 \geq 0.0875$ or not.

### 7.3.3 Non-exceptional case ($\lambda_1 \geq 0.0875$)

For this subsection, we assume $\lambda_1 \geq 0.0875$. Thus, we have no additional information as to whether $\rho_1$ is real or not, or whether $\chi_1$ is real or not.

Recall $\eta > 0$ is arbitrary and sufficiently small, say $\eta < 10^{-3}$ at least. Assume $\mathscr{L}$ is sufficiently large, depending only on $\eta$; we will frequently use this fact throughout this subsection without further mention. Suppose

$$B \geq 693.5, \qquad \ell = \lfloor \eta\mathscr{L} \rfloor, \qquad \text{and} \qquad A = \frac{4}{\mathscr{L}}.$$

Thus $B, \ell$, and $A$ satisfy (7.36) and $B' := B - 2\ell A > 693$. By Lemma 7.3.8, establishing Theorem 1.3.2 in the non-exceptional case is therefore reduced to verifying $\frac{|G|}{|C|}\mathscr{L}^{-1}S \gg 1$. Now, assume the fixed parameter $\lambda^\star > 0$ satisfies

$$\lambda^\star < \min\{\lambda', \lambda_2\},$$

where $\lambda'$ and $\lambda_2$ are defined in Section 7.3.1 with $T_\star = 1$. For a non-trivial zero $\rho$ of a Hecke $L$-function, as usual, write $\rho = \beta + i\gamma = (1 - \frac{\lambda}{\mathscr{L}}) + i\frac{\mu}{\mathscr{L}}$. Let $m(\rho_1) = 1$ if $\rho_1$ is real and $m(\rho_2) = 2$ if $\rho_1$ is complex. Thus, from Lemma 7.3.10 with $T_\star = 1$ and $R = R(\eta) \geq 1$ sufficiently large, it follows that

$$\frac{|G|}{|C|}\mathscr{L}^{-1}S \geq 1 - m(\rho_1)|F((1-\rho_1)\mathscr{L})| - \sum_{\chi}\sum_{\rho}{}^{\dagger} |F((1-\rho)\mathscr{L})| - \eta,$$

where the marked sum $\sum^\dagger$ runs over non-trivial zeros $\rho \neq \rho_1$ (or $\rho \neq \rho_1, \overline{\rho_1}$ if $\rho_1$ is complex) of $L(s,\chi)$, counted with multiplicity, satisfying $\lambda^\star \leq \lambda \leq R$ and $|\gamma| \leq 1$. Note we have used that $|F((1-\rho_1)\mathscr{L})| = |F((1-\overline{\rho_1})\mathscr{L})|$. By Lemma 7.1.1, this implies that

$$\frac{|G|}{|C|}\mathscr{L}^{-1}S \geq 1 - 2e^{-B'\lambda_1} - \sum_{\chi} \sum_{\substack{\lambda^\star \leq \lambda \leq R \\ |\gamma| \leq 1}} e^{-B'\lambda} - \eta. \tag{7.31}$$

Let $\Lambda > 0$ be a fixed parameter to be specified later. To bound the remaining sum over zeros, we will apply partial summation using the quantity $\mathcal{N}(\lambda)$, defined in (7.27), over two different ranges: (i) $\lambda^\star \leq \lambda \leq \Lambda$ and (ii) $\Lambda < \lambda \leq R$.

For (i), partition the interval $[\lambda^\star, \Lambda]$ into $M$ subintervals with sample points

$$\lambda^\star = \Lambda_0 < \Lambda_1 < \Lambda_2 < \cdots < \Lambda_M = \Lambda.$$

By partial summation, we see

$$\begin{aligned}\sum_{\chi} \sum_{\substack{\lambda^\star < \lambda \leq \Lambda \\ |\gamma| \leq 1}} e^{-B'\lambda} &= \sum_{j=1}^{M} \sum_{\chi} \sum_{\Lambda_{j-1} < \lambda \leq \Lambda_j} e^{-B'\lambda} \\ &\leq e^{-B'\Lambda_{M-1}} \mathcal{N}(\Lambda_M) + \sum_{j=1}^{M-1} \big( e^{-B'\Lambda_{j-1}} - e^{-B'\Lambda_j} \big) \mathcal{N}(\Lambda_j) =: Z_1,\end{aligned}$$

say. By Theorem 7.3.2, we may choose $\lambda^\star = 0.2866$. Furthermore, select

$$\Lambda = 1, \qquad M = 32, \qquad \Lambda_r = \begin{cases} 0.286 + 0.001r & 1 \leq r \leq 14, \\ 0.300 + 0.025(r - 14) & 15 \leq r \leq 22, \\ 0.5 + 0.05(r - 22) & 23 \leq r \leq 32. \end{cases}$$

By Theorem 7.3.7, we may use Table 5.1 to bound $\mathcal{N}(\,\cdot\,)$, yielding $Z_1 \leq 0.9926$.

For (ii), apply partial summation along with Theorem 7.3.7. Since $B' \geq 693 > 162$ and $R = R(\eta)$ is sufficiently large, it follows that

$$\begin{aligned}\sum_{\chi} \sum_{\substack{\Lambda < \lambda \leq R \\ |\gamma| \leq 1}} e^{-B'\lambda} &\leq \int_{\Lambda}^{R} e^{-B'\lambda} d\mathcal{N}(\lambda) \\ &\leq e^{-(B'-162)R+188} + \int_{\Lambda}^{\infty} B' e^{-(B'-162)\lambda+188} d\lambda \\ &\leq \frac{B'}{B'-162} e^{188-(B'-162)\Lambda} + \eta =: Z_2 + \eta,\end{aligned}$$

say. Evaluating the right hand side with $B' \geq 693$ and $\Lambda = 1$, we deduce $Z_2 \leq e^{-300}$. Incorporating (i) and (ii) into (7.31), we see that

$$\frac{|G|}{|C|} \mathscr{L}^{-1} S \geq 1 - 2e^{-B'\lambda_1} - 0.9926 - e^{-300} - 2\eta \geq 0.0073 - 2\eta,$$

as $\lambda_1 > 0.0875$ and $B' \geq 693$. Since $\eta < 10^{-3}$, we conclude $\frac{|G|}{|C|} \mathscr{L}^{-1} S \gg 1$ as desired. This completes the proof of Theorem 1.3.2 in the non-exceptional case. $\square$

### 7.3.4 Exceptional case ($\lambda_1 < 0.0875$)

For this subsection, we assume $\lambda_1 < 0.0875$, in which case $\rho_1$ is an exceptional real zero by Theorem 7.3.2. Thus, $\rho_1$ is a simple real zero and $\chi_1$ is a real Hecke character. Recall $\eta > 0$ is arbitrary and sufficiently small, say $\eta < 10^{-3}$ at least. Assume $\mathscr{L}$ is sufficiently large, depending only on $\eta$; we will frequently use this fact throughout this subsection without further mention. We will proceed in a similar fashion as the non-exceptional case, but need a less refined analysis due to the strength of the Deuring-Heilbronn phenomenon. Suppose

$$B \geq 163, \qquad \ell = \lfloor \eta \mathscr{L} \rfloor, \quad \text{and} \quad A = \frac{4}{\mathscr{L}}.$$

Thus, $B, \ell,$ and $A$ satisfy (7.36) and $B' := B - 2\ell A > 162$. For the moment, we do not make any additional assumptions on the minimum size of $B$ and hence $B'$. To prove Theorem 1.3.2 when $\rho_1$ is an exceptional zero, it suffices to show, by Lemma 7.3.8, that $\frac{|G|}{|C|}\mathscr{L}^{-1}S \gg_\eta \min\{1, \lambda_1\}$ for $B \geq 593$ and $\mathscr{L}$ sufficiently large depending on $\eta$.

For a non-trivial zero $\rho$ of a Hecke $L$-function, write $\rho = \beta + i\gamma = (1 - \frac{\lambda}{\mathscr{L}}) + i\gamma$ so by Lemma 7.1.1, $|F((1-\rho)\mathscr{L})| \leq e^{-B'\lambda}$. From Lemma 7.3.10 with $T_\star \geq 1$ and $1 \leq R \leq \mathscr{L}$ arbitrary, it follows that, if we define

$$\Delta = \begin{cases} \eta & \text{when } T_\star = 1 \text{ and } R = R(\eta) \text{ is sufficiently large,} \\ O(e^{-(B'-162)R} + e^{-78\mathscr{L}}) & \text{when } T_\star = T_\star(\eta) \text{ is sufficiently large and } 1 \leq R \leq \mathscr{L}, \end{cases} \tag{7.32}$$

then

$$\frac{|G|}{|C|}\mathscr{L}^{-1}S \geq 1 - \chi_1(g_C)e^{-B'\lambda_1} - \sum_{\chi} \sum_{\rho}{}^{\star} e^{-B'\lambda} - \Delta, \tag{7.33}$$

where the restricted sum $\sum^\star$ is over zeros $\rho \neq \rho_1$, counted with multiplicity, satisfying $0 < \lambda \leq R$ and $|\gamma| \leq T_\star$. Suppose the arbitrary parameter $\lambda^\star > 0$ satisfies

$$\lambda > \lambda^\star \quad \text{for every zero } \rho \text{ occurring in the restricted sum of (7.33).} \tag{7.34}$$

It remains for us to divide into cases according to the range of $\lambda_1$ and value of $\chi_1(g_C) \in \{\pm 1\}$. In each case, we make a suitable choice for $\lambda^\star$.

**Moderate exceptional zero ($\eta \leq \lambda_1 < 0.0875$ or $\chi_1(g_C) = -1$)**

For the moment, we do not make any assumptions on the size of $\lambda_1$ other than $0 < \lambda_1 < 0.0875$. Select $T_\star = 1$ and $R = R(\eta)$ sufficiently large so $\Delta = \eta$ according to (7.32). By partial summation, our choice of $\lambda^\star$ in (7.33), and Theorem 7.3.7, it follows that

$$\sum_{\chi} \sideset{}{^\star}\sum_{\rho} e^{-B'\lambda} \le \int_{\lambda^\star}^{R} e^{-B'\lambda} d\mathcal{N}(\lambda) \le e^{-(B'-162)R+188} + \int_{\lambda^\star}^{\infty} B' e^{-(B'-162)\lambda+188} d\lambda.$$

As $R = R(\eta)$ is sufficiently large and $B' > 162$, the above is $\le \frac{B'}{B'-162} e^{188-(B'-162)\lambda^\star} + \eta$. Comparing with (7.33), we have that

$$\frac{|G|}{|C|}\mathscr{L}^{-1}S \ge 1 - \chi_1(g_C)e^{-B'\lambda_1} - \tfrac{B'}{B'-162}e^{-(B'-162)\lambda^\star+188} - 2\eta. \tag{7.35}$$

Finally, we further subdivide into cases according to the size of $\lambda_1$ and value of $\chi_1(g_C) \in \{\pm 1\}$. Recall $\eta > 0$ is sufficiently small and arbitrary.

$\lambda_1$ *medium* ($10^{-3} \le \lambda_1 < 0.0875$). Here we assume $B \ge 593$ in which case $B' \ge 592$. Select $\lambda^\star = 0.44$ which, by Theorem 7.3.3, satisfies (7.34) for the specified range of $\lambda_1$. Substituting this estimate in (7.35) and noting $|\chi_1(g_C)| \le 1$, we deduce that

$$\frac{|G|}{|C|}\mathscr{L}^{-1}S \ge 1 - e^{-592\times 10^{-3}} - \frac{592}{430}e^{-430\times 0.44+188} - 2\eta \ge 0.032 - 2\eta$$

for $\lambda \in [10^{-3}, 0.0875]$. Hence, for $\eta$ sufficiently small, $\frac{|G|}{|C|}\mathscr{L}^{-1}S \gg 1$ in this subcase, as desired.

$\lambda_1$ *small* ($\eta \le \lambda_1 < 10^{-3}$). Here we assume $B \ge 297$ in which case $B' \ge 296.5$. Select $\lambda^\star = 0.2103\log(1/\lambda_1)$, which, by Theorem 7.3.3, satisfies (7.34). For $\lambda < 10^{-3}$, this implies $\lambda^\star > 1.45$. Applying both of these facts in (7.35) and noting $|\chi_1(g_C)| \le 1$, we see

$$\frac{|G|}{|C|}\mathscr{L}^{-1}S \ge 1 - e^{-296.5\lambda_1} - \frac{296}{134}e^{-(134.5-188/1.45)\lambda^\star} - 2\eta \ge 1 - e^{-296.5\lambda_1} - \frac{296}{134}\lambda_1 - 2\eta,$$

since $4.84 \times 0.2103 = 1.017\cdots > 1$. As $1 - e^{-x} \ge x - x^2/2$ for $x \ge 0$, the above is

$$\ge 296.5\lambda_1 - \frac{(296.5)^2}{2}\lambda_1^2 - \frac{296}{134}\lambda_1 - 2\eta \ge 294.2\lambda_1\big(1 - 150\lambda_1\big) - 2\eta \ge 250\eta,$$

because $\eta \le \lambda_1 < 10^{-3}$. Therefore, $\frac{|G|}{|C|}\mathscr{L}^{-1}S \gg_\eta 1$ completing the proof of this subcase.

$\lambda_1$ *very small* ($\lambda_1 < \eta$) *and* $\chi_1(g_C) = -1$. Here we also assume $B \ge 163$ in which case

$B' > 162.5$. From (7.35), it follows that

$$\frac{|G|}{|C|}\mathscr{L}^{-1}S \geq 1 + e^{-162.5\lambda_1} - 325e^{-0.5\lambda^\star+188} - 2\eta \geq 2 - O\big(e^{-0.5\lambda^\star} + \eta + \lambda_1\big).$$

By Theorem 7.3.4, the choice $\lambda^\star = \frac{1}{81}\log(c_1/\lambda_1)$ satisfies (7.34) for some absolute constant $c_1 > 0$. Since $\lambda_1 < \eta$, the above is therefore $\geq 2 - O(\eta^{0.5/81} + \eta) \geq 2 - O(\eta^{1/162})$. As $\eta$ is sufficiently small, we conclude $\frac{|G|}{|C|}\mathscr{L}^{-1}S \gg 1$ as desired. This completes the proof for a "moderate" exceptional zero.

**Truly exceptional zero ($\lambda_1 < \eta$ and $\chi_1(g_C) = 1$)**

Select $T_\star = T_\star(\eta)$ sufficiently large and let $R = \frac{1}{80.1}\log(c_1/\lambda_1)$, where $c_1 > 0$ is a sufficiently small absolute constant. By Theorem 7.3.4, it follows that the restricted sum over zeros $\rho$ in (7.33) is empty and therefore by (7.33) and (7.32),

$$\frac{|G|}{|C|}\mathscr{L}^{-1}S \geq 1 - e^{-B'\lambda_1} - O(\lambda_1^{(B'-162)/80.1} + e^{-78\mathscr{L}})$$

as $\chi_1(g_C) = 1$. Additionally assuming $B \geq 243$ in which case $B' \geq 242.2$ and noting $1 - e^{-x} \geq x - x^2/2$ for $x \geq 0$, we conclude that

$$\frac{|G|}{|C|}\mathscr{L}^{-1}S \geq 242.2\lambda_1 - O(\lambda_1^2 + \lambda_1^{80.2/80.1} + e^{-78\mathscr{L}}) \geq \lambda_1(242.2 - O(\lambda_1^{0.001} + e^{-73\mathscr{L}})).$$

In the last inequality, we use that $\lambda_1 \gg e^{-4.8\mathscr{L}}$ by Theorem 7.3.5. As $\lambda_1 \leq \eta$ for $\eta > 0$ sufficiently small, we conclude $\frac{|G|}{|C|}\mathscr{L}^{-1}S \gg \lambda_1$ as desired.

Comparing all subcases of the exceptional case, we see that the most stringent condition is $B \geq 593$. By Lemma 7.3.8, this completes the proof of Theorem 1.3.2. □

**Remark.** The "truly exceptional" subcase is analogously considered in Chapter 9 in the language of ray class groups. In particular, when $L/K$ corresponds to a *primitive* congruence class group $H$ of $K$, this subcase is implied by Theorems 9.1.1 and 9.1.2 which are numerically much stronger results and use entirely different methods.

## 7.4 Absolutely bounded degree

In this section, we improve upon Theorem 1.3.2 when $n_K = [K : \mathbb{Q}]$ is uniformly bounded by an absolute constant. We will proceed as in Section 7.3 and establish the following theorem.

**Theorem 7.4.1.** *Let $L/F$ be a Galois extension of number fields with Galois group $G$ and let $C \subset G$ be a conjugacy class. Let $H \subset G$ be an abelian subgroup such that $H \cap C$ is nonempty, $K = L^H$ be the fixed field of $H$, and $\mathcal{Q} = \mathcal{Q}(L/K)$ be defined by* (1.22)*. If $n_K \leq 10^{49}$ then*

$$\pi_C(x, L/F) \gg \frac{1}{D_K^5 \mathcal{Q}^4} \frac{|C|}{|G|} \frac{x}{\log x}$$

*for $x \geq D_K^{455} \mathcal{Q}^{342}$ and $D_K \mathcal{Q}$ sufficiently large. In particular,*

$$P(C, L/F) \ll D_K^{455} \mathcal{Q}^{342}.$$

**Remark.** Our primary goal was to minimize the exponents of $D_K$ and $\mathcal{Q}$. Our secondary goal was to allow the largest range of $n_K$ possible without sacrificing the exponents of $D_K$ and $\mathcal{Q}$.

This section is dedicated to proving Theorem 7.4.1. We will assume all of the contents and notation from Sections 7.1 and 7.3.

### 7.4.1 Additional preliminaries

For $\lambda > 0$, define $N(\lambda) = N(\lambda, L/K)$ to be the number of Hecke characters $\chi$ attached to $L/K$ such that $L(s, \chi, L/K)$ has a zero in the region

$$\mathrm{Re}\{s\} > 1 - \frac{\lambda}{\mathscr{L}}, \qquad |\mathrm{Im}\{s\}| \leq T_\star.$$

Recall $\mathscr{L}$ is given by (7.21) and $T_\star \geq 1$ is arbitrary. From Theorem 7.3.2, we have that $N(0.0875) \leq 1$ and $N(0.2866) \leq 2$. The source of our improvement will be our ability to use $N(\lambda)$ in place of $\mathcal{N}(\lambda)$ from Theorem 7.3.7 when $\lambda$ is small.

**Lemma 7.4.2.** *If $\mathscr{L}$ is sufficiently large depending on $T_\star$ then $N(0.569) \leq 3365$.*

*Proof.* This follows from Corollary 4.5.3 since $\mathscr{L} \geq \mathcal{L}$ for $\mathscr{L}$ sufficiently large. Here, $\mathcal{L}$ is given by (3.3). □

We reduce the proof of Theorem 7.4.1 to verifying the following lemma.

**Lemma 7.4.3.** *Let $\eta > 0$ be sufficiently small and arbitrary. Assume $n_K \leq 10^{49}$ and $\mathscr{L}$ is sufficiently large depending only on $\eta > 0$. Let $A = \frac{1}{4\ell}$ and $\ell = \lceil \frac{81n+162}{4} \rceil + 1$. If every $B \geq 454$ defining* (7.4) *implies $\frac{|G|}{|C|}\mathscr{L}^{-1} S \gg_\eta \min\{1, \lambda_1\}$ then*

$$\pi_C(x, L/F) \gg \frac{1}{D_K^5 \mathcal{Q}^4} \frac{|C|}{|G|} \frac{x}{\log x}$$

*for $x \geq D_K^{455}\mathcal{Q}^{342}$ and $D_K\mathcal{Q}$ sufficiently large.*

*Proof.* The argument is similar to Lemma 7.3.8. □

### 7.4.2 A sum over low-lying zeros

We will require Lemma 7.3.9 as well as two additional lemmas. One lemma addresses the low-lying zeros for our new choices of $A$ and $\ell$.

**Lemma 7.4.4.** *Let $T_\star \geq 1, \eta \in (0,1)$ and $1 \leq R \leq \mathscr{L}$ be arbitrary. Suppose*

$$B - 2\ell A > 162, \qquad A = \frac{1}{4\ell}, \qquad \ell = \lceil \frac{81n_K + 162}{4} \rceil + 1. \tag{7.36}$$

*If $\mathscr{L}$ is sufficiently large depending only on $T_\star$ and $\eta$ then*

$$\begin{aligned}\Big|\tfrac{|G|}{|C|}\mathscr{L}^{-1}S - F(0) + \overline{\chi_1}(g_C)F((1-\rho_1)\mathscr{L})\Big| \leq \sum_{\chi}\sum_{\rho}{}^{\star} |F((1-\rho)\mathscr{L})| \\ + O_{n_K}(e^{-(B-2\ell A-162)R} + \mathscr{L}^{-122}),\end{aligned}$$

*where the marked sum $\sum^\star$ runs over zeros $\rho \neq \rho_1$ of $L(s,\chi,L/K)$, counting with multiplicity, satisfying $1 - \frac{R}{\mathscr{L}} < \mathrm{Re}\{\rho\} < 1$ and $|\mathrm{Im}\{\rho\}| \leq T_\star$.*

*Proof.* This result is motivated by Lemma 7.3.10, so the arguments are similar. For $\mathscr{L}$ sufficiently large depending on $\eta$, the quantities $B, A$ and $\ell$ satisfy the assumptions of Lemma 7.3.9. Denote $B' = B - 2\ell A$. First, the bound (7.30) is established exactly as in Lemma 7.3.9. Thus, we again need only show that the error term in Lemma 7.3.9 is bounded by $\mathcal{L}^{-122}$. For $\mathscr{L}$ sufficiently large depending on $T_\star$, notice by our choices of $A$ and $\ell$ that

$$\Big(\frac{2}{AT_\star\mathscr{L}}\Big)^{2\ell} T_\star^{40.5n_K+81} \ll_{n_K} \Big(\frac{1}{T_\star\mathscr{L}}\Big)^{40.5n_K+82} T_\star^{40.5n_K+81} \ll_{n_K} \mathscr{L}^{-122},$$

as desired. □

The second lemma addresses the sum over zeros for a single Hecke character. There is no corresponding lemma in Section 7.3.2.

**Lemma 7.4.5.** *Let $\eta > 0, T_\star \geq 1$ and $R \geq 1$ be arbitrary. Let $\chi$ be a Hecke character attached to $L/K$. For $A > 0$ and $\ell \geq 1$ arbitrary, define*

$$\tilde{F}_\ell(z) := \Big(\frac{1 - e^{-Az}}{Az}\Big)^{2\ell}. \tag{7.37}$$

*Suppose $L(s, \chi, L/K)$ is non-zero in the region*

$$\mathrm{Re}\{s\} \geq 1 - \frac{\lambda}{\mathscr{L}}, \qquad |\mathrm{Im}\{s\}| \leq T_\star$$

*for some $0 < \lambda \leq 10$. Then, provided $\mathscr{L}$ is sufficiently large depending on $\eta, T_\star, R$, and $A$,*

$$\sum_{\rho}{}' |\tilde{F}_\ell((1-\rho)\mathscr{L})| \leq \Big(\frac{1-e^{-A\lambda}}{A\lambda}\Big)^{2(\ell-1)} \cdot \Big\{\phi\Big(\frac{1-e^{-2A\lambda}}{A^2\lambda}\Big) + \frac{2A\lambda - 1 + e^{-2A\lambda}}{2A^2\lambda^2} + \eta\Big\} \quad (7.38)$$

*where $\phi = 1/4$ and the marked sum $\sum'$ indicates a restriction to zeros $\rho = \beta + i\gamma$ of $\zeta_L(s)$ satisfying*

$$\beta \geq 1 - \frac{R}{\mathscr{L}}, \qquad |\gamma| \leq T_\star.$$

*In particular, as $\lambda \to 0$, the bound in* (7.38) *becomes $\frac{2\phi}{A} + 1 + \eta$.*

*Proof.* This result is motivated by [HB92, Lemma 13.3] and Lemma 7.2.6. The arguments are analogous except we replace the application of [KN12, Theorem 3] with Proposition 3.4.2. □

Lemma 7.4.5 poses two issues whereby we must fix the degree $n_K$ to obtain improvements over Theorem 1.3.2. First, it has a condition that $\mathscr{L}$ is sufficiently large depending on $A$ (amongst other quantities). Second, the quality of the bound in (7.4.5) is $O(1/A)$. Thus, if $A$ is not uniformly bounded below by an absolute constant, then we cannot apply Lemma 7.4.5 to obtain a degree uniform result like Theorem 1.3.2. However, our arguments seem to force $A \ll n_K^{-1}$. This is because $B - 2\ell A > 162$ implies $\ell A \ll 1$ to permit bounded values of $B$. Since $\ell \gg n_K$ is a seemingly necessary condition due to applications of the log-free zero density estimate (e.g. Theorem 7.3.6 and its relatives), it follows that one must impose $A \ll \ell^{-1} \ll n_K^{-1}$. If one could circumvent these two issues, then we expect the quality of exponents in Theorem 7.4.1 to carry over to Theorem 1.3.2. We did not attempt to pursue such a strategy.

### 7.4.3 Proof of Theorem 7.4.1

For the entirety of this subsection, assume

$$1 \leq n_K \leq 10^{49}.$$

Recall the definition of $\rho_1$ in Section 7.3.1. From the proof of Theorem 1.3.2 in Sections 7.3.3 and 7.3.4, it follows that we need only consider the cases $\lambda_1$ large ($\lambda_1 \geq 0.0875$) and $\lambda_1$

medium ($10^{-3} \leq \lambda_1 < 0.0875$). In all other cases, the values of $B$ are shown to be small enough. Thus, we assume $\lambda_1 \geq 10^{-3}$.

Recall $\eta > 0$ is arbitrary and sufficiently small, say $\eta < 10^{-3}$ at least. Assume $\mathscr{L}$ is sufficiently large, depending only on $\eta$; we will frequently use this fact throughout this subsection without further mention. Suppose

$$B \geq 164, \qquad \ell = \left\lceil \frac{81 n_K + 162}{4} \right\rceil + 1, \quad \text{and} \quad A = \frac{1}{4\ell}.$$

Thus $B, \ell$, and $A$ satisfy (7.36) and $B' := B - 2\ell A > 163$. For the moment, we do not make any additional assumptions on the minimum size of $B$ and hence $B'$. By Lemma 7.4.3, establishing Theorem 1.3.2 when $\lambda_1 \geq 10^{-3}$ is therefore reduced to verifying $\frac{|G|}{|C|}\mathscr{L}^{-1}S \gg 1$. Now, assume the fixed parameter $\lambda^\star > 0$ satisfies

$$\lambda^\star < \min\{\lambda', \lambda_2\},$$

where $\lambda'$ and $\lambda_2$ are defined in Section 7.3.1 with $T_\star = 1$. For a non-trivial zero $\rho_\chi$ of a Hecke $L$-function $L(s,\chi)$, as usual, write $\rho_\chi = \rho = \beta + i\gamma = (1 - \frac{\lambda}{\mathscr{L}}) + i\frac{\mu}{\mathscr{L}}$. Let $m(\rho_1) = 1$ if $\rho_1$ is real and $m(\rho_2) = 2$ if $\rho_1$ is complex. Thus, from Lemma 7.4.4 with $T_\star = 1$ and $R = R(\eta) \geq 1$ sufficiently large, it follows that

$$\frac{|G|}{|C|}\mathscr{L}^{-1}S \geq 1 - m(\rho_1)|F((1-\rho_1)\mathscr{L})| - \sum_{\chi}\sum_{\rho_\chi}{}^{\dagger} |F((1-\rho_\chi)\mathscr{L})| - \eta,$$

where the marked sum $\sum^\dagger$ runs over non-trivial zeros $\rho_\chi \neq \rho_1$ (or $\rho_\chi \neq \rho_1, \overline{\rho_1}$ if $\rho_1$ is complex) of $L(s,\chi)$, counted with multiplicity, satisfying $\lambda^\star \leq \lambda \leq R$ and $|\gamma| \leq 1$. Note we have used that $|F((1-\rho_1)\mathscr{L})| = |F((1-\overline{\rho_1})\mathscr{L})|$. For each character $\chi$, consider the corresponding inner sum over zeros. By Lemma 7.1.1 and (7.37), we have that

$$\sum_{\rho_\chi}{}^{\dagger} |F((1-\rho_\chi)\mathscr{L})| \leq \sum_{\rho_\chi}{}^{\dagger} e^{-B'\lambda}|\tilde{F}_\ell((1-\rho_\chi)\mathscr{L})| \leq e^{-B'\lambda^\star}\sum_{\rho}{}^{\dagger} |\tilde{F}_\ell((1-\rho_\chi)\mathscr{L})|.$$

Applying Lemma 7.4.5 (using $\lambda \to 0$) to the remaining sum, it follows that, for any given $\chi$,

$$\sum_{\rho_\chi}{}^{\dagger} |F((1-\rho_\chi)\mathscr{L})| \leq (41 n_K + 84)e^{-B'\lambda^\star},$$

since $\frac{2\phi}{A} + 1 < 41 n_K + 84$ . For $\Lambda^\star > 0$ fixed, let $\mathcal{M}(\Lambda^\star)$ be the set of characters $\chi$ (including the trivial character) with a zero $\rho$ satisfying $\lambda \le \Lambda^\star$ and $|\gamma| \le 1$. Thus, we have shown

$$\sum_{\chi \in \mathcal{M}(\Lambda^\star)} \sum_{\rho_\chi}{}^{\dagger} |F((1-\rho)\mathscr{L})| \le N_0(\Lambda^*) \cdot 166 e^{-B'\lambda^\star},$$

where $N_0(\Lambda^*) = \#\mathcal{M}(\Lambda^*)$. Observe that

$$N_0(\Lambda^*) = \begin{cases} N(\Lambda^*) + 1 & \text{if } \chi_0 \text{ has a zero } \rho \text{ satisfying } \lambda \le \Lambda^* \text{ and } |\gamma| \le 1, \\ N(\Lambda^*) & \text{otherwise,} \end{cases}$$

where $N(\,\cdot\,)$ is defined in Section 7.4.1. By Lemma 7.1.1, this implies that

$$\frac{|G|}{|C|}\mathscr{L}^{-1}S \ge 1 - m(\rho_1)e^{-B'\lambda_1} - Z_0 - \sum_{\chi \notin \mathcal{M}(\Lambda^*)} \sum_{\substack{\Lambda^\star \le \lambda \le R \\ |\gamma| \le 1}} e^{-B'\lambda} - \eta, \tag{7.39}$$

where

$$Z_0 := (41 n_K + 84)\, N_0(\Lambda^*) e^{-B'\lambda^*}.$$

Let $\Lambda \ge \Lambda^\star > 0$ be a fixed parameter to be specified later. To bound the remaining sum over zeros, we follow the proof in Section 7.3.3. Namely, we will apply partial summation using the quantity $\mathcal{N}(\lambda)$, defined in (7.27), over two different ranges: (i) $\Lambda^\star < \lambda \le \Lambda$, and (ii) $\Lambda < \lambda \le R$. For (i), partition the interval $[\Lambda^\star, \Lambda]$ into $M$ subintervals with sample points

$$\Lambda^\star = \Lambda_0 < \Lambda_1 < \Lambda_2 < \cdots < \Lambda_M = \Lambda.$$

By partial summation, we again see

$$\begin{aligned} \sum_{\chi \notin \mathcal{M}(\Lambda^*)} \sum_{\substack{\Lambda^\star < \lambda \le \Lambda \\ |\gamma| \le 1}} e^{-B'\lambda} &= \sum_{j=1}^{M} \sum_{\chi} \sum_{\Lambda_{j-1} < \lambda \le \Lambda_j} e^{-B'\lambda} \\ &\le e^{-B'\Lambda_{M-1}} \mathcal{N}(\Lambda_M) + \sum_{j=1}^{M-1} \big(e^{-B'\Lambda_{j-1}} - e^{-B'\Lambda_j}\big) \mathcal{N}(\Lambda_j) =: Z_1, \end{aligned}$$

say. If $M = 0$, we set $Z_1 = 0$ trivially. For (ii), we similarly apply partial summation along with Theorem 7.3.7. Since $B' > 163$ and $R = R(\eta)$ is sufficiently large, it follows that

$$\sum_{\chi \notin \mathcal{M}(\Lambda^*)} \sum_{\substack{\Lambda \le \lambda \le R \\ |\gamma| \le 1}} e^{-B'\lambda} \le \frac{B'}{B' - 162} e^{188 - (B'-162)\Lambda} + \eta =: Z_2 + \eta,$$

say. Incorporating (i) and (ii) into (7.39), we conclude that

$$\frac{|G|}{|C|}\mathscr{L}^{-1}S \geq 1 - m(\rho_1)e^{-B'\lambda_1} - Z_0 - Z_1 - Z_2 - \eta. \tag{7.40}$$

Finally, we divide into two cases. Recall that we need only consider $\lambda_1 \geq 10^{-3}$.

$\lambda_1$ **large** $(\lambda_1 \geq 10^{-2})$

Here we assume $B \geq 454$ so $B' > 453.5$. By Theorem 7.3.2, we may choose[5] $\lambda^\star = 0.2866$. Furthermore, select

$$\Lambda^\star = 0.569, \qquad \Lambda = 1, \qquad M = 9, \qquad \Lambda_r = 0.55 + 0.05r \quad (1 \leq r \leq 9).$$

By Lemma 7.4.2, it follows that

$$Z_0 \leq (41n_K + 84)\cdot 3366 \cdot e^{-453.5\times 0.2866} \leq e^{-118}n_K + e^{-117}.$$

For $Z_1$, we apply Theorem 7.3.7, using Table 5.1 to bound $\mathcal{N}(\,\cdot\,)$, yielding $Z_1 \leq 0.9649$. For $Z_2$, we see that $Z_2 \leq e^{-100}$. If $\lambda_1 \geq 0.0875$ then $m(\rho_1)e^{-B'\lambda_1} \leq 2e^{-453.5\times 0.0875} \leq 10^{-16}$. Otherwise, if $10^{-2} \leq \lambda_1 < 0.0875$ then, by Theorem 7.3.2, $m(\rho_1) = 1$ implying $m(\rho_1)e^{-B'\lambda_1} \leq e^{-435.5\times 0.01} \leq 0.011$. Combining all of these observations into (7.40), we conclude that

$$\begin{aligned}\frac{|G|}{|C|}\mathscr{L}^{-1}S &\geq 1 - 0.011 - e^{-118}n_K - e^{-117} - 0.9649 - e^{-100} - \eta \\ &\geq 0.008 - \eta\end{aligned}$$

for $n_K \leq 10^{49} \leq e^{114}$. Since $\eta < 10^{-3}$, we conclude $\frac{|G|}{|C|}\mathscr{L}^{-1}S \gg 1$ as desired.

$\lambda_1$ **small** $(10^{-3} \leq \lambda_1 < 10^{-2})$

Here we assume $B \geq 358$ so $B' > 357$. By Theorem 7.3.2, $\rho_1$ is a simple real zero attached to a real Hecke character $\chi_1$ so $m(\rho_1) = 1$. By Theorem 7.3.3, we may take $\lambda^* = 0.968 > 0.2103\log(1/0.01)$. Select

$$\Lambda = \Lambda^\star = \lambda^\star = 0.968, \quad \text{and} \quad M = 0.$$

[5]One could use Theorem 7.3.3 when $10^{-2} \leq \lambda_1 \leq 0.0875$ but this does not seem to lead to any significant improvements in Theorem 7.4.1.

With these choices, we see that $m(\rho_1)e^{-B'\lambda_1} \leq e^{-357\times 0.01} \leq 0.029$. Moreover, $Z_0 \leq 10^{-5}$ for $n_K \leq e^{320}$, $Z_1 = 0$ by definition, and $Z_2 \leq 0.8562$. Thus,

$$\begin{aligned}\frac{|G|}{|C|}\mathscr{L}^{-1}S &\geq 1 - 0.029 - 10^{-5} - 0 - 0.8562 - \eta \\ &\geq 0.11 - \eta.\end{aligned}$$

Since $\eta < 10^{-3}$, we conclude $\frac{|G|}{|C|}\mathscr{L}^{-1}S \gg 1$ as desired. This completes the proof of Theorem 7.4.1 in all cases. $\square$

# Chapter 8

# Brun–Titchmarsh

*"That is brand new information!"*

– Phoebe Buffay, Friends.

Throughout this chapter, let $L/F$ be a Galois extension of number fields with Galois group $G := \mathrm{Gal}(L/F)$ and let $C$ be a conjugacy class of $G$. Our aim is to upper bound

$$\pi_C(x, L/F) = \#\{\mathfrak{p} : \mathrm{N}\mathfrak{p} < x, \mathfrak{p} \text{ prime ideal of } F \text{ unramified in } L, \left[\frac{L/F}{\mathfrak{p}}\right] = C\},$$

where $\mathrm{N} = \mathrm{N}^F_{\mathbb{Q}}$ is the absolute norm of $F$. This chapter contains the proofs of Theorems 1.3.3 and 1.3.4, which are two of the main results of this thesis. The material here has substantive intersections and connections with Section 7.3 and can be viewed as a complementary perspective. Again, notation from Section 2.5 will be used throughout this chapter.

## 8.1 Setup

### 8.1.1 Choice of weight

Let us define a weight function and describe its properties. It will be used to count prime ideals with norm between $x^{1/2}$ and $x$.

**Lemma 8.1.1.** *For any $x \geq 3, \epsilon \in (0, 1/4)$, and positive integer $\ell \geq 1$, select*

$$A = \frac{\epsilon}{2\ell \log x}.$$

*There exists a real-variable function $f(t) = f(t; x, \ell, \epsilon)$ such that:*

*(i)* $0 \leq f(t) \leq 1$ *for all $t \in \mathbb{R}$, and $f(t) \equiv 1$ for $\frac{1}{2} \leq t \leq 1$.*

*(ii) The support of $f$ is contained in the interval $[\frac{1}{2} - \frac{\epsilon}{\log x}, 1 + \frac{\epsilon}{\log x}]$.*

*(iii) Its Laplace transform $F(z) = \int_{\mathbb{R}} f(t)e^{-zt}dt$ is entire and is given by*

$$F(z) = e^{-(1+2\ell A)z} \cdot \Big(\frac{1 - e^{(\frac{1}{2}+2\ell A)z}}{-z}\Big)\Big(\frac{1 - e^{2Az}}{-2Az}\Big)^{\ell}. \tag{8.1}$$

*(iv) Let $s = \sigma + it \in \mathbb{C}, \sigma > 0$ and $\alpha$ be any real number satisfying $0 \leq \alpha \leq \ell$. Then*

$$|F(-s \log x)| \leq \frac{e^{\sigma\epsilon}x^{\sigma}}{|s| \log x} \cdot \big(1 + x^{-\sigma/2}\big) \cdot \Big(\frac{2\ell}{\epsilon|s|}\Big)^{\alpha}.$$

*(v) If $s = \sigma + it \in \mathbb{C}$ and $\sigma > 0$, then*

$$|F(-s \log x)| \leq e^{\sigma\epsilon}x^{\sigma}.$$

*Moreover,*

$$1/2 < F(0) < 3/4, \qquad F(-\sigma \log x) \leq \frac{e^{\epsilon}x^{\sigma}}{\sigma \log x}.$$

*(vi) Let $s = -\frac{1}{2} + it \in \mathbb{C}$. Then*

$$|F(-s \log x)| \leq \frac{5x^{-1/4}}{\log x}\Big(\frac{2\ell}{\epsilon}\Big)^{\ell}(1/4 + t^2)^{-\ell/2}.$$

**Remark.** This choice of weight can be regarded as a smoothed version of Maynard's weight [May13, Equation (5.6)]. It is motivated by the choice of weight in Chapter 7 on the least prime ideal. See the remark following Lemma 7.1.1 for details.

*Proof.*

- For parts (i) and (ii), let $\mathbf{1}_S(\,\cdot\,)$ be an indicator function for the set $S \subseteq \mathbb{R}$. For $j \geq 1$, define

  $$w(t) := \frac{1}{2A}\mathbf{1}_{[-A,A]}(t), \quad g_0(t) := \mathbf{1}_{[\frac{1}{2}-\ell A, 1+\ell A]}(t), \quad \text{and} \quad g_j(t) := (w * g_{j-1})(t).$$

  Since $\int_{\mathbb{R}} w(t)dt = 1$, one can verify that $f = g_\ell$ satisfies (i) and (ii).

- For part (iii), observe the Laplace transform $W(z)$ of $w$ is given by

  $$W(z) = \frac{e^{Az} - e^{-Az}}{2Az} = e^{-Az} \cdot \Big(\frac{1 - e^{2Az}}{-2Az}\Big),$$

and the Laplace transform $G_0(z)$ of $g_0$ is given by

$$G_0(z) = \frac{e^{-(1/2-\ell A)z} - e^{-(1+\ell A)z}}{z} = e^{-(1+\ell A)z} \cdot \Big(\frac{1 - e^{(\frac{1}{2}+2\ell A)z}}{-z}\Big).$$

Thus (iii) follows as $F(z) = G_0(z) \cdot W(z)^\ell$.

- For part (iv), we see by (iii) and the definition of $A$ that

$$|F(-s\log x)| \leq \frac{e^{\sigma\epsilon}x^\sigma}{|s|\log x} \cdot \big(1 + e^{-\sigma\epsilon}x^{-\sigma/2}\big)\Big|\frac{1 - e^{-2As\log x}}{2As\log x}\Big|^\ell. \tag{8.2}$$

To bound the above quantity, we observe that

$$\Big|\frac{1 - e^{-w}}{w}\Big|^2 \leq \Big(\frac{1 - e^{-a}}{a}\Big)^2 \leq 1 \tag{8.3}$$

for $w = a + ib$ with $a > 0$ and $b \in \mathbb{R}$. This observation can be checked in a straightforward manner (cf. Lemma 7.1.2). Using (8.3), it follows that

$$\Big|\frac{1 - e^{-2As\log x}}{2As\log x}\Big|^\ell = \Big|\frac{1 - e^{-2As\log x}}{2As\log x}\Big|^\alpha \cdot \Big|\frac{1 - e^{-2As\log x}}{2As\log x}\Big|^{\ell-\alpha} \quad \leq \Big(\frac{1 + x^{-2A\sigma}}{2A|s|\log x}\Big)^\alpha \cdot 1 \leq \Big(\frac{2\ell}{\epsilon|s|}\Big)^\alpha.$$

In the last step, we noted $1 + x^{-2A\sigma} \leq 2$ and used the definition of $A$. Combining this with (8.2) and observing $e^{-\sigma\epsilon} \leq 1$, we deduce the desired bound.

- For part (v), we see by (iii) that

$$\begin{aligned} |F(-s\log x)| &\leq \Big(\frac{1}{2} + 2\ell A\Big)e^{\sigma\epsilon}x^\sigma \cdot \Big|\frac{1 - e^{-(\frac{1}{2}+2\ell A)s\log x}}{(\frac{1}{2} + 2\ell A)s\log x}\Big| \cdot \Big|\frac{1 - e^{-2As\log x}}{2As\log x}\Big|^\ell \\ &\leq e^{\sigma\epsilon}x^\sigma, \end{aligned}$$

where the second inequality follows from an application of (8.3) and the observation that $\frac{1}{2} + 2\ell A < \frac{1}{2} + \epsilon < 1$. For $s = \sigma > 0$, observe that $F(-\sigma\log x)$ is real and positive. Thus, by (iii) and (8.3),

$$\begin{aligned} F(-\sigma\log x) &\leq e^{\sigma\epsilon}x^\sigma \cdot \Big(\frac{1 - x^{-(\frac{1}{2}+2\ell A)\sigma}}{\sigma\log x}\Big) \cdot \Big(\frac{1 - x^{-2A\sigma}}{2A\sigma\log x}\Big)^\ell \\ &\leq \frac{e^{\sigma\epsilon}x^\sigma}{\sigma\log x} \cdot \Big(\frac{1 - x^{-2A\sigma}}{2A\sigma\log x}\Big)^\ell \\ &\leq \frac{e^{\sigma\epsilon}x^\sigma}{\sigma\log x}. \end{aligned}$$

This completes the proof of all cases of (iv).

- For part (vi), we shall argue as in (iv). Rearranging (iii), notice that

$$|F(z)| = \Big| e^{(-\frac{1}{2}+2\ell A)z} \cdot \Big(\frac{1-e^{-(\frac{1}{2}+2\ell A)z}}{z}\Big)\Big(\frac{1-e^{-2Az}}{2Az}\Big)^{\ell}\Big|.$$

If $r := \mathrm{Re}\{z\} > 0$, then

$$\begin{aligned}|F(z)| &\le e^{(-\frac{1}{2}+2\ell A)r} \cdot \frac{1+e^{-(\frac{1}{2}+2\ell A)r}}{|z|} \cdot \Big(\frac{1+e^{-2Ar}}{2A|z|}\Big)^{\ell} \\ &\le \frac{2e^{(-\frac{1}{2}+2\ell A)r}}{|z|}\Big(\frac{1}{A|z|}\Big)^{\ell}.\end{aligned}$$

If we substitute $z = -s\log x = (\frac{1}{2} - it)\log x$, then it follows by the definition of $A$ that

$$|F(-s\log x)| \le \frac{2e^{\epsilon/2}x^{-1/4}}{|\frac{1}{2}+it|\log x}\Big(\frac{2\ell}{\epsilon|\frac{1}{2}+it|}\Big)^{\ell} \le \frac{4e^{\epsilon/2}x^{-1/4}}{\log x}\Big(\frac{2\ell}{\epsilon}\Big)^{\ell}(1/4+t^2)^{-\ell/2}.$$

This yields (vi) since $4e^{\epsilon/2} < 5$ for $\epsilon < 1/4$.

$\square$

### 8.1.2 A weighted sum of prime ideals

Recall $L/F$ is a Galois extension of number fields with Galois group $G$ and $C$ is a conjugacy class of $G$. Furthermore, recall the notation and discussion in Section 2.5.

For $x > 3, \epsilon \in (0, 1/4)$ and integer $\ell \ge 1$, use the compactly-supported weight $f(\,\cdot\,) = f(\,\cdot\,; x, \ell, \epsilon)$ defined in Lemma 8.1.1 and set

$$S(x) = S_{\ell,\epsilon}(x) := \sum_{\mathfrak{n}\subseteq\mathcal{O}_F} \Lambda_F(\mathfrak{n})\Theta_C(\mathfrak{n}) f\Big(\frac{\log \mathrm{N}\mathfrak{n}}{\log x}\Big), \tag{8.4}$$

where $\Theta_C$ is given by (2.27). We reduce our estimation of $\pi_C(x, L/F)$ given by (2.28) to the smoothed version $S(x)$.

**Lemma 8.1.2.** *Let $x_0 > e^4$. Suppose there exist constants $a, b \ge 0$ and $0 \le c \le 1/2$, all of which are independent of $x$, such that $S(x) < \{a + bx^{-c}\}\frac{|C|}{|G|}x$ for all $x \ge x_0$. Then, for all $x \ge x_0$,*

$$\pi_C(x, L/F) < \Big\{a + 2bx^{-c} + O\Big(\frac{n_L}{x^{1/2}} + \frac{n_L x_0 \log x}{x}\Big)\Big\}\frac{|C|}{|G|}\mathrm{Li}(x).$$

*Proof.* If $t > 1$, then

$$\psi_C(t) = \sum_{t^{1/2} \leq \mathrm{N}\mathfrak{n} < t} \Theta_C(\mathfrak{n})\Lambda_K(\mathfrak{n}) + \psi_C(t^{1/2}). \tag{8.5}$$

The sum in (8.5) is bounded by $S(t)$ in (8.4) because of Lemma 8.1.1(i), while the secondary term in (8.5) is estimated much like (2.33). Thus, we have that

$$\psi_C(t) \leq S(t) + O(n_F t^{1/2}). \tag{8.6}$$

We substitute (8.6) into Lemma 2.5.1 and deduce that

$$\pi_C(x, L/F) \leq \frac{S(x)}{\log x} + \int_{x_0}^{x} \frac{S(t)}{t \log^2 t} dt + O\Big(\frac{n_F x^{1/2}}{\log x} + n_F x_0\Big).$$

From our assumption on $S(t)$ for $t \geq x_0$, it follows that

$$\pi_C(x, L/F) < a\frac{|C|}{|G|}\mathrm{Li}(x) + b\frac{|C|}{|G|}\Big[\frac{x^{1-c}}{\log x} + \int_{x_0}^{x} \frac{t^{-c}}{\log^2 t} dt\Big] + O\Big(\frac{n_F x^{1/2}}{\log x} + n_F x_0\Big). \tag{8.7}$$

Note that if $0 \leq c \leq 1/2$, then $t^{1-c}/\log^2 t$ is an increasing function of $t$ for $t > e^4$. Since $x_0 > e^4$ and $\mathrm{Li}(x) > \frac{x}{\log x}$ for $x > e^4$, we conclude that

$$\int_{x_0}^{x} \frac{t^{-c}}{\log^2 t} dt = \int_{x_0}^{x} \frac{t^{1-c}}{\log^2 t}\frac{dt}{t} \leq \frac{x^{1-c}}{\log^2 x}\int_{x_0}^{x} \frac{dt}{t} \leq \frac{x^{1-c}}{\log x} < x^{-c}\,\mathrm{Li}(x). \tag{8.8}$$

The desired result follows from (8.7), (8.8), and the identity $n_L = [L : F]n_F = |G|n_F$. □

By Mellin inversion, (8.4), and (2.26), it follows that

$$S(x) = \frac{\log x}{2\pi i}\int_{2-i\infty}^{2+i\infty} Z_C(s)F(-s\log x)ds. \tag{8.9}$$

To shift the contour, we use Deuring's reduction with an abelian subgroup $H$ of $G$ such that $H \cap C$ is non-empty. This is described in Section 2.5. In particular, using (2.34), this yields the following lemma.

**Lemma 8.1.3.** *Let $H$ be any abelian subgroup of $G$ such that $H \cap C$ is non-empty. Let $K = L^H$ be the fixed field of $L$ by $H$ and let $g_C \in H \cap C$. If $S = S(x)$ is defined by* (8.4) *and $F$ is the*

*Laplace transform of $f$ in Lemma 8.1.1 then*

$$S(x) = \frac{|C|}{|G|} \sum_{\chi} \overline{\chi}(g_C) \frac{\log x}{2\pi i} \int_{2-i\infty}^{2+i\infty} -\frac{L'}{L}(s, \chi, L/K) F(-s \log x) ds,$$

*where the sum is over all Hecke characters $\chi$ attached to the abelian extension $L/K$. Here $L(s, \chi, L/K)$ is the (primitive) Hecke $L$-function attached to $\chi$.*

Next we shift the contour in Lemma 8.1.3 and bound $S(x)$ in terms of the non-trivial zeros of Hecke $L$-functions. Henceforth write $S = S(x)$ for simplicity. Recall $f$ depends on the arbitrary quantities $x > 3, \epsilon \in (0, 1/4)$ and an integer $\ell \geq 1$.

**Lemma 8.1.4.** *Assume $\ell \geq 2$. Then*

$$\frac{|G|}{|C|} \frac{S}{e^\epsilon x} \leq 1 + \frac{\log x}{e^\epsilon x} \sum_{\chi} \sum_{\rho_\chi} |F(-\rho_\chi \log x)| + O\Big(n_L x^{-1} \log x + x^{-5/4} (2\ell/\epsilon)^\ell \log D_L\Big), \tag{8.10}$$

*where the outer sum is over all Hecke characters $\chi$ of the abelian extension $L/K$ and the inner sum runs over all non-trivial zeros $\rho_\chi$ of $L(s, \chi, L/K)$, counted with multiplicity.*

*Proof.* Shift the contour in Lemma 8.1.3 to the line $\mathrm{Re}\{s\} = -\frac{1}{2}$. This picks up the non-trivial zeros of $L(s, \chi, L/K)$, the simple pole at $s = 1$ when $\chi$ is trivial, and the trivial zero at $s = 0$ of $L(s, \chi, L/K)$ of order $r(\chi)$. Overall, we see that

$$\begin{aligned} \frac{|G|}{|C|} S = \log x \Big[ & F(-\log x) - \sum_{\chi} \overline{\chi}(g_C) \sum_{\rho_\chi} F(-\rho_\chi \log x) + O\Big( \sum_{\chi} r(\chi) |F(0)| \Big) \Big] \\ & + \log x \sum_{\chi} \frac{\overline{\chi}(g_C)}{2\pi i} \int_{-1/2-i\infty}^{-1/2+i\infty} -\frac{L'}{L}(s, \chi, L/K) F(-s \log x) ds, \end{aligned} \tag{8.11}$$

where the sum over $\rho = \rho_\chi$ is over all non-trivial zeros of $L(s, \chi, L/K)$, counted with multiplicity. From (2.5) and (2.8), we see that $r(\chi) \leq n_K$. Hence, it follows by Lemma 8.1.1(v) that

$$F(-\log x) \leq \frac{e^\epsilon x}{\log x}, \quad \text{and} \quad \sum_{\chi} r(\chi) |F(0)| \leq [L : K] n_K = n_L.$$

For the remaining contour, by [LO77, Lemma 6.2] and the primitivity of $\chi$, we have that

$$-\frac{L'}{L}(s, \chi, L/K) \ll \log D_\chi + n_K \log(|s| + 3)$$

for $\mathrm{Re}\{s\} = -1/2$, where $D_\chi$ is defined in (2.2). It follows by Lemma 8.1.1(vi) that

$$\begin{aligned}&\frac{\log x}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty} -\frac{L'}{L}(s,\chi,L/K)F(-s\log x)ds\\ &\ll x^{-1/4}\Big(\frac{2\ell}{\epsilon}\Big)^{\ell}\int_{-\infty}^{\infty}\frac{\log D_\chi + n_K\log(|t|+3)}{(1/4+t^2)^{\ell/2}}dt \ll x^{-1/4}\Big(\frac{2\ell}{\epsilon}\Big)^{\ell}\log D_\chi,\end{aligned}$$

because $n_K \ll \log D_K \leq \log D_\chi$ and $\ell \geq 2$. Summing over $\chi$ and using the conductor-discriminant formula (2.21) yields

$$\log x\sum_\chi \frac{\overline{\chi}(g_C)}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty} -\frac{L'}{L}(s,\chi,L/K)F(-s\log x)ds \ll x^{-1/4}\Big(\frac{2\ell}{\epsilon}\Big)^{\ell}\log D_L.$$

Taking absolute value of both sides in (8.11), multiplying both sides by $(e^\epsilon x)^{-1}$, and combining all of these observations yields the desired result. □

### 8.1.3 A sum over low-lying zeros

This subsection is dedicated to analyzing the sum in Lemma 8.1.4 over all non-trivial zeros of all Hecke $L$-functions $L(s,\chi,L/K)$. We will reduce our estimation to a sum over low-lying zeros by exploiting information about the distribution of zeros of Hecke $L$-functions. We will utilize[1] the results and notation of Section 7.3.1 for the Hecke $L$-functions associated to the abelian extension $L/K$. In particular, the quantity $\mathscr{L}$ in (7.21) and the zeros $\rho_1, \rho'$, and $\rho_2$ are defined exactly as in the aforementioned subsection.

We will demonstrate that the contribution of zeros is negligible if the zeros are either high-lying or far from the line $\mathrm{Re}\{s\} = 1$. Throughout this chapter (unlike Chapter 7), we assume $1 \leq B \leq 1000$ is a fixed absolute constant. Recall that $x > 3, \epsilon \in (0, 1/4)$, and $\ell \geq 1$ are arbitrary parameters used in the definition of $S = S(x)$ given by (8.4). Moreover, any sum $\sum_\chi$ is over all Hecke characters $\chi$ attached to $L/K$. We begin by considering high-lying zeros.

**Lemma 8.1.5.** *Let $T_\star \geq 1$ be arbitrary. Let $0 < E < \frac{2}{3}B$ be fixed. Let*

$$B > 162 + E, \quad \ell \geq 82n_K + 162, \quad \tfrac{1}{4} > \epsilon \geq 4\ell x^{-E/(B\ell)}. \tag{8.12}$$

*For $x \geq e^{B\mathscr{L}}$,*

$$\frac{\log x}{x}\sum_\chi \sum_{\substack{\rho \\ |\mathrm{Im}\{\rho\}|>T_\star}} |F(-\rho\log x)| \ll \frac{1}{T_\star}. \tag{8.13}$$

[1]Only Lemma 7.3.8 will be ignored.

*Proof.* Write $\rho = \beta + i\gamma$ with $\beta = 1 - \frac{\lambda}{\mathscr{L}}$. If $T \geq 1$, then Lemma 8.1.1(iv) with $\alpha = \ell(1-\beta)$ and our choices of our conditions on $\epsilon, \ell$, and $x$ imply that

$$\frac{\log x}{x}|F(-\rho\log x)| \leq \frac{2e^{\epsilon}x^{\beta-1}}{T}\Big(\frac{2\ell}{\epsilon T}\Big)^{\ell(1-\beta)} \leq \frac{4}{T}e^{-(B-E)\lambda}(2T)^{-(82n_K+162)\lambda/\mathscr{L}}. \tag{8.14}$$

Using Theorem 5.1.2 via partial summation, we see that

$$\begin{aligned}\frac{T\log x}{x}&\sum_{\chi}\sum_{\substack{\rho\\ T\leq|\mathrm{Im}\{\rho\}|\leq 2T}}|F(-\rho\log x)|\\ &\ll \frac{e^{-(B-E-162)\mathscr{L}}}{(2T)^{n_K}} + \Big(B-E+\frac{n_K\log(2T)}{\mathscr{L}}\Big)\int_0^{\mathscr{L}} e^{-(B-E-162)\lambda}(2T)^{-n_K\lambda/\mathscr{L}}d\lambda \ll 1,\end{aligned}$$

since $B > 162 + E$. Overall, this implies that the LHS of (8.13) is

$$\leq \frac{\log x}{x}\sum_{\chi}\sum_{k=0}^{\infty}\sum_{\substack{\rho\\ 2^kT_\star\leq\mathrm{Im}\{\rho\}<2^{k+1}T_\star}}|F(-\rho\log x)| \ll \frac{1}{T_\star}\sum_{k=0}^{\infty}\frac{1}{2^k} \ll \frac{1}{T_\star},$$

as desired. □

As we shall see in the next section, an appropriate combination of Lemmas 8.1.4 and 8.1.5 and Theorem 5.1.2 suffices to establish Theorem 1.3.3. For Theorem 1.3.4, we must also show low-lying zeros far to the left of $\mathrm{Re}\{s\} = 1$ contribute a negligible amount.

**Lemma 8.1.6.** *Let $0 \leq R \leq \frac{1}{2}\mathscr{L}$ be arbitrary. Assume* (8.12) *holds. For $x \geq e^{B\mathscr{L}}$,*

$$\frac{\log x}{x}\sum_{\chi}\sum_{\rho}{}' |F(-\rho\log x)| \ll x^{-(B-E-162)R/B\mathscr{L}},$$

*where the marked sum $\sum'$ runs over zeros $\rho = \beta + i\gamma$ of $L(s,\chi,L/K)$, counting with multiplicity, satisfying $0 < \beta \leq 1 - R/\mathscr{L}$ and $|\gamma| \leq \epsilon^{-1}$.*

*Proof.* From our choices of $\epsilon, \ell$ in (8.12) and Theorem 5.1.2, it follows that

$$N(1-\tfrac{\lambda}{\mathscr{L}}, \epsilon^{-1}) \ll e^{162\lambda}(1/\epsilon)^{(81n_K+162)\lambda/\mathscr{L}} \ll e^{162\lambda}x^{E\lambda/B\mathscr{L}} \ll x^{(162+E)\lambda/B\mathscr{L}}$$

for $0 < \lambda < \mathscr{L}$, where $N(\sigma, T)$ is given by (7.25). Write $\rho = \beta + i\gamma$ with $\beta = 1 - \frac{\lambda}{\mathscr{L}}$ for some non-trivial zero $\rho$ appearing in the marked sum. By Lemma 8.1.1(iv) with $\alpha = 0$ and

Lemma 8.1.1(v), it follows that

$$\frac{\log x}{x}|F(-\rho\log x)| \ll \begin{cases} x^{-\lambda/\mathscr{L}} & \text{for } |\rho| \geq 1/4, \\ x^{-3/4}\log x & \text{for } |\rho| \leq 1/4. \end{cases} \tag{8.15}$$

To clarify the second inequality, we observe by Lemma 8.1.1(v) that $|F(-\rho\log x)| \ll x^{\beta} \ll x^{1/4}$ for $|\rho| \leq 1/4$. Thus, by (8.15) and partial summation, we have that

$$\begin{aligned} \frac{\log x}{x}\sum_{\chi}\sideset{}{'}\sum_{|\rho|\geq 1/4}|F(-\rho\log x)| &\ll x^{\frac{-(B-E-162)}{B}} + \frac{\log x}{\mathscr{L}}\int_R^{\mathscr{L}} x^{\frac{-(B-E-162)\lambda}{B\mathscr{L}}}d\lambda \\ &\ll x^{-(B-E-162)R/B\mathscr{L}}. \end{aligned}$$

Moreover, by (8.15), a crude application of [LMO79, Lemma 2.1], and Lemma 7.3.1, it follows that

$$\frac{\log x}{x}\sum_{\chi}\sideset{}{'}\sum_{\substack{\rho \\ |\rho|\leq 1/4}}|F(-\rho\log x)| \ll [L:K]\mathscr{L}x^{-3/4}\log x \ll x^{-3/4}e^{2\mathscr{L}}\log x \ll x^{-\frac{3}{4}+\frac{3}{B}}.$$

Combining these estimates yields the desired result since, by our assumptions on $B$ and $R$, $x^{-(B-E-162)R/B\mathscr{L}} \gg x^{-(B-E-162)/2B} \gg x^{-1/2} \gg x^{-3/4+3/162} \gg x^{-3/4+3/B}$. □

We package these lemmas into the following convenient proposition.

**Proposition 8.1.7.** *Let $0 \leq R \leq \frac{1}{2}\mathscr{L}$ be arbitrary. Let $0 < E < \frac{2}{3}B$ be fixed. Assume that*

$$B > 162 + E, \qquad \ell \geq 82n_K + 162, \qquad \tfrac{1}{4} > \epsilon \geq 4\ell x^{-E/(B\ell)}. \tag{8.16}$$

*If $x \geq e^{B\mathscr{L}}$ and $S(x)$ is given by (8.4), then*

$$\frac{|G|}{|C|}\frac{S(x)}{e^{\epsilon}x} \leq 1 + \frac{\log x}{e^{\epsilon}x}\sum_{\chi}\sideset{}{^\star}\sum_{\rho}|F(-\rho\log x)| + O\big(\epsilon + x^{-(B-E-162)R/B\mathscr{L}}\big), \tag{8.17}$$

*where the sum $\sum^{\star}$ indicates a restriction to non-trivial zeros $\rho$ of $L(s,\chi,L/K)$, counted with multiplicity, satisfying $1 - R/\mathscr{L} < \mathrm{Re}\{\rho\} < 1$ and $|\mathrm{Im}\{\rho\}| \leq \epsilon^{-1}$.*

*Proof.* Let $T_\star = 1/\epsilon$. It follows from our hypothesis (8.16) along with Lemmas 7.3.10, 8.1.4

and 8.1.5 that

$$\begin{aligned}\frac{|G|}{|C|}\frac{S}{e^{\epsilon}x} \leq 1 + \frac{\log x}{e^{\epsilon}x}\sum_{\chi}\sum_{\rho}{}^{\star}\,|F(-\rho\log x)| \\ + O\Big(\epsilon + x^{-(B-E-162)R/B\mathscr{L}} + n_L x^{-1}\log x + x^{-5/4}(2\ell/\epsilon)^{\ell}\log D_L\Big).\end{aligned} \tag{8.18}$$

It remains to bound the third and fourth expressions in the error term by $\epsilon$. Since $E < B$ and $\ell \geq 244$, we see that

$$\epsilon > x^{-E/B\ell} > x^{-1/\ell} > x^{-1/244}.$$

Moreover, $n_L = n_K[L:K] \ll \mathscr{L}e^{2\mathscr{L}} \ll x^{3/162}$ by Lemma 7.3.1 and (7.22). Similarly, since $\log D_L = \sum_{\chi}\log D_{\chi} \leq [L:K]\log(D_K\mathcal{Q})$, it follows that

$$(2\ell/\epsilon)^{\ell}\log D_L \ll x^{E/B}\mathscr{L}[L:K] \ll x^{2/3}\mathscr{L}e^{2\mathscr{L}} \ll x^{2/3+3/162}.$$

Applying these estimates in (8.18) yields (8.17). □

## 8.2 Proofs of Brun–Titchmarsh

Finally, we have arrived at the proofs of Theorems 1.3.3 and 1.3.4. In comparison to Theorem 1.3.4, the proof of Theorem 1.3.3 is quite simple, requiring only the log-free zero density estimate of Hecke $L$-functions given by Theorem 7.3.6. Recall this result is uniform over all extensions $L/F$ and therefore we do not assume $\mathscr{L}$ is sufficiently large in the proof of Theorem 1.3.3.

The proof of Theorem 1.3.4 is divided into cases depending on how close the zero $\rho_1$, defined in Section 7.3.1, is to $\mathrm{Re}\{s\} = 1$. Namely, for $\eta > 0$ arbitrary and sufficiently small, if $\lambda_1 < \eta$ then we refer to $\rho_1$ as an $\eta$-Siegel zero. The cases depend on whether an $\eta$-Siegel zero exists. The main steps are similar to the proof for Theorem 1.3.3 but we need a more refined analysis involving zero-free regions and Deuring–Heilbronn phenomenon.

### 8.2.1 Proof of Theorem 1.3.3

Select

$$B = 244.5, \quad E = 82.1, \quad \ell = 82n_K + 162, \quad \epsilon = 1/8, \quad \text{and } R = 0. \tag{8.19}$$

Let $M_0 > 0$ be a sufficiently large absolute constant. For $x \geq x_0 := e^{244.5\mathscr{L}} + M_0 n_K^{244.5n_K}$, we claim these are valid choices to invoke Proposition 8.1.7. It suffices to check $\epsilon = \frac{1}{8} \geq 4\ell x^{-E/B\ell}$ for $x \geq x_0$. We need only show $(32\ell)^{B\ell/E} \leq x_0$. This is visible from the fact that

$$(32\ell)^{B\ell/E} \ll n_K^{\frac{244.5}{82.1}(82n_K+162)} e^{O(n_K)} \ll n_K^{244.5n_K} \leq x_0,$$

after enlarging $M_0$ if necessary. This proves the claim.

Therefore, by Proposition 8.1.7, we have that $S(x) \ll \frac{|C|}{|G|}x$ for $x \geq x_0$, because the corresponding restricted sum $\sum^{\star}$ is empty whenever $R = 0$. Let $M \geq 1$ denote the implicit absolute constant in the above estimate for $S(x)$. Thus, by Lemma 8.1.2 with $x_0 = e^{244.5\mathscr{L}} + M_0 n_K^{244.5n_K}$, $a = M$ and $b = c = 0$, we have that

$$\pi_C(x, L/F) < \Big\{ M + O\Big(n_L x^{-1/2} + \frac{n_L \log x}{x}(e^{244.5\mathscr{L}} + n_K^{244.5n_K})\Big)\Big\} \frac{|C|}{|G|}\mathrm{Li}(x)$$

for $x \geq x_0$. By Lemma 7.3.1 and (7.21), notice that $n_L \ll e^{4\mathscr{L}/3} \ll D_K^2 \mathcal{Q}^2 n_K^{n_K}$. Thus, the desired result follows for $x \gg e^{245.9\mathscr{L}} + D_K^2 \mathcal{Q}^2 n_K^{246n_K}$, completing the proof. □

**Remark.**

1) If one wishes to minimize the value of $B$ and hence minimize the exponents of $D_K$ and $\mathcal{Q}$ in (1.31) then one may alternatively select

$$B = 162.01, \quad E = 0.95, \quad \ell = 82n_K + 162, \quad \epsilon = 1/8, \quad \text{and } R = 0$$

in place of (8.19). Taking $x_0 = e^{162.01\mathscr{L}} + M_0 n_K^{13{,}999n_K}$, it follows that

$$(32\ell)^{B\ell/E} \ll n_K^{\frac{162.01}{0.95}(82n_K+162)} e^{O(n_K)} \ll n_K^{13{,}999n_K} \leq x_0.$$

Arguing as above, one deduces $\pi_C(x, L/F) \ll \frac{|C|}{|G|}\mathrm{Li}(x)$ for $x \gg e^{163.5\mathscr{L}} + D_K^2 \mathcal{Q}^2 n_K^{14{,}000n_K}$ as claimed in the remark following Theorem 1.3.3 based on (7.21).

2) Similarly, to minimize the exponents of $n_K^{n_K}$ in (1.31), one may alternatively select

$$B = 359.5, \quad E = 197, \quad \ell = 82n_K + 162, \quad \epsilon = 1/8, \quad \text{and } R = 0$$

in place of (8.19). Taking $x_0 = e^{359.5\mathscr{L}}$, it follows by (7.22) that

$$(32\ell)^{B\ell/E} \ll n_K^{\frac{359.5}{197}(82n_K+162)} e^{O(n_K)} \ll n_K^{149.65n_K} \leq x_0,$$

since $359.5 \times \frac{5}{12} > 149.7$. Arguing as above, one deduces $\pi_C(x, L/F) \ll \frac{|C|}{|G|}\mathrm{Li}(x)$ for $x \gg e^{360.9\mathscr{L}} \geq e^{4\mathscr{L}/3}e^{359.5\mathscr{L}}$ as claimed in the remark following Theorem 1.3.3.

### 8.2.2 Proof of Theorem 1.3.4: $\eta$-Siegel zero does not exist

Let $\eta > 0$ be arbitrary and sufficiently small and let $\mathscr{L}$ be sufficiently large depending only on $\eta$. We will frequently use these properties without further mention. Recall that the proof of Theorem 1.3.4 is divided according whether $\rho_1$ is an $\eta$-Siegel zero or not.

In this subsection, we assume an $\eta$-Siegel zero does not exist; that is, $\lambda_1 \geq \eta$. We will show Theorem 1.3.4 holds with no error term. Assume $\lambda^\star > 0$ satisfies

$$\lambda^\star < \min\{\lambda', \lambda_2\}, \tag{8.20}$$

where $\lambda'$ and $\lambda_2$ are defined in Section 7.3.1 with $T_\star = \eta^{-2}$. Select

$$B > 360, \qquad E = 198, \qquad \ell = 82n_K + 162, \qquad \epsilon = \eta^2, \tag{8.21}$$

and let $R = R(\eta)$ be sufficiently large. We claim these choices satisfy the assumptions of Proposition 8.1.7. Since $\mathscr{L}$ is sufficiently large depending only on $\eta$, it suffices to show, for $x \geq e^{B\mathscr{L}}$, that $4\ell x^{-E/B\ell} = o(1)$ as $\mathscr{L} \to \infty$. If $n_K$ is bounded while $\mathscr{L} \to \infty$ then this is immediate, so we may assume $n_K \to \infty$. By (7.22), notice that $\ell = 82n_K + 162 \leq \{196.8 + o(1)\}\frac{\mathscr{L}}{\log n_K} \leq 197\frac{\mathscr{L}}{\log n_K}$ for $n_K$ sufficiently large. Thus, for $n_K$ sufficiently large and $x \geq e^{B\mathscr{L}}$, we have that

$$4\ell x^{-E/B\ell} \ll n_K e^{-198\mathscr{L}/\ell} \ll n_K e^{-\frac{198}{197}\log n_K} \ll n_K^{-1/197}.$$

Hence, $4\ell x^{-E/B\ell} = o(1)$ for $x \geq e^{B\mathscr{L}}$, as $n_K \to \infty$. This proves the claim.

Therefore, by Proposition 8.1.7, it follows that

$$\frac{|G|}{|C|}\frac{S(x)}{e^\epsilon x} \leq 1 + \frac{\log x}{e^\epsilon x}\sum_\chi \sideset{}{^\star}\sum_\rho |F(-\rho\log x)| + O(\eta^2)$$

for $x \geq e^{B\mathscr{L}}$, where the sum $\sum^\star$ runs over non-trivial zeros $\rho$ of $L(s,\chi)$, counted with multiplicity, satisfying $\beta > 1 - R/\mathscr{L}$ and $|\gamma| \leq \eta^{-2}$. For a non-trivial zero $\rho$ of a Hecke $L$-function, write $\rho = \beta + i\gamma = 1 - \frac{\lambda}{\mathscr{L}} + i\frac{\mu}{\mathscr{L}}$. By Lemma 8.1.1, we see that

$$\frac{\log x}{e^\epsilon x}|F(-\rho\log x)| \leq x^{-(1-\beta)} \leq e^{-B\lambda},$$

since $x \geq e^{B\mathscr{L}}$. Extracting $\rho_1$ and $\overline{\rho_1}$ (or simply $\rho_1$ if $\rho_1$ is real) from $\sum^\star$, we deduce by our choice of $\lambda^\star$ in (8.20) that

$$\frac{|G|}{|C|}\frac{S(x)}{e^\epsilon x} \leq 1 + m(\rho_1)e^{-B\lambda_1} + \sum_{\chi} \sum_{\substack{\lambda^\star \leq \lambda \leq R \\ |\gamma| \leq \eta^{-2}}} e^{-B\lambda} + O(\eta^2), \tag{8.22}$$

where $m(\rho_1) = 2$ if $\rho_1$ is complex and $m(\rho_1) = 1$ if $\rho_1$ is real. To bound the remaining quantities[2], we must select $\lambda^\star$ for which we further subdivide into cases.

### $\lambda_1$ small ($\eta \leq \lambda_1 < 10^{-3}$)

By Theorem 7.3.2, $\rho_1$ is a simple real zero attached to a real character $\chi_1$, implying $m(\rho_1) = 1$. Select $B = 361$ and choose $\lambda^\star = 0.2103\log(1/\lambda_1)$, which satisfies (8.20) by Theorem 7.3.3 with $T_\star = \eta^{-2}$. Arguing as in[3] Section 7.3.4 (with $T_\star = \eta^{-2}$ instead) and using Theorem 7.3.7, we may conclude by (8.22) that

$$S(x) < \{2 - \eta + O(\eta^2)\}\frac{|C|}{|G|}x$$

for $x \geq e^{361\mathscr{L}}$. By Lemmas 7.3.1 and 8.1.2, we conclude that

$$\pi_C(x, L/F) < \{2 - \eta + O\big(\eta^2 + \mathscr{L}e^{1.4\mathscr{L}}(x^{-1/2} + e^{361\mathscr{L}}x^{-1}\log x)\big)\}\frac{|C|}{|G|}\mathrm{Li}(x)$$

for $x \geq e^{361\mathscr{L}}$. Hence, in this subcase, Theorem 1.3.4 (with no error term) follows for $x \geq e^{363\mathscr{L}}$ after fixing $\eta > 0$ sufficiently small and recalling $\mathscr{L}$ is sufficiently large.

### $\lambda_1$ medium ($10^{-3} < \lambda_1 \leq 0.0875$)

One argues similar to the previous case with some minor changes. Namely, select $B = 593$ and choose $\lambda^\star = 0.44$. Following the corresponding subcase in Section 7.3.4 (using $T_\star = \eta^{-2}$ instead) allows us to deduce Theorem 1.3.4 for $x \geq e^{595\mathscr{L}}$.

### $\lambda_1$ large ($\lambda_1 \geq 0.0875$)

Select $B = 693$ and $\lambda^\star = 0.2866$ as per Theorem 7.3.2 with $T_\star = \eta^{-2}$. Noting $m(\rho_1) \leq 2$ unconditionally, one may argue similarly as per the previous cases and follow Section 7.3.3

[2]At this stage, one may wish to compare (8.22) with its "least prime" counterparts (7.33) and (7.31). It is apparent that the arguments will be very similar.

[3]Observe $361 > 297$ so the same estimates hold.

(using $T_\star = \eta^{-2}$ instead) to deduce Theorem 1.3.4 for $x \geq e^{694.9\mathscr{L}}$. As $\delta_0$ in (7.21) is sufficiently small, this yields the desired range of $x$ in Theorem 1.3.4, completing the proof in all cases when an $\eta$-Siegel zero does not exist. $\square$

### 8.2.3 Proof of Theorem 1.3.4: $\eta$-Siegel zero exists

For this subsection, we consider the case when $\lambda_1 < \eta$. By Theorem 7.3.2, it follows that $\rho_1 = \beta_1 = 1 - \frac{\lambda_1}{\mathscr{L}}$ is a simple real zero and $\chi_1$ is a real Hecke character. Suppose

$$B = 692, \qquad E = 344, \qquad \ell = 82n_K + 162, \qquad 4\ell x^{-344/692\ell} \leq \epsilon < 1/4. \tag{8.23}$$

With these choices, we claim for $x \geq e^{692\mathscr{L}}$ that $4\ell x^{-344/692\ell} = o(1)$ as $\mathscr{L} \to \infty$. If $n_K$ is uniformly bounded while $\mathscr{L} \to \infty$ then this is immediate, so we may assume $n_K \to \infty$. By (7.22), notice that $\ell = 82n_K + 162 \leq \{196.8 + o(1)\}\frac{\mathscr{L}}{\log n_K} \leq 197\frac{\mathscr{L}}{\log n_K}$ for $n_K$ sufficiently large. Thus, for $n_K$ sufficiently large and $x \geq e^{692\mathscr{L}}$, we have that

$$4\ell x^{-344/692\ell} \ll n_K e^{-344\mathscr{L}/\ell} \ll n_K e^{\frac{-344}{197}\log n_K} \ll n_K^{-0.7}.$$

Hence, $4\ell x^{-344/692\ell} = o(1)$ as $n_K \to \infty$. This proves the claim, which implies the condition on $\epsilon$ in (8.23) is non-empty for $\mathscr{L}$ sufficiently large.

Now, let $1 \leq R \leq \frac{1}{2}\mathscr{L}$ be arbitrary. By Proposition 8.1.7, for $x \geq e^{692\mathscr{L}}$, we have that

$$\frac{|G|}{|C|}\frac{S(x)}{e^{\epsilon}x} \leq 1 + \frac{x^{-(1-\beta_1)}}{\beta_1} + \frac{\log x}{e^{\epsilon}x}\sum_{\chi}\sideset{}{^\star}\sum_{\rho\neq\rho_1}|F(-\rho\log x)| + O\big(\epsilon + x^{-186R/692\mathscr{L}}\big), \tag{8.24}$$

where $\sum^\star$ runs over non-trivial zeros $\rho \neq \rho_1$ of $L(s,\chi)$, counted with multiplicity, satisfying

$$1 - R/\mathscr{L} < \mathrm{Re}\{\rho\} < 1, \qquad |\mathrm{Im}\{\rho\}| \leq \epsilon^{-1}.$$

Note that the $\beta_1$ term in (8.24) arises from bounding $F(-\sigma\log x)$ in Lemma 8.1.1(v) with $\sigma = \beta_1$. We further subdivide our arguments depending on the range of $\lambda_1$.

**$\lambda_1$ very small ($\frac{2\eta\mathscr{L}}{\log x} \leq \lambda_1 < \eta$)**

Here select $\epsilon = \eta^2$ and $R = \min\{\frac{1}{82}\log(c_1/\lambda_1), \frac{1}{2}\mathscr{L}\}$ for some fixed sufficiently small $c_1 > 0$. Since $4\ell x^{-344/692\ell} = o(1)$ as $\mathscr{L} \to \infty$, it follows that this choice of $\epsilon$ satisfies (8.23) for $\mathscr{L}$ sufficiently large depending only on $\eta$.

Hence, by Theorem 7.3.4 with $T_\star = \eta^{-2}$, these choices imply that the restricted sum $\sum^\star$ in

(8.24) is empty for $\mathscr{L}$ sufficiently large depending only on $\eta$. Moreover, we see that

$$x^{-186R/693\mathscr{L}} \leq e^{-\frac{186}{82}\log(c_1/\lambda_1)} \ll \lambda_1^2 \ll \eta^2,$$

as $x \geq e^{692\mathscr{L}}$ and $186/82 > 2$. Further, we have that

$$\frac{x^{-(1-\beta_1)}}{\beta_1} = e^{-\lambda_1 \log x/\mathscr{L}}\{1 + O(\lambda_1/\mathscr{L})\} < 1 - \eta + O(\eta^2),$$

since $\frac{2\eta\mathscr{L}}{\log x} \leq \lambda_1 < \eta$ and $e^{-t} < 1 - t/2$ for $0 \leq t \leq 1$. Overall, we conclude that

$$S(x) < \{2 - \eta + O(\eta^2)\}\frac{|C|}{|G|}x$$

for $x \geq e^{692\mathscr{L}}$. By Lemmas 7.3.1 and 8.1.2, we conclude that

$$\pi_C(x, L/F) < \{2 - \eta + O\big(\eta^2 + \mathscr{L}e^{1.4\mathscr{L}}(x^{-1/2} + e^{693\mathscr{L}}x^{-1}\log x)\big)\}\frac{|C|}{|G|}\mathrm{Li}(x)$$

for $x \geq e^{692\mathscr{L}}$. Hence, in this subcase, Theorem 1.3.4 (with no error term) follows for $x \geq e^{694.5\mathscr{L}}$ after fixing $\eta > 0$ sufficiently small and recalling $\mathscr{L}$ is sufficiently large.

**$\lambda_1$ extremely small ($\lambda_1 < \frac{2\eta\mathscr{L}}{\log x} \leq \eta$)**

Here select

$$\epsilon = 4\ell x^{-344/692\ell} \quad \text{and} \quad R = \min\Big\{\frac{\mathscr{L}}{81\mathscr{L} + 25n_K\log(1/\epsilon)}\log\Big(\frac{c_1}{\lambda_1}\cdot\frac{\mathscr{L}}{\mathscr{L} + n_K\log(1/\epsilon)}\Big), \frac{1}{2}\mathscr{L}\Big\}$$

for some sufficiently small $c_1 > 0$. Again, since $4\ell x^{-344/692\ell} = o(1)$ as $\mathscr{L} \to \infty$, it follows that $\epsilon < 1/4$ for $\mathscr{L}$ sufficiently large so this choice of $\epsilon$ satisfies (8.23).

Now, from our choice of $R$ and Theorem 7.3.4, the restricted sum in (8.24) is empty. For the main term, observe for $\mathscr{L}$ sufficiently large and $\eta > 0$ sufficiently small that

$$\frac{x^{-(1-\beta_1)}}{\beta_1} < \Big(1 - \frac{\lambda_1\log x}{2\mathscr{L}}\Big)\Big(1 + \frac{\lambda_1}{\mathscr{L}}\Big) \leq 1 - \frac{\lambda_1\log x}{3\mathscr{L}},$$

as $\lambda_1 < \frac{2\eta\mathscr{L}}{\log x}$ and $e^{-t} < 1 - t/2$ for $0 \leq t \leq 1$. To bound the error term in (8.24), notice that

$$81\mathscr{L} + 25n_K\log(1/\epsilon) \leq \frac{81}{692}\log x + \frac{344\cdot 25n_K}{692(82n_K + 162)}\log x < \frac{185.9}{692}\log x,$$

by our choice of $\epsilon$ and $\ell$ and since $x \geq e^{693\mathscr{L}}$. Consequently, $R \geq \frac{692\mathscr{L}}{185.9\log x}\log(\frac{c_1'\mathscr{L}}{\lambda_1\log x})$ for some sufficiently small $c_1' > 0$, implying

$$x^{-186R/692\mathscr{L}} \ll \Big(\frac{\lambda_1 \log x}{\mathscr{L}}\Big)^{\frac{186}{185.9}} \ll \eta^{1/2000}\Big(\frac{\lambda_1 \log x}{\mathscr{L}}\Big),$$

since $\lambda_1 < \frac{2\eta\mathscr{L}}{\log x}$ and $\frac{0.1}{185.9} < \frac{1}{2000}$. Combining these observations into (8.24) implies that

$$\frac{|G|}{|C|}\frac{S(x)}{e^{\epsilon}x} < 2 - \frac{\lambda_1 \log x}{3\mathscr{L}} + O\Big(\epsilon + \eta^{1/2000}\cdot\frac{\lambda_1 \log x}{\mathscr{L}}\Big) < 2 - 100\lambda_1 + O(\epsilon),$$

as $\eta$ is sufficiently small. Rearranging and substituting the choice of $\epsilon$ and $\ell$, we see that

$$S(x) < \Big\{2 - 100\lambda_1 + O\big(n_K x^{-\frac{1}{166n_K+327}}\big)\Big\}\frac{|C|}{|G|}x$$

for $x \geq e^{692\mathscr{L}}$. Now, if $x \geq e^{694.9\mathscr{L}}$ then, by Lemma 7.3.1, we have that

$$n_L e^{692\mathscr{L}} x^{-1}\log x \ll n_K e^{693.4\mathscr{L}} x^{-1}\log x \ll n_K x^{-1.5/694.9}\log x \ll n_K x^{-1/(166n_K+327)}.$$

Similarly, $n_L x^{-1/2} \ll n_K x^{-1/(166n_K+327)}$. Thus, by the previous inequality and Lemma 8.1.2, it follows that

$$\pi_C(x, L/F) < \Big\{2 - 100\lambda_1 + O\big(n_K x^{-\frac{1}{166n_K+327}}\big)\Big\}\frac{|C|}{|G|}\mathrm{Li}(x) \tag{8.25}$$

for $x \geq e^{694.9\mathscr{L}}$. As $\delta_0$ in (7.21) is sufficiently small, this completes the proof of Theorem 1.3.4 in all cases. □

**Remark.**

1) In (8.23), we could instead take $B = 502$ and $E = 198$ to establish (8.25) except with an error term of $O(n_K x^{-1/(208n_K+411)})$. To improve the error term, we chose the largest values of $B$ and $E$ which did not reduce the valid range of $x$ in Theorem 1.3.4. This range of $x$ is limited by the $\lambda_1 \geq 0.0875$ case addressed in Section 8.2.2.

2) As stated in Theorem 1.3.4, we obtain the sharper bound $\pi_C(x, L/F) < 2\frac{|C|}{|G|}\mathrm{Li}(x)$ from (8.25) with good effective lower bounds for $\lambda_1$. To see this, notice the error term in (8.25) is $\ll \lambda_1^{1.001}$ provided

$$x \gg \Big(\frac{c_1 n_K}{\lambda_1^{1.001}}\Big)^{166n_K+327} =: x_1,$$

where $c_1 > 0$ is some absolute constant. If the above holds then (8.25) becomes

$$\pi_C(x, L/F) < \Big\{2 - 100\lambda_1 + O(\lambda_1^{1.001})\Big\}\frac{|C|}{|G|}\mathrm{Li}(x).$$

As $\lambda_1 \leq \eta$, this implies $\pi_C(x, L/F) < 2\frac{|C|}{|G|}\mathrm{Li}(x)$ by fixing $\eta$ sufficiently small. Hence, any effective upper bound on $x_1$ translates to a range of $x$ where the sharper bound for $\pi_C(x, L/F)$ holds. From the proof of Theorem 1' in Stark [Sta74], we have that $\lambda_1 \gg \min\{g(n_K)^{-1}, D_K^{-1/n_K}\mathcal{Q}^{-1/2n_K}\}$, where $g(n_K)$ equals 1 if $K$ has a normal tower over $\mathbb{Q}$ and equals $(2n_K)!$ otherwise. If $n_K \leq 10$ and $D_K\mathcal{Q}$ is sufficiently large then we have that

$$x_1 \ll (1/\lambda_1)^{167n_K+328} \ll D_K^{167+328/n_K}\mathcal{Q}^{84+164/n_K} \ll D_K^{495}\mathcal{Q}^{248} \ll x,$$

for $x$ satisfying (1.33), as desired. Thus, we may assume $n_K \geq 10$ in which case we have that

$$\begin{aligned} x_1 &\ll n_K^{167n_K}(1/\lambda_1)^{167n_K+328} \\ &\ll D_K^{167+328/n_K}\mathcal{Q}^{84+164/n_K}n_K^{167n_K} + n_K^{167n_K}g(n_K)^{167n_K+328} \\ &\ll D_K^{200}\mathcal{Q}^{101}n_K^{167n_K} + n_K^{167n_K}g(n_K)^{167n_K+328}. \end{aligned}$$

Therefore, if $K$ has a normal tower over $\mathbb{Q}$ or $(2n_K)! \ll D_K^{1/n_K}\mathcal{Q}^{1/2n_K}$ then

$$x_1 \ll D_K^{200}\mathcal{Q}^{101}n_K^{167n_K}e^{O(n_K)} \ll D_K^{200}\mathcal{Q}^{101}n_K^{168n_K} \ll x,$$

for $x$ satisfying (1.33) and $D_K\mathcal{Q}n_K^{n_K}$ sufficiently large. Otherwise, $g(n_K) \leq (2n_K)! \leq (2n_K)^{2n_K}$ which implies that

$$x_1 \ll D_K^{200}\mathcal{Q}^{101}n_K^{167n_K} + n_K^{333n_K^2}$$

unconditionally. Thus, imposing $x \gg n_K^{334n_K^2}$ in addition to (1.33) also yields the sharper estimate for $\pi_C(x, L/F)$ claimed in the remark after Theorem 1.3.4.

3) Just as Theorem 7.4.1 improves over Theorem 1.3.2 when $n_K$ is absolutely bounded, one could likely improve Theorem 1.3.4 via the same arguments. We have omitted such an argument for the sake of brevity.

# Chapter 9

# Siegel zeros and the least prime ideal

*"Once you do something, you never forget. Even if you can't remember."*

– Zeniba, Spirited Away.

In this chapter, we establish a bound (in an exceptional case when a so-called Siegel zero exists) for the prime ideal of least norm in a ray class of a number field $K$. The proof techniques are based on sieve methods and are completely different than those found in Chapter 7. Furthermore, the exposition will be in the language of ray class groups though one can translate the main theorems into a Chebotarev variant like Theorem 1.3.2. For these reasons, we keep this chapter almost entirely self-contained aside from the notation and results of Chapter 2. Moreover, we will repeat some contents and historical information of Chapter 1 in the language of ray class groups for the sake of clarity.

## 9.1 Introduction

Let $K$ be a number field, $\mathcal{O} = \mathcal{O}_K$ be its ring of integers, and $\mathfrak{q} \subseteq \mathcal{O}$ be an integral ideal. Let $H \pmod{\mathfrak{q}}$ be a congruence class group of $K$. Define the (narrow) ray class group of $K$ modulo $H$, denoted $\mathrm{Cl}(H)$, to be the quotient of fractional ideals of $K$ relatively prime to $\mathfrak{q}$ and $H$. In other words, $\mathrm{Cl}(H) := I(\mathfrak{q})/H$. If $H = P_{\mathfrak{q}}$ is the group of principal ideals $(\alpha)$ such that $\alpha \equiv 1 \pmod{\mathfrak{q}}$ and $\alpha$ is totally positive then $\mathrm{Cl}(P_{\mathfrak{q}}) = \mathrm{Cl}(\mathfrak{q})$ is the usual narrow ray class group of $K$ modulo $\mathfrak{q}$. Recall $Q = Q_H = \max\{\mathrm{N}^K_{\mathbb{Q}}\mathfrak{f}_\chi : \chi \pmod{H}\}$.

For any class $\mathcal{C} \in \mathrm{Cl}(H)$, it has long been known that there are infinitely many prime ideals $\mathfrak{p} \in \mathcal{C}$. Therefore, it is natural to ask:

*What is the least norm of a prime ideal $\mathfrak{p} \in \mathcal{C}$?*

We refer to this question as the *least prime ideal* problem. The Generalized Riemann Hypothesis (GRH) for Hecke $L$-functions implies for $\delta > 0$,

$$\mathrm{N}\mathfrak{p} \ll_\delta (D_K Q)^\delta \cdot h_H^{2+\delta}, \tag{9.1}$$

where $D_K = |\mathrm{disc}(K/\mathbb{Q})|$ is the absolute discriminant of $K$, $\mathrm{N} = \mathrm{N}^K_{\mathbb{Q}}$ is the absolute norm of $K$, and $h_H = \#\mathrm{Cl}(H)$ is the size of the ray class group. Fogels [Fog62b] was the first to give an unconditional answer when $H = P_{\mathfrak{q}}$ (and hence $Q = \mathrm{N}\mathfrak{q}$)) showing

$$\mathrm{N}\mathfrak{p} \ll_{n_K} (D_K Q)^{c(n_K)},$$

where $n_K = [K : \mathbb{Q}]$ is the degree of $K$ over $\mathbb{Q}$ and $c(n_K) > 0$ is a constant depending only on $n_K$. This bound is not entirely satisfactory because the implied constant and exponent depend on $n_K$ in an unspecified manner. In his Ph.D. thesis work, Weiss [Wei83] proved a $K$-uniform version of Fogels' result; that is, unconditionally for any congruence class group $H$ of $K$

$$\mathrm{N}\mathfrak{p} \ll n_K^{A n_K} \cdot D_K^B \cdot Q^C, \tag{9.2}$$

where $A, B, C > 0$ are absolute constants. Assuming GRH as in (9.1) and estimating $h_H$ using Lemma 2.4.6, one may take $(A, B, C) = (\delta, 1+\delta, 2+\delta)$ for $\delta > 0$. The focus of this chapter is, in an exceptional case, to exhibit a bound like (9.2) with explicit exponents.

Specializing to $K = \mathbb{Q}, \mathfrak{q} = (q)$ and $H = P_{\mathfrak{q}} = \{(n) : n \geq 1, n \equiv 1 \pmod q\}$, the least prime ideal problem naturally corresponds to the least prime $p$ in an arithmetic progression $a \pmod q$. Linnik [Lin44a] famously showed unconditionally that

$$p \ll q^L$$

for some absolute constant $L > 0$ known as "Linnik's constant" and where the implicit constant is effective. Conjecturally, $L = 1 + \delta$ for any $\delta > 0$ is admissible and GRH implies $L = 2 + \delta$ is acceptable. The current world record is $L = 5$ by Xylouris [Xyl11b] building upon suggestions of Heath-Brown [HB92].

Thus far, a crucial ingredient to all proofs computing Linnik's constant is the handling of a putative real zero

$$\beta = 1 - \frac{1}{\eta \log q}$$

of a Dirichlet $L$-function attached to a quadratic Dirichlet character $\psi \pmod q$. If $\eta \geq 3$ we refer to this scenario as the *exceptional case* and the zero $\beta$ as an *exceptional zero*. If additionally $1/\eta = o(1)$, then we call $\beta$ a *Siegel zero* which conjecturally does not exist. Most authors adapted Linnik's original proof and established a quantitative Deuring–Heilbronn phenomenon which is a strong form of zero repulsion for $\beta$. However, in the exceptional case, the best bound thus far on Linnik's constant involves sieve methods and was pioneered by Heath-Brown [HB90]. He showed, with effective implicit constants, that $L = 3 + \delta$ is an admissible value provided $\eta \geq \eta(\delta)$ which bests the aforementioned unconditional $L = 5$. Even more astonishingly, Heath-Brown showed that the GRH bound $L = 2 + \delta$ is an admissible value provided $\eta \geq \eta(\delta)$ although the implied constants are ineffective. Sieve techniques are indeed very advantageous in the exceptional case. To further emphasize this point, we remark that Friedlander and Iwaniec [FI03] proved, under some additional technical assumptions, that $L = 2 - \frac{1}{59}$ is admissible when a Siegel zero exists. This surpasses GRH!

Now, let us describe the exceptional case in the context of the least prime ideal problem for a congruence class group $H \pmod{\mathfrak{q}}$ for a number field $K$. Let $\chi \pmod{H}$ be a Hecke character and recall its associated Hecke $L$-function is given by

$$L(s,\chi) = \sum_{\mathfrak{n} \subseteq \mathcal{O}} \chi(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{-s} = \prod_{\mathfrak{p}} \Big(1 - \frac{\chi(\mathfrak{p})}{(\mathrm{N}\mathfrak{p})^s}\Big)^{-1}$$

for $\sigma > 1$, where $s = \sigma + it$. Recall that $L(s,\chi)$ possesses a (nearly) zero-free region of the form

$$\sigma \geq 1 - \frac{c_1}{\log(n_K^{n_K} D_K Q)}, \qquad |t| \leq 1,$$

where $c_1 > 0$ is an absolute constant. However, just as with Dirichlet $L$-functions, exactly one real zero $\beta$ attached to a real Hecke character $\psi \pmod{H}$ cannot be eliminated from this region – no matter how small $c_1$ is chosen. Note that Theorem 4.1.1 implies $c_1 = 0.0875$ is admissible for $n_K^{n_K} D_K Q$ sufficiently large. We emphasize that $\psi$ may be quadratic or principal.

For the remainder of this chapter, suppose $H \pmod{\mathfrak{q}}$ is a *primitive* congruence class group of $K$ and $\psi \pmod{H}$ is a real Hecke character with a real zero

$$\beta = 1 - \frac{1}{\eta \log(n_K^{n_K} D_K Q)}, \tag{9.3}$$

where $\eta \geq 20$; that is, $\beta$ is an *exceptional zero* of the *exceptional character* $\psi$. If $1/\eta = o(1)$ then we shall call $\beta$ a *Siegel zero*. Note that we did not attempt to relax the assumption that $H$ is primitive; it is conceivable that one could obtain similar results without this condition.

For a ray class $\mathcal{C} \in \mathrm{Cl}(H)$ satisfying $\psi(\mathcal{C}) = 1$, we establish an explicit effective field-uniform bound for the size of the least prime ideal $\mathfrak{p} \in \mathcal{C}$ provided $\beta$ is a Siegel zero.

**Theorem 9.1.1.** *Let $H \pmod{\mathfrak{q}}$ be a primitive congruence class group of a number field $K$. Suppose $\psi \pmod{H}$ is a real Hecke character such that $L(s,\psi)$ has a real zero $\beta$ as in* (9.3)*. Let $\mathcal{C} \in \mathrm{Cl}(H)$ satisfy $\psi(\mathcal{C}) = 1$ and $\delta > 0$ be given. Then there exists a prime ideal $\mathfrak{p} \in \mathcal{C}$ satisfying*

$$\mathrm{N}\mathfrak{p} \ll_\delta \{ n_K^{A n_K} \cdot D_K^B \cdot Q^C \cdot h_H^2 \}^{1+\delta}$$

*provided $\eta \geq \eta(\delta)$, where*

$$(A,B,C) = \begin{cases} (16, 6 + \frac{5}{n_K}, 5 + \frac{2}{n_K}) & \text{if } \psi \text{ is quadratic}, \\ (6, 3 + \frac{4}{n_K}, 3) & \text{if } \psi \text{ is principal}. \end{cases} \tag{9.4}$$

*All implicit constants are effective.*

**Remark.**

1) The factor of $h_H^2$ is natural in light of (9.1) but one may prefer a bound similar to (9.2). Using Lemma 2.4.6 allows us to give the alternative bound

$$\mathrm{N}\mathfrak{p} \ll_\delta \left\{ n_K^{A' n_K} \cdot D_K^{B'} \cdot Q^{C'} \right\}^{1+\delta},$$

where

$$(A', B', C') = \begin{cases} (16, 7 + \frac{5}{n_K}, 7 + \frac{2}{n_K}) & \text{if } \psi \text{ is quadratic,} \\ (6, 4 + \frac{4}{n_K}, 5) & \text{if } \psi \text{ is principal.} \end{cases}$$

Even more simply, $(A', B', C') = (16, 9.5, 9)$ is admissible in all cases.

2) For a point of reference, consider the estimate in the special case $K = \mathbb{Q}$ and $\mathfrak{q} = (q)$. If there exists a quadratic Dirichlet character $\psi \,(\mathrm{mod}\, q)$ with real zero $\beta = 1 - \frac{1}{\eta \log q}$ and $\psi(a) = 1$ for $(a, q) = 1$, then Theorem 9.1.1 implies there exists a prime $p \equiv a \,(\mathrm{mod}\, q)$ such that

$$p \ll_\delta q^{9+\delta}$$

provided $\eta \geq \eta(\delta)$. The exponent $L = 9 + \delta$ is comparable to the unconditional $L = 5$ [Xyl11b] and to the effective Siegel zero case $L = 3 + \delta$ [HB90].

3) By a straightforward modification, one can improve Theorem 9.1.1 by appealing to the Brauer-Siegel Theorem (see Theorem 9.3.4) from which it follows that

$$(A, B, C) = \begin{cases} (6, 6, 5) & \text{if } \psi \text{ is quadratic,} \\ (2, 3, 3) & \text{if } \psi \text{ is principal,} \end{cases}$$

or as in Remark 1,

$$(A', B', C') = \begin{cases} (6, 7, 7) & \text{if } \psi \text{ is quadratic,} \\ (2, 4, 5) & \text{if } \psi \text{ is principal,} \end{cases}$$

but the implicit constants are *ineffective*.

Theorem 9.1.1 is a straightforward consequence of the following quantitative lower bound for the number of prime ideals in a given ray class. Here $\kappa_K$ is the residue at $s = 1$ of the Dedekind zeta function $\zeta_K(s)$ and

$$\varphi_K(\mathfrak{q}) = \#(\mathcal{O}/\mathfrak{q})^\times = \mathrm{N}\mathfrak{q} \prod_{\mathfrak{p} | \mathfrak{q}} \Big(1 - \frac{1}{\mathrm{N}\mathfrak{p}}\Big)$$

is the generalized Euler $\varphi$-function of $K$.

**Theorem 9.1.2.** *Let $H \,(\mathrm{mod}\, \mathfrak{q})$ be a primitive congruence class group of a number field $K$. Suppose $\psi \,(\mathrm{mod}\, H)$ is a real Hecke character such that $L(s, \psi)$ has a real zero $\beta$ as in* (9.3). *Let $\mathcal{C} \in \mathrm{Cl}(H)$*

*satisfy* $\psi(\mathcal{C}) = 1$. *For* $\delta > 0$, *assume* $\eta \geq \eta(\delta)$ *and* $M_\delta > 0$ *is sufficiently large. Further assume*

$$M_\delta \cdot \{n_K^{An_K} \cdot D_K^B \cdot Q^C \cdot h_H^2\}^{1+\delta} \leq x \leq M_\delta \cdot (n_K^{n_K} D_K Q)^{100}, \tag{9.5}$$

*where* $(A, B, C)$ *are given by* (9.4). *Then*

$$\#\{\mathfrak{p} \in \mathcal{C} \text{ prime} : \mathrm{N}\mathfrak{p} < x\} \geq c_\psi \Delta_\psi \cdot \kappa_K \frac{\varphi_K(\mathfrak{q})}{\mathrm{N}\mathfrak{q}} \cdot \frac{x}{h_H} \tag{9.6}$$

*where*

$$\Delta_\psi = \begin{cases} L(1,\psi) \prod\limits_{\psi(\mathfrak{p})=1} \Big(1 - \dfrac{3}{\mathrm{N}\mathfrak{p}^2} + \dfrac{2}{\mathrm{N}\mathfrak{p}^3}\Big) \prod\limits_{\psi(\mathfrak{p})=-1} \Big(1 - \dfrac{1}{\mathrm{N}\mathfrak{p}^2}\Big) & \text{if } \psi \text{ is quadratic}, \\ \prod\limits_{\mathfrak{p}\nmid\mathfrak{q}} \Big(1 - \dfrac{1}{\mathrm{N}\mathfrak{p}^2}\Big) & \text{if } \psi \text{ is principal}, \end{cases}$$

*and*

$$c_\psi = \begin{cases} 0.00466 & \text{if } \psi \text{ is quadratic}, \\ 0.0557 & \text{if } \psi \text{ is principal}. \end{cases}$$

*All implicit constants are effectively computable.*

**Remark.**

1) Bounding $h_H$ by Lemma 2.4.6, we see that (9.5) contains the interval

$$M'_\delta \{n_K^{A'n_K} \cdot D_K^{B'} \cdot Q^{C'}\}^{1+2\delta} \leq x \leq M_\delta (n_K^{n_K} D_K Q)^{100}.$$

   where $(A', B', C')$ are given by Remark 1 following Theorem 9.1.1 and $M'_\delta > 0$ is sufficiently large.

2) According to Remark 3 following Theorem 9.1.1, one can widen the lower bound of interval (9.5) using the ineffective Brauer-Siegel Theorem.

3) By obvious modifications to the proof, one can easily obtain an upper bound of the same form as (9.6). That is, for the same range as (9.5), one can show

$$\#\{\mathfrak{p} \in \mathcal{C} \text{ prime} : \mathrm{N}\mathfrak{p} < x\} \leq \tilde{c}_\psi \Delta_\psi \cdot \kappa_K \frac{\varphi_K(\mathfrak{q})}{\mathrm{N}\mathfrak{q}} \cdot \frac{x}{h_H}$$

   where

$$\tilde{c}_\psi = \begin{cases} 8.62 & \text{if } \psi \text{ is quadratic}, \\ 4.02 & \text{if } \psi \text{ is principal}. \end{cases}$$

   Upper bounds for even wider ranges of $x$ could potentially also be established by allowing for a constant larger than $\tilde{c}_\psi$.

4) The constant $c_\psi$ is likely subject to improvement which we do not seriously pursue here as that is not our aim.

5) One can also establish a variant of Theorem 9.1.2 which holds for larger values of $x$. For instance, one could instead assume

$$(M_\delta \cdot n_K^{n_K} D_K \mathrm{N}\mathfrak{q})^\ell \leq x \leq (M_\delta \cdot n_K^{n_K} D_K \mathrm{N}\mathfrak{q})^{100\ell}$$

for any integer $\ell \geq 20$, say. Adapting the argument in Section 9.5.2, one can deduce the same lower bound with

$$c_\psi = \begin{cases} 0.0275 - O(\frac{e^\ell}{\ell!}) & \text{if } \psi \text{ is quadratic,} \\ 0.0749 - O(\frac{e^\ell}{\ell!}) & \text{if } \psi \text{ is principal,} \end{cases}$$

provided $\eta \geq \eta(\delta, \ell)$.

The primary objective of this chapter is to prove Theorem 9.1.2. The arguments involved are motivated by the sieve-based techniques employed for the classical case $K = \mathbb{Q}$, including Heath-Brown's aforementioned foundational paper [HB90] and an elegant modern proof by Friedlander and Iwaniec [FI10, Chapter 24]. To be more specific, let us sketch the main components and, for concreteness, temporarily suppose that $\psi \pmod{H}$ is quadratic. First, we establish the Fundamental Lemma (Theorem 9.2.1) for zero-dimensional sieves in number fields and aim to apply it a sequence $\{a_\mathfrak{n}\}_{\mathfrak{n} \subseteq \mathcal{O}}$, where

$$a_\mathfrak{n} \approx \mu_K^2(\mathfrak{n}) \mathbf{1}\{\mathfrak{n} \in \mathcal{C}\} \cdot \sum_{\mathfrak{d} \mid \mathfrak{n}} \psi(\mathfrak{d}),$$

$\mu_K(\,\cdot\,)$ is the Möbius function defined by (9.7), and $\mathbf{1}\{\,\cdot\,\}$ is an indicator function. Roughly speaking, the sum $\sum_{\mathfrak{d}|\mathfrak{n}} \psi(\mathfrak{d})$ pretends to be an indicator function for integral ideals $\mathfrak{n}$ satisfying $\mathfrak{p} \mid \mathfrak{n} \implies \psi(\mathfrak{p}) = 1$. After computing local densities, we show that our sieve problem is zero-dimensional because $\psi(\mathcal{C}) = 1$ and a Siegel zero is assumed to exist. Then we use a Buchstab identity and apply the Fundamental Lemma to lower bound terms with no small prime ideal factors and upper bound terms with large prime ideal factors. An appropriate choice of the relevant sieve parameters and a Tauberian-type argument finishes the proof.

The numerical values in (9.4) and corresponding bounds in Theorem 9.1.1 are ultimately based on estimates for Hecke $L$-functions inside the critical strip. Similarly, in the classical case, Heath-Brown [HB90] uses Montgomery's mean value theorem for Dirichlet $L$-functions [Mon71] and bounds for their fourth moments inside the critical strip. As far as the author is aware, a suitable mean value theorem for Hecke $L$-functions with complete uniformity over all number fields has not yet been established. We instead employ Rademacher's convexity estimate [Rad60] for Hecke $L$-functions due to its complete uniformity in all aspects. In certain cases, such as the narrow class group for imaginary quadratic fields, one could improve on the numerical values in (9.4) using subconvexity estimates for Hecke $L$-functions contained, for example, in the deep works of Fouvry and Iwaniec [FI01] and Duke, Friedlander, and

Iwaniec [DFI02].

Proving a version of Theorem 9.1.2 for the non-residue case $\psi(\mathcal{C}) = -1$ would certainly be desirable but it is not immediately clear how to do so by sieve-based techniques. In the classical case $K = \mathbb{Q}$, the corresponding sieve problem is one-dimensional leading to an excellent value for Linnik's constant which was first established by Heath-Brown [HB90]. For a general number field $K$ of degree $n_K$, if most small rational primes split then the sieve problem could at worst have dimension $n_K$. Since we seek a bound like (9.2) with absolute exponents, this high dimension issue therefore poses a difficulty when $\psi(\mathcal{C}) = -1$.

Finally, we summarize the organization of this chapter. Section 9.2 sets up a sieve in number fields and proves the Fundamental Lemma for zero-dimensional sieves. The discussion therein is a close adaptation of [FI10, Chapters 5 & 6] but is included for completeness as many variations of number field sieves exist. Section 9.3 consists of notation and elementary estimates related to the exceptional character $\psi$. Section 9.4 computes the key components of our sieve problem – local densities and dimension – and estimates terms with small prime factors and large prime factors. Section 9.5 contains the proof of Theorem 9.1.2.

## 9.2 Sieve theory in number fields

### 9.2.1 Notation

Begin with a sequence $\mathcal{A} = \{a_\mathfrak{n}\}_{\mathfrak{n} \subseteq \mathcal{O}}$ of non-negative real numbers such that

$$|\mathcal{A}| := \sum_{\mathfrak{n} \subseteq \mathcal{O}} a_\mathfrak{n}$$

converges[1]. For an integral ideal $\mathfrak{d} \subseteq \mathcal{O}$, define

$$\mathcal{A}_\mathfrak{d} = \{a_\mathfrak{n} : \mathfrak{d} \mid \mathfrak{n}\}, \qquad |\mathcal{A}_\mathfrak{d}| := \sum_{\mathfrak{d} \mid \mathfrak{n}} a_\mathfrak{n},$$

and suppose

$$|\mathcal{A}_\mathfrak{d}| = g(\mathfrak{d})X + r_\mathfrak{d}$$

for some multiplicative function $g(\mathfrak{d})$ called the *density function* and *remainders* $r_\mathfrak{d}$. The *local densities* $g(\mathfrak{d})$ satisfy

$$0 \leq g(\mathfrak{p}) < 1$$

[1]For instance, one could take $a_\mathfrak{n} = e^{-\mathrm{N}\mathfrak{n}/x}$ with $x \geq 1$.

for all prime ideals $\mathfrak{p}$ of $\mathcal{O}$. Given a set of prime ideals $\mathcal{P}$ and *sifting level* $z \geq 2$, define

$$\mathfrak{P} = \mathfrak{P}(z) := \prod_{\substack{\mathfrak{p} \in \mathcal{P} \\ \mathrm{N}\mathfrak{p} < z}} \mathfrak{p}, \qquad V(z) := \prod_{\substack{\mathfrak{p} \in \mathcal{P} \\ \mathrm{N}\mathfrak{p} < z}} (1 - g(\mathfrak{p})),$$

and

$$S(\mathcal{A}, \mathcal{P}, z) = S(\mathcal{A}, z) := \sum_{(\mathfrak{n}, \mathfrak{P}(z))=1} a_{\mathfrak{n}},$$

where we suppress the dependence on $\mathcal{P}$ or $z$ when it is understood. Recall the Möbius function $\mu_K(\,\cdot\,)$ on integral ideals is defined by

$$\mu_K(\mathfrak{n}) = \begin{cases} (-1)^r & \text{if } \mathfrak{n} = \mathfrak{p}_1 \cdots \mathfrak{p}_r \text{ where } \mathfrak{p}_i \text{ are distinct prime ideals,} \\ 0 & \text{otherwise,} \end{cases} \tag{9.7}$$

or equivalently

$$\sum_{\mathfrak{d} \mid \mathfrak{n}} \mu_K(\mathfrak{d}) = \begin{cases} 1 & \text{if } \mathfrak{n} = (1), \\ 0 & \text{otherwise.} \end{cases} \tag{9.8}$$

Sifting $\mathcal{A}$ according to $\mathcal{P}$ amounts to estimating $S(\mathcal{A}, z)$. It is therefore natural to introduce a function, called the *sieve weight*,

$$\Lambda = (\lambda_{\mathfrak{d}})_{\mathfrak{d}}, \qquad \text{for } \mathfrak{d} \mid \mathfrak{P}(z) \text{ and } \mathrm{N}\mathfrak{d} < D,$$

which acts as a finite approximation to the Möbius function with *level of distribution* $D$. From (9.8), one can easily see that

$$S(\mathcal{A}, z) = \sum_{\mathfrak{d} \mid \mathfrak{P}(z)} \mu(\mathfrak{d}) |\mathcal{A}_{\mathfrak{d}}|,$$

so our approximation takes the form

$$S^{\Lambda}(\mathcal{A}, z) := \sum_{\mathfrak{d}} \lambda_{\mathfrak{d}} |\mathcal{A}_{\mathfrak{d}}| = \sum_{\mathfrak{n}} a_{\mathfrak{n}} \Big( \sum_{\mathfrak{d} \mid \mathfrak{n}} \lambda_{\mathfrak{d}} \Big).$$

Of special importance are weights $\Lambda^+ = (\lambda_{\mathfrak{d}}^+)$ and $\Lambda^- = (\lambda_{\mathfrak{d}}^-)$ satisfying

$$\sum_{\mathfrak{d} \mid \mathfrak{n}} \lambda_{\mathfrak{d}}^- \leq \sum_{\mathfrak{d} \mid \mathfrak{n}} \mu_K(\mathfrak{d}) \leq \sum_{\mathfrak{d} \mid \mathfrak{n}} \lambda_{\mathfrak{d}}^+ \tag{9.9}$$

and therefore implying

$$S^-(\mathcal{A}, z) \leq S(\mathcal{A}, z) \leq S^+(\mathcal{A}, z), \tag{9.10}$$

where the *lower bound sieve* $S^-$ and the *upper bound sieve* $S^+$ correspond to $\Lambda^-$ and $\Lambda^+$ respectively.

In keeping with notation, we naturally define the main term sums by

$$V^+(D,z) = \sum_{\substack{\mathfrak{d}|\mathfrak{P}(z) \\ \mathrm{N}\mathfrak{d}<D}} \lambda_{\mathfrak{d}}^+ g(\mathfrak{d}), \qquad V^-(D,z) = \sum_{\substack{\mathfrak{d}|\mathfrak{P}(z) \\ \mathrm{N}\mathfrak{d}<D}} \lambda_{\mathfrak{d}}^- g(\mathfrak{d}),$$

and remainder terms by

$$R^+(D,z) = \sum_{\substack{\mathfrak{d}|\mathfrak{P}(z) \\ \mathrm{N}\mathfrak{d}<D}} \lambda_{\mathfrak{d}}^+ r_{\mathfrak{d}}, \qquad R^-(D,z) = \sum_{\substack{\mathfrak{d}|\mathfrak{P}(z) \\ \mathrm{N}\mathfrak{d}<D}} \lambda_{\mathfrak{d}}^- r_{\mathfrak{d}}.$$

The conditions under the sums may be dropped in light of the definition of the sieve weights, but we include them for emphasis and clarity.

We will be concerned with sieves satisfying

$$\frac{V(w)}{V(z)} = \prod_{w \le \mathrm{N}\mathfrak{p} < z} \big(1 - g(\mathfrak{p})\big)^{-1} \le C\Big(\frac{\log z}{\log w}\Big)^{\kappa} \qquad \text{for } 2 \le w < z, \tag{9.11}$$

where $C > 1$ is a constant and $\kappa \ge 0$ is the *sieve dimension*.

### 9.2.2 Buchstab iterations

Fix a norm-based total ordering "$\prec$" of prime ideals of $\mathcal{O}$; that is, for prime ideals $\mathfrak{p}$ and $\mathfrak{p}'$,

$$\mathfrak{p} \prec \mathfrak{p}' \implies \mathrm{N}\mathfrak{p} \le \mathrm{N}\mathfrak{p}'.$$

Abusing notation, for $y \in \mathbb{R}$, write $y \prec \mathfrak{p}$ (resp. $\mathfrak{p} \prec y$) if $y < \mathrm{N}\mathfrak{p}$ (resp. $\mathrm{N}\mathfrak{p} < y$). Similarly, write $y \preceq \mathfrak{p}$ (resp. $\mathfrak{p} \preceq y$) if $y \le \mathrm{N}\mathfrak{p}$ (resp. $\mathrm{N}\mathfrak{p} \le y$). Observe that

$$\mathrm{N}\mathfrak{p} \preceq \mathfrak{p} \text{ and } \mathfrak{p} \preceq \mathrm{N}\mathfrak{p}, \qquad \text{but} \qquad \mathrm{N}\mathfrak{p} \not\prec \mathfrak{p} \text{ and } \mathfrak{p} \not\prec \mathrm{N}\mathfrak{p} \tag{9.12}$$

with this choice. Further abusing notation, for a prime ideal $\mathfrak{m}$, define

$$\mathfrak{P}(\mathfrak{m}) := \prod_{\substack{\mathfrak{p} \in \mathcal{P} \\ \mathfrak{p} \prec \mathfrak{m}}} \mathfrak{p}, \qquad V(\mathfrak{m}) := \prod_{\substack{\mathfrak{p} \in \mathcal{P} \\ \mathfrak{p} \prec \mathfrak{m}}} (1 - g(\mathfrak{p})),$$

and

$$S(\mathcal{A}, \mathfrak{m}) := \sum_{(\mathfrak{n}, \mathfrak{P}(\mathfrak{m}))=1} a_{\mathfrak{n}}.$$

Comparing with notation from the previous subsection and using (9.12), notice that

$$\mathfrak{P}(\mathrm{N}\mathfrak{m}) \mid \mathfrak{P}(\mathfrak{m}), \qquad V(\mathfrak{m}) \le V(\mathrm{N}\mathfrak{m}), \qquad \text{and} \qquad S(\mathcal{A}, \mathfrak{m}) \le S(\mathcal{A}, \mathrm{N}\mathfrak{m}).$$

Note that the results herein are independent of the choice of ordering.

Now, choose sieve weights $\Lambda^+ = (\lambda_{\mathfrak{d}}^+)$ and $\Lambda^- = (\lambda_{\mathfrak{d}}^-)$ defined to be the Möbius function truncated to sets of the type

$$\begin{aligned} \mathcal{D}^+ &:= \{\mathfrak{d} = \mathfrak{p}_1 \cdots \mathfrak{p}_\ell : \mathfrak{p}_m \prec y_m \quad \text{for } m \text{ odd}\} \\ \mathcal{D}^- &:= \{\mathfrak{d} = \mathfrak{p}_1 \cdots \mathfrak{p}_\ell : \mathfrak{p}_m \prec y_m \quad \text{for } m \text{ even}\} \end{aligned} \tag{9.13}$$

where $\mathfrak{d}$ is written as a product of distinct prime ideals enumerated in decreasing order,

$$\mathfrak{d} = \mathfrak{p}_1 \cdots \mathfrak{p}_\ell \quad \text{with } z \succ \mathfrak{p}_1 \succ \cdots \succ \mathfrak{p}_\ell.$$

By convention, $\mathcal{D}^+$ and $\mathcal{D}^-$ both contain $\mathfrak{d} = (1)$. The real numbers $y_m$ are *truncation parameters* and by inclusion-exclusion, (9.9) is satisfied regardless of the choices for $y_m$.

Following the discussion on Buchstab iterations in [FI10, Section 6.2], one may similarly deduce

$$S(\mathcal{A}, z) = S^+(\mathcal{A}, z) - \sum_{n \text{ odd}} S_n(\mathcal{A}, z), \tag{9.14}$$

$$S(\mathcal{A}, z) = S^-(\mathcal{A}, z) + \sum_{n \text{ even}} S_n(\mathcal{A}, z), \tag{9.15}$$

where

$$S_n(\mathcal{A}, z) = \sum_{\substack{y_n \preceq \mathfrak{p}_n \prec \cdots \prec \mathfrak{p}_1 \\ \mathfrak{p}_m \prec y_m,\, m<n,\, m \equiv n(2)}} \cdots \sum S(\mathcal{A}_{\mathfrak{p}_1 \cdots \mathfrak{p}_n}, \mathfrak{p}_n). \tag{9.16}$$

Moreover, by the same procedure,

$$V(z) = V^+(D, z) - \sum_{n \text{ odd}} V_n(z), \tag{9.17}$$

$$V(z) = V^-(D, z) + \sum_{n \text{ even}} V_n(z), \tag{9.18}$$

where

$$V_n(z) = \sum_{\substack{y_n \preceq \mathfrak{p}_n \prec \cdots \prec \mathfrak{p}_1 \prec z \\ \mathfrak{p}_m \prec y_m,\, m<n,\, m \equiv n(2)}} \cdots \sum g(\mathfrak{p}_1 \cdots \mathfrak{p}_n) V(\mathfrak{p}_n). \tag{9.19}$$

From (9.14) and (9.15),

$$\begin{aligned} S(\mathcal{A}, z) &\leq S^+(\mathcal{A}, z) = XV^+(D, z) + R^+(D, z), \\ S(\mathcal{A}, z) &\geq S^-(\mathcal{A}, z) = XV^-(D, z) + R^-(D, z). \end{aligned}$$

Thus, to prove the "Fundamental Lemma" for a certain choice of truncation parameters $y_m$, it suffices to upper bound $V_n(z)$ in light of (9.17) and (9.18).

### 9.2.3 Fundamental Lemma for zero dimensional sieves

We assume the sieve dimension is zero, i.e. $\kappa = 0$ in (9.11). For the sets defined in (9.13), choose the truncation parameters

$$y_m = \frac{D}{\mathrm{N}(\mathfrak{p}_1 \cdots \mathfrak{p}_m)},$$

which is an instance of the beta-sieve independently due to Iwaniec and Rosser. Thus, $\lambda_{\mathfrak{d}}^{\pm}$ is a combinatorial weight truncated to $\mathcal{D}^{\pm}$ with level of support $D$. Define the *sifting variable*

$$\tau := \frac{\log D}{\log z}.$$

As previously remarked, it remains to upper bound $V_n(z)$ as defined in (9.19).

Suppose $n \leq \tau - 1$. By our choice of truncation parameters, the condition $y_n \preceq \mathfrak{p}_n$ in (9.19) implies that $D \leq (\mathrm{N}\mathfrak{p}_1)^{n+1} < z^{n+1} \leq z^{\tau} = D$, a contradiction. Thus,

$$V_n(z) = 0 \qquad \text{for } n \leq \tau - 1.$$

Now, suppose $n > \tau - 1$. Since the terms of $V_n(z)$ are non-negative and $V(\mathfrak{p}_n) \leq 1$, we deduce that

$$V_n(z) \leq \sum_{\mathfrak{p}_n \prec \cdots \prec \mathfrak{p}_1 \prec z} \cdots \sum g(\mathfrak{p}_1 \cdots \mathfrak{p}_n) \leq \frac{1}{n!}\Big( \sum_{\mathfrak{p} \prec z} g(\mathfrak{p}) \Big)^n \leq \frac{1}{n!}\big| \log V(z) \big|^n.$$

Using (9.11) with $\kappa = 0$, observe that

$$\frac{V_n(z)}{V(z)} \leq \frac{C(\log C)^n}{n!} \qquad \text{for } n > \tau - 1.$$

Summing over all $n$ of the same parity and using the power series for hyperbolic sine and cosine, observe

$$\begin{aligned}
\sum_{n \text{ odd}} V_n(z) &\leq V(z) \cdot \sum_{\substack{n > \tau - 1 \\ n \text{ odd}}} \frac{C(\log C)^n}{n!} = V(z) \cdot \Big[ \frac{C^2 - 1}{2} - C \sum_{\substack{1 \leq n < n_1(\tau) \\ n \text{ odd}}} \frac{(\log C)^n}{n!} \Big], \\
\sum_{n \text{ even}} V_n(z) &\leq V(z) \cdot \sum_{\substack{n > \tau - 1 \\ n \text{ even}}} \frac{C(\log C)^n}{n!} = V(z) \cdot \Big[ \frac{C^2 + 1}{2} - C \sum_{\substack{0 \leq n < n_0(\tau) \\ n \text{ even}}} \frac{(\log C)^n}{n!} \Big],
\end{aligned}$$

where $n_1(t)$ is the least odd integer $> t - 1$, and $n_0(t)$ is the least even integer $> t - 1$. We have therefore established the following theorem.

**Theorem 9.2.1** (Fundamental Lemma for zero dimensional sieves)**.** *Let $D \geq 1$ and $z \geq 2$. Suppose* (9.11) *holds with $\kappa = 0$ for all $w$ with $2 \leq w < z$ and some $C > 1$. Then*

$$\begin{aligned}
S(\mathcal{A}, z) &\leq XV(z)\Big\{1 + E_1(C;\tau)\Big\} + R^{+}(D, z), \\
S(\mathcal{A}, z) &\geq XV(z)\Big\{1 - E_0(C;\tau)\Big\} + R^{-}(D, z),
\end{aligned} \tag{9.20}$$

*where* $\tau = \frac{\log D}{\log z}$*,* $n_1(t)$ *is the least odd integer* $> t-1$*,* $n_0(t)$ *is the least even integer* $> t-1$*,*

$$E_1(C;\tau) = \frac{C^2-1}{2} - C \sum_{\substack{1 \le n < n_1(\tau) \\ n \text{ odd}}} \frac{(\log C)^n}{n!},$$
$$E_0(C;\tau) = \frac{C^2+1}{2} - C \sum_{\substack{0 \le n < n_0(\tau) \\ n \text{ even}}} \frac{(\log C)^n}{n!},$$

*and* $R^{\pm}(D,z)$ *are the remainders given by*

$$R^{\pm}(D,z) = \sum_{\substack{\mathfrak{d} \mid \mathfrak{P}(z) \\ \mathrm{N}\mathfrak{d} < D}} \lambda_{\mathfrak{d}}^{\pm} r_{\mathfrak{d}} \qquad \text{with} \qquad |\lambda_{\mathfrak{d}}^{\pm}| \le 1.$$

**Remark.** Of course, one could replace $E_0(C;\tau)$ and $E_1(C;\tau)$ by simpler expressions using Taylor's theorem but this results in slightly worse constants.

## 9.3 Exceptional character

In this section, we setup notation related to the central object of our study – the exceptional character $\psi$ – and subsequently prove various estimates by standard methods. Let $\psi \pmod{H}$ be a real character with real zero

$$\beta = 1 - \frac{1}{\eta \log(n_K^{n_K} D_K Q)} \qquad \text{with } \eta \ge 20. \tag{9.21}$$

For integral ideals $\mathfrak{n} \subseteq \mathcal{O}$, define

$$\lambda(\mathfrak{n}) := \begin{cases} \displaystyle\sum_{\substack{\mathfrak{m} \mid \mathfrak{n} \\ (\mathfrak{m},\mathfrak{q})=1}} \psi(\mathfrak{m}) & \text{if } \psi \text{ is quadratic,} \\ \chi_0(\mathfrak{n}) & \text{if } \psi \text{ is principal,} \end{cases} \tag{9.22}$$

and

$$\rho(\mathfrak{n}) := \mu_K^2(\mathfrak{n}) \lambda(\mathfrak{n}), \tag{9.23}$$

where $\mu_K(\,\cdot\,)$ is defined by (9.7) and $\chi_0 \pmod{H}$ is the principal Hecke character. That is, $\chi_0(\mathfrak{n}) = 1$ for all $(\mathfrak{n},\mathfrak{q}) = 1$ and equals zero otherwise. Recall that $\psi(\mathfrak{m}) = 0$ for $(\mathfrak{m},\mathfrak{q}) \neq 1$. Hence, restricting the sum in (9.22) to ideals $\mathfrak{m}$ coprime to $\mathfrak{q}$ is superfluous but added for clarity. Now, we first collect some simple observations about these functions which we state without proof.

**Lemma 9.3.1.** *Define* $\lambda(\mathfrak{n})$ *and* $\rho(\mathfrak{n})$ *as in* (9.22) *and* (9.23) *respectively. Then:*

*(i)* $\rho(\mathfrak{n})$ *and* $\lambda(\mathfrak{n})$ *are multiplicative functions of* $\mathfrak{n}$.

*(ii)* $\rho(\mathfrak{p}) = \lambda(\mathfrak{p}) = 1$ *or* 2 *if* $\psi(\mathfrak{p}) = 1$ *and* $\psi$ *is principal or quadratic respectively.*

*(iii)* $\rho(\mathfrak{n}) = 0$ *if there exists a prime ideal* $\mathfrak{p} \mid \mathfrak{n}$ *such that* $\psi(\mathfrak{p}) = -1$.

*(iv)* $0 \leq \rho(\mathfrak{n}) \leq \lambda(\mathfrak{n})$.

Next, define

$$F_\psi(s) := \sum_{\mathfrak{n} \subseteq \mathcal{O}} \frac{\lambda(\mathfrak{n})}{(\mathrm{N}\mathfrak{n})^s} \qquad \text{for } \mathrm{Re}\{s\} > 1. \tag{9.24}$$

We highlight some basic properties of $F_\psi(s)$ in the following lemma.

**Lemma 9.3.2.** *Define* $F_\psi(s)$ *as in* (9.24). *Then:*

*(i)* $F_\psi(s)$ *extends meromorphically to all of* $\mathbb{C}$ *with only a simple pole at* $s = 1$.

*(ii)* $F_\psi(\beta) = 0$ *where* $\beta$ *is the real zero associated to* $\psi \pmod{\mathfrak{q}}$.

*(iii) For* $\delta \in (0, \frac{1}{2})$ *and* $s = \sigma + it$,

$$F_\psi(s) \leq \begin{cases} Q^\delta \big\{ D_K^2 Q (2 + |t|)^{2n_K} \big\}^{(1-\sigma+\delta)/2} e^{O_\delta(n_K)} & \text{if } \psi \text{ is quadratic}, \\ Q^\delta \big\{ D_K (2 + |t|)^{n_K} \big\}^{(1-\sigma+\delta)/2} e^{O_\delta(n_K)} & \text{if } \psi \text{ is principal}, \end{cases}$$

*uniformly in region* $\delta \leq \sigma \leq 1 + \delta$ *with* $|s - 1| \geq \delta$.

*Proof.* By (9.22),

$$F_\psi(s) = \begin{cases} L(s, \chi_0) L(s, \psi) & \text{if } \psi \text{ is quadratic}, \\ L(s, \chi_0) & \text{if } \psi \text{ is principal}. \end{cases}$$

From this factorization, (i) follows from well-known properties of Hecke $L$-functions and (ii) is implied by $L(\beta, \psi) = 0$. For (iii), use Lemma 2.4.5 with $a = \delta/2$ for the "imprimitive" part of $F_\psi(s)$, i.e. Euler factors corresponding to $\mathfrak{p} \mid \mathfrak{q}$. This is bounded by $(\mathrm{N}\mathfrak{q})^{\delta/2} e^{O_\delta(n_K)}$. As $H \pmod{\mathfrak{q}}$ is primitive, we have by Lemma 2.4.7 that $\mathrm{N}\mathfrak{f}_H = \mathrm{N}\mathfrak{q} \leq Q^2$. Hence, the "imprimitive" part of $F_\psi(s)$ contributes at most $Q^\delta e^{O_\delta(n_K)}$. Second, apply Lemma 2.3.2 to the "primitive" part of $F_\psi(s)$ and note $\zeta_{\mathbb{Q}}(1+\delta) \ll_\delta 1$. Also observe that $D_\chi = D_K \mathrm{N}\mathfrak{f}_\chi \leq D_K Q$ by the definition of $Q$ in (2.2). Combining these estimates yields the claimed bound. □

In light of Lemma 9.3.2, we define some naturally-occurring quantities. First,

$$\kappa_\psi = \operatorname*{Res}_{s=1} F_\psi(s) = \begin{cases} \dfrac{\varphi_K(\mathfrak{q})}{\mathrm{N}\mathfrak{q}} \kappa_K L(1, \psi) & \text{if } \psi \text{ is quadratic}, \\ \dfrac{\varphi_K(\mathfrak{q})}{\mathrm{N}\mathfrak{q}} \kappa_K & \text{if } \psi \text{ is principal}, \end{cases} \tag{9.25}$$

where $\kappa_K$ is the residue of the Dedekind zeta function $\zeta_K(s)$ at $s = 1$ and

$$\varphi_K(\mathfrak{q}) = \#(\mathcal{O}/\mathfrak{q})^\times = \mathrm{N}\mathfrak{q} \prod_{\mathfrak{p} \mid \mathfrak{q}} \Big(1 - \frac{1}{\mathrm{N}\mathfrak{p}}\Big)$$

is the generalized Euler $\varphi$-function. Further, denote

$$W_\psi = \begin{cases} n_K^{2n_K} D_K^2 Q & \text{if } \psi \text{ is quadratic,} \\ n_K^{n_K} D_K & \text{if } \psi \text{ is principal.} \end{cases} \tag{9.26}$$

For the remainder of this section, we collect various well-known lower and upper bounds for $\kappa_\psi$ and establish other relevant estimates involving $\lambda(\mathfrak{n})$. The arguments are straightfoward with standard applications of Mellin inversion.

**Theorem 9.3.3** (Stark)**.** *For $\delta > 0$,*

$$\frac{1}{\kappa_\psi} \ll_\delta \begin{cases} n_K^{(2+\delta)n_K} D_K^{1/n_K} Q^{1/2n_K+\delta} & \text{if } \psi \text{ is quadratic,} \\ n_K^{(1+\delta)n_K} D_K^{1/n_K} Q^{\delta} & \text{if } \psi \text{ is principal,} \end{cases}$$

*where all implicit constants are effective.*

*Proof.* This is a weak rephrasing of [Sta74, Theorem 1] to our context. If $\psi$ is principal then $\kappa_\psi = \frac{\varphi_K(\mathfrak{q})}{\mathrm{N}\mathfrak{q}}\kappa_K$ so the result follows from [Sta74, Theorem 1] and by noting $\frac{\mathrm{N}\mathfrak{q}}{\varphi_K(\mathfrak{q})} \ll e^{O_\delta(n_K)}(\mathrm{N}\mathfrak{q})^{\delta/2} \ll_\delta n_K^{\delta n_K} Q^\delta$ from Lemmas 2.4.5 and 2.4.7. If $\psi$ is quadratic, then consider the quadratic extension of $K$ given by $M = K(\psi)$ implying $\kappa_M = \kappa_K L(1,\psi^*)$ where $\psi^*$ is the primitive character inducing $\psi$. Since $L(1,\psi) \geq \frac{\varphi(\mathfrak{q})}{\mathrm{N}\mathfrak{q}} L(1,\psi^*)$, it follows that $\kappa_\psi \geq \big(\frac{\varphi_K(\mathfrak{q})}{\mathrm{N}\mathfrak{q}}\big)^2 \kappa_M$ so we again apply [Sta74, Theorem 1] and Lemmas 2.4.5 and 2.4.7 to prove the claim. □

**Theorem 9.3.4** (Brauer–Siegel)**.** *For $\delta > 0$,*

$$\frac{1}{\kappa_\psi} \ll_\delta (n_K^{n_K} D_K Q)^\delta,$$

*where the implicit constant is ineffective.*

*Proof.* Similar to Theorem 9.3.3 but instead of using [Sta74, Theorem 1] to bound the residues of Dedekind zeta functions, we apply the celebrated Brauer-Siegel theorem:

$$\kappa_M \gg_\epsilon d_M^{-\epsilon}$$

for any number field $M$, where the implicit constant is ineffective. See [Bra47] for details. □

Theorem 9.3.4 is the only result with ineffective constants so we reiterate that, unless otherwise stated, *all implicit constants are effective and absolute*.

**Lemma 9.3.5** (Stark)**.** *For $\delta > 0$ arbitrary, $\kappa_\psi \gg_\delta (1-\beta)(n_K^{n_K} Q)^{-\delta}$.*

*Proof.* Arguing as in Theorem 9.3.3, this is an analogous rephrasing of [Sta74, Lemma 4] to our context. □

**Lemma 9.3.6.** *For $\delta > 0$,*

$$\sum_{\mathfrak{n}} \frac{\lambda(\mathfrak{n})}{(\mathrm{N}\mathfrak{n})^{\beta}} e^{-\mathrm{N}\mathfrak{n}/y} = \kappa_\psi \Gamma(1-\beta) y^{1-\beta} \{1 + O(\delta)\},$$

*provided*

$$y \geq M_\delta W_\psi^{1/2+\delta} Q^{\delta} \tag{9.27}$$

*for some sufficiently large constant $M_\delta \geq 1$.*

*Proof.* We apply Mellin inversion to see

$$S := \sum_{\mathfrak{n}} \frac{\lambda(\mathfrak{n})}{(\mathrm{N}\mathfrak{n})^{\beta}} e^{-\mathrm{N}\mathfrak{n}/y} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} F_\psi(s+\beta)\Gamma(s) y^s ds.$$

Shift the line of integration to $\mathrm{Re}\{s\} = 1/2 - \beta$, pick up the pole $s = 1-\beta$, and bound the remaining integral using Lemma 9.3.2(iii) and (2.14). Therefore,

$$S = \Big\{ \kappa_\psi \Gamma(1-\beta) + O_\delta\Big( \frac{W_\psi^{1/4+\delta/8} Q^{\delta/8} e^{O_\delta(n_K)}}{y^{1/2}} \Big) \Big\} y^{1-\beta}.$$

Note $W_\psi^{1/4+\delta/8} e^{O_\delta(n_K)} \ll_\delta W_\psi^{1/4+\delta/4}$ from the $n_K^{n_K}$ factor in the definition of $W_\psi$. Thus, by condition (9.27),

$$S = \Big\{ \kappa_\psi \Gamma(1-\beta) + O_\delta\Big( \frac{1}{M_\delta^{1/2} (n_K^{n_K} D_K Q)^{\delta/8}} \Big) \Big\} y^{1-\beta}.$$

From Lemma 9.3.5, it follows that the main term dominates the error provided $M_\delta$ is sufficiently large. This yields the desired result. □

**Lemma 9.3.7.** *For $\delta > 0$ and $y_2 \geq 3y_1$,*

$$\sum_{\mathfrak{n}} \frac{\lambda(\mathfrak{n})}{\mathrm{N}\mathfrak{n}} \big( e^{-\mathrm{N}\mathfrak{n}/y_2} - e^{-\mathrm{N}\mathfrak{n}/y_1} \big) \ll \kappa_\psi \log(y_2/y_1),$$

*provided*

$$y_1 \geq M_\delta \cdot \kappa_\psi^{-1-\delta} W_\psi^{1/2+\delta} Q^{\delta}$$

*for some sufficiently large constant $M_\delta \geq 1$.*

*Proof.* By Mellin inversion,

$$S' := \sum_{\mathfrak{n}} \frac{\lambda(\mathfrak{n})}{\mathrm{N}\mathfrak{n}} \big( e^{-\mathrm{N}\mathfrak{n}/y_2} - e^{-\mathrm{N}\mathfrak{n}/y_1} \big) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} F_\psi(s+1)\Gamma(s) \{ y_2^s - y_1^s \} ds.$$

Shift the line of integration to $\mathrm{Re}\{s\} = -1 + \delta$, pick up the simple pole at $s = 0$, and bound the remaining integral using Lemma 9.3.2(iii) and (2.14). Thus, for $\delta > 0$,

$$S' = \kappa_\psi \log(y_2/y_1) + O_\delta\Big(\frac{W_\psi^{1/2+\delta/8} Q^{\delta/8} e^{O_\delta(n_K)}}{y_1^{1-\delta}}\Big).$$

Since $W_\psi^{1/2+\delta/8} e^{O_\delta(n_K)} \ll_\delta W_\psi^{1/2+\delta/4}$ and $\log(y_2/y_1) \gg 1$, the result follows from the condition on $y_1$. □

## 9.4 Application of the sieve

### 9.4.1 Sieve sequence

Let $\mathcal{C} \in \mathrm{Cl}(H)$ be any ray class satisfying $\psi(\mathcal{C}) = 1$ and retain the notation of Section 9.3. Recall that the Hecke $L$-function $L(s, \psi)$ is assumed to have a real zero

$$\beta = 1 - \frac{1}{\eta \log(n_K^{n_K} D_K \mathrm{N}\mathfrak{q})}$$

with $\eta \geq 20$. Let $2 \leq z \leq x$. During the course of our arguments, the parameter $z$ will be chosen and the valid range of $x$ will be specified. We wish to apply the sieve to the sequence

$$\mathcal{A} = \mathcal{A}(x) = \{a_\mathfrak{n}\}_{\mathfrak{n} \subseteq \mathcal{O}} \quad \text{with} \quad a_\mathfrak{n} = \rho(\mathfrak{n}) e^{-\mathrm{N}\mathfrak{n}/x} \cdot \mathbf{1}_\mathcal{C}(\mathfrak{n}), \tag{9.28}$$

where $\rho(\mathfrak{n})$ is defined in (9.23) and $\mathbf{1}_\mathcal{C}(\mathfrak{n})$ is an indicator function for $\mathcal{C}$. The choice of the smoothing weight $e^{-\mathrm{N}\mathfrak{n}/x}$ for $a_\mathfrak{n}$ was made for the sake of simplicity and without any claim to optimality for the resulting constants in (9.4). Other sufficiently smooth weights, such as $e^{-(\mathrm{N}\mathfrak{n}/x)^2}$ or $\big(1 - \frac{\mathrm{N}\mathfrak{n}}{x}\big)^{2n_K}$, could also potentially be used although we did not investigate these possibilities. However, for any suitable choice of weight, we expect the factor of $n_K^{An_K}$ to appear in Theorem 9.1.1 with a possibly different value for $A$.

Now, choose the set of prime ideals to be

$$\mathcal{P} = \{\mathfrak{p} \subseteq \mathcal{O} \text{ prime} : \psi(\mathfrak{p}) = 1\} \tag{9.29}$$

and denote

$$\mathcal{D} = \{\mathfrak{d} \subseteq \mathcal{O} \text{ square-free} : \mathfrak{p} \mid \mathfrak{d} \implies \psi(\mathfrak{p}) = 1\}. \tag{9.30}$$

### 9.4.2 Local densities

**Lemma 9.4.1.** *Let $\mathfrak{d} \in \mathcal{D}$. Then, for any $\delta > 0$,*

$$|\mathcal{A}_{\mathfrak{d}}| = \sum_{\substack{\mathfrak{n} \in \mathcal{C} \\ \mathfrak{d} | \mathfrak{n}}} \rho(\mathfrak{n}) e^{-\mathrm{N}\mathfrak{n}/x} = g(\mathfrak{d}) X + r_{\mathfrak{d}}$$

*with $X = b_\psi \kappa_\psi \cdot \dfrac{x}{h_H}$, where if $\psi$ is quadratic then*

$$\begin{aligned} b_\psi &= 2 \prod_{\psi(\mathfrak{p})=1} \Big(1 - \frac{3}{\mathrm{N}\mathfrak{p}^2} + \frac{2}{\mathrm{N}\mathfrak{p}^3}\Big) \prod_{\psi(\mathfrak{p})=-1} \Big(1 - \frac{1}{\mathrm{N}\mathfrak{p}^2}\Big), \\ g(\mathfrak{p}) &= \frac{2}{\mathrm{N}\mathfrak{p} + 2} \quad \textit{for } \mathfrak{p} \in \mathcal{P}, \\ |r_{\mathfrak{d}}| &\ll_\delta \frac{x^{1/2+\delta}}{(\mathrm{N}\mathfrak{d})^{1/2}} \cdot (n_K^{n_K} D_K Q)^{(1+\delta)/2}, \end{aligned}$$

*and if $\psi$ is principal then*

$$\begin{aligned} b_\psi &= \prod_{\mathfrak{p} \nmid \mathfrak{q}} \Big(1 - \frac{1}{\mathrm{N}\mathfrak{p}^2}\Big), \\ g(\mathfrak{p}) &= \frac{1}{\mathrm{N}\mathfrak{p} + 1} \quad \textit{for } \mathfrak{p} \in \mathcal{P}, \\ |r_{\mathfrak{d}}| &\ll_\delta \frac{x^{1/2+\delta}}{(\mathrm{N}\mathfrak{d})^{1/2}} \cdot (n_K^{n_K} D_K Q)^{(1+\delta)/4}. \end{aligned}$$

**Remark.** If $\mathfrak{d} \notin \mathcal{D}$, then $|A_{\mathfrak{d}}| = 0$ by Lemma 9.3.1. Thus, for prime ideals $\mathfrak{p} \notin \mathcal{P}$, set $g(\mathfrak{p}) = 0$ and multiplicatively extend the function $g$ to all integral ideals of $\mathcal{O}$.

*Proof.* We adapt the proof of [HB90, Lemma 1] with some modifications when bounding the remainder terms $r_{\mathfrak{d}}$. Write

$$f(s, \chi) := \sum_{\substack{\mathfrak{n} \subseteq \mathcal{O} \\ \mathfrak{d} | \mathfrak{n}}} \rho(\mathfrak{n}) \chi(\mathfrak{n}) (\mathrm{N}\mathfrak{n})^{-s} \qquad \text{for } \mathrm{Re}\{s\} > 1$$

so, by orthogonality and Mellin inversion,

$$\sum_{\substack{\mathfrak{n} \in \mathcal{C} \\ \mathfrak{d} | \mathfrak{n}}} \rho(\mathfrak{n}) e^{-\mathrm{N}\mathfrak{n}/x} = \frac{1}{h_H} \sum_{\chi \,(\mathrm{mod}\, H)} \overline{\chi}(\mathcal{C}) \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} f(s, \chi) \Gamma(s) x^s ds. \tag{9.31}$$

Alternatively, we may write $f(s, \chi)$ as an Euler product to see that

$$f(s, \chi) = \rho(\mathfrak{d}) \chi(\mathfrak{d}) (\mathrm{N}\mathfrak{d})^{-s} \times \prod_{\substack{\mathfrak{p} \nmid \mathfrak{d} \\ \psi(\mathfrak{p})=1}} \Big(1 + \rho(\mathfrak{p}) \frac{\chi(\mathfrak{p})}{(\mathrm{N}\mathfrak{p})^s}\Big) \times \prod_{\substack{\mathfrak{p} \nmid \mathfrak{d} \\ \psi(\mathfrak{p})=-1}} 1.$$

Note that prime ideals $\mathfrak{p} \mid \mathfrak{d}$ do not appear in the Euler product since $\rho(\mathfrak{n}) = 0$ for $\mathfrak{n}$ not square-free.

Including these analogous factors, we may write

$$f(s,\chi) = \prod_{\psi(\mathfrak{p})=1} \Big(1 + \rho(\mathfrak{p})\frac{\chi(\mathfrak{p})}{(\mathrm{N}\mathfrak{p})^s}\Big) \times \prod_{\psi(\mathfrak{p})=-1} 1 \times g_{\mathfrak{d}}(s,\chi), \tag{9.32}$$

where

$$g_{\mathfrak{d}}(s,\chi) = \rho(\mathfrak{d})\chi(\mathfrak{d})(\mathrm{N}\mathfrak{d})^{-s} \prod_{\substack{\mathfrak{p}\mid\mathfrak{d} \\ \psi(\mathfrak{p})=1}} \Big(1 + \rho(\mathfrak{p})\frac{\chi(\mathfrak{p})}{(\mathrm{N}\mathfrak{p})^s}\Big)^{-1}.$$

On the other hand,

$$\begin{aligned} L(s,\chi)L(s,\chi\psi) &= \prod_{\psi(\mathfrak{p})=1} \Big(1 - 2\frac{\chi(\mathfrak{p})}{(\mathrm{N}\mathfrak{p})^s} + \frac{\chi^2(\mathfrak{p})}{(\mathrm{N}\mathfrak{p})^{2s}}\Big)^{-1} \times \prod_{\psi(\mathfrak{p})=-1} \Big(1 - \frac{\chi^2(\mathfrak{p})}{(\mathrm{N}\mathfrak{p})^{2s}}\Big)^{-1}, \\ L(s,\chi) &= \prod_{\mathfrak{p}\nmid\mathfrak{q}} \Big(1 - \frac{\chi(\mathfrak{p})}{(\mathrm{N}\mathfrak{p})^s}\Big)^{-1}. \end{aligned}$$

Upon comparing with (9.32), we deduce

$$f(s,\chi) = g_{\mathfrak{d}}(s,\chi)g(s,\chi)G(s,\chi), \tag{9.33}$$

where

$$g(s,\chi) = \begin{cases} \displaystyle\prod_{\psi(\mathfrak{p})=1} \Big(1 - 3\frac{\chi^2(\mathfrak{p})}{(\mathrm{N}\mathfrak{p})^{2s}} + 2\frac{\chi^3(\mathfrak{p})}{(\mathrm{N}\mathfrak{p})^{3s}}\Big) \times \prod_{\psi(\mathfrak{p})=-1} \Big(1 - \frac{\chi^2(\mathfrak{p})}{(\mathrm{N}\mathfrak{p})^{2s}}\Big) & \text{if } \psi \text{ is quadratic,} \\ \displaystyle\prod_{\mathfrak{p}\nmid\mathfrak{q}} \Big(1 - \frac{\chi^2(\mathfrak{p})}{(\mathrm{N}\mathfrak{p})^{2s}}\Big) & \text{if } \psi \text{ is principal,} \end{cases}$$

and

$$G(s,\chi) = \begin{cases} L(s,\chi)L(s,\chi\psi) & \text{if } \psi \text{ is quadratic,} \\ \\ L(s,\chi) & \text{if } \psi \text{ is principal.} \end{cases}$$

Therefore, $f(s,\chi)$ has meromorphic continuation to $\mathbb{C}$ and is analytic in $\mathrm{Re}\{s\} > 1/2$, except possibly for a pole at $s = 1$ when $\chi$ or $\chi\psi$ is principal.

Furthermore, we claim

$$g_{\mathfrak{d}}(s,\chi) \ll_\delta e^{O_\delta(n_K)}(\mathrm{N}\mathfrak{d})^{-1/2}, \tag{9.34}$$

$$g(s,\chi) \ll_\delta e^{O_\delta(n_K)}, \tag{9.35}$$

uniformly in the region $\mathrm{Re}(s) \geq 1/2+\delta$ for any $\delta > 0$. Here we ignore $s$ in neighborhoods of poles arising from local factors of $g_{\mathfrak{d}}(s,\chi)$ with $\mathrm{N}\mathfrak{p} < 4$. To see the claim, notice (9.35) follows from Lemma 2.4.5(i). Estimate (9.34) follows from Lemma 2.4.5(iii) with $a = 1/2$ combined with the observation that

$$\rho(\mathfrak{d}) \ll \sum_{\mathfrak{p}|\mathfrak{d}} 1 \ll \log \mathrm{N}\mathfrak{d} \ll_\delta (\mathrm{N}\mathfrak{d})^\delta.$$

This proves the claim.

Now, we move the line of integration in (9.31) from $\mathrm{Re}\{s\} = 2$ to $\mathrm{Re}\{s\} = \frac{1}{2}+\delta$. This yields a main term of

$$R = \frac{x}{h_H} \sum_{\chi \,(\mathrm{mod}\, H)} \overline{\chi}(\mathcal{C}) \operatorname*{Res}_{s=1} f(s,\chi).$$

Before computing $R$, observe that since $\psi(\mathcal{C}) = 1$ and $\psi^2 = \chi_0$

$$f(s,\chi_0) = f(s,\psi), \qquad G(s,\chi_0) = F_\psi(s),$$
$$g(1,\chi_0) = g(1,\psi), \qquad g_{\mathfrak{d}}(1,\chi_0) = g_{\mathfrak{d}}(1,\psi) = g(\mathfrak{d}),$$

where $F_\psi(s)$ and $g(\mathfrak{d})$ are defined in (9.24) and the statement of Lemma 9.4.1 respectively. Therefore, if $\psi$ is quadratic, the main term $R$ picks up residues for $\chi = \chi_0$ and $\chi = \psi$. Namely,

$$\begin{aligned} R &= \frac{x}{h_H}\Big[\operatorname*{Res}_{s=1} f(s,\chi_0) + \overline{\psi}(\mathcal{C}) \operatorname*{Res}_{s=1} f(s,\psi)\Big] \\ &= \frac{x}{h_H}\cdot g(1,\chi_0)g(\mathfrak{d})\cdot\Big[2\operatorname*{Res}_{s=1} F_\psi(s)\Big] \\ &= \frac{x}{h_H}\cdot g(1,\chi_0)g(\mathfrak{d})\cdot 2\kappa_\psi \\ &= g(\mathfrak{d})X, \end{aligned}$$

since $b_\psi = 2g(1,\chi_0)$ when $\psi$ is quadratic. If $\psi$ is principal, the main term $R$ picks up a residue for $\chi = \chi_0$ only. In other words,

$$\begin{aligned} R &= \frac{x}{h_H}\cdot \operatorname*{Res}_{s=1} f(s,\chi_0) \\ &= \frac{x}{h_H}\cdot g(1,\chi_0)g(\mathfrak{d})\cdot \operatorname*{Res}_{s=1} L(s,\chi_0) \\ &= \frac{x}{h_H}\cdot g(1,\chi_0)g(\mathfrak{d})\cdot \kappa_\psi \\ &= g(\mathfrak{d})X, \end{aligned}$$

since $b_\psi = g(1, \chi_0)$ when $\psi$ is principal.

Thus far, we have shown

$$|\mathcal{A}_{\mathfrak{d}}| = g(\mathfrak{d})X + r_{\mathfrak{d}},$$

where

$$r_{\mathfrak{d}} = \frac{1}{h_H} \sum_{\chi \,(\mathrm{mod}\, H)} \overline{\chi}(\mathcal{C}) \frac{1}{2\pi i} \int_{1/2+\delta-i\infty}^{1/2+\delta+i\infty} f(s,\chi)\Gamma(s)x^s ds.$$

To bound the remainder, we factor $f(s,\chi)$ via (9.33) and apply the estimates (9.34), (9.35), and (2.14). This yields

$$|r_{\mathfrak{d}}| \ll_\delta \frac{x^{\frac{1}{2}+\delta} e^{O_\delta(n_K)}}{h_H (\mathrm{N}\mathfrak{d})^{1/2}} \sum_{\chi \,(\mathrm{mod}\, H)} \int_{-\infty}^{\infty} |G(\tfrac{1}{2}+\delta+it,\chi)| e^{-|t|} dt.$$

Hence, the desired result follows from the convexity bound for Hecke $L$-functions (Lemma 2.3.2), Lemmas 2.4.5 and 2.4.7, and observing as usual that $e^{O_\delta(n_K)} \ll_\delta n_K^{\delta n_K}$. □

Motivated by the bounds on the remainder terms $r_{\mathfrak{d}}$ in Lemma 9.4.1, we define

$$U_\psi = \begin{cases} (n_K^{n_K} D_K Q)^{1/2} & \text{if } \psi \text{ is quadratic,} \\ (n_K^{n_K} D_K Q)^{1/4} & \text{if } \psi \text{ is principal,} \end{cases} \tag{9.36}$$

so more simply

$$|r_{\mathfrak{d}}| \ll \frac{x^{1/2+\delta}}{(\mathrm{N}\mathfrak{d})^{1/2}} U_\psi^{1+\delta}.$$

### 9.4.3 Sieve dimension

We prove our sieve problem is zero-dimensional.

**Lemma 9.4.2.** *For $\delta > 0$,*

$$\sum_{\substack{\mathrm{N}\mathfrak{p}<z \\ \psi(\mathfrak{p})=1}} \frac{1}{\mathrm{N}\mathfrak{p}} \leq 1+\delta,$$

*provided $\eta \geq \eta(\delta)$ and $z \leq (n_K^{n_K} D_K Q)^{O_\delta(1)}$.*

*Proof.* According to Lemma 9.3.6, set

$$y = M_\delta W_\psi^{1/2+\delta} Q^\delta,$$

where $W_\psi$ is defined in (9.26). Using $\lambda(\mathfrak{n})$ defined in (9.22) and its properties described in Lemma 9.3.1, one can verify that $\lambda(\mathfrak{n}) \leq \lambda(\mathfrak{n}\mathfrak{p})$ for $\psi(\mathfrak{p}) = 1$ and $\mathfrak{n} \subseteq \mathcal{O}$. Thus,

$$\Big( \sum_{\substack{\mathrm{N}\mathfrak{p}<z \\ \psi(\mathfrak{p})=1}} \frac{1}{(\mathrm{N}\mathfrak{p})^\beta} \Big) \Big( \sum_{\mathfrak{n}} \frac{\lambda(\mathfrak{n})}{(\mathrm{N}\mathfrak{n})^\beta} e^{-\mathrm{N}\mathfrak{n}/y} \Big) \leq \sum_{\mathfrak{n}} \frac{\lambda(\mathfrak{n})}{(\mathrm{N}\mathfrak{n})^\beta} e^{-\mathrm{N}\mathfrak{n}/yz}, \tag{9.37}$$

which we write as $S_1 S_2 \leq S_3$, say. It suffices to show $S_1 \leq 1+\delta$. By our choice $y$, we may apply Lemma 9.3.6 to $S_2$ and $S_3$ deducing

$$S_1 \leq z^{1-\beta}\{1+O(\delta)\}.$$

Since $z \leq (n_K^{n_K} D_K Q)^{O_\delta(1)}$ by assumption, we conclude that

$$S_1 \leq 1 + O(\delta) + O_\delta(\eta^{-1}),$$

whence the result follows after rescaling $\delta$. □

**Corollary 9.4.3.** *Let $g(\mathfrak{d})$ be the multiplicative function defined in Lemma 9.4.1 and $\delta > 0$ be arbitrary. Then, provided $\eta \geq \eta(\delta)$,*

$$\frac{V(w)}{V(z)} = \prod_{w \leq \mathrm{N}\mathfrak{p} < z} \Big(1 - g(\mathfrak{p})\Big)^{-1} \leq C_\psi := \begin{cases} e^{2+\delta} & \text{if } \psi \text{ is quadratic}, \\ e^{1+\delta} & \text{if } \psi \text{ is principal}, \end{cases}$$

*for all $2 \leq w \leq z \leq (n_K^{n_K} D_K Q)^{O_\delta(1)}$. In particular,* (9.11) *holds with $C = C_\psi$ and $\kappa = 0$.*

### 9.4.4 Small prime ideal factors

With the local densities and dimension computed, we may now apply the Fundamental Lemma and sieve out small primes. Before doing so, we restrict the choice of sieve parameters for the remainder of the section. For $\delta > 0$, suppose

$$B_\delta \cdot \{\kappa_\psi^{-1} + 1\}^{1+\delta} \cdot W_\psi^{1/2+\delta} Q^\delta \leq z \leq (n_K^{n_K} D_K Q)^{O_\delta(1)} \tag{9.38}$$

for some sufficiently large constant $B_\delta > 0$. Define

$$D = \frac{x^{1-4\delta}}{h_H^2 U_\psi^{2+2\delta}}, \qquad \tau = \frac{\log D}{\log z}, \tag{9.39}$$

where $\kappa_\psi, W_\psi, U_\psi$ are defined in (9.25), (9.26), and (9.36) respectively.

**Proposition 9.4.4.** *For $\delta > 0$, suppose the sifting level $z$ satisfies* (9.38) *and define the level of distribution $D$ and sifting variable $\tau$ as in* (9.39)*. Assume $\eta \geq \eta(\delta)$ and*

$$x \geq M_\delta \big\{ (\kappa_\psi^{-1} + 1) W_\psi^{1/4} U_\psi \cdot h_H \big\}^{2+20\delta} \tag{9.40}$$

*for $M_\delta > 0$ sufficiently large. Then*

$$\begin{aligned} S(\mathcal{A}, z) &\leq XV(z)\Big\{1 + E_1(C_\psi; \tau) + O_\delta\big(\frac{1}{\log x}\big)\Big\}, \\ S(\mathcal{A}, z) &\geq XV(z)\Big\{1 - E_0(C_\psi; \tau) + O_\delta\big(\frac{1}{\log x}\big)\Big\}, \end{aligned} \tag{9.41}$$

*where $E_0$ and $E_1$ are defined in Theorem 9.2.1, and $C_\psi$ is defined in Corollary 9.4.3.*

*Proof.* We only prove the lower bound; the upper bound follows similarly. With the described choice of parameters, we employ the Fundamental Lemma for zero-dimensional sieves (Theorem 9.2.1) in conjunction with Lemma 9.4.1 and Corollary 9.4.3, yielding

$$S(\mathcal{A}, z) \geq XV(z)\Big\{1 - E_0(C_\psi; \tau)\Big\} + R^-(\mathcal{A}, D). \tag{9.42}$$

Since the sequence $\mathcal{A} = \{a_\mathfrak{n}\}_\mathfrak{n}$ is only supported on the set $\mathcal{D}$ (defined in Section 9.4.1),

$$R^-(\mathcal{A}, D) \ll \sum_{\substack{\mathrm{N}\mathfrak{d} < D \\ \mathfrak{d} \in \mathcal{D}}} |r_\mathfrak{d}|.$$

From Lemma 9.3.1, it follows $\mathbf{1}\{\mathfrak{d} \in \mathcal{D}\} \leq \lambda(\mathfrak{d})$ so by Lemma 9.4.1,

$$\begin{aligned} R^-(\mathcal{A}, D) &\ll_\delta x^{1/2+\delta} U_\psi^{1+\delta} \sum_{\mathrm{N}\mathfrak{d} < D} \lambda(\mathfrak{d})(\mathrm{N}\mathfrak{d})^{-1/2} \\ &\ll_\delta x^{1/2+\delta} U_\psi^{1+\delta} \sum_{\mathfrak{d}} \lambda(\mathfrak{d})(\mathrm{N}\mathfrak{d})^{-1/2} e^{-\mathrm{N}\mathfrak{d}/D}. \end{aligned} \tag{9.43}$$

By Mellin inversion, the sum over $\mathfrak{d}$ equals

$$\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} F_\psi(s + \tfrac{1}{2})\Gamma(s) D^s ds.$$

Pulling the contour to $\mathrm{Re}\{s\} = \delta/8$, we pick up a main term of $\kappa_\psi \Gamma(1/2) D^{1/2}$ and bound the resulting integral using Lemma 9.3.2 and (2.14). Applying these estimates in (9.43), we find

$$R^-(\mathcal{A}, D) \ll_\delta x^{1/2+\delta} U_\psi^{1+\delta}\Big(\kappa_\psi D^{1/2} + W_\psi^{1/4+\delta/8} Q^{\delta/8} e^{O_\delta(n_K)} D^{\delta/8}\Big)$$

By (9.40), the first term in the parentheses dominates whence

$$R^-(\mathcal{A}, D) \ll_\delta \kappa_\psi U_\psi^{1+\delta} x^{1/2+\delta} D^{1/2} \ll_\delta \frac{\kappa_\psi x^{1-\delta}}{h_H}.$$

Since $z$ satisfies the upper bound in (9.38), it follows from Corollary 9.4.3 and the definition of $X$ in Lemma 9.4.1 that

$$XV(z) \gg_\delta \frac{\kappa_\psi x}{h_H} \cdot \frac{1}{e^{O_\delta(n_K)}}$$

for $\eta \geq \eta(\delta)$. By these two observations, we conclude that

$$R^-(\mathcal{A}, D) \ll_\delta XV(z)x^{-\delta/2} \ll_\delta \frac{XV(z)}{\log x}$$

provided $x \geq e^{O_\delta(n_K)}$. This latter condition on $x$ is clearly implied by assumption (9.40). Substituting this estimate into (9.42) yields the desired result. □

### 9.4.5 Large prime ideal factors

**Lemma 9.4.5.** *Suppose $\mathfrak{p} \in \mathcal{P}$ satisfies $z \leq \mathrm{N}\mathfrak{p} < x^{1/2}$ and assume $z$ satisfies* (9.38). *Then for $\delta > 0$,*

$$S(\mathcal{A}_\mathfrak{p}, \mathfrak{p}) \ll_\delta \frac{XV(z)}{\mathrm{N}\mathfrak{p}},$$

*provided $\eta \geq \eta(\delta)$ and*

$$M_\delta\big\{(\kappa_\psi^{-1} + 1)W_\psi^{1/4}U_\psi \cdot h_H\big\}^{4+50\delta} \leq x \leq M_\delta(n_K^{n_K} D_K Q)^{100} \tag{9.44}$$

*for $M_\delta > 0$ sufficiently large.*

*Proof.* From Section 9.2, recall $S(\mathcal{A}_\mathfrak{p}, \mathfrak{p}) \leq S(\mathcal{A}_\mathfrak{p}, \mathrm{N}\mathfrak{p})$ so it suffices to bound the latter. Using Lemma 9.4.1 and Corollary 9.4.3, we apply the upper bound sieve from Theorem 9.2.1 to the sequence $\mathcal{A}_\mathfrak{p}$ with level of distribution $D' = D/\mathrm{N}\mathfrak{p}$, sifting level $z' = \mathrm{N}\mathfrak{p}$, and sifting variable $\tau' = \frac{\log D'}{\log z'}$. This application therefore yields

$$S(\mathcal{A}_\mathfrak{p}, \mathrm{N}\mathfrak{p}) \ll g(\mathfrak{p})XV(z) + \sum_{\substack{\mathfrak{d} \mid \mathfrak{P}(z') \\ \mathrm{N}\mathfrak{d} < D'}} |r_{\mathfrak{p}\mathfrak{d}}|$$

since $V(\mathrm{N}\mathfrak{p}) \leq V(z)$ for $\mathrm{N}\mathfrak{p} \geq z$. As $g(\mathfrak{p}) \ll (\mathrm{N}\mathfrak{p})^{-1}$ by Lemma 9.4.1, it suffices to bound the remainder sum. Following the same argument as in Proposition 9.4.4, we see that

$$\begin{aligned}
\sum_{\substack{\mathfrak{d} \mid \mathfrak{P}(z') \\ \mathrm{N}\mathfrak{d} < D'}} |r_{\mathfrak{p}\mathfrak{d}}| &\ll_\delta \frac{x^{1/2+\delta}}{(\mathrm{N}\mathfrak{p})^{1/2}} U_\psi^{1+\delta} \sum_{\mathfrak{d}} \frac{\lambda(\mathfrak{d})}{(\mathrm{N}\mathfrak{d})^{1/2}} e^{-\mathrm{N}\mathfrak{d}\mathfrak{p}/D} \\
&\ll_\delta \frac{x^{1/2+\delta}}{(\mathrm{N}\mathfrak{p})^{1/2}} U_\psi^{1+\delta} \Big(\kappa_\psi \big(\frac{D}{\mathrm{N}\mathfrak{p}}\big)^{1/2} + W_\psi^{1/4+\delta/8} Q^{\delta/8} e^{O_\delta(n_K)} \big(\frac{D}{\mathrm{N}\mathfrak{p}}\big)^{\delta/8}\Big) \\
&\ll_\delta \frac{1}{\mathrm{N}\mathfrak{p}} \cdot \kappa_\psi x^{1/2+\delta} U_\psi^{1+\delta} D^{1/2}
\end{aligned}$$

provided (9.44) holds. One can similarly show that the above is $\ll_\delta XV(z)(\mathrm{N}\mathfrak{p})^{-1}$ since $z$ satisfies the upper bound in (9.38) and $\eta \geq \eta(\delta)$. □

**Lemma 9.4.6.** *Let $\delta > 0$ and assume $z$ satisfies* (9.38)*. For $x > 2z$,*

$$\sum_{\substack{z \leq \mathrm{N}\mathfrak{p} < x \\ \psi(\mathfrak{p}) = 1}} \frac{1}{\mathrm{N}\mathfrak{p}} \ll_\delta (1-\beta)\log x$$

*provided $\eta \geq \eta(\delta)$.*

*Proof.* From Lemma 9.3.1 and the condition $x > 2z$, notice

$$\sum_{\substack{z \leq \mathrm{N}\mathfrak{p} < x \\ \psi(\mathfrak{p}) = 1}} \frac{1}{\mathrm{N}\mathfrak{p}} \ll \sum_{z \leq \mathrm{N}\mathfrak{p} < x} \frac{\lambda(\mathfrak{p})}{\mathrm{N}\mathfrak{p}} \big\{ e^{-\mathrm{N}\mathfrak{p}/x} - e^{-\mathrm{N}\mathfrak{p}/z} \big\} = S_1,$$

say, so we estimate $S_1$. Observe that

$$S_2 := \sum_{\mathfrak{n}} \frac{\lambda(\mathfrak{n})}{(\mathrm{N}\mathfrak{n})^\beta} e^{-2\mathrm{N}\mathfrak{n}/z} = \sum_{\mathfrak{n}} \frac{\lambda(\mathfrak{n})}{\mathrm{N}\mathfrak{n}} e^{-\mathrm{N}\mathfrak{n}/z} \cdot H_{1-\beta}\Big(\frac{\mathrm{N}\mathfrak{n}}{z}\Big) z^{1-\beta},$$

where $H_\epsilon(t) = t^\epsilon e^{-t}$ for $\epsilon > 0$ and $t > 0$. By calculus, $H_\epsilon(t)$ is maximized at $t = \epsilon$ and $H_\epsilon(\epsilon) \to 1$ as $\epsilon \to 0^+$. Moreover, $z^{1-\beta} = 1 + O((1-\beta)\log z)$. Therefore, by (9.38),

$$S_2 \ll_\delta \sum_{\mathfrak{n}} \frac{\lambda(\mathfrak{n})}{\mathrm{N}\mathfrak{n}} e^{-\mathrm{N}\mathfrak{n}/z}$$

for $\eta \geq \eta(\delta)$. Hence, using Lemma 9.3.1, we see that

$$\begin{aligned} S_1 S_2 &\ll \Big( \sum_{z \leq \mathrm{N}\mathfrak{p} < x} \frac{\lambda(\mathfrak{p})}{\mathrm{N}\mathfrak{p}} \big\{ e^{-\mathrm{N}\mathfrak{p}/x} - e^{-\mathrm{N}\mathfrak{p}/z} \big\} \Big) \Big( \sum_{\mathfrak{n}} \frac{\lambda(\mathfrak{n})}{\mathrm{N}\mathfrak{n}} e^{-\mathrm{N}\mathfrak{n}/z} \Big) \\ &\ll \sum_{\mathfrak{n}} \frac{\lambda(\mathfrak{n})}{\mathrm{N}\mathfrak{n}} \big\{ e^{-\mathrm{N}\mathfrak{n}/xz} - e^{-\mathrm{N}\mathfrak{n}/z} \big\} = S_3, \end{aligned}$$

say. By both the lower and upper bound of (9.38), we may lower bound $S_2$ using Lemma 9.3.6 and upper bound $S_3$ using Lemma 9.3.7. Combining these estimates and noting $z^{1-\beta} \geq 1$ yields the desired bound for $S_1$ provided $\eta \geq \eta(\delta)$. □

## 9.5 Proof of Theorem 9.1.2

We claim Theorem 9.1.2 is a consequence of the following result.

**Theorem 9.5.1.** *Let $H \pmod{\mathfrak{q}}$ be a primitive congruence class group of a number field $K$. Suppose $\psi \pmod{H}$ is an real Hecke character of the number field $K$ with associated real zero $\beta$ as in* (9.3)*. Let $\mathcal{C} \in \mathrm{Cl}(H)$ satisfy $\psi(\mathcal{C}) = 1$ and $\delta > 0$ be given. Denote $X$ as per Lemma 9.4.1. Assume $x$ satisfies* *both of the following*

$$x \leq M_\delta \cdot (n_K^{n_K} D_K Q)^{100}, \tag{9.45}$$
$$x \geq M_\delta \cdot \{(\kappa_\psi^{-1}+1)^4 W_\psi U_\psi^4 h_H^4\}^{1+50\delta}, \tag{9.46}$$

*for $M_\delta > 0$ sufficiently large. If $\psi$ is quadratic then*

$$\sum_{\substack{\mathrm{N}\mathfrak{p}<x \\ \mathfrak{p}\in\mathcal{C}}} \rho(\mathfrak{p}) \geq 0.00466 \cdot X$$

*provided $\eta \geq \eta(\delta)$ and additionally*

$$x \geq M_\delta \cdot \{(\kappa_\psi^{-1}+1)^5 W_\psi^{5/2} U_\psi^2 h_H^2\}^{1+50\delta}. \tag{9.47}$$

*Otherwise, if $\psi$ is principal then*

$$\sum_{\substack{\mathrm{N}\mathfrak{p}<x \\ \mathfrak{p}\in\mathcal{C}}} \rho(\mathfrak{p}) \geq 0.0557 \cdot X$$

*provided $\eta \geq \eta(\delta)$ and additionally*

$$x \geq M_\delta \cdot \{(\kappa_\psi^{-1}+1)^3 W_\psi^{3/2} U_\psi^2 h_H^2\}^{1+50\delta}. \tag{9.48}$$

**Remark.** Recall $\kappa_\psi, W_\psi, U_\psi$ are defined in (9.25), (9.26), and (9.36) respectively.

This section is dedicated to the proofs of Theorems 9.1.2 and 9.5.1.

### 9.5.1 Proof of Theorem 9.1.2 from Theorem 9.5.1

By comparing notation[2], one can verify that it suffices show that (9.5) is implied by (9.46) and (9.47) when $\psi$ is quadratic and similarly is implied by (9.46) and (9.48) when $\psi$ is principal. If $\psi$ is quadratic, then by Theorem 9.3.3 and Lemma 2.4.6,

$$(\kappa_\psi^{-1}+1)^4 W_\psi U_\psi^4 h_H^4 \ll_\delta \left\{ n_K^{12 n_K} D_K^{5+\frac{4}{n_K}} Q^{5+\frac{2}{n_K}} \cdot h_H^2 \right\}^{1+\delta},$$
$$(\kappa_\psi^{-1}+1)^5 W_\psi^{5/2} U_\psi^2 h_H^2 \ll_\delta \left\{ n_K^{16 n_K} D_K^{6+\frac{5}{n_K}} Q^{3.5+\frac{2.5}{n_K}} \cdot h_H^2 \right\}^{1+\delta}.$$

[2]Note that $\rho(\mathfrak{p}) = 1$ and $\Delta_\psi = b_\psi$ if $\psi$ is principal, and $\rho(\mathfrak{p}) = 2$ and $\Delta_\psi = L(1,\psi) b_\psi/2$ if $\psi$ is quadratic.

One can therefore see by inspection that (9.46) and (9.47) indeed imply (9.5). If $\psi$ is principal, then similarly

$$(\kappa_\psi^{-1}+1)^4 W_\psi U_\psi^4 h_H^4 \ll_\delta \{n_K^{6n_K} D_K^{3+\frac{4}{n_K}} Q^3 \cdot h_H^2\}^{1+\delta},$$
$$(\kappa_\psi^{-1}+1)^3 W_\psi^{3/2} U_\psi^2 h_H^2 \ll_\delta \{n_K^{5n_K} D_K^{2+\frac{3}{n_K}} Q^{0.5} \cdot h_H^2\}^{1+\delta}.$$

Again, one can see by inspection that (9.46) and (9.48) imply (9.5). □

### 9.5.2 Proof of Theorem 9.5.1

Let $y \in [1,10]$ be a parameter which is to be optimized later. Consider the sequence[3]

$$\mathcal{A}^{(y)} = \{a_\mathfrak{n}^{(y)}\}_\mathfrak{n} \qquad \text{given by} \qquad a_\mathfrak{n}^{(y)} = \rho(\mathfrak{n}) e^{-y\mathrm{N}\mathfrak{n}/x} \cdot \mathbf{1}_\mathcal{C}(\mathfrak{n}),$$

where $\rho(\mathfrak{n})$ is defined in (9.23). For $B_\delta > 0$ sufficiently large, choose

$$z = B_\delta \cdot \{\kappa_\psi^{-1}+1\}^{1+\delta} \cdot W_\psi^{1/2+\delta} Q^\delta,$$

so $z$ indeed satisfies (9.38). Analogous to (9.39), define

$$D_y = \frac{(x/y)^{1-4\delta}}{h_H^2 U_\psi^{2+2\delta}}, \qquad \tau_y = \frac{\log D_y}{\log z}.$$

Furthermore, according to the notation of Lemma 9.4.1, denote

$$X = b_\psi \kappa_\psi \frac{x}{h_H}, \qquad V(z) = \prod_{\mathrm{N}\mathfrak{p}<z} (1-g(\mathfrak{p})).$$

Now, by Lemma 9.3.1, $\mathcal{A}^{(y)}$ is supported on $\mathfrak{n}$ satisfying $\mathfrak{p} \mid \mathfrak{n} \implies \psi(\mathfrak{p}) = 1$. Thus, we have the following Buchstab identity:

$$S(\mathcal{A}^{(y)}, \sqrt{x}) = S(\mathcal{A}^{(y)}, z) - \sum_{\substack{z \le \mathrm{N}\mathfrak{p} < \sqrt{x} \\ \psi(\mathfrak{p})=1}} S(\mathcal{A}_\mathfrak{p}^{(y)}, \mathfrak{p}). \tag{9.49}$$

Noting $a_\mathfrak{n}^{(y)} \le a_\mathfrak{n}^{(1)}$, it follows $S(\mathcal{A}_\mathfrak{p}^{(y)}, \mathfrak{p}) \le S(\mathcal{A}_\mathfrak{p}^{(1)}, \mathfrak{p})$. Moreover, $(1-\beta)\log x \ll_\delta \eta^{-1}$ by (9.45). Thus, from (9.46) and Lemmas 9.4.5 and 9.4.6, it follows that

$$\sum_{\substack{z \le \mathrm{N}\mathfrak{p} < \sqrt{x} \\ \psi(\mathfrak{p})=1}} S(\mathcal{A}_\mathfrak{p}^{(y)}, \mathfrak{p}) \ll_\delta \eta^{-1} \cdot XV(z) \tag{9.50}$$

[3]Comparing with the notation of (9.28), notice $\mathcal{A}(x/y) = \mathcal{A}^{(y)}$.

provided $\eta \geq \eta(\delta)$. Assumption (9.46) allows us to apply Proposition 9.4.4 to $S(\mathcal{A}^{(y)}, z)$ so, combined with (9.49) and (9.50), we deduce that

$$S(\mathcal{A}^{(y)}, \sqrt{x}) \geq \frac{1}{y}\Big\{1 - E_0(C_\psi; \tau_y) + O(\delta) + O_\delta\big(\frac{1}{\log x}\big)\Big\} \cdot XV(z) \tag{9.51}$$

provided $\eta \geq \eta(\delta)$. It remains to convert the "exponentially-weighted sieve" to the usual "cutoff sieve". Observe that

$$\begin{aligned} S(\mathcal{A}^{(y)}, \sqrt{x}) &= \sum_{\substack{\mathfrak{p} \in \mathcal{C} \\ \sqrt{x} \leq \mathrm{N}\mathfrak{p} < x}} \rho(\mathfrak{p}) e^{-y\mathrm{N}\mathfrak{p}/x} + \sum_{\substack{\mathfrak{n} \in \mathcal{C} \\ (\mathfrak{n}, \mathfrak{P}(\sqrt{x}))=1 \\ \mathrm{N}\mathfrak{n} \geq x}} \rho(\mathfrak{n}) e^{-y\mathrm{N}\mathfrak{n}/x}, \\ &= S_1 + S_2 \end{aligned} \tag{9.52}$$

say. To complete the proof, it suffices to lower bound $S_1$ so we require an upper bound on $S_2$. As $y \geq 1, z \leq \sqrt{x}$ and $x$ satisfies (9.46), it follows by Proposition 9.4.4 that

$$S_2 \leq e^{-y+1} S(\mathcal{A}^{(1)}, z) \leq e^{-y+1}\Big\{1 + E_1(C_\psi; \tau_1) + O(\delta) + O_\delta\big(\frac{1}{\log x}\big)\Big\} \cdot XV(z).$$

Using the above, (9.51), and (9.52), we conclude for $\eta \geq \eta(\delta)$ that

$$S_1 \geq \frac{1}{C_\psi}\Big\{\frac{1}{y}\Big(1 - E_0(C_\psi; \tau_y)\Big) - e^{-y+1}\Big(1 + E_1(C_\psi; \tau_1)\Big) + O(\delta) + O_\delta\big(\frac{1}{\log x}\big)\Big\} \cdot X \tag{9.53}$$

after bounding $V(z)$ by Corollary 9.4.3. Finally, we consider cases.

### $\psi$ quadratic

Then (9.47) and our choice of $z$ imply $\tau_1 \geq \tau_y > 5$, so $n_0(\tau_y) \geq 6$ and $n_1(\tau_1) \geq 5$. Hence, by the definitions in Theorem 9.2.1,

$$\begin{aligned} E_0(C_\psi; \tau_y) &\leq \big(\tfrac{1}{2}e^4 - \tfrac{11}{3}e^2 + \tfrac{1}{2}\big)\{1 + O(\delta)\}, \\ E_1(C_\psi; \tau_1) &\leq \big(\tfrac{1}{2}e^4 - \tfrac{10}{3}e^2 - \tfrac{1}{2}\big)\{1 + O(\delta)\}, \end{aligned}$$

since $C_\psi = e^{2+\delta}$ by Corollary 9.4.3. Substituting these bounds into (9.53), choosing roughly optimally $y = 7.37$, and rescaling $\delta$ appropriately completes the proof of Theorem 9.5.1 when $\psi$ is quadratic.

### $\psi$ principal

Then (9.48) and our choice of $z$ imply $\tau_1 \geq \tau_y > 3$, so $n_0(\tau_y) \geq 4$ and $n_1(\tau_1) \geq 3$. Hence, by the definitions in Theorem 9.2.1,

$$\begin{aligned} E_0(C_\psi; \tau_y) &\leq \big(\tfrac{1}{2}e^2 - \tfrac{3}{2}e + \tfrac{1}{2}\big)\{1 + O(\delta)\}, \\ E_1(C_\psi; \tau_1) &\leq \big(\tfrac{1}{2}e^2 - e - \tfrac{1}{2}\big)\{1 + O(\delta)\}, \end{aligned}$$

since $C_\psi = e^{1+\delta}$ by Corollary 9.4.3. Substituting these bounds into (9.53), choosing roughly optimally $y = 4.54$, and rescaling $\delta$ appropriately completes the proof of Theorem 9.5.1 when $\psi$ is principal. $\square$

# Chapter 10

# Elliptic curves and modular forms

*"It is a lovely language, but it takes a very long time saying anything in it, because we do not say anything in it, unless it is worth taking a long time to say, and to listen to."*

– Treebeard, The Lord of the Rings.

In this chapter, we present the proofs for the applications to elliptic curves and modular forms found in Section 1.4. Background on elliptic curves and modular forms can be found in [ST94, Sil09] and [DS05, MM06, Ono04] respectively. Aside from basic definitions in Chapters 1 and 2, the notation here will be self-contained.

## 10.1 Reformulating Theorems 1.3.2 and 1.3.3

First, we state slightly weaker (but more convenient) reformulations of Theorems 1.3.2 and 1.3.3. For an abelian extension $L/K$, the max conductor quantity $\mathcal{Q} = \mathcal{Q}(L/K)$, defined by (1.22), and discriminant $D_K$ measures the ramification occurring in $L/K$ and $K/\mathbb{Q}$ respectively. However, it can be somewhat cumbersome to use these in certain arithmetic applications. To measure the ramification of $L/K$, we will therefore avoid using $\mathcal{Q}$ and instead use the set

$$\mathcal{P}(L/K) = \{p \text{ prime} : \mathfrak{p} \text{ prime ideal of } K \text{ with } \mathfrak{p} \mid (p) \text{ and } \mathfrak{p} \text{ ramifies in } L\}. \tag{10.1}$$

The following proposition allows us to reformulate our main results in terms of $\mathcal{P}(L/K)$.

**Proposition 10.1.1** (Murty–Murty–Saradha)**.** *If $L/K$ is an abelian extension of number fields then*

$$\mathcal{Q}(L/K) \leq \Big([L:K] \prod_{p \in \mathcal{P}(L/K)} p\Big)^{2n_K}.$$

*Proof.* See [MMS88, Proposition 2.5], which proves a more general result. □

Next, we record an alternate bound for $D_K = |\mathrm{disc}(K/\mathbb{Q})|$ using $n_K = [K:\mathbb{Q}]$ and the squarefree part of $D_K$. For positive integers $n$, let $\omega(n) = \#\{p : p \mid n\}$ and $\mathrm{rad}(n) = \prod_{p \mid n} p$.

**Proposition 10.1.2** (Serre). *For any number field $K$,*

$$D_K \leq (n_K)^{n_K \omega(D_K)} \mathrm{rad}(D_K)^{n_K - 1}.$$

*If $K$ is Galois over $\mathbb{Q}$ then $\omega(D_K)$ may be replaced by* 1.

*Proof.* See [Ser81, Proposition 6]. □

Combining these two propositions yields the following lemma.

**Lemma 10.1.3.** *Let $A, B, C \geq 0$. If $L/K$ is an abelian extension of number fields with max conductor $\mathcal{Q} = \mathcal{Q}(L/K)$ defined by* (7.20) *then*

$$D_K^A \mathcal{Q}^B (n_K^{n_K})^C \leq \Big[ [L:K]^B n_K^{A\omega(D_K)+C} \mathrm{rad}(D_L)^{A+2B} \Big]^{n_K}.$$

*Proof.* Since every prime $p \in \mathcal{P}(L/K)$ divides the discriminant of $L/\mathbb{Q}$, we have that $\prod_{p \in \mathcal{P}(L/K)} p \leq \mathrm{rad}(D_L)$. Furthermore, $\mathrm{rad}(D_K) \leq \mathrm{rad}(D_L)$ since $D_K \mid D_L$. Combining these observations with Propositions 10.1.1 and 10.1.2 yields the desired result. □

We may now reformulate two of our main theorems.

**Theorem 10.1.4.** *Let $L/F$ be a Galois extension of number fields with Galois group $G$, and let $C$ be any conjugacy class of $G$. Let $H$ be an abelian subgroup of $G$ such that $H \cap C$ is nonempty, and let $K \neq \mathbb{Q}$ be the subfield of $L$ fixed by $H$. Then*

$$\pi_C(x, L/F) \gg \frac{1}{\big([L:K]^4 \mathrm{rad}(D_L)^{13} n_K^{5\omega(D_K)+3}\big)^{n_K}} \cdot \frac{|C|}{|G|} \frac{x}{\log x}.$$

*for $x \geq \big\{[L:K]^{521} \mathrm{rad}(D_L)^{1736} n_K^{694\omega(D_K)+290}\big\}^{n_K}$ and $([L:K] n_K \mathrm{rad}(D_L))^{n_K}$ sufficiently large. In particular,*

$$P(C, L/F) \ll \big\{[L:K]^{521} \mathrm{rad}(D_L)^{1736} n_K^{694\omega(D_K)+290}\big\}^{n_K}.$$

*Proof.* Aside from the “sufficiently large” condition, this is an immediate consequence of Theorem 1.3.2 and Lemma 10.1.3. It remains to show that $D_K \mathcal{Q} n_K^{n_K} \to \infty$ if and only if $([L:K] n_K \mathrm{rad}(D_L))^{n_K} \to \infty$. The “only if” direction follows from Lemma 10.1.3. Now consider the“if” direction. If $n_K \to \infty$ or $\mathrm{rad}(D_K) \to \infty$ then we are done. By Lemma 7.3.1, if $[L:K] \to \infty$ then we are done. Thus, we may assume that $\displaystyle\prod_{\substack{p \in \mathcal{P}(L/K) \\ p \nmid D_K}} p \to \infty$, where $\mathcal{P}(L/K)$ is given by (10.1). By the conductor-discriminant formula (2.21) and the definition of $\mathcal{Q}$ in (7.20), any prime $p \in \mathcal{P}(L/K)$ with $p \nmid D_K$ must divide the norm of a conductor $\mathfrak{f}_\chi$ for some Hecke character $\chi$ attached to $L/K$. By the definition of $\mathcal{Q}$ in (7.20), this implies that

$$\prod_{\substack{p \in \mathcal{P}(L/K) \\ p \nmid D_K}} p \leq \prod_{p \leq \mathcal{Q}} p.$$

Thus, if the former quantity is unbounded then so is the max conductor $\mathcal{Q}$. This completes the verification of the "sufficiently large" condition. $\square$

**Remark.**

- For comparison, if one uses [Ser81, Proposition 6] to bound $D_L$, then Theorem 1.3.1 implies that $P(C, L/F) \ll (n_L^{\omega(D_L)} \operatorname{rad}(D_L))^{40n_L}$. Thus, the above theorem gives an asymptotic improvement when $|H| = [L : K]$ is large.
- The arguments in the above proof may be used to quantify the "sufficiently large" condition in terms of $D_K \mathcal{Q} n_K^{n_K}$. We omit these details for brevity.

**Theorem 10.1.5.** *Let $L/F$ be a Galois extension of number fields with Galois group $G$, and let $C$ be any conjugacy class of $G$. Let $H$ be an abelian subgroup of $G$ such that $H \cap C$ is non-empty, and let $K$ be the subfield of $L$ fixed by $H$. Define*

$$M(L/K) = [L : K] D_K^{1/n_K} \prod_{p \in \mathcal{P}(L/K)} p. \tag{10.2}$$

*If* $\log x \gg n_K \log(M(L/K) n_K)$ *then*

$$\pi_C(x, L/F) \ll \frac{|C|}{|G|} \operatorname{Li}(x).$$

**Remark.** Theorems 1.3.2 and 10.1.4 can be restated using $M(L/K)$ as well; that is,

$$P(C, L/F) \ll (n_K M(L/K))^{1050 n_K},$$

since $1050 > \max\{694, 521 \cdot 2, 290\}$.

*Proof.* Using the definition of $M(L/K)$, by Proposition 10.1.1, we see that (1.31) is

$$\ll (D_K \mathcal{Q}(L/K) n_K^{n_K})^{246} \ll (n_K M(L/K))^{500 n_K}.$$

The claimed result now follows immediately from Theorem 1.3.3. $\square$

## 10.2 Proofs of Theorems 1.4.5 to 1.4.7

### $\mathrm{GL}_2$ extensions

We will now review some facts about $\mathrm{GL}_2$ extensions of $\mathbb{Q}$ and class functions to prove Theorems 1.4.5 to 1.4.7. Let $f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z} \in \mathbb{Z}[[e^{2\pi i z}]]$ be a non-CM newform of even weight $k \geq 2$ and level $N \geq 1$. By Deligne [Del71], there exists a representation

$$\rho_{f,\ell} : \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_\ell)$$

with the property that if $p \nmid \ell N$ and $\sigma_p$ is a Frobenius element at $p$ in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, then $\rho_{f,\ell}$ is unramified at $p$, $\mathrm{tr}\ \rho_{f,\ell}(\sigma_p) \equiv a_f(p) \pmod{\ell}$, and $\det \rho_{f,\ell}(\sigma_p) \equiv p^{k-1} \pmod{\ell}$. If $\ell$ is sufficiently large (depending on $f$) then the representation is surjective. Let $L = L_\ell$ be the subfield of $\overline{\mathbb{Q}}$ fixed by the kernel of $\rho_{f,\ell}$. Then $L/\mathbb{Q}$ is a Galois extension unramified outside $\ell N$ whose Galois group is $\ker \rho_{f,\ell}$, which is isomorphic to a subgroup of

$$G = G_\ell = \{A \in \mathrm{GL}_2(\mathbb{F}_\ell)\colon \det A \text{ is a } (k-1)\text{-th power in } \mathbb{F}_\ell^\times\}.$$

If $\ell \gg_f 1$ then the representation is surjective, in which case

$$\ker \rho_{f,\ell} \cong G. \tag{10.3}$$

When $k = 2$ and the level is $N$, $f$ is necessarily the newform of a non-CM elliptic curve $E/\mathbb{Q}$ of conductor $N$. In this case, we write $\rho_{f,\ell} = \rho_{E,\ell}$, and $L$ is the $\ell$-division field $\mathbb{Q}(E[\ell])$. It is conjectured that $\ker \tilde{\rho}_{E,\ell} \cong \mathrm{GL}_2(\mathbb{F}_\ell)$ for all $\ell > 37$. When $E/\mathbb{Q}$ is non-CM and has squarefree level, it follows from the work of Mazur [Maz78] that $\ker \tilde{\rho}_{E,\ell} \cong \mathrm{GL}_2(\mathbb{F}_\ell)$ for all $\ell \geq 11$.

**Lemma 10.2.1.** *Let $L/\mathbb{Q}$ be a $\mathrm{GL}_2(\mathbb{F}_\ell)$ extension which is unramified outside of $\ell N$ for some $N \geq 1$. Let $C \subset \mathrm{GL}_2(\mathbb{F}_\ell)$ be a conjugacy class intersecting the subgroup $D$ of diagonal matrices. There exists a prime $p \nmid \ell N$ so that $[\frac{L/\mathbb{Q}}{p}] = C$ and*

$$p \ll \left\{\ell^{2778}\mathrm{rad}(N)^{1736}(\ell(\ell+1))^{694\omega(N)+984}\right\}^{\ell(\ell+1)}.$$

*Proof.* If $K = L^D$ is the subfield of $L$ fixed by $D$, then $[L : K] = (\ell-1)^2$ and $[K : \mathbb{Q}] = \ell(\ell+1)$. Moreover, $\mathrm{rad}(D_L) \mid \ell\,\mathrm{rad}(N)$ and $\omega(D_K) \leq \omega(D_L) \leq 1 + \omega(N)$. The result now follows immediately from Theorem 10.1.4. $\square$

**Proof of Theorem 1.4.5**

It follows from the proof of [Mur94, Theorem 4] and Mazur's torsion theorem [Maz78] that it suffices to consider $\ell \geq 11$. Let $L = \mathbb{Q}(E[\ell])$ be the $\ell$-division field of $E/\mathbb{Q}$. For $p \nmid \ell N_E$, we have that $E(\mathbb{F}_p)$ has an element of order $\ell$ if and only if

$$\mathrm{tr}\ \rho_{\ell,E}(\sigma_p) \equiv \det \rho_{\ell,E}(\sigma_p) + 1 \pmod{\ell}, \tag{10.4}$$

where $\sigma_p$ is Frobenius automorphism at $p$ in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. If $\mathrm{Gal}(L/\mathbb{Q}) \cong \mathrm{GL}_2(\mathbb{F}_\ell)$, then the $\rho_{\ell,E}(\sigma_p) \in \mathrm{GL}_2(\mathbb{F}_\ell)$ which satisfy (10.4) form a union of conjugacy classes in $\mathrm{GL}_2(\mathbb{F}_\ell)$ which includes the identity element. The subgroup $D$ of diagonal matrices is a maximal abelian subgroup of $\mathrm{GL}_2(\mathbb{F}_\ell)$. Thus $\pi_{\{\mathrm{id}\}}(x, L/\mathbb{Q})$ is a lower bound for the function that counts the primes $p \leq x$ such that $p \nmid \ell N_E$ and $\ell \mid \#E(\mathbb{F}_p)$. Since $\mathrm{rad}(D_L) \mid \ell\,\mathrm{rad}(N)$, Lemma 10.2.1 exhibits a prime $p$ with the desired arithmetic properties satisfying

$$p \ll \left\{\ell^{2778}\mathrm{rad}(N)^{1736}(\ell(\ell+1))^{694\omega(N)+984}\right\}^{\ell(\ell+1)}.$$

Since $\ell \geq 11$, we have that $\ell(\ell+1) \leq \ell^{2.04}$ and $\ell(\ell+1) \leq \frac{12}{11}\ell^2$. Appropriately applying both of these facts to the above bound yields the desired result.

Suppose now that $\mathrm{Gal}(L/\mathbb{Q})$ is not isomorphic to $\mathrm{GL}_2(\mathbb{F}_\ell)$. The possible cases are described in the proof of [Mur94, Theorem 4]. Applying similar analysis to all of these cases, one sees that the above case gives the largest upper bound for the least prime $p$ such that $\ell \mid \#E(\mathbb{F}_p)$. This completes the proof. □

**Class functions**

Next, we require some basic results on class functions (cf. [Ser81, Zyw15]) for the proof of Theorem 1.4.7. Let $L/F$ be a Galois extension of number fields with Galois group $G$, and let $\phi : G \to \mathbb{C}$ be a class function. For each prime ideal $\mathfrak{p}$ of $F$, choose any prime ideal $\mathfrak{P}$ of $L$ dividing $\mathfrak{p}$. Let $D_{\mathfrak{P}}$ and $I_{\mathfrak{P}}$ be the decomposition and inertia subgroups of $G$ at $\mathfrak{p}$, respectively. We then have a distinguished Frobenius element $\sigma_{\mathfrak{P}} \in D_{\mathfrak{P}}/I_{\mathfrak{P}}$. For each $m \geq 1$, define

$$\phi(\mathrm{Frob}_{\mathfrak{p}}^m) := \frac{1}{|I_{\mathfrak{P}}|} \sum_{\substack{g \in D_{\mathfrak{P}} \\ gI_{\mathfrak{P}} = \sigma_{\mathfrak{P}}^m \in D_{\mathfrak{P}}/I_{\mathfrak{P}}}} \phi(g).$$

Note that $\phi(\mathrm{Frob}_{\mathfrak{p}}^m)$ is independent of the aforementioned choice of $\mathfrak{P}$. If $\mathfrak{p}$ is unramified in $L$, this definition agrees with the value of $\phi$ on the conjugacy class $\mathrm{Frob}_{\mathfrak{p}}$ of $G$. For $x \geq 2$, we define

$$\pi_\phi(x) = \sum_{\substack{\mathfrak{p} \text{ unramified in } L \\ \mathrm{N}_{F/\mathbb{Q}}\,\mathfrak{p} \leq x}} \phi(\mathrm{Frob}_{\mathfrak{p}}), \qquad \widetilde{\pi}_\phi(x) = \sum_{\substack{\mathfrak{p} \text{ unramified in } L \\ \mathrm{N}_{F/\mathbb{Q}}\,\mathfrak{p}^m \leq x}} \frac{1}{m}\phi(\mathrm{Frob}_{\mathfrak{p}}^m).$$

Let $C \subset G$ be stable under conjugation, and let $\mathbf{1}_C : G \to \{0,1\}$ be the class function given by the indicator function of $C$. Now, define[1] $\pi_C(x, L/F) = \pi_{\mathbf{1}_C}(x)$ and $\widetilde{\pi}_C(x, L/F) = \widetilde{\pi}_{\mathbf{1}_C}(x)$. Serre [Ser81, Proposition 7] proved that if $x \geq 2$, then

$$|\pi_C(x, L/F) - \widetilde{\pi}_C(x, L/F)| \leq 4n_F((\log D_L)/n_L + \sqrt{x}). \tag{10.5}$$

[1]This agrees with our usual definition of $\pi_C(x, L/F)$ in (1.15).

**Proof of Theorem 1.4.6**

This is an immediate consequence of Theorem 1.4.7 in the case $k_f = 2$. $\square$

**Proof of Theorem 1.4.7**

Let $\ell$ be an odd prime such that (10.3) is satisfied. Assuming $\gcd(k-1, \ell-1) = 1$, we have $G \cong \mathrm{GL}_2(\mathbb{F}_\ell)$. To prove the theorem, we consider

$$\pi_f(x; a, \ell) := \#\{p \le x \colon p \nmid \ell N,\, a_f(p) \equiv a \,(\mathrm{mod}\, \ell),\, \ell \text{ splits in } \mathbb{Q}((a_f(p)^2 - 4p^{k-1})^{1/2})\}.$$

Note that for $p \nmid \ell N$, $a_f(p)^2 - 4p^{k-1} = \mathrm{tr}(\rho_{f,\ell}(\sigma_p)) - 4\det(\rho_{f,\ell}(\sigma_p))^2$, where $\sigma_p$ is Frobenius at $p$ in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The subset $C \subset G$ given by

$$C = \{A \in G \colon \mathrm{tr}(A) \equiv a \,(\mathrm{mod}\, \ell),\, \mathrm{tr}(A)^2 - 4\det(A) \text{ is a square in } \mathbb{F}_\ell^\times\}$$

is a conjugacy-invariant subset of $G$, so we bound $\widetilde{\pi}_C(x, L/\mathbb{Q})$. Let $B \subset G$ denote the subgroup of upper triangular matrices; the condition that $\mathrm{tr}(A)^2 - 4\det(A)$ is a square in $\mathbb{F}_\ell^\times$ means that $\sigma_p$ is conjugate to an element in $B$. If $\Gamma$ is a maximal set of elements $\gamma \in B$ which are non-conjugate in $G$ with $\mathrm{tr}(\gamma) \equiv a \,(\mathrm{mod}\, q)$, then $C = \bigsqcup_{\gamma\in\Gamma} C_G(\gamma)$, where $C_G(\gamma)$ denotes the conjugacy class of $\gamma$ in $G$. Since $B$ is a subgroup of $G$ with the property that every element of $C$ is conjugate to an element of $B$, it follows from [Zyw15, Lemma 2.6] that

$$\widetilde{\pi}_C(x, L/\mathbb{Q}) = \sum_{\gamma\in\Gamma} \frac{\widetilde{\pi}_{C_B(\gamma)}(x, L/L^B)}{[\mathrm{Cent}_G(\gamma) : \mathrm{Cent}_B(\gamma)]},$$

where $\mathrm{Cent}_G(\gamma)$ is the centralizer of $\gamma$ in $G$ (and similarly for $B$). If $C_1 = \bigsqcup_{\gamma \,\in\, \Gamma \text{ non-scalar}} C_B(\gamma)$, then it follows that $\widetilde{\pi}_C(x; L/\mathbb{Q}) \ge \frac{1}{|G|}\widetilde{\pi}_{C_1}(x, L/L^B)$ for all $x \ge 2$.

*Case 1: $\ell N$ sufficiently large, $a \not\equiv 0 \,(\mathrm{mod}\, \ell)$*

Let $U$ be the normal subgroup of $B$ consisting of the matrices whose diagonal entries are both 1. We observe that $U \cdot C_1 \subset C_1$; therefore, using arguments from [Zyw15, Lemma 2.6], we have that $\widetilde{\pi}_{C_1}(x, L/L^B) = \widetilde{\pi}_{C_2}(x, L^U/L^B)$ for $x \ge 2$, where $C_2$ is the image of $C_1 \cap B$ in $B/U$. It follows from (10.5) and Theorem 10.1.4 that if $\ell N$ is sufficiently large and $x$ is bounded below as in Theorem 10.1.4, then

$$\widetilde{\pi}_{C_2}(x, L^U/L^B) > 0 \text{ if and only if } \pi_{C_2}(x, L^U/L^B) > 0. \tag{10.6}$$

It is straightforward to compute $n_{L^B} = \ell + 1$ and $[L^U : L^B] = (\ell-1)^2$. Since $L^U/L^B$ is abelian and all of primes $p$ ramifying in $L^U$ divide $\ell N$, we therefore obtain a prime $p$ from Theorem 10.1.4 with the desired arithmetic properties and which satisfies

$$p \ll \left\{\ell^{1042}(\ell\,\mathrm{rad}(N))^{1736}(\ell+1)^{694\omega(N)+984}\right\}^{\ell+1}.$$

For $\ell \geq 3$, we have $\ell+1 \leq \ell^{1.27}$. Appropriately applying this fact to the above yields the desired bound.

*Case 2: $\ell N$ sufficiently large, $a \equiv 0 \pmod{\ell}$*

Let $H$ be the normal subgroup of $B$ consisting of matrices whose eigenvalues are both equal. We have that $H \cdot C_1 \subset C_1$ since multiplying a trace zero matrix by a scalar does not change the trace. Let $C_3$ be the image of $C_1 \cap B$ in $B/H$. The arguments are now the same as in the previous case, with $L^H$ replacing $L^U$. In fact, since $B/H \cong \mathbb{F}_\ell^\times$ is abelian of order $\ell-1$ and $C_3$ is a singleton, we obtain a slightly better exponent than what is stated in Theorem 1.4.7 when $a \equiv 0 \pmod{\ell}$.

*Case 3: $\ell N$ not sufficiently large*

Let $A_2 = U$ and $A_3 = H$. When $\ell N$ is not sufficiently large (in which case $\ell N \ll 1$), then the lower bound for $\pi_{C_i}(x, L^{A_i}/L^B)$ ($i = 2$ or $3$) in Theorem 10.1.4 may have an implied constant that is so small that (10.6) becomes false in the range of $x$ given by Theorem 10.1.4. For these finitely many exceptional cases, we use Weiss' lower bound on $\pi_{C_i}(x, L^{A_i}/L^B)$ that follows [Wei83, Theorem 5.2], which holds uniformly for all choices of $N$ and $\ell$. Continuing the proof as in Case 1 (this requires us to take $c_{10}$ sufficiently small and $c_{11}$ to be sufficiently large in [Wei83, Theorem 5.2]), we see that the least prime $p \nmid \ell N$ such that $a_f(p) \equiv a \pmod{\ell}$ is absolutely bounded in all of the finitely many exceptional cases. This proves Theorem 1.4.7. □

## 10.3 Lang–Trotter conjectures

For this subsection, fix a newform

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z}$$

of even integral weight $k_f \geq 2$, level $N_f$, and trivial nebentypus with integral Fourier coefficients. For each prime $p$, we define $\omega_p = (a_f(p)^2 - 4p^{k_f-1})^{1/2}$. From Deligne's proof of the Weil conjectures, we have that $|a_f(p)| \leq 2p^{(k_f-1)/2}$ for all $p$, so $\mathbb{Q}(\omega_p)$ is an imaginary quadratic extension of $\mathbb{Q}$.

For a fixed integer $a \in \mathbb{Z}$ and fixed imaginary quadratic field $k$, we will estimate

$$\pi_f(x,a) := \#\{p \leq x \colon p \nmid N_E,\, a_f(p) = a\},$$
$$\pi_f(x,k) := \#\{p \leq x \colon p \nmid N_E,\, \mathbb{Q}(\omega_p) \cong k\}.$$

The arguments found in this subsection closely follow the prior works of [MMS88, Mur97] and especially [Zyw15]. We begin by giving an upper bound for $\pi_f(x,a)$.

**Proof of Theorem 1.4.8**

For any prime $\ell \geq 3$, set

$$\pi_f(x,a;\ell) := \#\{p \leq x \colon a_f(p) \equiv a \pmod{\ell} \text{ and } \ell \text{ splits in } \mathbb{Q}(\omega_p)\}.$$

Let $\ell_1 < \ell_2 < \cdots < \ell_t$ be any $t$ odd primes, each less than $\exp(\frac{\log x}{2t})$. By [Wan90, Corollary 4.2], if $t \sim \frac{4}{\log 2} \log\log x$, then

$$\pi_f(x,a) \ll \sum_{j=1}^{t} \pi_f(x,a;\ell_j) + \frac{x}{(\log x)^2} \ll (\log\log x) \max_{1\leq j\leq t} \pi_f(x,a;\ell_j) + \frac{x}{(\log x)^2}. \tag{10.7}$$

We proceed to bound $\pi_f(x,a;\ell)$, where $\ell \leq \exp((\log 2)(\log x)/(8\log\log x))$.

Let $\ell$ be prime, let $\mathbb{F}_\ell$ be the field of $\ell$ elements, and let $\mathrm{Frob}_p$ be the Frobenius automorphism of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ at $p$. For each $\ell$, there is a representation

$$\rho_{f,\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_\ell) \tag{10.8}$$

unramified outside $N_f\ell$, such that for all primes $p \nmid N_f\ell$, we have that $\mathrm{tr}(\rho_{f,\ell}(\mathrm{Frob}_p)) \equiv a_f(p) \pmod{\ell}$ and $\det(\rho_{f,\ell}(\mathrm{Frob}_p)) \equiv p^{k_f-1} \pmod{\ell}$. We have that $\rho_{f,\ell}$ is surjective for all but finitely many $\ell$. Let $L = L_\ell$ be the subfield of $\overline{\mathbb{Q}}$ fixed by $\ker \rho_{f,\ell}$. If $\ell$ is sufficiently large, then $L/\mathbb{Q}$ is a Galois extension, unramified outside of $N_f\ell$, whose Galois group is $G = \{g \in \mathrm{GL}_2(\mathbb{F}_\ell) : \det g \in (\mathbb{F}_\ell^\times)^{k_f-1}\}$.

Define $C = \{A \in G \colon \mathrm{tr}(A) \equiv a \pmod{\ell} \text{ and } \mathrm{tr}(A)^2 - 4\det(A) \in \mathbb{F}_\ell \text{ is a square}\}$. Let $B$ denote the upper triangular matrices in $\mathrm{GL}_2(\mathbb{F}_\ell) \cap G$, and let $L^B$ be the subfield of $L$ fixed by $B$. Let $U$ be the unipotent elements of $B$, and let $L^U$ be the subfield of $L$ fixed by $U$. Note that $U$ is a normal subgroup of $B$ and that $B/U \cong \mathrm{Gal}(L^U/L^B)$ is abelian. Let $C'$ be the image of $C \cap B$ in $B/U$. If $x$ is sufficiently large, then by [Zyw15, Lemmas 2.7 and 4.3],

$$\pi_f(x,a;\ell) \ll \pi_{C'}(x, L^U/L^B) + n_{L^B}\Big(\frac{\sqrt{x}}{\log x} + \log M(L^U/L^B)\Big).$$

Applying Theorem 10.1.5 to the Chebotarev prime counting functions for each conjugacy class in $C'$, we have that if $\log x \gg n_{L^B} \log(M(L^U/L^B)n_{L^B})$, then

$$\pi_f(x,a;\ell) \ll \frac{|C'|}{|B/U|}\frac{x}{\log x} + n_{L^B}\Big(\frac{\sqrt{x}}{\log x} + \log M(L^U/L^B)\Big).$$

By [Zyw15, Lemma 4.4], we have $|C'|/|B/U| \ll 1/\ell$, $n_{L^B} \ll \ell$, and $\log M(L^U/L^B) \ll_{N_f} \log \ell$. Combining all of our estimates, we find that

$$\pi_f(x,a;\ell) \ll \frac{1}{\ell}\frac{x}{\log x} + \frac{\ell\sqrt{x}}{\log x} + \ell \log N_f \ell, \qquad \log x \gg \ell \log N_f \ell. \tag{10.9}$$

Thus, taking $\ell \sim c' \log x/\log(N_f \log x)$ for some sufficiently small absolute constant $c' > 0$,

$$\pi_f(x,a;\ell) \ll \frac{x\log(N_f \log x)}{(\log x)^2}. \tag{10.10}$$

Now, as before, let $t \in \mathbb{Z}$ satisfy $t \sim \frac{4}{\log 2}\log\log x$, and let $\ell_1 < \ell_2 < \cdots < \ell_t$ be $t$ consecutive primes with $\ell_1 \sim c' \log x/\log(N_f \log x)$. By the Prime Number Theorem, $\ell_j \in [\ell_1, 2\ell_1]$ for all $1 \le j \le t$. Therefore, if $c'$ is made sufficiently small, we have that

$$\max_{1\le j\le t} \pi_f(x,a;\ell_j) \ll \frac{x\log(N_f \log x)}{(\log x)^2}. \tag{10.11}$$

Theorem 1.4.8 now follows from inserting the inequality (10.11) into the inequality (10.7). □

**Remark.** The source of our improvement over [Mur97] stems solely from the application of Theorem 10.1.5. See the end of Section 9.1 in [TZ17a] for further discussion.

**Proof of Theorem 1.4.9**

In this case, we are estimating $\pi_f(x,k)$ for a fixed imaginary quadratic field $k$. The proof of Theorem 1.4.9 is nearly identical to the proof of [Zyw15, Theorem 1.3(ii)] except that we use Theorem 10.1.5 to bound the ensuing Chebotarev prime counting function instead of using [Zyw15, Theorem 2.1(ii)]. The analytic details are very similar to the above proof of Theorem 1.4.8, but the particular Galois extension to which Theorem 1.3.3 is applied is different. Following [Zyw15, Section 5.2], we apply Theorem 1.3.3 instead of [Zyw15, Theorem 2.1(ii)], which allows us to choose

$$y = \frac{c}{h_k}\frac{\log x}{\log(\frac{D_k}{h_k}\log x)}$$

for some sufficiently small absolute constant $c > 0$. Here $D_k$ is the absolute discriminant of $k$ and $h_k$ is the (broad) class number of $k$. This yields the claimed result. □

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