SUBLINEARLY MORSE BOUNDARY I: CAT(0) SPACES

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Abstract. To every Gromov hyperbolic space X one can associate a space at infinity called the Gromov boundary of X. Gromov showed that quasi-isometries of hyperbolic metric spaces induce homeomorphisms on their boundaries, thus giving rise to a well-defined notion of the boundary of a hyperbolic group. Croke and Kleiner showed that the visual boundary of non-positively curved (CAT(0)) groups is not well-defined, since quasi-isometric CAT(0) spaces can have non-homeomorphic boundaries. For any sublinear function κ, we consider a subset of the visual boundary called the κ–Morse boundary and show that it is QI-invariant and metrizable. This is to say, the κ–Morse boundary of a CAT(0) group is well-defined. In the case of Right-angled Artin groups, it is shown in the Appendix that the Poisson boundary of random walks is naturally identified with the $(\sqrt{t \log t})$–boundary.

1. Introduction

To every Gromov hyperbolic space X one can associate a space at infinity ∂X called the *Gromov boundary* of X. The space ∂X consists of equivalence classes of geodesic rays, where two rays are equivalent if they stay within bounded distance of each other, and is equipped with the visual topology. This boundary is a fundamental tool for studying hyperbolic groups and hyperbolic spaces (for example, see [BK02]). As shown by Gromov [Gro87], quasi-isometries between hyperbolic metric spaces induce homeomorphisms between their boundaries, thus giving rise to a well-defined notion of the boundary of a hyperbolic group.

However, this is not true under weaker assumptions. In particular, for CAT(0) spaces, Croke and Kleiner [CK00] showed that visual boundaries of CAT(0) spaces are generally not quasi-isometrically invariant and hence one cannot talk about the visual boundary of a CAT(0) group. In [Qin16] Qing showed that even if we restrict our attention to *rank-1* geodesics, the space of all rank-1 geodesics is still not quasi-isometry invariant. In [Cas16] Cashen showed that the subset of the visual boundary consisting of only the Morse geodesics (equipped with the usual cone topology) is not in general preserved by quasi-isometries.

In this paper, we introduce a boundary for CAT(0) spaces that is strictly larger than the set of Morse geodesics and is equipped with a coarse notion of cone topology that makes it invariant under quasi-isometries. The points in this boundary are geodesics rays that behave like geodesics in a Gromov hyperbolic space with a sublinear error term. More precisely, they satisfy one of the following two equivalent characterizations.

Given a base-point $\mathfrak{o}$ in X, define the norm of a point x to be $\|x\| = d_X(\mathfrak{o}, x)$. Now, fixing a sublinear function κ, we say a geodesic ray $b\colon [0,\infty) \to X$ starting from $\mathfrak{o}$ is κ–Morse if there is a Morse gauge function $\mathsf{m}_b\colon \mathbb{R}_+^2 \to \mathbb{R}_+$ such that if ζ is a (q, Q)–quasi-geodesic segment

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with end points on b then, for every point x on ζ, we have

$$d_X(x,b) \leq \mathsf{m}_b(\mathsf{q},\mathsf{Q}) \cdot \kappa(\|x\|).$$

Alternatively, we say b is κ–contracting if there exists a constant c_b such that, for any metric ball B centered at x that is disjoint from b, the projection of B to b has diameter at most $\mathsf{c}_b \cdot \kappa(\|x\|)$.

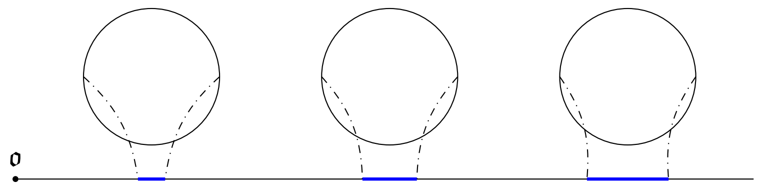


FIGURE 1. Along a κ–contracting geodesic ray, the diameter of the projection of a disjoint ball is allowed to grow at a rate comparable to κ.

Recall that geodesic rays in Gromov hyperbolic spaces are κ–contracting and κ–Morse for $\kappa = 1$.

Theorem A. *A geodesic ray is κ–Morse if and only if it is κ–contracting.*

We define the *κ–Morse boundary* of X, which we denote by $\partial_\kappa X$, to be the space of all such geodesic rays and we equip this space with a notion of visual topology on quasi-geodesics (see Section 4). In the case where X is a Gromov hyperbolic space, $\partial_\kappa X$ is the same as the Gromov boundary of X for every function κ.

Theorem B. *If $\Phi\colon X \to Y$ is a quasi-isometry between proper* CAT(0) *metric spaces X and Y, then Φ induces a homeomorphism $\Phi^\star\colon \partial_\kappa X \to \partial_\kappa Y$.*

Therefore one can define the κ–Morse boundary for any group that acts geometrically on a CAT(0) space or generally any space that is quasi-isometric to a CAT(0) space.

Corollary C. *If G acts quasi-isometrically, discretely and co-compactly on two* CAT(0) *spaces X_1 and X_2, then for any κ, the space $\partial_\kappa X_1$ is homeomorphic to $\partial_\kappa X_2$. Hence, the κ–Morse boundary $\partial_\kappa G$ of G is well defined.*

Our choice of topology seems to be a natural one, especially since $\partial_\kappa X$ has good topological properties.

Theorem D. *For every proper* CAT(0) *space X, $\partial_\kappa X$ is metrizable.*

We also show that the κ–boundaries associated to different sublinear functions are topological subspaces of each other.

Theorem E. *If X is a* CAT(0) *metric space and $\kappa \leq \kappa'$ are two sublinear functions then*

$$\partial_\kappa X \subseteq \partial_{\kappa'} X$$

where the topology of $\partial_\kappa X$ is the subspace topology associated to the inclusion.

A motivation for this definition of the boundary is the study of random walks on CAT(0) groups. Given a group G and a probability measure μ on it, the *Poisson boundary* of (G, μ) is a canonical measurable G-space which classifies all possible asymptotic behaviours of a random walk on G driven by μ (see the Appendix for precise definitions).

The boundary depends on the choice of measure, and it is an important open problem [Ka96, page 153] whether two finitely supported generating measures on the same group give rise to isomorphic boundaries. This question is the probabilistic analog of the quasi-isometry invariance question: indeed, two generating sets for G give rise to both two quasi-isometric metrics on G and to two finitely supported measures.

In the case G is a right-angled Artin group, in the Appendix we prove the following:

Theorem F. *The $\sqrt{t \log t}$–boundary of $A(\Gamma)$ is a QI-invariant topological model for the Poisson boundary of $A(\Gamma)$ associated to any random walk with finite support.*

To our knowledge, the κ–Morse boundary defined in this paper is the first boundary that is both invariant under quasi-isometries and a model for the Poisson boundary. By comparison, the visual boundary is known to be a model of the Poisson boundary for CAT(0) groups but it is not QI-invariant, while the Morse boundary ([CS15]) is quasi-isometrically invariant but has zero measure with respect to random walks, hence, in general, it is not a model for the Poisson boundary.

The function $\sqrt{t \log t}$ arises from the fact that a generic trajectory of the random walk spends a logarithmic amount of time in each flat ([ST18], see also Theorem A.17). As shown in [QT18], the same logarithmic excursion property also holds for generic elements with respect to the uniform measure on balls in the Cayley graph of G. This suggests that the κ–Morse boundary should have full measure not only with respect to the hitting measure for random walks but also with respect to the Patterson-Sullivan type measure obtained as a weak limit of uniform measures on balls.

History. Our work builds on previous attempts to construct a boundary for a CAT(0) group that is quasi-isometry invariant. Charney and Sultan [CS15] defined a contracting geodesic ray in X to be one such that all disjoint balls project to sets of diameter at most D for some $\mathsf{D} \geq 0$. They call the set of all such geodesic rays the *contracting boundary* or the *Morse boundary of X*. They equip this space with a *direct limit topology* and show that it is invariant under quasi-isometries. But this space does not have good topological properties, for example, it is not first countable. Cashen-Mackay [CM19], following the work of Arzhantseva-Cashen-Gruber-Hume [ACGH17], defined a different topology on the Morse boundary of X. They showed that it is Hausdorff and when there is a geometric action by a countable group, it is also metrizable. In fact, their definition works for every geodesic metric space.

The approach in [ACGH17, CM19] uses a different notion of sublinearly contracting geodesic. In [ACGH17, CM19], the contraction is sublinear with respect to the radii of the disjoint balls. This a natural extension of the notion of a Morse geodesic to the setting of general metric spaces. But this boundary is smaller than the one defined in this paper and, in particular, cannot be used as a model for the Poisson boundary.

It is likely that, when $\kappa = 1$, $\partial_\kappa X$ is the same topological space as the Morse-boundary equipped with the topology defined in [CM19]. If so, Theorem D would imply that the Morse boundary of every CAT(0) space is metrizable.

Remark 1.1. In a sequel to this paper, we also plan to extend the results of this paper to the setting of general geodesic metric spaces. However, we prefer to present the CAT(0) setting first as many of the arguments are simpler, so the main ideas are more visible. Also, it is unlikely that one can define a notion of κ–Morse boundary for every κ. Instead, we will define the sublinearly-Morse boundary of a space as the *union* of κ–boundaries over all possible κ.

Outline of the paper. Section 2 contains some needed properties of CAT(0) geometry. In Section 3, we give several equivalent definitions for the notion of κ–contracting geodesic. In Section 4, we define a topology for $\partial_\kappa X$ and establish some topological properties, including the metrizability. In Section 5, we define the boundary of a CAT(0) group, in particular we show that $\partial_\kappa X$ is invariant under quasi-isometry. In the last section we examine the group $A = \mathbb{Z} \star \mathbb{Z}^2$ to illustrate in full detail the properties of sublinearly Morse boundaries for this example. In particular, we show that the log–boundary is a metric model for the Poisson boundary of A. The Poisson boundary of right-angled Artin groups in general is treated in the Appendix.

Acknowledgments. We thank Jason Behrstock, Ruth Charney, Matthew Cordes, Talia Fernos, Joseph Maher, Sam Taylor and Giulio Tiozzo for helpful conversations. We especially thank Chris Cashen and Giulio Tiozzo on comments on earlier versions of this paper.

2. Preliminaries

Quasi-Isometry and Quasi-Isometric Embeddings.

Definition 2.1 (Quasi Isometric embedding)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. For constants $\mathsf{k} \geq 1$ and $\mathsf{K} \geq 0$, we say a map $\Phi\colon X \to Y$ is a (k, K)–*quasi-isometric embedding* if, for all $x_1, x_2 \in X$

$$\frac{1}{\mathsf{k}} d_X(x_1, x_2) - \mathsf{K} \leq d_Y\big(\Phi(x_1), \Phi(x_2)\big) \leq \mathsf{k}\, d_X(x_1, x_2) + \mathsf{K}.$$

If, in addition, every point in Y lies in the K–neighbourhood of the image of Φ, then f is called a (k, K)–quasi-isometry. When such a map exists, X and Y are said to be *quasi-isometric*.

A quasi-isometric embedding $\Phi^{-1}\colon Y \to X$ is called a *quasi-inverse* of Φ if for every $x \in X$, $d_X(x, \Phi^{-1}\Phi(x))$ is uniformly bounded above. In fact, after replacing k and K with larger constants, we assume that Φ^{-1} is also a (k, K)–quasi-isometric embedding,

$$\forall x \in X \quad d_X\big(x, \Phi^{-1}\Phi(x)\big) \leq \mathsf{K} \qquad \text{and} \qquad \forall y \in Y \quad d_Y\big(y, \Phi\,\Phi^{-1}(x)\big) \leq \mathsf{K}.$$

Definition 2.2 (Quasi-Geodesics)**.** A *geodesic ray* in X is an isometric embedding $b\colon [0, \infty) \to X$. We fix a base-point $\mathfrak{o} \in X$ and always assume that $b(0) = \mathfrak{o}$, that is, a geodesic ray is always assumed to start from this fixed base-point. A *quasi-geodesic ray* is a continuous quasi-isometric embedding $\beta\colon [0, \infty) \to X$ again starting from $\mathfrak{o}$. The additional assumption that quasi-geodesics are continuous is not necessary, but it is added for convenience and to make the exposition simpler.

If $\beta\colon [0, \infty) \to X$ is a (q, Q)–quasi-isometric embedding, and $\Phi\colon X \to Y$ is a (k, K)–quasi-isometry then the composition $\Phi \circ \beta\colon [t_1, t_2] \to Y$ is a quasi-isometric embedding, but it may not be continuous. However, one can adjust the map slightly to make it continuous (see [BH99, Lemma III.1.11]). Abusing notation, we denote the new map again by $\Phi \circ \beta$. Following [BH99, Lemma III.1.11], we have that $\Phi \circ \beta$ is a $(\mathsf{kq}, 2(\mathsf{kq} + \mathsf{kQ} + \mathsf{K}))$–quasi-geodesic.

Similar to above, a *geodesic segment* is an isometric embedding $b\colon [t_1, t_2] \to X$ and a *quasi-geodesic segment* is a continuous quasi-isometric embedding $\beta\colon [t_1, t_2] \to X$.

Basic properties of $\mathrm{CAT}(0)$ **spaces.** A proper geodesic metric space (X, d_X) is $\mathrm{CAT}(0)$ if geodesic triangles in X are at least as thin as triangles in Euclidean space with the same side lengths. To be precise, for any given geodesic triangle $\triangle pqr$, consider the unique triangle $\triangle \overline{pqr}$ in the Euclidean plane with the same side lengths. For any pair of points x, y on edges $[p, q]$ and $[p, r]$ of the triangle $\triangle pqr$, if we choose points $\overline{x}$ and $\overline{y}$ on edges $[\overline{p}, \overline{q}]$ and $[\overline{p}, \overline{r}]$ of the triangle $\triangle \overline{pqr}$ so that $d_X(p, x) = d_{\mathbb{E}}(\overline{p}, \overline{x})$ and $d_X(p, y) = d_{\mathbb{E}}(\overline{p}, \overline{y})$ then,

$$d_X(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y}).$$

For the remainder of the paper, we assume X is a proper $\mathrm{CAT}(0)$ space. A metric space X is *proper* if closed metric balls are compact. Here, we list some properties of proper $\mathrm{CAT}(0)$ spaces that are needed later (see [BH99]).

Lemma 2.3. *A proper* $\mathrm{CAT}(0)$ *space* X *has the following properties:*

i. It is uniquely geodesic, that is, for any two points x, y in X, there exists exactly one geodesic connecting them. Furthermore, X is contractible via geodesic retraction to a base point in the space.

ii. The nearest-point projection from a point x to a geodesic line b is a unique point denoted x_b. In fact, the closest-point projection map

$$\pi_b\colon X \to b$$

is Lipschitz.

Remark 2.4. Let Z be a closed subset of X. For $x \in X$, we often denote the set of the nearest points in Z to x by x_Z. We also write $d_X(x, Z)$ to mean the distance between x and the set Z, that is $d_X(x, Z) = d_X(x, y)$ for any $y \in x_Z$. We often think of a geodesic or a quasi-geodesic as a subset of X instead of a map. For example, for $x \in X$ and a quasi-geodesic β, we write $d_X(x, \beta)$ to mean the distance between x and the image of β in X.

We show that if a geodesic segment is "perpendicular" to a quasi-geodesic, then the concatenation of the geodesic segment with the quasi-geodesic is also quasi-geodesic. Given a quasi-geodesic β, we use $[\cdot, \cdot]_\beta$ to denote the segment of β between two specified points.

Lemma 2.5. *Consider a point $x \in X$ and a (q, Q)–quasi-geodesic segment β connecting a point $z \in X$ to a point $w \in X$. Let y be a point in x_β, and let γ be the concatenation of the geodesic segment $[x, y]$ and the quasi-geodesic segment $[y, z]_\beta \subset \beta$. Then $\gamma = [x, y] \cup [y, z]_\beta$ is a $(3\mathsf{q}, \mathsf{Q})$–quasi-geodesic.*

Proof. Consider $\gamma\colon [t_0, t_2] \to X$ and let $t_1 \in [t_0, t_2]$ be the time when $\gamma(t_1) = y$, the restriction of γ to $[t_0, t_1]$ is the parametrization of $[x, y]$ given by arc length and the restriction of γ to $[t_1, t_2]$ is the parametrization of $[y, z]_\beta$ given by β. To show that γ is a quasi-geodesic, we need to estimate the distance between a point in $[x, y]$ and a point in $[y, z]_\beta$. However, it is enough to show that $d_X(x, z)$ is comparable to $|t_2 - t_0|$ because the argument for any other points along $[x, y]$ and along $[y, z]_\beta$ is the same. We argue in two cases.

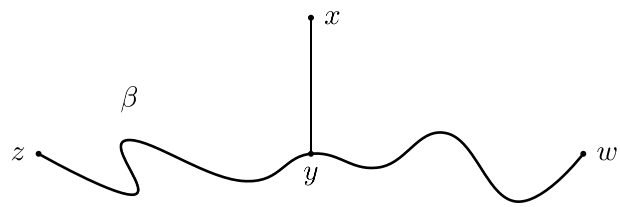


FIGURE 2. For $y \in x_\beta$, the concatenation of the geodesic segment $[x, y]$ and the quasi-geodesic segment $[y, z]_\beta$ is a quasi-geodesic.

Case 1. Suppose $2d_X(x, y) \geq d_X(z, y)$. Then,

$$3d_X(x, y) \geq d_X(z, y) + d_X(x, y)$$

Therefore,

$$\begin{aligned} d_X(x, z) \geq d_X(x, y) &\geq \frac{1}{3}\big(d_X(z, y) + d_X(x, y)\big) \\ &\geq \frac{1}{3}\left(\frac{1}{\mathsf{q}}|t_2 - t_1| - \mathsf{Q} + |t_1 - t_0|\right) \\ &\geq \frac{1}{3\mathsf{q}}|t_2 - t_0| - \frac{\mathsf{Q}}{3}. \end{aligned}$$

Case 2. Suppose $2d_X(x, y) < d_X(z, y)$, then

$$3d_X(x, y) \leq d_X(z, y) + d_X(x, y) \qquad \Longrightarrow \qquad 2d_X(x, y) \leq \frac{2}{3}\big(d_X(z, y) + d_X(x, y)\big).$$

We have

$$\begin{aligned} d_X(x, z) &\geq d_X(z, y) - d_X(x, y) = d_X(z, y) + d_X(x, y) - 2d_X(x, y) \\ &\geq \big(d_X(z, y) + d_X(x, y)\big) - \frac{2}{3}\big(d_X(z, y) + d_X(x, y)\big) \\ &\geq \frac{1}{3}(d_X(z, y) + d_X(x, y)) \\ &\geq \frac{1}{3}\left(\frac{1}{\mathsf{q}}|t_2 - t_1| - \mathsf{Q} + |t_1 - t_0|\right) \geq \frac{1}{3\mathsf{q}}|t_2 - t_0| - \frac{\mathsf{Q}}{3}. \end{aligned}$$

This established the lower-bound. The upper-bound follows from the triangle inequality:

$$d_X(x, z) \leq d_X(x, y) + d_X(y, z) \leq |t_1 - t_0| + \mathsf{q}|t_2 - t_1| + \mathsf{Q} \leq \mathsf{q}|t_2 - t_0| + \mathsf{Q}.$$

It follows that γ is a $(3\mathsf{q}, \mathsf{Q})$–quasi-geodesic. □

The boundaries of CAT(0) **spaces.** A proper CAT(0) space X can be compactified via the *visual boundary*. The points of the visual boundary $\partial_\infty X$ of X are geodesic rays (starting from $\mathfrak{o}$). Set $\overline{X} = X \bigcup \partial_\infty X$ where points in $\overline{X}$ can be thought of as geodesic rays or geodesic segments starting from $\mathfrak{o}$. The space $\overline{X}$ is usually equipped with the *cone topology* where two geodesics are considered nearby if they fellow travel each other for a long time (see [BH99] for more details).

3. The κ–Morse geodesics of X

The goal of this section is to prove Theorem 3.8 which gives several equivalent characterizations of the notion of a κ–Morse geodesic (or quasi-geodesic) ray.

3.1. **Sublinear functions.**

We fix a function

$$\kappa\colon [0,\infty) \to [1,\infty)$$

that is monotone increasing, concave and sublinear, that is

$$\lim_{t\to\infty} \frac{\kappa(t)}{t} = 0.$$

Note that using concavity, for any $a > 1$, we have

$$\kappa(at) \leq a\left(\frac{1}{a}\,\kappa(at) + \left(1 - \frac{1}{a}\right)\kappa(0)\right) \leq a\,\kappa(t). \tag{1}$$

We say a quantity D *is small compared to a radius* $\mathsf{r} > 0$ if

$$\mathsf{D} \leq \frac{\mathsf{r}}{2\kappa(\mathsf{r})}. \tag{2}$$

Remark 3.1. The assumption that κ is increasing and concave makes certain arguments cleaner, otherwise they are not really needed. One can always replace any sublinear function κ, with another sublinear function $\overline{\kappa}$ so that $\kappa(t) \leq \overline{\kappa}(t) \leq \mathsf{C}\,\kappa(t)$ for some constant C and $\overline{\kappa}$ is monotone increasing and concave. For example, define

$$\overline{\kappa}(t) = \sup\left\{\lambda\kappa(u) + (1-\lambda)\kappa(v) \;\middle|\; 0 \leq \lambda \leq 1,\ u, v > 0,\ \text{and } \lambda u + (1-\lambda)v = t\right\}.$$

The requirement $\kappa(t) \geq 1$ is there to remove additive errors in the definition of κ–contracting geodesics.

Lemma 3.2. *For any $\mathsf{D}_0 > 0$, there exists $\mathsf{D}_1, \mathsf{D}_2 > 0$ depending on D_0 and κ so that, for $x, y \in X$,*

$$d(x,y) \leq \mathsf{D}_0 \cdot \kappa(x) \qquad \textit{implies} \qquad \mathsf{D}_1\kappa(x) \leq \kappa(y) \leq \mathsf{D}_2\kappa(x).$$

Proof. Since κ is sublinear, there is a constant A such that, for every $u > 0$,

$$\kappa(u) \leq \frac{u}{2\mathsf{D}_0} + \mathsf{A}.$$

Then

$$\Big|\|x\| - \|y\|\Big| \leq d_X(x,y) \leq \mathsf{D}_0 \cdot \kappa(x) \leq \mathsf{D}_0 \cdot \left(\frac{\|x\|}{2\mathsf{D}_0} + \mathsf{A}\right) \leq \frac{1}{2}\|x\| + \mathsf{D}_0\mathsf{A}. \tag{3}$$

We argue in two cases. Suppose $\|x\| \geq \|y\|$. Then, Equation (3) implies

$$\|x\| \leq 2\|y\| + 2\mathsf{D}_0\mathsf{A},$$

and from Equation (1), we get

$$\kappa(x) \leq (2 + 2\mathsf{D}_0\mathsf{A}) \cdot \kappa(y).$$

Thus

$$(2 + 2\mathsf{D}_0\mathsf{A})^{-1}\kappa(x) \leq \kappa(y) \leq \kappa(x).$$

On the other hand, if $\|x\| < \|y\|$, then Equation (3) implies

$$\|y\| \leq \frac{3}{2}\|x\| + \mathsf{D}_0\mathsf{A}.$$

Again, by Equation (1) we have

$$\kappa(y) \leq \left(\frac{3}{2} + \mathsf{D}_0\mathsf{A}\right) \cdot \kappa(x)$$

and hence

$$\kappa(x) < \kappa(y) \leq \left(\frac{3}{2} + \mathsf{D}_0\mathsf{A}\right) \cdot \kappa(x).$$

Combining the two cases, we get

$$(2 + 2\mathsf{D}_0\mathsf{A})^{-1}\kappa(x) \leq \kappa(y) \leq \left(\frac{3}{2} + \mathsf{D}_0\mathsf{A}\right) \cdot \kappa(x).$$

That is, the lemma holds for $\mathsf{D}_1 = (2 + 2\mathsf{D}_0\mathsf{A})^{-1}$ and $\mathsf{D}_2 = \frac{3}{2} + \mathsf{D}_0\mathsf{A}$. □

Definition 3.3 (κ–neighborhood)**.** For a closed set Z and a constant n define the (κ, n)–neighbourhood of Z to be

$$\mathcal{N}_\kappa(Z, \mathsf{n}) = \left\{ x \in X \;\middle|\; d_X(x, Z) \leq \mathsf{n} \cdot \kappa(x) \right\}.$$

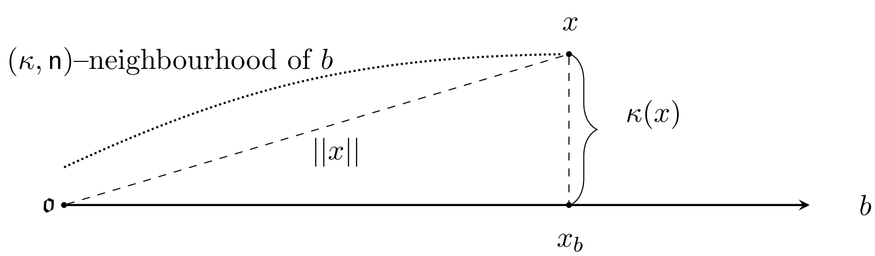


Figure 3. A κ-neighbourhood of a geodesic ray b

In view of Remark 2.4, a geodesic or a quasi-geodesic can take the place of the set Z in the above definitions. That is, we can write $\mathcal{N}_\kappa(b, \mathsf{n})$ to mean the (κ, n)–neighborhood of the image of the geodesic ray b. Or, we can use phrases like "the quasi-geodesic β is κ–contracting" or "the geodesic b is in a (κ, n)–neighbourhood of the geodesic c".

Definition 3.4. Let β and γ be two quasi-geodesic rays in X. If β is in some κ–neighbourhood of γ and γ is in some κ–neighbourhood of β, we say that β and γ *κ–fellow travel* each other. This defines an equivalence relation on the set of quasi-geodesic rays in X (to obtain transitivity, one needs to change n of the associated (κ, n)–neighbourhood). We denote the equivalence class that contains β by $[\beta]$ or we use the notation $\mathbf{b}$ for such an equivalence class when no quasi-geodesic in the class is given.

Lemma 3.5. *Let $b\colon [0, \infty) \to X$ be a geodesic ray in X. Then b is the unique geodesic ray in any (κ, n)–neighbourhood of b for any n. That is to say, distinct geodesic rays do not κ–fellow travel each other.*

Proof. Consider any other geodesic ray $c\colon [0,\infty) \to X$ emanating from the same base-point. Then, there is a time t_0 where $b(t_0) \neq c(t_0)$. For a given $t \geq t_0$, let t' be the time so that

$$d_X(c(t), b) = d_X\big(c(t), b(t')\big).$$

That is, $b(t')$ is the projection of $c(t)$ to b. Since X is a CAT(0) space, we have

$$d_X\big(c(t), b(t')\big) \geq \frac{t}{t_0} \cdot d_X\left(c(t_0), b\left(\frac{t'\, t_0}{t}\right)\right) \geq \frac{d_X\big(c(t_0), b\big)}{t_0} \cdot t.$$

This means that the distance from $c(t)$ to b grows linearly with t and hence c is not contained in any (κ, n)–neighborhood of b. □

3.2. κ–Morse and κ–contracting sets.

Definition 3.6 (κ–Morse)**.** We say a closed subset Z of X is *κ–Morse* if there is a function

$$\mathsf{m}_Z\colon \mathbb{R}_+^2 \to \mathbb{R}_+$$

so that if $\beta\colon [s,t] \to X$ is a (q, Q)–quasi-geodesic with end points on Z then

$$\beta[s,t] \subset \mathcal{N}_\kappa\big(Z, \mathsf{m}_Z(\mathsf{q}, \mathsf{Q})\big).$$

We refer to m_Z as the *Morse gauge* for Z. We always assume

$$\mathsf{m}_Z(\mathsf{q}, \mathsf{Q}) \geq \max(\mathsf{q}, \mathsf{Q}). \tag{4}$$

Definition 3.7 (κ–contracting)**.** For $x \in X$, define $\|x\| = d_X(\mathfrak{o}, x)$. For a closed subspace Z of X, we say Z is *κ–contracting* if there is a constant c_Z so that, for every $x, y \in X$

$$d_X(x,y) \leq d_X(x, Z) \quad \Longrightarrow \quad \operatorname{diam}_X\big(x_Z \cup y_Z\big) \leq \mathsf{c}_Z \cdot \kappa(\|x\|).$$

In fact, to simplify notation, we often drop $\|\cdot\|$. That is, for $x \in X$, we define

$$\kappa(x) := \kappa(\|x\|).$$

Theorem 3.8. *Let $\boldsymbol{b}$ be an equivalence class of quasi-geodesics in X. The following properties of $\boldsymbol{b}$ are equivalent.*

1. *The class $\boldsymbol{b}$ contains a geodesic ray b that is κ–contracting.*
2. *Every quasi-geodesic $\beta \in \boldsymbol{b}$ is κ–contracting.*
3. *Every quasi-geodesic $\beta \in \boldsymbol{b}$ is κ–Morse.*
4. *There exists a quasi-geodesic $\beta \in \boldsymbol{b}$ that is κ–Morse.*
5. *The class $\boldsymbol{b}$ contains a geodesic ray b that is κ–Morse for $(32, 0)$–quasi-geodesics.*

Note that the implication (3) $\Longrightarrow$ (4) is immediate. Later in this section, we will prove (4) $\Longrightarrow$ (5) $\Longrightarrow$ (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) in separate statements. To prepare for the first statement, we study the finite geodesic segments connecting points of the κ–Morse quasi-geodesic.

Proposition 3.9. *Let $\beta\colon [0,\infty) \to X$ be a (q, Q)–quasi-geodesic ray in X that is κ–Morse with m_β as its Morse gauge. For any given $T \in (0, \infty)$, let $b = b_T$ be the finite geodesic segment connecting $\beta(0) = \mathfrak{o}$ and $\beta(T)$. Then b is κ–Morse and the Morse-gauge of b is independent of T. That is, there exists $\mathsf{m}\colon \mathbb{R}^2 \to \mathbb{R}$ such that for every $T \in [0, \infty)$ and for every $(\mathsf{q}', \mathsf{Q}')$–quasi-geodesic $\zeta\colon [s,t] \to X$ with end points in $b = b_T$, we have*

$$\zeta[s,t] \subset \mathcal{N}_\kappa\big(b, \mathsf{m}(\mathsf{q}', \mathsf{Q}')\big).$$

Proof. We parametrize $b\colon [0,\mathsf{d}] \to X$ by arc length so $\mathsf{d} = d_X(\beta(0),\beta(T))$. The geodesic segment b can be considered as a $(1,0)$–quasi-geodesic with end points on β. Hence, for every $0 \le s \le \mathsf{d}$, there is $t_s \in [0,\infty)$ so that

$$d_X\big(b(s),\beta(t_s)\big) \le \mathsf{m}_\beta(1,0)\cdot\kappa(s). \tag{5}$$

We take $t_0 = 0$ and $t_d = T$. We show that $\beta[0,T]$ stays in some uniform κ–neighborhood of b by arguing that the times t_s nearly cover the interval $[0,T]$. Let $0 = s_0, s_1, \dots, s_k = d$ be a set of times so that $|s_i - s_{i+1}| \le 1$. Then, for every $t \in [0,T]$ we have $t_{s_0} \le t \le t_{s_k}$. Hence there is an index i such that $t_{s_{i-1}} \le t$ and $t_{s_i} \ge t$.

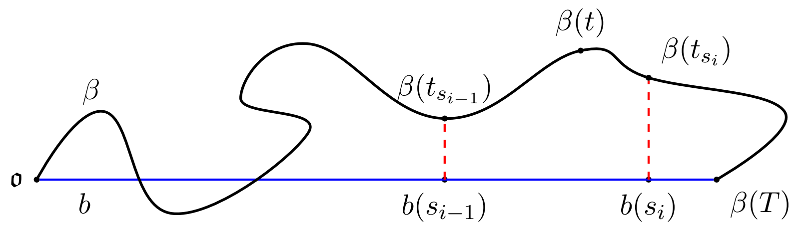


FIGURE 4. The index i is chosen so that $t_{s_{i-1}} \le t \le t_{s_i}$.

We have

$$\begin{aligned} d_X(\beta(t_{s_{i-1}}),\beta(t_{s_i})) &\le d_X(\beta(t_{s_{i-1}}),b(s_{i-1})) + d_X(b(s_{i-1}),b(s_i)) + d_X(b(s_i),\beta(t_{s_i}))\\ &\le \mathsf{m}_\beta(1,0)\cdot\kappa(s_{i-1}) + 1 + \mathsf{m}_\beta(1,0)\cdot\kappa(s_i). \end{aligned}$$

Using the lower-bound condition for a (q,Q)–quasi-geodesic we have

$$|t_{s_i} - t_{s_{i-1}}| \le \mathsf{q} d_X(\beta(t_{s_{i-1}}),\beta(t_{s_i})) + \mathsf{q}\mathsf{Q} \le \mathsf{q}\big(2\mathsf{m}_\beta(1,0)\kappa(s_i)+1\big) + \mathsf{q}\mathsf{Q}.$$

From this and using the upper-bound condition, we get

$$\begin{aligned} d_X\big(\beta(t_{s_i}),\beta(t)\big) &\le \mathsf{q}|t_{s_i} - t| + \mathsf{Q}\\ &\le \mathsf{q}|t_{s_i} - t_{s_{i-1}}| + \mathsf{Q}\\ &\le \mathsf{q}^2(2\mathsf{m}_\beta(1,0)\,\kappa(s_i)+1) + \mathsf{q}^2\mathsf{Q} + \mathsf{Q}. \end{aligned}$$

Combining this with Equation (5), we get that there is a function $\mathsf{m}_1\colon \mathbb{R}^2 \to \mathbb{R}$ depending only on the value of $\mathsf{m}_\beta(1,0)$ so that

$$d_X(\beta(t),b(s_i)) \le \mathsf{m}_1(\mathsf{q},\mathsf{Q})\cdot\kappa(s_i). \tag{6}$$

By Lemma 3.2, there exists m_2 depending only on $\mathsf{m}_1(\mathsf{q},\mathsf{Q})$ and κ such that

$$\kappa(s_i) = \kappa(b(s_i)) \le \mathsf{m}_2\cdot\kappa(\beta(t)).$$

Thus we have

$$\beta[0,T] \subset \mathcal{N}_\kappa\big(b,\mathsf{m}_2(\mathsf{q},\mathsf{Q})\big). \tag{7}$$

Now consider a $(\mathsf{q}',\mathsf{Q}')$–quasi-geodesic $\zeta\colon [s,t] \to X$ with end points on b. To show that ζ stays near b, we modify ζ to a $(9\mathsf{q}',\mathsf{Q}')$–quasi-geodesic ζ' with end points on β which implies that ζ' stays near β since β is κ–Morse. The Equation (7) then implies that ζ stays near b as well.

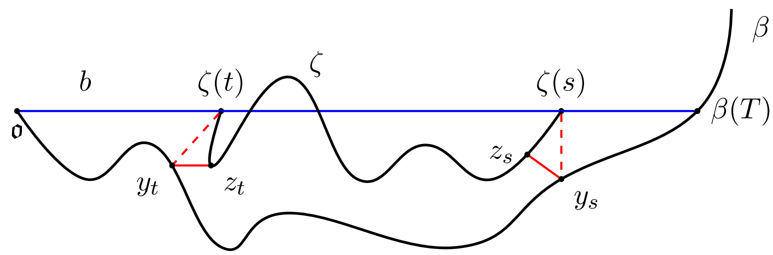


FIGURE 5. The concatenation of $[y_s, z_s]$, $[z_s, z_t]_\zeta$ and $[z_t, y_t]$ is a quasi-geodesic with end points on β.

Let $y_s \in \beta$ be the closest-point in β to $\zeta(s)$ and let z_s be the closest point in ζ to y_s. By Lemma 2.5 the concatenation of the geodesic segment $[y_s, z_s]$ and the quasi-geodesic segment $[z_s, \zeta(t)]_\zeta$ forms a $(3\mathsf{q}', \mathsf{Q}')$–quasi-geodesic. Similarly we can find points $y_t \in \beta$ and $z_t \in \zeta$ and apply Lemma 2.5 again. Denote the concatenation of the geodesic segment $[y_s, z_s]$, the quasi-geodesic segment $[z_s, z_t]_\zeta$ and the geodesic segment $[z_t, y_t]$ by ζ' which is a $(9\mathsf{q}', \mathsf{Q}')$–quasi-geodesic. Then

$$\zeta' \subset \mathcal{N}_\kappa\big(\beta, \mathsf{m}_\beta(9\mathsf{q}', \mathsf{Q}')\big). \tag{8}$$

We say x is κ–close to y, if there is a constant c depending on $\mathsf{q}, \mathsf{Q}, \mathsf{q}', \mathsf{Q}'$ and m_β such that $d_X(x, y) \le \mathsf{c} \cdot \kappa(x)$. It follows from Lemma 3.2 that if x is κ–close to y and y is κ–close to z then x is κ–close z. Thus every point in ζ is κ–close to a point in ζ'. Now Equation (8) and Equation (7) imply that

$$\zeta \subset \mathcal{N}_\kappa\big(b, \mathsf{m}(\mathsf{q}', \mathsf{Q}')\big)$$

for some $\mathsf{m}\colon \mathbb{R}^2 \to \mathbb{R}$ depending on q, Q and m_β only. □

Proposition 3.10 ((4) $\Longrightarrow$ (5))**.** *If $\beta\colon [0, \infty) \to X$ is a κ–Morse quasi-geodesic ray then*

(1) the class $\boldsymbol{b} = [\beta]$ contains a geodesic b, and
(2) the geodesic b is κ–Morse (in particular, for $(32, 0)$–quasi-geodesics).

Proof. For $n \in \mathbb{N}$, let b_n be the geodesic segment connecting $\mathfrak{o}$ to $\beta(n)$. Up to taking a subsequence, we can assume the geodesic segments b_n converge to a geodesic ray b in X. Since β is κ–Morse, $b_n \subset \mathcal{N}_\kappa\big(\beta, \mathsf{m}_\beta(1, 0)\big)$ which means $b \subset \mathcal{N}_\kappa\big(\beta, \mathsf{m}_\beta(1, 0)\big)$. That is, $b \in [\beta]$. But the class $[\beta]$ contains only one geodesic (Lemma 3.5) hence any other subsequence of b_n has to also converge to b. In particular, every point in b is the limit of points in b_n and every limit point of a sequence $x_n \in b_n$ is on b.

The second part follows almost immediately from Proposition 3.9. For every quasi-geodesic ζ with end points on b, there is n_0 so that for $n \ge n_0$, the end points of ζ are distance 1 from some point in b_n. Then ζ can be modified slightly to have end points in b_n. Proposition 3.9 implies that ζ stays in a κ–neighborhood of b_n. But this is true for every $n \ge n_0$. Hence ζ stays in some κ–neighborhood of b. □

To prepare for the next step, we recall a construction of quasi-geodesics from [CS15].

Proposition 3.11 ([CS15])**.** *Given a geodesic segment (possibly infinite) b and points $x, y \in X$ such that $d_X(x, y) < d_X(x, b)$, there exists a $(32, 0)$–quasi-geodesic $\zeta\colon [s_0, s_1] \to X$ with* *endpoints on b such that $\zeta(s_0) = x_b$,*

$$\frac{1}{4} d_X(x_b, y_b) \le d_X(\zeta(s_0), \zeta(s_1)) < d_X(x_b, y_b)$$

and there is a point $p = \zeta(t)$ on ζ so that

$$d_X(p, b) \ge \frac{1}{80} d_X(x_b, y_b). \tag{9}$$

Outline of the proof of Proposition 3.11. The proof of this statement is contained in the proof of [CS15, Theorem 2.9]. We now give the outline of the argument and a detailed reference to that proof. Given a geodesic b and points x and y that satisfy the assumptions, consider the following quadrilateral:

$$Q_1 = [x, x_b] \cup [x_b, y_b] \cup [y, y_b] \cup [x, y].$$

We first construct a smaller quadrilateral inside Q_1 out of two points x', y' where x' on the segments $[x, x_b]$ and y' is either in the interior of the geodesic segment connecting x to y (Theorem 2.9, Case (2)) or on $[y, y_b]$ (Theorem 2.9, Case (1)) and consider the quadrilateral

$$Q_2 = [x', x_b] \cup [x_b, y_b] \cup [y', y_b] \cup [x', y']$$

with the property (in all cases) that

$$d_X(x'_b, y'_b) \ge \frac{1}{4} d_X(x_b, y_b).$$

Let $\mathsf{D} = d_X(x'_b, y'_b)$, and let $\mathsf{a}, \mathsf{b}, \mathsf{c} > 0$ be real numbers such that

$$\begin{aligned} d_X(x', x'_b) &= \mathsf{a}\,\mathsf{D} \\ d_X(x', y') &= \mathsf{b}\,\mathsf{D} \\ d_X(y', y'_b) &= \mathsf{c}\,\mathsf{D} \end{aligned}$$

The quadrilateral Q_2 also satisfies the condition that $\mathsf{a} + \mathsf{c} - \mathsf{b} > 0.1$ and $\mathsf{a} + \mathsf{b} + \mathsf{c} < 8$ (worst case is Case (1); in Case (3) it is shown that $\mathsf{a} + \mathsf{c} - \mathsf{b} > 0.2$).

Next we construct a quasi-geodesics $\zeta(t)$ that starts from x'_b follows along the segment $[x'_b, x']$ until it is close to the segment $[x', y']$, then travels to $[x', y']$ and follows $[x', y']$ until it is close to $[y'_b, y']$, then it travels to $[y'_b, y']$ and finally follows $[y'_b, y']$ until y'_b. [CS15, Lemma 2.7] establishes that $\zeta(t)$ is a $(4(\mathsf{a} + \mathsf{b} + \mathsf{c}), 0)$–quasi-geodesic, that is, ζ is a $(32, 0)$–quasi-geodesic. Let p be a point on $\zeta(t)$ on the segment between x' and y'. Equation (4) of [CS15] states that

$$d_X(p, b) \ge \frac{\mathsf{a} + \mathsf{c} - \mathsf{b}}{2} \mathsf{D}.$$

Combining this with $\mathsf{a} + \mathsf{c} - \mathsf{b} > 0.1$ we have

$$d_X(p, b) \ge \frac{1}{20} d_X(x'_b, y'_b) \ge \frac{1}{80} d_X(x_b, y_b).$$

This finishes the proof. □

Theorem 3.12 ((5) $\Longrightarrow$ (1))**.** *Let b be a geodesic ray in X that is κ–Morse for $(32, 0)$–quasi-geodesics. Then b is κ–contracting. In fact, $\mathsf{c}_b = 82000\, \mathsf{m}_b(32, 0)$.*

Proof. Given points x, y such that $d_X(x,y) < d_X(x,b)$ let $\zeta\colon [s_0, s_1] \to X$ and $p = \zeta(t)$ be as in Proposition 3.11. Since b is κ–Morse for $(32,0)$–quasi-geodesics, we have

$$d_X(p,b) \leq \mathsf{m}_b(32,0) \cdot \kappa(p).$$

On the other hand,

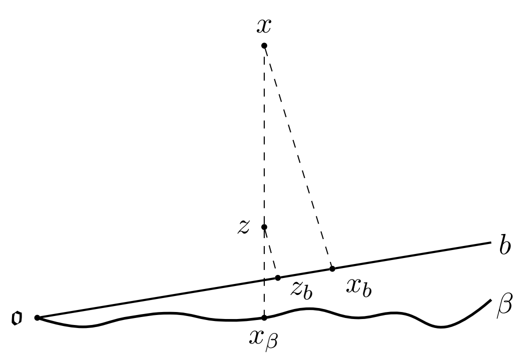
$$\begin{aligned} \|p\| &\leq \|x_b\| + d_X\big(\zeta(s_0), \zeta(t)\big) \\ &\leq \|x_b\| + 32 \cdot |s_1 - s_0| && \zeta \text{ is a (32, 0)–quasi-geodesic} \\ &\leq \|x_b\| + (32)^2 \cdot d_X\big(\zeta(s_0), \zeta(s_1)\big) && \zeta \text{ is a (32, 0)–quasi-geodesic} \\ &\leq \|x_b\| + 1024 \cdot d_X(x_b, y_b) \\ &\leq \|x_b\| + 1024 \cdot d_X(x, y) && \text{Projection to } b \text{ is Lipschitz.} \\ &\leq \|x_b\| + 1024 \cdot d_X(x, x_b) \\ &\leq 1025 \cdot \|x_b\|. \end{aligned}$$

Therefore,

$$\begin{aligned} d_X(x_b, y_b) &\leq 80 \cdot d_X(p, b) \\ &\leq 80 \cdot \mathsf{m}(32,0) \cdot \kappa(p) \\ &\leq 80 \cdot \mathsf{m}(32,0) \cdot \kappa(1025\|x\|) \\ &\leq 82000 \cdot \mathsf{m}(32,0) \cdot \kappa(x). \end{aligned}$$

That is, b is a κ–contracting geodesic with $\mathsf{c}_b = 82000 \cdot \mathsf{m}_b(32,0)$. □

Proposition 3.13 ((1) $\Longrightarrow$ (2))**.** *Le b be a geodesic ray and let β be a quasi-geodesic ray in $\boldsymbol{b} = [b]$. Suppose that b is κ–contracting. Then β is also κ–contracting.*



Proof. Since β and b are in the same class, there exists n such that

$$\beta \subset \mathcal{N}_\kappa(b, \mathsf{n}) \qquad \text{and} \qquad b \subset \mathcal{N}_\kappa(\beta, \mathsf{n}).$$

Let x, y be points in X so that $d_X(x,y) \leq d_X(x,\beta)$. We need to find an upper-bound for $d_X(x', y')$, where $x' \in \pi_\beta(x), y' \in \pi_\beta(y)$. For the remainder of the proof, we use x_β to denote a point in the set $\pi_\beta(x)$ and y_β to denote a point in $\pi_\beta(y)$. The upper-bound certainly exists if $x \in \mathcal{N}_\kappa(\beta, \mathsf{n})$. Thus assume $d(x,\beta) \geq \mathsf{n}\kappa(x)$.

We claim that there is a point z along the geodesic segment $[x, x_\beta]$ such that

$$d_X(x,z) \leq d_X(x,b) \qquad \text{and} \qquad d_X(z,b) \leq 3\mathsf{n} \cdot \kappa(x).$$

To see this, note that

$$d_X(x,\beta) \le d_X(x,x_b) + d_X(x_b,\beta) \le d_X(x,x_b) + \mathsf{n}\cdot\kappa(x_b). \tag{10}$$

Meanwhile, the projection of the segment $[\mathfrak{o}, x]$ to the geodesic b is the segment $[\mathfrak{o}, x_b]$. Since projections in CAT(0) spaces are Lipschitz, $\|x_b\| \le \|x\|$. Thus $\kappa(x_b) \le \kappa(x)$. Therefore, if we choose z to have distance $\mathsf{n}\cdot\kappa(x)$ from x_β, we are sure to have $d_X(z,x) \le d_X(x,b)$. Also,

$$d_X(z,b) \le d_X(z,z_\beta) + d_X(z_\beta,b) \le \mathsf{n}\cdot\kappa(x) + \mathsf{n}\cdot\kappa(x_\beta). \tag{11}$$

Now, note that

$$\|x_\beta\| \le \|x\| + d_X(x,x_\beta) \le 2\|x\|.$$

Hence, $\kappa(x_\beta) \le 2\kappa(x)$. This and Equation (11) imply the second assertion in the claim.

Now, since b is contracting,

$$d_X(z_b,x_b) \le \mathsf{c}_b\cdot\kappa(x).$$

Therefore,

$$\begin{aligned} d_X(x_b,x_\beta) &\le d_X(x_b,z) + d_X(z,x_\beta) \\ &\le 3\mathsf{n}\cdot\kappa(x) + \mathsf{n}\cdot\kappa(x) = 4\mathsf{n}\cdot\kappa(x). \end{aligned} \tag{12}$$

Now let $x,y \in X$ be such that $d_X(x,y) \le d_X(x,\beta)$. Note that,

$$\|y\| \le \|x\| + d_X(x,y) \le \|x\| + d_X(x,\beta) \le 2\|x\|.$$

Hence, applying Equation (12) to x and y we have

$$d_X(x_b,x_\beta) \le 4\mathsf{n}\cdot\kappa(x) \qquad \text{and} \qquad d_X(y_b,y_\beta) \le 4\mathsf{n}\cdot\kappa(y) \le 8\mathsf{n}\cdot\kappa(x).$$

Also, from Equation (10), we have

$$d_X(x,b) \ge d_X(x,\beta) - \mathsf{n}\cdot\kappa(x_b) \ge d_X(x,y) - \mathsf{n}\cdot\kappa(x).$$

Therefore, there is a point $y' \in [x,y]$ with

$$d_X(x,y') \le d_X(x,b) \qquad \text{and} \qquad d(y,y') \le \mathsf{n}\cdot\kappa(x).$$

Thus, since closest-point projection is distance non-increasing,

$$\begin{aligned} d_X(x_\beta,y_\beta) &\le d_X(x_b,y_b) + d_X(x_b,x_\beta) + d_X(y_b,y_\beta) \\ &\le d_X(x_b,y'_b) + d_X(y'_b,y_b) + 12\cdot\mathsf{n}\cdot\kappa(x) \\ &\le \mathsf{c}_b\cdot\kappa(x) + \mathsf{n}\cdot\kappa(x) + 12\cdot\mathsf{n}\cdot\kappa(x) \\ &\le (\mathsf{c}_b + 13\mathsf{n})\cdot\kappa(x). \end{aligned}$$

That is β is κ–contracting with $\mathsf{c}_\beta = (\mathsf{c}_b + 13\mathsf{n})$. □

The property of being κ–contracting implies a stronger version of κ–Morse. Namely, if the end point of η, a quasi-geodesic segment, is only assumed to be sublinearly close to Z for any other sublinear function κ', then conclusion of being Morse still holds with the same functions m_Z, however, in a smaller sub-interval. Here, we prove this stronger version which will be used in the next section and, in particular, proves (2) $\Longrightarrow$ (3).

Theorem 3.14 (Strongly Morse). *Let Z be a closed subspace that is κ–contracting. Then, there is a function $\mathsf{m}_Z\colon \mathbb{R}^2 \to \mathbb{R}$ such that, for every constants $\mathsf{r} > 0$, $\mathsf{n} > 0$ and every sublinear function κ', there is an $\mathsf{R} = \mathsf{R}(Z, \mathsf{r}, \mathsf{n}, \kappa') > 0$ where the following holds: Let $\eta\colon [0, \infty) \to X$ be a (q, Q)–quasi-geodesic ray so that $\mathsf{m}_Z(\mathsf{q}, \mathsf{Q})$ is small compared to r, let t_r be the first time $\|\eta(t_\mathsf{r})\| = \mathsf{r}$ and let t_R be the first time $\|\eta(t_\mathsf{R})\| = \mathsf{R}$. Then*

$$d_X\big(\eta(t_\mathsf{R}), Z\big) \le \mathsf{n} \cdot \kappa'(\mathsf{R}) \quad \Longrightarrow \quad \eta[0, t_\mathsf{r}] \subset \mathcal{N}_\kappa\big(Z, \mathsf{m}_Z(\mathsf{q}, \mathsf{Q})\big).$$

Proof. Let c_Z be the contracting constants for Z. Set

$$\mathsf{m}_0 = \mathsf{q}\big((\mathsf{q}+1) + \mathsf{q}\mathsf{c}_Z + \mathsf{Q}\big) \qquad \text{and} \qquad \mathsf{m}_1 = q\mathsf{c}_Z + \mathsf{q} + \mathsf{Q}. \tag{13}$$

Claim. Consider a time interval $[s, s']$ during which η is outside of $\mathcal{N}_\kappa(Z, \mathsf{m}_0)$. Then

$$|s' - s| \le \mathsf{m}_1\big(d_X\big(\eta(s), Z\big) + d_X\big(\eta(s'), Z\big)\big). \tag{14}$$

Proof of the Claim. Let

$$s = t_0 < t_1 < t_2 < \cdots < t_\ell = s'$$

be a sequence of times such that, for $i = 0, \dots, \ell - 2$, we have t_{i+1} is a first time after t_i where

$$d_X\big(\eta(t_i), \eta(t_{i+1})\big) = d_X(\eta(t_i), Z) \quad \text{and} \quad d_X\big(\eta(t_{\ell-1}), \eta(t_\ell)\big) \le d_X(\eta(t_{\ell-1}), Z).$$

To simplify the notation, we define

$$\eta_i = \eta(t_i), \qquad \mathsf{r}_i = \|\eta(t_i)\|, \qquad \mathsf{d}_i = d_X(\eta_i, Z) \qquad \text{and} \qquad \pi_i = (\eta_i)_Z.$$

Recall that $(\eta_i)_Z$ is the set of the closest points in Z to η_i. Note that, by assumption

$$\mathsf{d}_i \ge \mathsf{m}_0 \cdot \kappa(\mathsf{r}_i).$$

Since Z is contracting,

$$d_X\big(\pi_0, \pi_\ell\big) \le \sum_{i=0}^{\ell-1} \operatorname{diam}_X\big(\pi_i, \pi_{i+1}\big) \le \sum_{i=0}^{\ell-1} \mathsf{c}_Z \cdot \kappa(\mathsf{r}_i).$$

But η is a (q, Q)–quasi-geodesic, hence,

$$\begin{aligned} |s' - s| &\le \mathsf{q}\, d_X(\eta_0, \eta_\ell) + \mathsf{Q} \\ &\le \mathsf{q}\left(\mathsf{d}_0 + d_X\big(\pi_0, \pi_\ell\big) + \mathsf{d}_\ell\right) + \mathsf{Q} \\ &\le \mathsf{q}\,\mathsf{c}_Z\left(\sum_{i=1}^{\ell-1} \kappa(\mathsf{r}_i)\right) + \mathsf{q}\left(\mathsf{d}_0 + \mathsf{d}_\ell\right) + \mathsf{Q}. \end{aligned} \tag{15}$$

On the other hand,

$$|s' - s| = \sum_{i=0}^{\ell-1} |t_{i+1} - t_i| \ge \sum_{i=0}^{\ell-1} \left(\frac{1}{\mathsf{q}} d_X(\eta_i, \eta_{i+1}) - \mathsf{Q}\right).$$

But, for $i = 0, \dots, (\ell - 2)$ we have $d_X(\eta_i, \eta_{i+1}) = \mathsf{d}_i$ and

$$d_X(\eta_{\ell-1}, \eta_\ell) \ge \mathsf{d}_{\ell-1} - \mathsf{d}_\ell.$$

Hence,

$$|s' - s| \ge \sum_{i=0}^{\ell-1} \left(\frac{\mathsf{m}_0}{\mathsf{q}} \cdot \kappa(\mathsf{r}_i) - \mathsf{Q}\right) - \frac{\mathsf{d}_\ell}{\mathsf{q}}. \tag{16}$$

Combining Equation (15) and Equation (16) we get

$$\mathsf{q}\,(\mathsf{d}_0 + \mathsf{d}_\ell) + \mathsf{Q} + \frac{\mathsf{d}_\ell}{\mathsf{q}} \geq \left(\frac{\mathsf{m}_0}{\mathsf{q}} - \mathsf{q}\,\mathsf{c}_Z - \mathsf{Q}\right) \sum_{i=0}^{\ell-1} \kappa(\mathsf{r}_i).$$

But, from (13), we have $\mathsf{Q} \leq \mathsf{m}_0 \leq \mathsf{r}_0$ and

$$\mathsf{q}\,(\mathsf{d}_0 + \mathsf{d}_\ell) + \mathsf{Q} + \frac{\mathsf{d}_\ell}{\mathsf{q}} \leq (\mathsf{q}+1)(\mathsf{d}_0 + \mathsf{d}_\ell) \qquad \text{and} \qquad \left(\frac{\mathsf{m}_0}{\mathsf{q}} - \mathsf{q}\,\mathsf{c}_Z - \mathsf{Q}\right) \geq (\mathsf{q}+1).$$

Which implies

$$\sum_{i=0}^{\ell-1} \kappa(\mathsf{r}_i) \leq \mathsf{d}_0 + \mathsf{d}_\ell \qquad \text{and by Equation (15)} \qquad |s' - s| \leq \mathsf{m}_1(\mathsf{d}_0 + \mathsf{d}_\ell).$$

This proves the claim. ∎

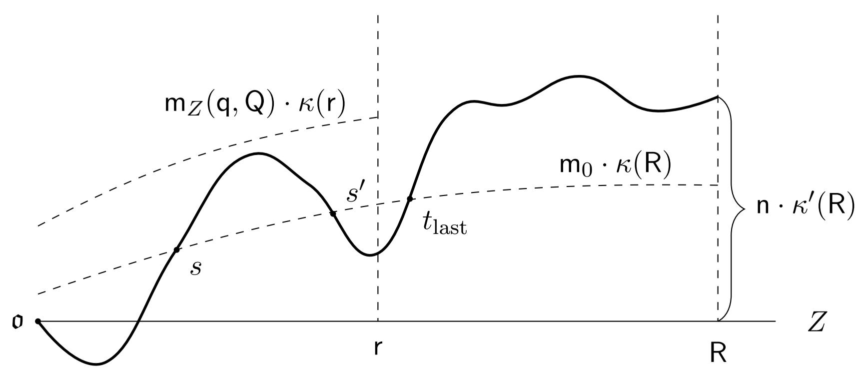


FIGURE 6. The concatenation of a geodesic segment $[x, y]$ and the quasi-geodesic segment $[y, z_1]$ is a quasi-geodesic.

Now let t_{last} be the last time η is in $\mathcal{N}_\kappa(Z, \mathsf{m}_0)$ and consider the quasi-geodesic path $\eta[t_{\text{last}}, t_{\mathsf{R}}]$. Since this path is outside of $\mathcal{N}_\kappa(Z, \mathsf{m}_0)$, we can use Equation (14) to get

$$|\mathsf{R} - t_{\text{last}}] \leq \mathsf{m}_1\big(d_X(\eta(t_{\text{last}}), Z) + d_X(\eta(t_{\mathsf{R}}), Z)\big).$$

But

$$d_X(\eta(t_{\text{last}}), Z) \leq \mathsf{m}_0 \cdot \kappa(\eta(t_{\text{last}})) \leq \mathsf{m}_0 \cdot \kappa(\mathsf{R}) \qquad \text{and} \qquad d_X(\eta(t_{\mathsf{R}}), Z) \leq \mathsf{n} \cdot \kappa'(\mathsf{R}).$$

Therefore,

$$|\mathsf{R} - t_{\text{last}}] \leq \mathsf{m}_0 \cdot \mathsf{m}_1 \cdot \kappa(\mathsf{R}) + \mathsf{n} \cdot \kappa'(\mathsf{R})$$

Since m_0, m_1 and n are given and κ and κ' are sublinear, there is a value of R depending on m_0, m_1, n, r, κ and κ' such that

$$\mathsf{m}_0 \cdot \mathsf{m}_1 \cdot \kappa(\mathsf{R}) + \mathsf{n} \cdot \kappa'(\mathsf{R}) \leq \mathsf{R} - \mathsf{r}.$$

For any such R, we then have

$$t_{\text{last}} \geq \mathsf{r}.$$

We show that $\eta[0, t_{\text{last}}]$ stays in a larger κ–neighborhood of Z. Consider any other subinterval $[s, s'] \subset [0, t_{\text{last}}]$ where η exits $\mathcal{N}_\kappa(Z, \mathsf{m}_0)$. By taking $[s, s']$ as large as possible, we can assume $\eta(s), \eta(s') \in \mathcal{N}_\kappa(Z, \mathsf{m}_0)$. In this case,

$$d_X(\eta(s), Z) \le \mathsf{m}_0 \cdot \kappa(\eta(s)) \qquad \text{and} \qquad d_X(\eta(s'), Z) \le \mathsf{m}_0 \cdot \kappa(\eta(s')).$$

again applying Equation (14), we get

$$|s' - s| \le \mathsf{m}_0\, \mathsf{m}_1 \cdot \big(\kappa(\eta(s)) + \kappa(\eta(s'))\big).$$

and thus

$$\begin{aligned}\Big| \|\eta(s')\| - \|\eta(s)\| \Big| &\le \mathsf{q}\, \mathsf{m}_0\, \mathsf{m}_1 \cdot \big(\kappa(\eta(s)) + \kappa(\eta(s'))\big) + \mathsf{Q} \\ &\le (\mathsf{q}\, \mathsf{m}_0\, \mathsf{m}_1 + \mathsf{Q}) \cdot \big(\kappa(\eta(s)) + \kappa(\eta(s'))\big) \\ &\le 2(\mathsf{q}\, \mathsf{m}_0\, \mathsf{m}_1 + \mathsf{Q}) \cdot \max\big(\kappa(\eta(s)), \kappa(\eta(s'))\big).\end{aligned}$$

Applying Lemma 3.2, we have that

$$\kappa(\eta(s')) \le \mathsf{m}_2 \cdot \kappa(\eta(s)),$$

for some m_2 depending on c_Z, q, Q and κ. Therefore, for any $t \in [s, s']$

$$|t - s| \le \mathsf{m}_0\, \mathsf{m}_1 (1 + \mathsf{m}_2) \cdot \kappa(\eta(s)). \tag{17}$$

As before, this implies,

$$\Big| \|\eta(t)\| - \|\eta(s)\| \Big| \le \mathsf{q}\, \mathsf{m}_0\, \mathsf{m}_1 (1 + \mathsf{m}_2) \cdot \kappa(\eta(s)) + \mathsf{Q} \le (\mathsf{q}\, \mathsf{m}_0\, \mathsf{m}_1 (1 + \mathsf{m}_2) + \mathsf{Q}) \cdot \kappa(\eta(s)).$$

Applying Lemma 3.2 again, we have

$$\kappa(\eta(s)) \le \mathsf{m}_3 \cdot \kappa(\eta(t)), \tag{18}$$

for some m_3 depending on c_Z, q, Q and κ.

Now, for any $t \in [s, s']$ we have

$$\begin{aligned} d_X(\eta(t), Z) &\le d_X(\eta(t), \eta(s)) + \mathsf{r}_0 \\ &\le \mathsf{q}|t - s| + \mathsf{Q} + \mathsf{m}_0 \cdot \kappa(\eta(s)) \\ \text{(Equation (17))} \qquad &\le (\mathsf{q}\mathsf{m}_0\, \mathsf{m}_1 (1 + \mathsf{m}_2) + \mathsf{Q} + \mathsf{m}_0) \cdot \kappa(\eta(s)) \\ \text{(Equation (18))} \qquad &\le (\mathsf{q}\mathsf{m}_0\, \mathsf{m}_1 (1 + \mathsf{m}_2) + \mathsf{Q} + \mathsf{m}_0)\, \mathsf{m}_3 \cdot \kappa(\eta(t)). \end{aligned}$$

Now setting

$$\mathsf{m}_Z(\mathsf{q}, \mathsf{Q}) = (\mathsf{q}\mathsf{m}_0\, \mathsf{m}_1 (1 + \mathsf{m}_2) + \mathsf{Q} + \mathsf{m}_0)\, \mathsf{m}_3 \tag{19}$$

we have that

$$\eta[s, s'] \subset \mathcal{N}_\kappa\big(Z, \mathsf{m}_Z(\mathsf{q}, \mathsf{Q})\big) \qquad \text{and hence} \qquad \eta[0, t_{\text{last}}] \subset \mathcal{N}_\kappa\big(Z, \mathsf{m}_Z(\mathsf{q}, \mathsf{Q})\big).$$

The R we have chosen depends on the value of q and Q. However, the assumption that $\mathsf{m}_Z(\mathsf{q}, \mathsf{Q})$ is small compared to r (see Equation (2)) gives an upper-bound for the values of q and Q. Hence, we can choose R to be the radius associated to the largest possible value for q and the largest possible value for Q. This finishes the proof.

Note that, the assumption that $\mathsf{m}_Z(\mathsf{q}, \mathsf{Q})$ is small compared to r is not really needed here and any upper-bound on the values of q and Q would suffice. But this is the assumption we will have later on and hence it is natural to state the theorem this way. □

Remark 3.15. The above argument in particular proves (2) $\Longrightarrow$ (3). First of all, the proof also works for quasi-geodesic segments $\eta\colon [s,t] \to X$ as long as $\eta(s)$ lies on Z. Secondly, if $\eta(t)$ is also assumed to be on Z, by definition, $t_{\text{last}} = t$. Which means the entire segment $\eta[s,t]$ is shown to be contained in $\mathcal{N}_\kappa\big(Z, \mathsf{m}_Z(\mathsf{q},\mathsf{Q})\big)$.

We finish with a couple of corollaries of Theorem 3.14. Recall that, a (q,Q)–quasi-geodesic β is in $\mathbf{b}$ if β is contained in some (κ,n)–neighborhood of the geodesic ray $b \in \mathbf{b}$. A priori, it might be possible for the constant n to go to infinity even as q and Q remain bounded. However, this does not happen.

Corollary 3.16. *Let b be a κ–contracting geodesic ray and let m_b be as in Theorem 3.14 (where Z is the image of b). Then, for any (q,Q)–quasi-geodesic $\beta \in [b]$, we have*

$$\beta \subset \mathcal{N}_\kappa\big(b, \mathsf{m}_b(\mathsf{q},\mathsf{Q})\big) \qquad \textit{and} \qquad b \subset \mathcal{N}_\kappa\big(\beta, 2\mathsf{m}_b(\mathsf{q},\mathsf{Q})\big).$$

Proof. Since $\beta \in [b]$, there is a constant n so that $\beta \subset \mathcal{N}_\kappa(b,\mathsf{n})$. For every r, let t_r be the first time when $\beta(t_\mathsf{r})$ has norm r. We have

$$d_X(\beta(t_\mathsf{R}), b) \le \mathsf{n} \cdot \kappa(\mathsf{R})$$

for every R. Now Theorem 3.14 implies that

$$\beta[0, t_\mathsf{r}] \subset \mathcal{N}_\kappa\big(b, \mathsf{m}_b(\mathsf{q},\mathsf{Q})\big)$$

for every r. This proves the first assertion.

To see the second assertion, consider a point $b_\mathsf{r} = b(\mathsf{r})$, let $\beta_\mathsf{r} = \beta(t_\mathsf{r})$ and let $q = \pi_b(\beta_\mathsf{r})$. Then, the first assertion implies

$$d_X(\beta_\mathsf{r}, q) \le \mathsf{m}_b(\mathsf{q},\mathsf{Q}) \cdot \kappa(\mathsf{r}).$$

Hence,

$$\begin{aligned} d_X(b_\mathsf{r}, q) &\le \mathsf{r} - d_X(\mathfrak{o}, q) \\ &\le \mathsf{r} - \big(d_X(\mathfrak{o}, \beta_\mathsf{r}) - d_X(\beta_\mathsf{r}, q)\big) \le \mathsf{m}_b(\mathsf{q},\mathsf{Q}) \cdot \kappa(\mathsf{r}). \end{aligned}$$

Therefore,

$$d_X(b_\mathsf{r}, \beta) \le d_X(b_\mathsf{r}, \beta_\mathsf{r}) \le d_X(b_\mathsf{r}, q) + d_X(q, \beta_\mathsf{r}) \le 2\mathsf{m}_b(\mathsf{q},\mathsf{Q}) \cdot \kappa(\mathsf{r}),$$

which implies $b \subset \mathcal{N}_\kappa\big(\beta, 2\mathsf{m}_b(\mathsf{q},\mathsf{Q})\big)$. □

Corollary 3.17. *If $\beta \in \boldsymbol{b}$ is a (q,Q)–quasi-geodesic, then the function*

$$\mathsf{m}_\beta(\cdot,\cdot) \le \mathsf{m}_b(\cdot,\cdot) + 2\mathsf{m}_b(\mathsf{q},\mathsf{Q})$$

is a Morse gauge for β. In particular, the Morse gauge depends only on m_b, q and Q and not on the particular quasi-geodesic β.

Proof. Let $\beta' \in \mathbf{b}$ be a $(\mathsf{q}',\mathsf{Q}')$–quasi-geodesic. Let β'_r be a point along β' with norm r, let $p = \pi_b(\beta'_\mathsf{r})$ and let q be the closest point in β to p. Note that $\|p\| \le \mathsf{r}$. Hence,

$$\begin{aligned} d_X(\beta'_\mathsf{r}, \beta) &\le d_X(\beta'_\mathsf{r}, p) + d_X(p, q) \\ &\le \mathsf{m}_b(\mathsf{q}',\mathsf{Q}') \cdot \kappa(\mathsf{r}) + 2\mathsf{m}_b(\mathsf{q},\mathsf{Q}) \cdot \kappa(p) \le \big(\mathsf{m}_b(\mathsf{q}',\mathsf{Q}') + 2\mathsf{m}_b(\mathsf{q},\mathsf{Q})\big) \cdot \kappa(\mathsf{r}) \end{aligned}$$

This finishes the proof. □

4. κ–Morse boundary

Recall the definition of κ–fellow traveling (Definition 3.4) which defines an equivalence relation on the set of all quasi-geodesic rays in X. Recall also that all geodesic rays and quasi-geodesic rays are assumed to start from the fixed base-point $\mathfrak{o}$.

Definition 4.1 (κ–Morse boundary set)**.** The κ–Morse boundary of X, $\partial_\kappa X$, is the set of κ–fellow traveling classes of κ–contracting quasi-geodesic rays in X. Since each class contains a unique geodesic which is also κ–contracting (Theorem 3.8) we could also define $\partial_\kappa X$ to be the set of κ–contracting geodesic rays in X.

We equip $\partial_\kappa X$ with a topology which is a coarse version of the visual topology. Roughly speaking, we think of a point $\mathbf{a} \in \partial_\kappa X$ as being in a small neighborhood of $\mathbf{b} \in \partial_\kappa X$ if, for some large radius r, every (q, Q)–quasi-geodesic $\alpha \in \mathbf{a}$, where $\mathsf{m}_b(\mathsf{q}, \mathsf{Q})$ is small compared to the radius r, fellow travels the geodesic $b \in \mathbf{b}$ up to the radius r. As we shall see, this is strictly stronger than assuming that the geodesics $a \in \mathbf{a}$ and $b \in \mathbf{b}$ fellow travel each other for a long time.

We introduce the following notations. Let β be a (q, Q)–quasi-geodesic ray that is κ–Morse and let m_β be the associated Morse gauge functions as in Theorem 3.14. For $\mathsf{r} > 0$, let t_r be the first time where $\|\beta(t)\| = \mathsf{r}$ and define:

$$\beta_\mathsf{r} = \beta(t_\mathsf{r}) \qquad \text{and} \qquad \beta|_\mathsf{r} = \beta[0, t_\mathsf{r}]$$

which we consider as a subset of X.

Definition 4.2 (neighbourhoods)**.** Let $\mathbf{b} \in \partial_\kappa X$ and $b \in \mathbf{b}$ be the unique geodesic in the class $\mathbf{b}$. Define $\mathcal{U}_\kappa(\mathbf{b}, \mathsf{r})$ to be the set of points $\mathbf{a} \in \partial_\kappa X$ such that, for any (q, Q)–quasi-geodesic $\alpha \in \mathbf{a}$ where $\mathsf{m}_b(\mathsf{q}, \mathsf{Q})$ is small compared to r (see Equation (2)) we have

$$\alpha|_\mathsf{r} \subset \mathcal{N}_\kappa\big(b, \mathsf{m}_b(\mathsf{q}, \mathsf{Q})\big).$$

4.1. Neighborhood system.

In this sub-section we show that the sets $\mathcal{U}_\kappa(\gamma, \mathsf{r})$ generate a neighborhood system which can be used to define a topology for $\partial_\kappa X$. We start with a technical lemma.

Lemma 4.3. *Let b be a geodesic ray and γ be a (q, Q)–quasi-geodesic ray. For $\mathsf{r} > 0$, assume that $d_X(b_\mathsf{r}, \gamma) \le \mathsf{r}/2$. Then, there exists a $(9\mathsf{q}, \mathsf{Q})$–quasi-geodesic γ' so that*

$$\gamma' \in [b], \qquad \text{and} \qquad \gamma|_{\mathsf{r}/2} = \gamma'|_{\mathsf{r}/2}.$$

Proof. Let q be a point in γ that is closest to b_r and let $\mathsf{R} > 0$ be such that the ball of radius R centered at $\mathfrak{o}$ contains $[\mathfrak{o}, q]_\gamma$. Now let q' be the point in $[0, q]_\gamma$ closest to b_R. Then

$$\begin{aligned}\|q'\| &\ge \|b_\mathsf{R}\| - d_X(b_\mathsf{R}, q') \\ &\ge \mathsf{R} - d_X(b_\mathsf{R}, q) \\ &\ge \mathsf{R} - \Big(d_X(b_\mathsf{R}, b_\mathsf{r}) + d_X(b_\mathsf{r}, q)\Big) \\ &\ge \mathsf{R} - (\mathsf{R} - \mathsf{r}) - \frac{\mathsf{r}}{2} = \frac{\mathsf{r}}{2}\end{aligned}$$

Applying Lemma 2.5, we have a $(3\mathsf{q}, \mathsf{Q})$–quasi-geodesic segment

$$\zeta = [\mathfrak{o}, q']_\gamma \cup [q', b_\mathsf{R}].$$

Furthermore, by construction, $\|q'\| \leq \mathsf{R} = \|b_{\mathsf{R}}\|$. Therefore, the projection of any point on the geodesic $b[\mathsf{R}, \infty)$ to ζ is the point b_{R}. Applying Lemma 2.5 again we have that the concatenation

$$\gamma' = \zeta \cup b[\mathsf{R}, \infty)$$

is a $(9\mathsf{q}, \mathsf{Q})$–quasi-geodesic ray.

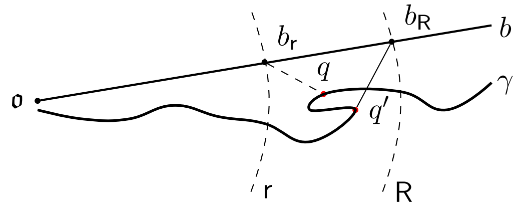


FIGURE 7. The red point denotes a point in the projection set.

Lastly, since $\|q'\| \geq \mathsf{r}/2$, we have $\gamma|_{\mathsf{r}/2} = \zeta|_{\mathsf{r}/2} = \gamma'|_{\mathsf{r}/2}$. □

Proposition 4.4. *For each $\boldsymbol{b} \in \partial_\kappa X$ and $\mathsf{r} > 0$, there exists a radius $\mathsf{r}_{\boldsymbol{b}}$ such that*

(1) for any point $\boldsymbol{a}$ there exists $\mathsf{r}_{\boldsymbol{a}}$ so that

$$\boldsymbol{a} \in \mathcal{U}_\kappa(\boldsymbol{b}, \mathsf{r}_{\boldsymbol{b}}) \qquad \Longrightarrow \qquad \mathcal{U}_\kappa(\boldsymbol{a}, \mathsf{r}_{\boldsymbol{a}}) \subset \mathcal{U}_\kappa(\boldsymbol{b}, \mathsf{r}).$$

(2) for any point $\boldsymbol{a}$ there exists $\mathsf{r}_{\boldsymbol{a}}$ so that

$$\mathcal{U}_\kappa(\boldsymbol{a}, \mathsf{r}_{\boldsymbol{a}}) \cap \mathcal{U}_\kappa(\boldsymbol{b}, \mathsf{r}_{\boldsymbol{b}}) \neq \emptyset. \qquad \Longrightarrow \qquad \boldsymbol{a} \in \mathcal{U}_\kappa(\boldsymbol{b}, \mathsf{r}).$$

Proof. For the rest of this proof, we assume q and Q are such that if $\mathsf{m}_a(\mathsf{q}', \mathsf{Q}')$ is small compared to r then $\mathsf{q}' \leq \mathsf{q}$ and $\mathsf{Q}' \leq \mathsf{Q}$. Hence, if we prove a statement for all (q, Q)–quasi-geodesics, we have also shown the statement for all $(\mathsf{q}', \mathsf{Q}')$–quasi-geodesics where $\mathsf{m}_a(\mathsf{q}', \mathsf{Q}')$ is small compared to r.

Let $b \in \mathbf{b}$ be the unique geodesic ray in $\mathbf{b}$. We choose $\mathsf{r}_{\mathbf{b}}$ such that

$$\mathsf{r}_{\mathbf{b}} \geq 2\mathsf{r} \qquad \text{and} \qquad \mathsf{m}_b(9\mathsf{q}, \mathsf{Q}) \leq \frac{\mathsf{r}_{\mathbf{b}}}{2\kappa(\mathsf{r}_{\mathbf{b}})}.$$

Also, letting $\mathsf{n}(\mathsf{q}, \mathsf{Q}) = \mathsf{m}_b(9\mathsf{q}, \mathsf{Q})$, we require that $\mathsf{r}_{\mathbf{b}} \geq \mathsf{R}$ where $\mathsf{R} = \mathsf{R}(b, \mathsf{r}, \mathsf{n}, \kappa)$ is as in Theorem 3.14.

Proof of Part (1). Let $\mathbf{a} \in \mathcal{U}_\kappa(\mathbf{b}, \mathsf{r}_{\mathbf{b}})$ and let $a \in \mathbf{a}$ be the unique geodesic ray in $\mathbf{a}$. Choose $\mathsf{r}_{\mathbf{a}}$ such that,

$$\mathsf{r}_{\mathbf{a}} \geq 2\mathsf{r}_{\mathbf{b}} \qquad \text{and} \qquad \mathsf{m}_a(\mathsf{q}, \mathsf{Q}) \leq \frac{\mathsf{r}_{\mathbf{a}}}{4\kappa(\mathsf{r}_{\mathbf{a}})}.$$

Now consider $\mathbf{c} \in \mathcal{U}_\kappa(\mathbf{a}, \mathsf{r}_{\mathbf{a}})$ and let $\gamma \in \mathbf{c}$ be a (q, Q)–quasi-geodesic. By the second assertion in Corollary 3.16

$$d_X(a(\mathsf{r}_{\mathbf{a}}), \gamma|_{\mathsf{r}_{\mathbf{a}}}) \leq 2\mathsf{m}_a(\mathsf{q}, \mathsf{Q}) \cdot \kappa(\mathsf{r}_{\mathbf{a}}) \leq \frac{\mathsf{r}_{\mathbf{a}}}{2}.$$

We apply Lemma 4.3, with radius being $\mathsf{r}_{\mathbf{a}}$, to modify γ to a $(9\mathsf{q}, \mathsf{Q})$–quasi-geodesic $\gamma' \in \mathbf{a}$. Since, $\mathsf{r}_{\mathbf{b}} \leq \mathsf{r}_{\mathbf{a}}/2$, we have $\gamma|_{\mathsf{r}_{\mathbf{b}}} = \gamma'|_{\mathsf{r}_{\mathbf{b}}}$. Also, $\mathbf{a} \in \mathcal{U}_\kappa(\mathbf{b}, \mathsf{r}_{\mathbf{b}})$ and $\mathsf{m}_b(9\mathsf{q}, \mathsf{Q})$ is small compare to $\mathsf{r}_{\mathbf{b}}$, therefore

$$\gamma|_{\mathsf{r}_{\mathbf{b}}} = \gamma'|_{\mathsf{r}_{\mathbf{b}}} \subset \mathcal{N}_\kappa\big(a, \mathsf{m}_a(9\mathsf{q}, \mathsf{Q})\big).$$

But $\gamma|_{\mathsf{r}_{\mathbf{a}}}$ is actually a (q,Q)–quasi-geodesic. Hence, Theorem 3.14 (with $\mathsf{n}(\mathsf{q},\mathsf{Q}) = \mathsf{m}_a(9\mathsf{q},\mathsf{Q})$) implies that

$$\gamma|_{\mathsf{r}} \subseteq \mathcal{N}_\kappa\big(a, \mathsf{m}_a(\mathsf{q},\mathsf{Q})\big).$$

This holds for every such $\gamma \in \mathbf{c}$, thus $\mathbf{c} \in \mathcal{U}_\kappa(\mathbf{b},\mathsf{r})$. And this argument holds for every $\mathbf{c} \in \mathcal{U}_\kappa(\mathbf{a},\mathsf{r}_{\mathbf{a}})$, therefore $\mathcal{U}_\kappa(\mathbf{a},\mathsf{r}_{\mathbf{a}}) \subset \mathcal{U}_\kappa(\mathbf{b},\mathsf{r})$.

Proof of Part (2). In view of Corollary 3.17, there exists a constant $\mathsf{u} > 0$, depending on q and Q, such that, for any (q,Q)–quasi-geodesic $\alpha \in \mathbf{a}$ we have

$$\mathsf{m}_\alpha(1,0) + 2\mathsf{m}_a(\mathsf{q},\mathsf{Q}) \leq \mathsf{u}.$$

Choose $\mathsf{r}_{\mathbf{a}}$ large enough so that

$$\mathsf{r}_{\mathbf{a}} \geq \max\big(2\mathsf{u}\cdot\kappa(\mathsf{r}_{\mathbf{a}}), 2\mathsf{r}_{\mathbf{b}}\big).$$

Assume $\mathcal{U}_\kappa(\mathbf{a},\mathsf{r}_{\mathbf{a}}) \cap \mathcal{U}_\kappa(\mathbf{b},\mathsf{r}_{\mathbf{b}})$ is non-empty and consider a point $\mathbf{c}$ in this set. Let $c \in \mathbf{c}$ be the unique geodesic ray in this class. We have to show $\mathbf{a} \in \mathcal{U}_\kappa(\mathbf{b},\mathsf{r})$.

Consider a (q,Q)–quasi-geodesic $\alpha \in \mathbf{a}$. Since, $\mathbf{c} \in \mathcal{U}_\kappa(\mathbf{a},\mathsf{r}_{\mathbf{a}})$,

$$d_X(c(\mathsf{r}_{\mathbf{a}}), a) \leq \mathsf{m}_a(1,0)\cdot\kappa(\mathsf{r}_{\mathbf{a}}).$$

Defining $p = \pi_a(c(\mathsf{r}_{\mathbf{a}}))$, we have $\|p\| \leq \mathsf{r}_{\mathbf{a}}$. Therefore, the second assertion in Corollary 3.16 implies

$$d_X(p,\alpha) \leq 2\mathsf{m}_a(\mathsf{q},\mathsf{Q})\cdot\kappa(p) \leq 2\mathsf{m}_a(\mathsf{q},\mathsf{Q})\cdot\kappa(\mathsf{r}_{\mathbf{a}}).$$

Hence,

$$d_X(c(\mathsf{r}_{\mathbf{a}}),\alpha) \leq d_X(c(\mathsf{r}_{\mathbf{a}}),p) + d_X(p,\alpha) \leq \mathsf{u}\cdot\kappa(\mathsf{r}_{\mathbf{a}}) \leq \frac{\mathsf{r}_{\mathbf{a}}}{2}.$$

We can now apply Lemma 4.3 to α and c with radius $\mathsf{r}_{\mathbf{a}}$ to obtain a $(9\mathsf{q},\mathsf{Q})$–quasi-geodesic $\alpha' \in \mathbf{c}$ where (using $\frac{\mathsf{r}_{\mathbf{a}}}{2} \geq \mathsf{r}_{\mathbf{b}}$), $\alpha'|_{\mathsf{r}_{\mathbf{b}}} = \alpha|_{\mathsf{r}_{\mathbf{b}}}$.

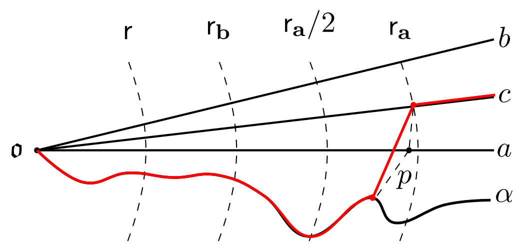


FIGURE 8. The quasi-geodesic α' (in red) is in the class $\mathbf{c}$ which is contained in $\mathcal{U}_\kappa(\mathbf{b},\mathsf{r})$. Therefore, $\alpha|_{\mathsf{r}} = \alpha'|_{\mathsf{r}}$ is near $b \in \mathbf{b}$.

Since $\mathbf{c} \in \mathcal{U}_\kappa(\mathbf{b},\mathsf{r}_{\mathbf{b}})$, we have

$$\alpha|_{\mathsf{r}_{\mathbf{b}}} = \alpha'|_{\mathsf{r}_{\mathbf{b}}} \subset \mathcal{N}_\kappa\big(b, \mathsf{m}_b(9\mathsf{q},\mathsf{Q})\big).$$

But $\alpha|_{\mathsf{r}_{\mathbf{b}}}$ is really a (q,Q)–quasi-geodesic. Hence, letting $\mathsf{n}(\mathsf{q},\mathsf{Q}) = \mathsf{m}_b(9\mathsf{q},\mathsf{Q})$, Theorem 3.14 implies that

$$\alpha|_{\mathsf{r}} \subset \mathcal{N}_\kappa\big(b, \mathsf{m}_b(\mathsf{q},\mathsf{Q})\big).$$

But this holds for every such α, thus $\mathbf{a} \in \mathcal{U}_\kappa(\mathbf{b},\mathsf{r})$. This finishes the proof. $\square$

Remark 4.5. Let $\phi\colon \partial_\kappa X \times \mathbb{R} \to \mathbb{R}$ be a map so that $\mathsf{r}_{\mathbf{b}} = \phi(\mathbf{b}, \mathsf{r})$ as above. We can define a similar map for $\mathsf{r}_{\mathbf{a}}$. Note that, in either part of Proposition 4.4, the radius $\mathsf{r}_{\mathbf{a}}$ does not really depend on $\mathbf{b}$ or r. It depends on $\mathbf{a}$, $\mathsf{r}_{\mathbf{b}}$ and the maximum value of q and Q so that $\mathsf{m}_b(\mathsf{q}, \mathsf{Q})$ is small compared to r. But such an upper-bound always exists, for example, $\mathsf{q}, \mathsf{Q} \leq \mathsf{m}_b(\mathsf{q}, \mathsf{Q}) \leq \mathsf{r} \leq \mathsf{r}_{\mathbf{b}}$. Hence, there are maps $\psi_1, \psi_2\colon \partial_\kappa X \times \mathbb{R} \to \mathbb{R}$ where $\mathsf{r}_{\mathbf{a}} = \psi_1(\mathbf{a}, \mathsf{r}_{\mathbf{b}})$ in the first part of Proposition 4.4 and $\mathsf{r}_{\mathbf{a}} = \psi_2(\mathbf{a}, \mathsf{r}_{\mathbf{b}})$ in the second part. These maps make the dependence of constants more clear and we will refer to these map in the proof of Theorem 4.9. Using this notation, the part of Proposition 4.4 can be written as

$$\mathbf{a} \in \mathcal{U}_\kappa\big(\mathbf{b}, \phi(\mathbf{b}, \mathsf{r})\big) \qquad \Longrightarrow \qquad \mathcal{U}_\kappa\Big(\mathbf{a}, \psi_1\big(\mathbf{a}, \phi(\mathbf{b}, \mathsf{r})\big)\Big) \subset \mathcal{U}_\kappa(\mathbf{b}, \mathsf{r}). \tag{20}$$

and

$$\mathcal{U}_\kappa\Big(\mathbf{a}, \psi_2\big(\mathbf{a}, \phi(\mathbf{b}, \mathsf{r})\big)\Big) \cap \mathcal{U}_\kappa\big(\mathbf{b}, \phi(\mathbf{b}, \mathsf{r})\big) \neq \emptyset. \qquad \Longrightarrow \qquad \mathbf{a} \in \mathcal{U}_\kappa(\mathbf{b}, \mathsf{r}). \tag{21}$$

A fundamental system of neighborhoods. We will show that the sets $\mathcal{U}_\kappa(\mathbf{b}, \mathsf{r})$ form a fundamental system of neighborhoods for $\partial_\kappa X$ that can be used to define a topology on $\partial_\kappa X$. For $\mathbf{b} \in \partial_\kappa X$, define

$$\mathcal{B}(\mathbf{b}) = \Big\{ \mathcal{V} \subset \partial_\kappa X \;\Big|\; \mathcal{U}_\kappa(\mathbf{b}, \mathsf{r}) \subset \mathcal{V} \quad \text{for some } \mathsf{r} > 0 \Big\}.$$

We would like to equip $\partial_\kappa X$ with a topology where $\mathcal{B}(\mathbf{b})$ is the set of neighborhoods of $\mathbf{b}$. Recall that $\mathcal{V}$ is a neighborhood of $\mathbf{b}$ if it contains an open set that includes $\mathbf{b}$. We need to check that $\mathcal{B}(\mathbf{b})$ has certain properties.

Lemma 4.6. *For every $\boldsymbol{b} \in \partial_\kappa X$, the set $\mathcal{B}(\boldsymbol{b})$ satisfies the following properties:*

(i) Every subset of $\partial_\kappa X$ which contains a set belonging to $\mathcal{B}(\boldsymbol{b})$ itself belongs to $\mathcal{B}(\boldsymbol{b})$.
(ii) Every finite intersection of sets of $\mathcal{B}(\boldsymbol{b})$ belongs to $\mathcal{B}(\boldsymbol{b})$.
(iii) The element $\boldsymbol{b}$ is in every set of $\mathcal{B}(\boldsymbol{b})$.
(iv) If $\mathcal{V} \in \mathcal{B}(\boldsymbol{b})$ then there is $\mathcal{W} \in \mathcal{B}(\boldsymbol{b})$ such that, for every $\boldsymbol{a} \in \mathcal{W}$, we have $\mathcal{V} \in \mathcal{B}(\boldsymbol{a})$.

Proof. Property (i) is immediate from the definition of $\mathcal{B}(\mathbf{b})$. To see (ii), consider sets $\mathcal{V}_1, \dots, \mathcal{V}_k \in \mathcal{B}(\mathbf{b})$ and ler r_i be such that $\mathcal{U}_\kappa(\mathbf{b}, \mathsf{r}_i) \subset \mathcal{V}_i$ and let $\mathsf{r} = \max \mathsf{r}_i$. Note that $\mathcal{U}_\kappa(\mathbf{b}, \mathsf{r}) \subset \mathcal{U}_\kappa(\mathbf{b}, \mathsf{r}_i)$ by definition. Therefore,

$$\mathcal{U}_\kappa(\mathbf{b}, \mathsf{r}) \subset \bigcap_i \mathcal{V}_i$$

and hence the intersection is in $\mathcal{B}(\mathbf{b})$. Property (iii) holds since, by Corollary 3.16, every (q, Q)–quasi-geodesic $\beta \in \mathbf{b}$ lies inside $\mathcal{N}_\kappa\big(b, \mathsf{m}_b(\mathsf{q}, \mathsf{Q})\big)$ and hence $\mathbf{b} \in \mathcal{U}_\kappa(\mathbf{b}, \mathsf{r})$ for every r. Property (iv) follows from the first part of Proposition 4.4. □

These properties for $\mathcal{B}(\mathbf{b})$ are characteristic of the set of neighborhoods of $\mathbf{b}$. That is,

Proposition 4.7 ([Bou98] Proposition 2)**.** *If to each elements $\boldsymbol{b} \in \partial_\kappa X$ there corresponds a set $\mathcal{B}(\boldsymbol{b})$ of subsets of $\partial_\kappa X$ such that properties (i) to (iv) above are satisfied, then there is a unique topological structure on $\partial_\kappa X$ such that for each $\boldsymbol{b} \in \partial_\kappa X$, $\mathcal{B}(\boldsymbol{b})$ is the set of neighborhoods of $\boldsymbol{b}$ in this topology.*

We now equip $\partial_\kappa X$ with this topological structure. Then a set $\mathcal{W} \subset \partial_\kappa X$ is open if for every $\mathbf{b} \in \mathcal{W}$ there is $\mathsf{r} > 0$ such that $\mathcal{U}_\kappa(\mathbf{b}, \mathsf{r}) \subset \mathcal{W}$. We refer to this topology as the *visual topology on quasi-geodesics* and from now on we consider $\partial_\kappa X$ to be a topological space.

Properties of the topology. In this section, we establish some topological properties of $\partial_\kappa X$. We will show that $\partial_\kappa X$ is metrizable and, for $\kappa' \prec \kappa$, we show that the inclusion $\partial_{\kappa'} X \subset \partial_\kappa X$ is a topological embedding.

We make use the following criterion for a topological space to be metrizable.

Theorem 4.8 (Theorem 3, [Fri37]). *Assume, for every point $\boldsymbol{b}$ of a topological space, there exists a monotonic decreasing sequence $\mathcal{U}_1(\boldsymbol{b}), \mathcal{U}_2(\boldsymbol{b}), \cdots, \mathcal{U}_i(\boldsymbol{b}), \cdots$ of neighborhoods whose intersection is $\boldsymbol{b}$ and such that the following holds: For every point $\boldsymbol{b}$ of the neighborhood space and every integer i, there exists an integer $j = j(\boldsymbol{b}, i) > i$ such that if $\boldsymbol{a}$ is any point for which $\mathcal{U}_j(\boldsymbol{a})$ and $\mathcal{U}_j(\boldsymbol{b})$ have a point in common then $\mathcal{U}_j(\boldsymbol{a}) \subset \mathcal{U}_i(\boldsymbol{b})$. Then the space is homeomorphic to a metric space.*

We check this condition for $\partial_\kappa X$.

Theorem 4.9. *The space $\partial_\kappa X$ is metrizable.*

Proof. Recall the maps $\phi, \psi_1, \psi_2 \colon \partial_\kappa X \times \mathbb{R} \to \mathbb{R}$ from Remark 4.5. For $i \in \mathbb{N}$ and $\mathbf{a} \in \partial_\kappa X$, define

$$\mathcal{U}_i(\mathbf{a}) = \mathcal{U}_\kappa(\mathbf{a}, \mathsf{r}_i(\mathbf{a})), \qquad \text{where} \qquad \mathsf{r}_i(\mathbf{a}) = \max\big(i, \psi_1(\mathbf{a}, i), \psi_2(\mathbf{a}, i)\big).$$

Also, given $\mathbf{b}$ and i, we define

$$j = j(\mathbf{b}, i) = \Big\lceil \phi\big(\mathbf{b}, \phi(\mathbf{b}, \mathsf{r}_i(\mathbf{b}))\big) \Big\rceil.$$

Assume $\mathcal{U}_j(\mathbf{a})$ and $\mathcal{U}_j(\mathbf{b})$ have a point in common, that is,

$$\mathcal{U}_\kappa\big(\mathbf{a}, \mathsf{r}_j(\mathbf{a})\big) \cap \mathcal{U}_\kappa\big(\mathbf{b}, \mathsf{r}_j(\mathbf{b})\big) \neq \emptyset.$$

Since,

$$\mathsf{r}_j(\mathbf{a}) \geq \psi_2(\mathbf{a}, j) \geq \psi_2\Big(\mathbf{a}, \phi\big(\mathbf{b}, \phi(\mathbf{b}, \mathsf{r}_i(\mathbf{b}))\big)\Big) \qquad \text{and} \qquad \mathsf{r}_j(\mathbf{b}) \geq j \geq \phi(\mathbf{b}, \phi(\mathbf{b}, \mathsf{r}_i(\mathbf{b})))$$

Equation (21) implies

$$\mathbf{a} \in \mathcal{U}_\kappa\Big(\mathbf{b}, \phi\big(\mathbf{b}, \mathsf{r}_i(\mathbf{b})\big)\Big).$$

Now, Equation (20) implies

$$\mathcal{U}_\kappa\Big(\mathbf{a}, \psi_1\big(\mathbf{a}, \phi(\mathbf{b}, \mathsf{r}_i(\mathbf{b}))\big)\Big) \subset \mathcal{U}_\kappa\big(\mathbf{b}, \mathsf{r}_i(\mathbf{b})\big).$$

But

$$\mathsf{r}_j(\mathbf{a}) \geq \psi_1(\mathbf{a}, \phi(\mathbf{b}, \phi(\mathbf{b}, \mathsf{r}_i(\mathbf{b})))) \geq \psi_1(\mathbf{a}, \phi(\mathbf{b}, \mathsf{r}_i(\mathbf{b}))).$$

Therefore,

$$\mathcal{U}_\kappa\big(\mathbf{a}, \mathsf{r}_j(\mathbf{a})\big) \subset \mathcal{U}_\kappa\big(\mathbf{b}, \mathsf{r}_i(\mathbf{b})\big).$$

Which is to say $\mathcal{U}_j(\mathbf{a}) \subset \mathcal{U}_i(\mathbf{b})$. The theorem follows from Theorem 4.8. □

Lastly, we prove that different boundaries associated with different sublinear functions are nested.

Proposition 4.10. *Let κ, κ' be sublinear functions such that, for some $\mathsf{M} > 0$,*

$$\kappa'(t) \leq \mathsf{M} \cdot \kappa(t), \qquad \forall t > 0. \tag{22}$$

Then, $\partial_{\kappa'} X \subseteq \partial_\kappa X$ as a subspace with the subspace topology.

Proof. It is immediate from the definition that $\partial_{\kappa'} X$ is a subset of $\partial_\kappa X$. First we have to show that the intersection of an open set in $\partial_\kappa X$ with $\partial_{\kappa'} X$ is open in $\partial_{\kappa'} X$.

Let $\mathcal{V}$ be an open set in $\partial_\kappa X$ and consider $\mathbf{b} \in \mathcal{V} \cap \partial_{\kappa'} X$. Let m_b be the κ–Morse gauge for b and let m'_b be the κ'–Morse gauge for b. Let radius $\mathsf{r} > 0$ be such that $\mathcal{U}_\kappa(\mathbf{b}, \mathsf{r}) \subset \mathcal{V}$. We need to find radius R so that $\mathcal{U}_{\kappa'}(\mathsf{b}, \mathsf{R}) \subset \mathcal{U}_\kappa(\mathbf{b}, \mathsf{r})$. For any q, Q, where $\mathsf{m}_b(\mathsf{q}, \mathsf{Q})$ is small compared to r, there is $\mathsf{R} = \mathsf{R}(b, \mathsf{r}, \mathsf{m}'_b, \kappa'(\mathsf{q}, \mathsf{Q}))$ as in Theorem 3.14. We denote the maximum such radius again with R.

Let $\mathbf{a} \in \mathcal{U}_{\kappa'}(\mathsf{b}, \mathsf{R})$ and let $\alpha \in \mathbf{a}$ be a (q, Q)–quasi-geodesic such that $\mathsf{m}_b(\mathsf{q}, \mathsf{Q})$ is small compared to r. Taking R even larger if needed, we can assume that $\mathsf{m}'_b(\mathsf{q}, \mathsf{Q})$ is small compare to R. Then, $\mathbf{a} \in \mathcal{U}_{\kappa'}(\mathsf{b}, \mathsf{R})$ implies that

$$d_X(\alpha_{\mathsf{R}}, b) \leq \mathsf{m}'_b(\mathsf{q}, \mathsf{Q}) \cdot \kappa'(\mathsf{R}).$$

By Theorem 3.14,

$$\alpha|_{\mathsf{r}} \subset \mathcal{N}_\kappa\big(b, \mathsf{m}_b(\mathsf{q}, \mathsf{Q})\big).$$

Since this holds for every such $\alpha \in \mathbf{a}$, we have $\mathbf{a} \in \mathcal{U}_\kappa(\mathbf{b}, \mathsf{r})$. Therefore,

$$\mathcal{U}_{\kappa'}(\mathbf{b}, \mathsf{R}) \subset \mathcal{U}_\kappa(\mathbf{b}, \mathsf{r}) \subset \mathcal{V}.$$

That is, every such point $\mathbf{b}$ is in the interior of $\mathcal{V} \cap \partial_{\kappa'} X$ and $\mathcal{V} \cap \partial_{\kappa'} X$ is open in $\partial_{\kappa'} X$.

Next we show that every open set in $\partial_{\kappa'} X$ is the intersection of an open set of $\partial_\kappa X$ with $\partial_{\kappa'} X$. It suffices to show that given an open set $\mathcal{V} \in \partial_{\kappa'} X$, and a point $\mathbf{c} \in \mathcal{V}$, there exists a neighbourhood $\mathcal{U}_\kappa(\mathbf{c}, \mathsf{r}) \subset \mathcal{V}$. By definition of the topology there exists an open set $\mathcal{U}_{\kappa'}(\mathbf{c}, \mathsf{r}_c)$ such that

$$\mathbf{c} \in \mathcal{U}'_\kappa(\mathbf{c}, \mathsf{r}_c) \subset \mathcal{V}.$$

Now apply Theorem 3.14, there exists R such that for any (q, Q)–quasi-geodesic η where (q, Q) is small compared to r,

$$d_X\big(\eta(t_{\mathsf{R}}), c\big) \leq \kappa(\mathsf{R}) \quad \Longrightarrow \quad \eta[0, t_{\mathsf{r}_c}] \subset \mathcal{N}_{\kappa'}\big(c, \mathsf{m}_c(\mathsf{q}, \mathsf{Q})\big).$$

That is to say, $\mathcal{U}_\kappa(c, \mathsf{R}) \subset \mathcal{U}_{\kappa'}(c, \mathsf{r}_c)$, which also that implies

$$\mathcal{U}_\kappa(c, \mathsf{R}) \subset \big(\mathcal{U}_\kappa(c, \mathsf{R}) \cap \partial_{\kappa'} X\big).$$

Consider the union:

$$\bigcup_{\mathbf{c} \in \mathcal{U}_{\kappa'}(\mathbf{b}, \mathsf{r})} \mathcal{U}_\kappa(c, \mathsf{R}) \subset \bigcup_{\mathbf{c} \in \mathcal{U}_{\kappa'}(\mathbf{b}, \mathsf{r})} \big(\mathcal{U}_\kappa(c, \mathsf{R}) \cap \partial_{\kappa'} X\big) = \left(\bigcup_{\mathbf{c} \in \mathcal{U}_{\kappa'}(\mathbf{b}, \mathsf{r})} \mathcal{U}_\kappa(c, \mathsf{R})\right) \cap \partial_{\kappa'} X.$$

□

5. Boundary of a CAT(0) group

Let G be a finitely generated group that acts geometrically on X, that is, properly discontinuously, co-compactly and by isometries. Let $\mathfrak{o}$ denote the base-point of X. Equip G with the word length associated to some generating set. Also, given an element $g \in G$, denote the image of $\mathfrak{o}$ under the action of g by $g\mathfrak{o}$. Then the map

$$\Psi \colon G \to X, \qquad \Psi(g) = g\mathfrak{o}$$

defines a quasi-isometry between G and X which means there is an association between quasi-geodesics in G and in X. Hence, we can define $\partial_\kappa G$ to be $\partial_\kappa X$. Namely, consider a path $P = \{g_i\}_{i=0}^\infty$ in G such that $g_0 = id$ and g_i and g_{i+1} differ by a generator. Define β_P

to be the ray in X that is a concatenation of geodesic segments $[g_i, g_{i+1}]$. If β_P is a κ-Morse quasi-geodesic in X, then we say $g_i \to [\beta_P]$. In other words, $\partial_\kappa G$ is the set of κ–equivalence classes of quasi-geodesic rays in G so that the associated quasi-geodesic in X is κ-contracting.

However, G may act geometrically on different CAT(0) spaces. To show $\partial_\kappa G$ is well defined, we need to show different such spaces give the same boundary for G. We show, more generally, that $\partial_\kappa X$ is invariant under quasi-isometry.

Theorem 5.1. *Consider proper* CAT(0) *metric spaces X and Y and let $\Phi\colon X \to Y$ be a (k, K)–quasi-isometry. Then Φ induces a homeomorphism $\Phi^\star\colon \partial_\kappa X \to \partial_\kappa Y$ for every sublinear function κ where, for $\boldsymbol{b} \in \partial_\kappa X$ and $\beta \in \boldsymbol{b}$,*

$$\Phi^\star(\boldsymbol{b}) = [\Phi \circ \beta].$$

Proof. For a quasi-geodesic ray $\zeta\colon [0,\infty) \to X$ in X let $\Phi\zeta$ be a quasi-geodesic ray in Y constructed from the composition of ζ and Φ as in Definition 2.2. It is immediate from the definition that two quasi-geodesics ζ and ξ in X κ–fellow travel each other if and only if $\Phi\zeta$ and $\Phi\xi$ κ–fellow travel each other in Y. Also (again immediate from the definition) the property of being κ–Morse is preserved under a quasi-isometry. Hence, $[\zeta] \in \partial_\kappa X$ if and only if $[\Phi\zeta] \in \partial_\kappa Y$. Therefore, $\Phi^\star$ defined as above gives a bijection between $\partial_\kappa X$ and $\partial_\kappa Y$. We need to show that $(\Phi^\star)^{-1}$ is continuous. Then, the same argument applied in the other direction will show that $\Phi^\star$ is also continuous which means $\Phi^\star$ is a homeomorphism.

Let $\mathcal{V}$ be an open set in $\partial_\kappa X$, $\mathbf{b}_X \in \mathcal{V}$ and $\mathcal{U}_\kappa(\mathbf{b}_X, \mathsf{r})$ be a neighborhood $\mathbf{b}$ that is contained in $\mathcal{V}$. Let $\mathbf{b}_Y = \Phi^\star(\mathbf{b}_X)$. We need to show that there is a constant r' such that, for every point $\mathbf{a}_Y \in \mathcal{U}_\kappa(\mathbf{b}_Y, \mathsf{r}')$, we have

$$\mathbf{a}_X = (\Phi^\star)^{-1}(\mathbf{a}_Y) \in \mathcal{U}_\kappa(\mathbf{b}_X, \mathsf{r}).$$

Let q' and Q' be constants (depending on $\mathsf{q}, \mathsf{Q}, \mathsf{k}$ and K) such that if ζ is a (q, Q)–quasi-geodesic where $\mathsf{m}_{b_X}(\mathsf{q}, \mathsf{Q})$ is small compared to r then $\Phi\zeta$ is a $(\mathsf{q}', \mathsf{Q}')$–quasi-geodesic. Let b_X be the unique geodesic ray in $\mathbf{b}_X$, let b_Y be the unique geodesic ray in $\mathbf{b}_Y$ and let m_{b_X} and m_{b_Y} be their Morse gauges respectively. By Corollary 3.16, there is a constant n_1 depending on k, K and m_{b_Y} such that

$$\Phi b_X \subset \mathcal{N}_\kappa(b_Y, \mathsf{n}_1).$$

For

$$\mathsf{n} = \mathsf{k}\big(\mathsf{m}_{b_Y}(\mathsf{q}', \mathsf{Q}') + \mathsf{n}_1\big)(\mathsf{k} + \mathsf{K}) + \mathsf{K}$$

let $\mathsf{R} = \mathsf{R}(b_X, \mathsf{r}, \mathsf{n}, \kappa)$ as in Theorem 3.14 and choose r' such that $\mathsf{r}' \geq \mathsf{k}\,\mathsf{R} + \mathsf{K}$ and $\mathsf{m}_{b_Y}(\mathsf{q}', \mathsf{Q}')$ is small compare to r'.

Let $\alpha \in \mathbf{a}_X$ be a (q, Q)–quasi-geodesic where $\mathsf{m}_{b_X}(\mathsf{q}, \mathsf{Q})$ is small compared to r such that $\Phi\alpha \in \mathcal{U}_\kappa(\mathbf{b}_Y, \mathsf{r}')$. By our choice of r', $\mathsf{m}_{b_Y}(\mathsf{q}', \mathsf{Q}')$ is small compared to r'. Hence,

$$\Phi\alpha|_{\mathsf{r}'} \subset \mathcal{N}_\kappa\big(b_Y, \mathsf{m}_{b_Y}(\mathsf{q}', \mathsf{Q}')\big)$$

Pick $x \in \alpha_X|_{\mathsf{R}}$. Then $\Phi x \in \Phi\alpha|_{\mathsf{r}'}$ and we have

$$\begin{aligned} d_X(x, b_X) &\leq \mathsf{k}(d_Y(\Phi(x), \Phi b_X) + \mathsf{K} \\ &\leq \mathsf{k}\Big(d_Y(\Phi(x), b_Y) + \mathsf{n}_1 \cdot \kappa(\Phi x)\Big) + \mathsf{K} \\ &\leq \mathsf{k}\big(\mathsf{m}_{b_Y}(\mathsf{q}', \mathsf{Q}') + \mathsf{n}_1\big) \cdot \kappa(\Phi x) + \mathsf{K} \end{aligned}$$

This and

$$\kappa(\Phi x) \leq \mathsf{k}\kappa(x) + \mathsf{K} \leq (\mathsf{k} + \mathsf{K})\kappa(x)$$

imply that

$$\alpha|_{\mathsf{R}} \subset \mathcal{N}_\kappa(b_X, \mathsf{n}).$$

Now, Theorem 3.14 implies that

$$\alpha|_{\mathsf{r}} \subset \mathcal{N}_\kappa(b_X, \mathsf{m}_{b_X}).$$

Therefore, $\mathbf{a}_X \in \mathcal{U}_\kappa(\mathbf{b}_X, \mathsf{r})$ and

$$(\Phi^\star)^{-1}\mathcal{U}_\kappa(\mathbf{b}_Y, \mathsf{r}') \subset \mathcal{U}_\kappa(\mathbf{b}_X, \mathsf{r}).$$

But $\mathcal{U}_\kappa(\mathbf{b}_Y, \mathsf{r}')$ contains an open neighborhood of $\mathbf{b}_Y$, therefore, $\mathbf{b}_Y$ is in the interior of $\Phi\mathcal{V}$. This finishes the proof. □

6. Examples

A tree of flats. In this section we examine κ–boundaries of a few simple examples to illustrate several typical properties of κ–boundaries of CAT(0) spaces. Consider the right-angled Artin group

$$A = \mathbb{Z}^2 * \mathbb{Z} = \left\langle g_1, g_2, g_3 \;\middle|\; [g_1, g_2] \right\rangle.$$

Let X_A be the universal cover of the Salvetti complex of $\mathbb{Z}^2 * \mathbb{Z}$, or simply the *universal Salvetti complex*, as in Definition A.4. We observe that X_A is a tree of flats. The flats are associated to orbits of conjugate copies of the subgroup

$$\langle g_1, g_2 \mid [g_1, g_2] \rangle \simeq \mathbb{Z}^2.$$

The oriented edges that are outside of these flats are labelled g_3.

We equip X_A with a metric so that each flat is isometric to the Euclidean plane $\mathbb{E}^2$, the axes of g_1 and g_2 intersect at a 90-degree angle and edges labeled g_3 are attached at the lattice points. The space is simply connected and the metric on X_A is CAT(0). The closest-point projection of any flat to any other flat is a single point. Also, since flats are convex subspaces, given a geodesic ray in X_A, there is a well-defined *itinerary of flats* that the geodesic passes through. Choose a base-point $\mathfrak{o}$ where an edge labeled g_3 is attached to a flat and let Y_0 be the flat that contains $\mathfrak{o}$. As before, we always assume a geodesic ray starts at $\mathfrak{o}$.

We give a characterization of the κ-contracting rays in X_A. First we need the following lemma:

Lemma 6.1. *Let b be a geodesic ray in X_A. Given any ball B disjoint from b. The projection $\pi_b(B)$ of B to b lies inside a unique flat.*

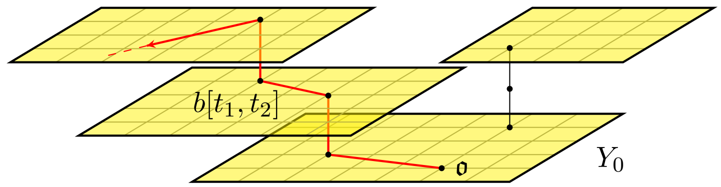
Proof. Assume for contradiction that $\pi_b(B)$ contains a point w in the interior of an edge $e = (v_1, v_2)$ labelled by g_3. Then b traverses e. The point w is a cut-point of X_A. Let v_1 be the vertex of the edge e that is in the same component as the center of the ball B. Since $\pi_b(B)$ contains a point w, there exists a point $x \in B$ where $w = x_b$. However, we have

$$d(x, w) > d(x, v_1) \geq d(x, b). \tag{23}$$

This contradicts the assumption that w is a nearest-point projection from x to b. This holds for every edge labeled g_3. Hence $\pi_b(B)$ in contained a single flat. □

Lemma 6.2. *A unit speed geodesic ray b in X_A is κ–contracting, if and only if, there exists a constant c such that if $b[t_1, t_2]$ is contained in a flat, then*

$$|t_1 - t_2| \leq \mathsf{c} \cdot \kappa(t_1).$$



Proof. First we consider the "if" direction. Let $\{Y_i\}$ be the sequence of flats visited by b. By Lemma 6.1, if a ball B is disjoint from b then $\pi_b(B)$ is contained in some Y_i. Let $[t_1, t_2]$ be the interval of time where the image of b is in Y_i.

Let x be the center of the ball B and y be any other point in B. By Lemma 6.1, $x_b = b(t)$ for $t \in [t_1, t_2]$. Therefore,

$$\|x\| \geq \|x_b\| \geq t_1.$$

Thus we have

$$d_{X_A}(x_b, y_b) \leq |t_2 - t_1| \leq \mathsf{c} \cdot \kappa(t_1),$$

which means b is κ–contracting and we can set $\mathsf{c}_b = \mathsf{c}$.

For the "only if" direction, assume b is κ-contracting with c_b as a contracting constant. For an interval $b[t_1, t_2]$ that stays in a flat, consider the ball B whose center x is at a distance $|t_2 - t_1|$ from the point $b(t_1)$ in a perpendicular direction from the segment $b[t_1, t_2]$ and with radius $(t_2 - t_1)$. Then $\pi_b(B) = b[t_1, t_2]$. The definition of κ-contracting geodesic ray dictates that

$$|t_2 - t_1| \leq \mathsf{c}_b \cdot \kappa(x) \leq \mathsf{c}_b \cdot \kappa(t_1 + (t_2 - t_1)) = \mathsf{c}_b \cdot \kappa(t_2).$$

By Lemma 3.2,

$$\kappa(t_2) \leq \mathsf{c}' \cdot \kappa(t_1),$$

for some c' depending on κ and c_b. Thus we have

$$|t_2 - t_1| \leq \mathsf{c}_b \cdot \kappa(t_2) \leq \mathsf{c}_b \mathsf{c}' \cdot \kappa(t_1).$$

□

Proposition 6.3. *If $\kappa(t), \kappa'(t)$ are two sublinear functions such that,*

$$\lim_{t \to \infty} \frac{\kappa'(t)}{\kappa(t)} = 0$$

Then $\partial_{\kappa'} X_A \subsetneq \partial_\kappa X_A$, that is to say, $\partial_\kappa X_A$ strictly contains $\partial_{\kappa'} X_A$.

Proof. The fact that $\partial_{\kappa'} X \subseteq \partial_\kappa X$ follows Proposition 4.10. We give a specific construction of a geodesic ray b that is in $\partial_\kappa X$ but not $\partial_{\kappa'} X$. The ray b is the concatenation of vertical segments v_i consisting of edges labeled g_3 and horizontal segments h_i that are contained in a single flat.

Let i_0 be an integer so that $2^{i_0} \geq \kappa(2^{i_0})$ and let b_{i_0} be a vertical segment of length 2^{i_0}. For $i > i_0$, assume a segment b_{i-1} is given. Continue b_{i-1} along a horizontal segment h_i of length $\lfloor \kappa(2^i) \rfloor$, then along a vertical segment v_i of length $\lceil 2^i - \kappa(2^i) \rceil$ and denote the resulting segment by b_i. We see inductively that $|b_i| = 2^{i-1}$ because,

$$|b_i| = |b_{i-1}| + |h_i| + |v_i| = 2^i + \lfloor \kappa(2^i) \rfloor + \lceil 2^n - \kappa(2^i) \rceil = 2^{i+1}.$$

Also,

$$\lfloor \kappa(2^i) \rfloor = |h_i| \leq \kappa(|b_{i-1}|).$$

That is, if we let the ray b be the union of the segments b_i, then b satisfies Lemma 6.2 for the sublinear function κ but not for κ'. Hence, $[b] \in (\partial_\kappa X - \partial_{\kappa'} X)$. $\square$

As an easy consequence, we have

Corollary 6.4. *There exists two* CAT(0) *spaces that are not distinguishable by their Charney-Sultan contracting boundaries* [CS15] *but are distinguishable by their sublinear Morse boundaries.*

Proof. We can adjust the metric on X_A by changing the lengths of the edges. We say a flat Y is at *height* n, if the geodesic segment connecting o to any point in Y traverses through n edges labeled by g_3 (in either direction). Let $X_{\sqrt{\cdot}}$ be the space obtained from X_A where the side lengths of unit squares in flats at height n is scaled to $\sqrt{n}$. Similarly, let $X_{\log}$ be the space obtained from X_A the side lengths of unit squares in flats at height n is scaled to $\log(n)$.

Since $\sqrt{n}$ grows faster than $\log n$, the log–boundary of $X_{\sqrt{\cdot}}$ is a set that contains geodesic rays that eventually cannot travel even one edge in any flat. That is to say, after a finite time, this geodesic ray will travel along the g_3 direction only. The number of such geodesic rays is countable.

On the other hand, we see from Lemma 6.2 that the log–boundary of $X_{\log}$ consists of geodesic rays whose projections to any flat are bounded by log of the time they enter the flat. A geodesic in this boundary therefore can travel in infinitely many flats. Therefore, the log–boundary of $X_{\log}$ is an uncountable set.

Meanwhile, the Morse boundaries of both $X_{\sqrt{\cdot}}$ and $X_{\log}$ consist of geodesic rays that eventually travel along the g_3 direction only. By the previous argument the Morse boundary of $X_{\sqrt{n}}$ and the Morse boundary of $X_{\log}$ are homeomorphic via the equivariant map of the group. $\square$

Random Walks. As mentioned in the introduction, one motivation for constructing the κ–Morse boundary is to study random walks on a group. (For details of construction of random walks on groups, see the Appendix.) In the setting of the group A acting on X_A, Theorem A.17 tells us that, for almost every sample path in X_A, the maximum amount of time spent on a given flat after n steps is bounded by $\mathsf{c} \cdot \log n$.

Furthermore, since X_A is CAT(0), by [KM99], almost every sample path $w = \{w_n\}$ tracks a geodesic ray in X_A which we denote b_w. That is, there is $\mathsf{u} > 0$ such that the distance between $w_n(\mathsf{o})$ and $b(\mathsf{u} \cdot n)$ grows sublinearly with n. Therefore, for every flat Y, the projection of $w_n(\mathsf{o})$ to Y is eventually the same as the projection of $b(\mathsf{u} \cdot n)$ to Y, which is the point in Y where b_w exits Y. In fact, if n is larger than a fixed multiple of $d_{X_A}(\mathsf{o}, Y)$, then $w_n(\mathsf{o})$ is closer to b_w than to Y, and hence the path connecting $w_n(\mathsf{o})$ to b_w is disjoint from Y and projects to a point in Y.

By the above theorem, $d_Y\big(\pi_Y(1), \pi_Y(w_n)\big)$ grows only logarithmically. Hence, the time b_w spends in Y is less than a multiple of the distance between Y and o. That is, b_w satisfies the condition of Lemma 6.2 and hence $[b_w] \in \partial_{\log} X_A$.

By [NS13], the visual boundary of the Salvetti complex of a right-angled Artin group together with the hitting measure constitutes a metric model for the Poisson boundaries of the group. Since almost every sample path converges to a point in $\partial_{\log} X_A$, we have:

Corollary 6.5. *Let μ be a symmetric, finitely supported probability measure on $A = \mathbb{Z} * \mathbb{Z}^2$. Then $\partial_{\log} X_A$ is a metric model for the Poisson boundary $(\mathbb{Z}^2 * \mathbb{Z}, \mu)$.*

In the Appendix, this is generalized to the class of all right-angled Artin groups.

Other topological properties of $\partial_\kappa X_A$. We have shown that $\partial_\kappa X$ is metrizable which implies that it is, Hausdorff, normal and paracompact. However, $\partial_\kappa X$ is often not compact. For the example given in this section, we have:

Proposition 6.6. *The topological space $\partial_\kappa X_A$ is non-compact, totally disconnected and with no isolated points.*

Proof. Let e be any vertical edge in X, i.e. an edge labeled g_3. Define $\mathcal{W}(e)$ to be the set all points $\mathbf{b} \in \partial_\kappa X$ where the geodesic ray $b \in \mathbf{b}$ traverses e. For any $\mathbf{b} \in \mathcal{W}(e)$ and r large enough, $\mathcal{U}_\kappa(\mathbf{b}, \mathsf{r}) \subset \mathcal{W}(e)$. This is because, if $\mathbf{a} \in \mathcal{U}_\kappa(\mathbf{b}, \mathsf{r})$, then the geodesic ray $a \in \mathbf{a}$ stays in a κ–neighborhood of b for distance r, namely $a|_{\mathsf{r}} \subset \mathcal{N}_\kappa\big(b, \mathsf{m}_b(1,0)\big)$, and hence has to also traverse e. Therefore, $\mathcal{W}(e)$ is an open set in $\partial_\kappa X$.

But $\partial_\kappa X - \mathcal{W}(e)$ is also open because it can be written as a union of sets of the form $\mathcal{W}(e')$. For any $b' \neq b$ in $\partial_\kappa X$, let e be an edge traversed by b and not by b'. Then $b \in \mathcal{W}(e)$ and $b' \in \partial_\kappa X - \mathcal{W}(e)$ which are both open. Thus, $\partial_\kappa X$ is totally disconnected.

All sets $\mathcal{W}(e)$ are homeomorphic to each other and contain more than one point in $\partial_\kappa X$. Let $\{e_i\}$ be the set of vertical segments along b and let $b_i \in \mathcal{W}(e_i)$ be a point not equal to b. Since $\cap_i \mathcal{W}(e_i) = b$, we have $b_i \to b$. That is, $\partial_\kappa X$ has no isolated points.

To see that $\partial_\kappa X$ is not compact, consider a sequence of geodesics $\{b_j\}$ where each b_j leaves the flat Y_0 at coordinate $(j, 0)$ and then follows the g_3–direction indefinitely. All geodesic rays b_j are κ–contracting. But, the point-wise limit of this sequence is the geodesic that lies in Y_0 which is not contracting for any κ. In fact, b_j has no limit point in $\partial_\kappa X$ because if $b_j \to b$ then infinity many b_j have to be contained $\mathcal{W}(e)$ for some e along b. But this does not hold for any e. Therefore, $\partial_\kappa X$ is not compact. $\square$

However, the boundary does not always have to be totally disconnected. In [Beh17], Behrstock constructed a family of right-angled Coxeter groups where the Morse boundary is not totally disconnected. And, since the Morse boundary is a topological subspace of $\partial_\kappa X$ (see Lemma 4.10), the same holds for $\partial_\kappa X$.

Appendix A. Poisson boundaries of Right-angled Artin groups

Yulan Qing[1] and Giulio Tiozzo[2]

As an application of sublinearly Morse boundaries, we show that when $\kappa = \sqrt{t \log t}$, the κ–Morse boundary of the universal Salvetti complex is a model for the Poisson boundary of a right-angled Artin group. This establishes Theorem F in the introduction.

Let Γ be a finite graph, and let $A(\Gamma)$ be the right-angled Artin group associated to Γ, which is defined by the presentation

$$A(\Gamma) := \big\langle v \text{ is a vertex in } \Gamma \mid [v,w]=1, (v,w) \text{ is an edge in } \Gamma \big\rangle.$$

That is to say, there is an infinite order generator for each vertex, and a pair of generators commute if and only if there is an edge between the two corresponding vertices in Γ.

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Each right-angled Artin group is associated with a cube complex known as its *Salvetti complex*, and its universal cover $X(\Gamma)$ is a proper CAT(0) space on which $A(\Gamma)$ acts cocompactly. We call $X(\Gamma)$ the *universal Salvetti complex*. The main theorem of this appendix is the following.

Theorem A.1. *Let μ be a finitely supported, generating measure on an irreducible right-angled Artin group $A(\Gamma)$. Then the $\sqrt{t \log t}$–Morse boundary of $X(\Gamma)$ is a QI-invariant topological model for the Poisson boundary of $(A(\Gamma), \mu)$.*

Let us now recall some background material and fundamental definitions.

Random walks and the Poisson boundary. Let G be a countable group of isometries of a metric space X, and let μ be a probability measure on G. A measure μ is *generating* if the semigroup generated by the support of μ equals G. We define the *random walk* associated to (G, μ) as the stochastic process

$$w_n := g_1 \dots g_n$$

where $(g_n)_{n \geq 1}$ is a sequence of G-valued i.i.d. random variables, each with distribution μ. Let us fix a base point $x \in X$. The sequence $(w_n x)_{n \geq 1}$ is called a *sample path* for the random walk.

In most interesting situations, almost every sample path converges to a point in a suitable boundary ∂X; in that case, we define the *hitting measure* ν on ∂X as

$$\nu(A) := \mathbb{P}\left(\lim_{n \to \infty} w_n x \in A\right).$$

A function $f : G \to \mathbb{R}$ is μ-harmonic if it satisfies a discrete version of the mean value property; namely, $f(g) = \sum_{h \in G} \mu(h) f(gh)$ for any $g \in G$. We denote the space of bounded, μ-harmonic functions as $H^\infty(G, \mu)$. Now, the *Poisson transform* $\Phi : L^\infty(\partial X, \nu) \to H^\infty(G, \mu)$ is defined as

$$\Phi(f)(g) := \int_{\partial X} f(g(x))\, d\nu(x)$$

and the space $(\partial X, \nu)$ is the *Poisson boundary* if Φ is an isomorphism.

That is, the Poisson boundary is the natural space where all bounded harmonic functions can be represented. It is well-defined as a measurable G-space. For groups acting on CAT(0) metric spaces, an identification of the Poisson boundary is given as follows.

Theorem A.2 ([KM99], [NS13])**.** *Let G be a countable group of isometries of a* CAT(0) *proper metric space such that its action has bounded exponential growth, and let μ be a nonelementary measure on G with finite first moment. Then almost every sample path converges to the visual boundary of X, and the visual boundary with the hitting measure is a model for the Poisson boundary of (G, μ).*

In this appendix, we prove:

Theorem A.3. *Let $G = A(\Gamma)$ be an irreducible right-angled Artin group, let μ be a finitely supported, generating measure on G, and let ν be the hitting measure for the corresponding random walk. Then the κ–Morse boundary with $\kappa(t) = \sqrt{t \log t}$ is a G-invariant subset of the visual boundary of full ν-measure.*

Theorem A.3 and Theorem A.2 immediately imply Theorem A.1, which is the same as Theorem F in the introduction.

Background on cube complexes. For all basic definitions related to right-angled Artin groups and the associated CAT(0) cube complex $X(\Gamma)$, we follow [Cha07].

Definition A.4. Associated to a right-angled Artin group $A(\Gamma)$ is an infinite and locally finite cube complex called the *Salvetti complex*, constructed as follows: associated to each vertex of $A(\Gamma)$ is a simple closed loop of unit length. If two vertices form an edge in $A(\Gamma)$ then attach to the two associated loops a square torus generated by the two loops intersecting at a right angle. More generally, given a complete subgraph on k vertices, consider a unit k-torus generated by k loops intersecting at right angles. The *universal Salvetti complex* associated to $A(\Gamma)$, denoted as $X(\Gamma)$, is then the universal cover of this tori-complex. Notice that the 0 and 1-skeleta of $X(\Gamma)$ are isomorphic, respectively, to the 0 and 1-skeleton of the Cayley graph of $A(\Gamma)$ with this specific presentation.

The universal Salvetti complex $X(\Gamma)$ is a CAT(0) *cube complex* [Hag14], which we discuss now. A *cube complex* is a polyhedral complex in which the cells are Euclidean cubes of side length one. The attaching maps are isometries identifying the faces of a given cube with cubes of lower dimension and the intersection of two cubes is a common face of each. Cubes of dimension 0, 1 and 2 are also referred to as vertices, edges and squares. A cube complex is finite dimensional if there is an upper bound on the dimension of its cubes. Finally, a *CAT(0) cube complex* is a cube complex in which the link of each vertex is a flag simplicial complex.

Hyperplanes and contact graph. In a CAT(0) cube complex, consider the equivalence relation on the set of mid-cubes generated by the rule that two mid-cubes are related if they share a face. Then a *hyperplane* H is defined as the union of the mid-cubes in a single equivalence class. Every hyperplane H is a geodesic subspace of $X(\Gamma)$ which separates $X(\Gamma)$ into two components. We shall refer to the each of these two components as a *half-space*, and denote them as $\{H^+, H^-\}$. Two hyperplanes provide four possible half-space intersections; the hyperplanes *intersect* if and only if each of these four half-space intersections is non-empty. In contrast, we say two convex subcomplexes F_1, F_2 are *parallel* (and we denote it as $F_1 \sim F_2$) if, given any other hyperplane H',

$$F_1 \cap H' \neq \emptyset \Leftrightarrow F_2 \cap H' \neq \emptyset.$$

We say a hyperplane H *separates* two hyperplanes H_1, H_2 if, given any pair of points $x \in H_1, y \in H_2$, all geodesics connecting x and y have non-empty intersection with H. Lastly, we say a (combinatorial) geodesic *crosses* a hyperplane H if there exists two consecutive vertices on the geodesic such that one belongs to H^+ and the other belongs to H^-.

Given a finite graph Γ, a *join* $J \subset \Gamma$ is an induced subgraph whose vertices can be partitioned into two sets A, B such that all edges of the form $\{(a,b) \ : \ a \in A, b \in B\}$ are edges of J. Recall a right-angled Artin group $A(\Gamma)$ is *irreducible* if Γ itself is not a join. Let $\mathcal{J}$ denote the set of all maximal joins of Γ, where maximality is defined by containment.

Remark A.5. By definition, every join between a vertex and its link is contained in a maximal join.

Definition A.6. The *contact graph* $\mathcal{C}(X)$ of a CAT(0) cube complex X is a graph whose vertex set is the set of hyperplanes of X. Moreover, two vertices are adjacent if the corresponding hyperplanes H_1, H_2 satisfy one of the following:

- either H_1 intersects H_2 nontrivially; or
- H_1 and H_2 are not separated by a third hyperplane.

It is known that the contact graph is always hyperbolic (in fact, a quasi-tree [Hag14]).

Gates and projections. Given a point x and a convex subset Z of X, the nearest-point projection of x to Z exists and is unique by CAT(0) geometry. We denote it as x_Z.

Definition A.7. If $K \subset X$ is convex, then for all $x \in X^{(0)}$, there exists a unique closest 0-cube $\mathfrak{g}_K(x) \in K$, called the *gate* of x in K.

The gate is characterized by the property that any hyperplane H separates $\mathfrak{g}_K(x)$ from x if and only if H separates x from K.

The convexity of K allows us to extend the map $x \to \mathfrak{g}_K(x)$ to a projection $\mathfrak{g}_K \colon X \to K$, which is a cubical map defined as follows. Let c be a d-dimensional cube of X and let $H_1, H_2 ..., H_d$ be the collection of (pairwise-crossing) hyperplanes which cross c. Suppose that these are labeled so that $H_1, H_2 ..., H_s$ cross K, for some $0 \leq s \leq d$, and that $H_{s+1}, H_{s+2} ..., H_d$ do not cross K. Then the 0-cubes of c map by $\mathfrak{g}_K$ to the 0-cubes of a uniquely determined s-dimensional cube $\mathfrak{g}_K(c)$ of K in which the hyperplanes $H_1, H_2 ..., H_s$ intersect, and there is a cubical collapsing map $c \simeq [-1,1]^d \to [-1,1]^s \simeq \mathfrak{g}_K(c)$ extending the gate map on the 0-skeleton.

Definition A.8 (Projection to the contact graph)**.** Let K be a convex subcomplex of X. Given a hyperplane H, let $\mathcal{N}_\kappa(H)$ denote its *carrier*, i.e., the union of all closed cubes intersecting H. For each 0-cube $k \in K$, let $\{H_i\}_{i \in \mathcal{I}}$ be the collection of hyperplanes such that $k \in \mathcal{N}_\kappa(H_i)$, and define $\rho_K \colon K \to 2^{\mathcal{C}(K)}$ by setting $\rho_K(k) = \{H_i \cap K\}_{i \in \mathcal{I}}$. Let us now define the projection map $\pi_K : X \to 2^{\mathcal{C}(K)}$ by setting $\pi_K := \rho_K \circ \mathfrak{g}_K$, where $\mathfrak{g}_K(x)$ is the gate of x in K.

The following version of the bounded geodesic image theorem is inspired by [BHS17, Proposition 4.2]. Given a set $S \subseteq C(X)$, we use the notation $B_1(S)$ to denote the 1-neighborhood of S.

Lemma A.9 (Bounded geodesic image theorem)**.** *Let X be a* CAT(0) *cube complex, let J be a join and let $K \subseteq J$ a sub-Salvetti complex. Then if a path γ in $C(X)$ satisfies $\gamma \cap B_1(\pi_X(J)) = \emptyset$, we have* $\operatorname{diam} \pi_K(\gamma) \leq 1$.

Proof. Let x, y be two points on γ. If $\pi_K(x) \neq \pi_K(y)$, then there exists a hyperplane H in K which separates $\mathfrak{g}_K(x)$ and $\mathfrak{g}_K(y)$. Let H' be a hyperplane in X such that $H = H' \cap K$. Then by convexity H also separates x and y, hence its projection to the contact graph $C(X)$ intersects the projection of γ. Since H also intersects J, this contradicts the condition $\gamma \cap B_1(\pi_X(J)) = \emptyset$. $\square$

We also recall the notion of *factor system* from [BHS17].

Definition A.10 ([BHS17], Definition 8.1)**.** (Factor system). Let $X = X(\Gamma)$. A set of sub-complexes of X, denoted $\mathfrak{F}$, which satisfies the following is called a *factor system* in X:

(1) $X \in \mathfrak{F}$.
(2) Each $F \in \mathfrak{F}$ is a nonempty convex sub-complex of X.
(3) There exists $\delta \geq 1$ such that for all $x \in X^{(0)}$, at most δ elements of $\mathfrak{F}$ contain x.
(4) Every nontrivial convex sub-complex parallel to a combinatorial hyperplane of X is in $\mathfrak{F}$.
(5) There exists $\xi \geq 0$ such that for all $F, F' \in \mathfrak{F}$, either $\mathfrak{g}_F(F') \in \mathfrak{F}$ or $\operatorname{diam}(\mathfrak{g}_F(F')) \leq \xi$.

Associated with a factor $F \in \mathfrak{F}$ is a *factored contact graph* $\hat{C}F$ defined as the contact graph of F with each subgraph that is the contact graph of some smaller element of $\mathfrak{F}$ coned off.

Lemma A.11 ([BHS17], Lemmas 2.6 and 8.19)**.** *Let F, F' be two convex subcomplexes. Then:*

i) $\mathfrak{g}_F(F')$ and $\mathfrak{g}_{F'}(F)$ are parallel subcomplexes.
ii) If F is not parallel to a subcomplex of F', then

$$\operatorname{diam}_{\hat{C}F}(\pi_F(F')) \le \xi + 2.$$

Let us remark that if F and F' are isometric, then ii) is true under the (seemingly weaker) assumption that F is not parallel to F'.

Excursion geodesics. It follows from Theorem A.2 that almost every sample path (w_n) of a random walk on an irreducible right-angled Artin group converges to exactly one point ξ in the visual boundary, and there is a unique CAT(0) geodesic ray γ which connects the base-point with ξ. In this case, we say that the sample path *tracks* the geodesic ray γ. To build the connection between a sample path and the associated geodesic ray, we characterize geodesics by bounding their excursions.

We say a geodesic ray $\gamma = \{g_0, g_1, g_2, \dots, g_n, \dots\}$ with respect to the word metric in $A(\Gamma)$ is a *κ-excursion geodesic* if there exists a function κ and a constant C such that its projection to every maximal join J subcomplex is bounded above by $C\kappa(t)$. That is, we have:

$$\sup_J d_{s(J)}(\mathfrak{g}_{s(J)}(g_0), \mathfrak{g}_{s(J)}(g_n)) \le C\kappa(\|g_n\|) \tag{24}$$

where the supremum is taken over all maximal join subcomplexes $J \subseteq X(\Gamma)$. The main result of this section is the following.

Proposition A.12. *For any sublinear function κ, a κ–excursion geodesic is also a κ–contracting geodesic.*

In order to discuss the proof of this Proposition, let us recall that two hyperplanes H_1, H_2 are *strongly separated* if there does not exist a hyperplane H that intersects both H_1 and H_2. Given two hyperplanes H_1 and H_2, the *bridge* B between them is the union of all geodesic segments of minimal length between H_1 and H_2. We need the following properties about hyperplanes in the Salvetti complex:

Lemma A.13 (Properties of Strongly Separated Hyperplanes)**.** *Let u, v, w be vertices of Γ, and let H_u, H_v, H_w be the associated hyperplanes that are dual to edges incident to the base-point of $X(\Gamma)$. Let L_v denote the stabilizer of H_v, i.e. the group generated by the link $lk(v)$.*

1) Let $H_1 = g_1 H_v$ and $H_2 = g_2 H_w$. Then,
   (a) H_1 intersects $H_2 \Leftrightarrow v, w$ commute and $g_1^{-1} g_2 \in L_v L_w$.
   (b) There exists H_3 intersecting both H_1 and $H_2 \Leftrightarrow \exists u \in st(v) \cap st(w)$ such that $g_1^{-1} g_2 \in L_v L_u L_w$.
2) Let H_1, H_2 be strongly separated hyperplanes in a universal Salvetti complex. The bridge B between H_1 and H_2 consists of a single geodesic from H_1 to H_2.
3) There is a universal constant $C > 1$, depending only on the dimension of $X(\Gamma)$, such that for any $x \in H_1$ and $y \in H_2$,

$$d(x, y) \ge \frac{1}{C}\left(d(x, B) + d(y, B)\right) - d(H_1, H_2) - 4.$$

Proof. 1) and 2) are proven in ([BC12], Lemma 2.2 and Lemma 3.1). 3) is proven for word-metric geodesics in ([BC12], Lemma 2.3). However, for every CAT(0) geodesic, there exists a word-metric geodesic that lies in a 1-neighbourhood of it and is a $(2,0)$-quasi-geodesic in the CAT(0) metric. Combined with the fact that a bridge is both a CAT(0) geodesic and a word-metric geodesic, 3) holds with a larger multiplicative constant. □

Lemma A.14. *Let γ be a geodesic ray. Let $\{S_i\}$ denote a maximal sequence of strongly separated hyperplanes crossed by γ. Then there exists a sequence of joins, denoted $\{J_k\}$, travelled by γ such that for all i, if $S_i \in J_k$, then $S_{i+1} \in \bigcup_{l=1,2,3} J_{k+l}$.*

Proof. Consider the sequence $(H_1, H_2, H_3, \dots)$ of hyperplanes crossed by γ. Let H_k be the first hyperplane that is strongly separated from $H_1 = H_v$. By Lemma A.131, suppose $g_{k-1}H_w = H_{k-1}$, then g_{k-1} lies in $L_vL_uL_w$ where $u \in st(v) \cap st(w)$. Since H_w is the next hyperplane, then g_k lies in $L_vL_uL_ws_w$. By Remark A.5, each link is contained in a join, thus there exists a sequence of joins travelled consecutively by γ such that if $H_i \in J_i$ then $H_k \in J_{i+2}$. Now repeat the process between H_k and the first hyperplane that is strongly separated from H_k, say $H_{k'}$. It is possible that in this case the three joins connecting H_k and and $H_{k'}$ do not overlap with the joins that connect H_1 and H_k. In that case, consider H_k to be the wall that is in both joins. Therefore from H_1 to $H_{k'}$ the ray γ crosses 6 joins, satisfying the claim that $S_{i+1} \in \bigcup_{l=1,2,3} J_{k+l}$. □

Corollary A.15. *Consider the sequence of joins produced in Lemma A.14 and denote it $\{J_i\}$. Then the projection of J_i to J_{i+5} is a point.*

Proof. By Lemma A.14 of J_i to J_{i+5} passes through at least 2 strongly separated hyperplanes. By Lemma A.11(i), the projections of a pair of strongly separated hyperplanes to one another are parallel. But strong separability implies that both projections consist of a single point. Therefore, the projection of J_i to J_{i+5} is a point. □

Lemma A.16 (An excursion geodesic travels close to bridges)**.** *Fix a sublinear function κ, and let γ be a κ-excursion geodesic ray with itinerary $\{J_i\}$ as produced in Lemma A.14. Let $\{S_i\}$ be the sequence of strongly separated hyperplanes in Corollary A.14, and let $\mathcal{B}_{i,j}$ be the bridge between S_i and S_j. Let $b_i(j)$ denote the intersection point of $\mathcal{B}_{i,j}$ with S_i. Also let x_i be any point in the intersection $\gamma \cap S_i$. Then, if $|i-j| = 1$ we have*

$$d(x_i, b_i(j)) \leq C\kappa(\|x_i\|).$$

Proof. By Lemma A.13(2), the bridge $\mathcal{B}_{i,i+1}$ is a geodesic segment. By definition the length of a bridge is shorter than the distance between any other pair of points in S_i and S_{i+1}. Since $\{J_i\}$ is a $\kappa(t)$-itinerary, the lengths of bridges are bounded above by the lengths $d(x_i, x_{i+1})$, which is bounded by a constant multiple of $\kappa(t)$. Let that constant be C. Since γ is a $\kappa(t)$-excursion geodesic, $d(x_i, x_{i+1}) \leq \kappa(\|x_i\|)$. By Lemma A.13(3),

$$\begin{aligned}
\frac{1}{C}(d(x_i, b_i(i+1)) + d(x_{i+1}, b_{i+1}(i)) - |\mathcal{B}_{i,i+1}| - 4 &\leq d(x_i, x_{i+1}) \leq \kappa(\|x_i\|) \\
\frac{1}{C}(d(x_i, b_i(i+1)) + d(x_{i+1}, b_{i+1}(i)) &\leq |\mathcal{B}_{i,i+1}| + 4 + \kappa(\|x_i\|) \\
&\leq C\kappa(\|x_i\|)
\end{aligned}$$

Therefore $d(x_i, b_i(i+1))$ and $d(x_{i+1}, b_{i+1}(i))$ are both bounded by $C\kappa(\|x_i\|)$. □

Now we are ready to prove that the set of all κ-excursion geodesics is a subset of the κ–Morse boundary. We first replace an excursion geodesic with a geodesic in the CAT(0) metric that enters and leaves each maximal join at the same pair of points.

Proof of Proposition A.12. Let γ be a $\kappa(t)$–excursion geodesic and let $\{J_i\}$ be the associated itinerary of joins produced in Lemma A.14. Let x be in a maximal join J_i with $x \notin \gamma$, let

$$A := \bigcup_{k=0}^{5} J_{i+k},$$

and $\overline{A} := A \cup J_{i-1} \cup J_{i+6}$. Consider now a metric ball $\Sigma := \{y \in X(\Gamma) \ : \ d(x,y) < d(x,\gamma)\}$ which is disjoint from γ. Our goal is to prove that for any $y \in \Sigma$ we have $d(x_\gamma, y_\gamma) \leq C\kappa(\|x\|)$, where x_γ denotes the closest-point projection of the point x to γ.

Let $y \in \Sigma$. If $y \in A$, then there exists a constant C_1 such that

$$d(x_\gamma, y_\gamma) \leq C_1\kappa(\|x\|).$$

Otherwise, consider y in J_{i+k}, $k > 6$. There exists points $p \in J_{i+k}$ and closest to x such that

$$d(x,y) = d(x,p) + d(p,y). \tag{25}$$

That is to say $p \in \mathfrak{g}_{J_{i+6}}(J_i)$. By Corollary A.15, p is unique and therefore $p = \mathcal{B}_{i+6}(i+5)$, thus by Lemma A.16, there exists constant C_2 such that

$$d(p,\gamma) \leq C_2\kappa(\|x\|).$$

By way of contradiction, suppose $d(p,y) \geq d(p,\gamma)$. Then

$$\begin{aligned} d(x,y) &= d(x,p) + d(p,y) \qquad \text{by eq. (25)} \\ &\geq d(x,p) + d(p,\gamma) \\ &\geq d(x,\gamma). \end{aligned}$$

This is contrary to our assumption that $y \in \Sigma$. Therefore, $d(p,y) < d(p,\gamma)$, hence $d(p,y) < C_2\kappa(\|x\|)$. By the contractibility of CAT(0) projections, we have

$$d(p_\gamma, y_\gamma) \leq C_2\kappa(\|x\|).$$

Since $d(x_\gamma, p_\gamma) \leq C_1\kappa(\|x\|)$, then

$$d(x_\gamma, y_\gamma) \leq d(x_\gamma, p_\gamma) + d(p_\gamma, y_\gamma) \leq (C_1 + C_2)\kappa(\|x\|).$$

□

Excursion of random geodesics. To show that the κ–Morse boundary has full measure, we need to control the excursion of the random walk in each sub-join of the Salvetti complex. We will use the following variation of the main theorem in [ST18] (we thank Sam Taylor for suggesting the argument).

Theorem A.17. *Let μ be a finitely supported, generating probability measure on an irreducible right-angled Artin group $A(\Gamma)$. Then for any $k > 0$ there exists $C > 0$ such that for all n we have*

$$\mathbb{P}\left(\sup_J d_{s(J)}(1, w_n) \geq C \log n\right) \leq Cn^{-k},$$

where the supremum is taken over all join subcomplexes of $X(\Gamma)$.

As a consequence, for almost every sample path there exists $C > 0$ such that for all n

$$\sup_J d_{s(J)}(1, w_n) \leq C \log n.$$

Proof. The idea of the proof is that in order to make progress in $s(J)$, the sample path must project close to the projection of J in the contact graph $C(X)$. However, linear progress with exponential decay implies that the sample path can stay close to the projection of J only for a time of order $\log n$, which completes the proof.

Let us see the details. By linear progress with exponential decay [Mah12, Theorem 1.2], there exists $L > 0$ and C_1 such that

$$\mathbb{P}(d_{C(X)}(1, w_n) \leq Ln) \leq Ce^{-n/C}$$

for all n, so for any $A > 0$ and $n \geq e^{2/LA}$ we have

$$\mathbb{P}(d_{C(X)}(1, w_{A \log n}) \leq 2) \leq Cn^{-A/C}. \tag{26}$$

By the distance formula [BHS17, Theorem 9.1], for any $B > 0$ there exist C_2 such that for any join J

$$d_{s(J)}(x, y) \leq C_2 \sum_{K \subseteq J} \{d_{C(K)}(x, y)\}_B + C_2$$

where $\{x\}_B = x$ if $x \geq B$, and $\{x\}_B = 0$ otherwise. In particular, by Lemma A.9 there exists C_2 such that if a path $\gamma = [x, y]$ projects far from J in $C(X)$ then

$$d_{s(J)}(x, y) \leq C_2.$$

Consider now the path of vertices $(w_i)_{i \leq n}$ in $X(\Gamma)$, and suppose that for a join J we have $d_{s(J)}(1, w_n) \geq C \log n$. Let

$$i_1 := \min\{0 \leq i \leq n \ : \ d_{C(X)}(w_i, J) \leq 1\},$$

$$i_2 := \max\{0 \leq i \leq n \ : \ d_{C(X)}(w_i, J) \leq 1\},$$

and

$$D := \max\{d_{X(\Gamma)}(1, g) \ : \ g \in \text{supp } \mu\}.$$

Then

$$C \log n \leq d_{s(J)}(1, w_n) \leq d_{s(J)}(1, w_{i_1}) + d_{s(J)}(w_{i_1}, w_{i_2}) + d_{s(J)}(w_{i_2}, w_n) \leq D(i_2 - i_1) + 2C_2$$

hence, for n large enough,

$$|i_1 - i_2| \geq \frac{C \log n}{2D}.$$

Hence

$$\mathbb{P}(\exists J : \ d_{s(J)}(1, w_n) \geq C \log n) \leq \mathbb{P}\left(\exists i_1 \leq i_2 \leq n, i_2 - i_1 \geq \frac{C}{2D} \log n \ : \ d_{C(X)}(w_{i_1}, w_{i_2}) \leq 2\right)$$

and by (26) this is bounded above by

$$\leq n^2 \cdot C_1 n^{-\frac{C}{2DC_1}}$$

which tends to 0 for $n \to \infty$ as long as $C > 4DC_1$.

The second claim follows immediately from the first one for $k = 2$ by Borel-Cantelli. $\square$

Sublinear deviation between geodesics and sample paths. What remains is to understand the Hausdorff distance between a sample path and the geodesic ray that it is tracking. We first recall the following result:

Theorem A.18 ([Sis17], Theorem 5.2)**.** *Let S be a connected, orientable surface of finite type, with empty boundary and complexity at least 2. Let $M(S)$ be its mapping class group and let $\{w_n\}$ be a random walk on $M(S)$ driven by a finitely supported measure μ. Then for any $k \geq 1$ there exists a constant C such that*

$$\mathbb{P}\left(\sup d_{\text{Haus}}(\{w_i\}_{i\leq n}, \gamma(w_n)) \geq C\sqrt{n\log n}\right) \leq Cn^{-k}$$

where the supremum is taken over all geodesics in a given word metric and hierarchy paths $\gamma(w_n)$ from 1 to w_n.

It turns out that the proof in [Sis17] uses all ingredients that are known for right-angled Artin groups, namely the bounded geodesic image theorem, the distance formula, and quadratic divergence (where the $\sqrt{n}$ function comes from). Hence the same proof as in [Sis17] yields:

Theorem A.19. *Let G be an irreducible right-angled Artin group and let $\{w_n\}$ be a random walk on G driven by a finitely supported, generating measure μ. Then for any $k \geq 1$ there exists a constant C such that*

$$\mathbb{P}\left(\sup d_{\text{Haus}}\big(\{w_i\}_{i\leq n}, \gamma(w_n)\big) \geq C\sqrt{n\log n}\right) \leq Cn^{-k}$$

where the supremum is taken over all geodesics in a given word metric from 1 to w_n.

As a consequence, we obtain the following tracking estimate.

Proposition A.20. *Let μ be a finitely supported, generating measure on an irreducible right-angled Artin group $G = A(\Gamma)$, with universal Salvetti complex $X = X(\Gamma)$. Then there exists $\ell > 0$ such that for almost every sample path (w_n) there exists a* CAT(0) *geodesic ray γ in X starting at the base-point such that*

$$\limsup_{n\to\infty} \frac{d_X(w_n, \gamma(\ell n))}{\sqrt{n\log n}} < +\infty.$$

Proof. Since $G = A(\Gamma)$ is non-amenable and its action on X is cocompact, there exists $\ell > 0$ such that for almost every sample path

$$\lim_{n\to\infty} \frac{d_X(1, w_n)}{n} = \ell. \tag{27}$$

Now, by Theorem A.19, there exists $C > 0$ such that for any n

$$\mathbb{P}\left(\sup_{i\leq n} d_X(w_i, \gamma_n) \geq C\sqrt{n\log n}\right) \leq Cn^{-2}$$

where γ_n is the CAT(0) geodesic joining 1 and w_n. Hence, by Borel-Cantelli for almost every sample path there exists a constant C' such that

$$\sup_{i\leq n} d_X(w_i, \gamma_n) \leq C'\sqrt{n\log n} \tag{28}$$

for any n. Now, let $n_k := e^k n$ and consider the triangle with vertices $\{w_{n_{k-1}}, 1, w_{n_k}\}$. For large n and any $k \geq 1$, we have $d_X(w_{n_{k-1}}, \gamma_{n_k}) \leq C' \sqrt{n_k \log n_k}$ by (28) and $d_X(1, w_{n_{k-1}}) \geq \frac{\ell}{2} n_{k-1}$ by (27), then by comparison with a euclidean triangle,

$$d_X(\gamma_{n_{k-1}}(\ell n), \gamma_{n_k}(\ell n)) \lesssim \ell n \frac{d_X(w_{n_{k-1}}, \gamma_{n_k})}{d_X(1, w_{n_{k-1}})} \lesssim \sqrt{n \log n}\sqrt{k} e^{-k/2}.$$

Hence

$$d_X(w_n, \gamma(\ell n)) \lesssim \sum_{k=1}^{\infty} d_X(\gamma_{n_{k-1}}(\ell n), \gamma_{n_k}(\ell n)) \lesssim \sqrt{n \log n}$$

which proves the claim. □

Proof of Theorem A.3 and Theorem F. Recall that almost every sample path (w_n) converges to a point ξ in the visual boundary: let γ be the infinite CAT(0) geodesic connecting the base-point to ξ, and let $\gamma' = \{g_0, g_1, \dots\}$ be a combinatorial geodesic in $X(\Gamma)$ which lies at distance at most 1 from γ.

By Theorem A.17, for almost every sample path there exists $C_1 > 0$ such that

$$d_{s(J)}(1, w_n) \leq C_1 \log n.$$

for any join J. Moreover, by Proposition A.20, for almost every sample path and any J,

$$d_{s(J)}(\gamma(\ell n), w_n) \leq d_{X(\Gamma)}(\gamma(\ell n), w_n) \leq C_2 \sqrt{n \log n}$$

hence, since g_n lies within distance 1 of $\gamma(n)$,

$$d_{s(J)}(1, g_n) \leq C_1 \log(n/\ell) + C_2 \sqrt{(n/\ell)\log(n/\ell)} \leq C_3 \sqrt{n \log n}.$$

Thus, the geodesic γ' is a κ-excursion geodesic with $\kappa = \sqrt{t \log t}$, hence it is also a κ-contracting geodesic. This proves Theorem A.3, hence also Theorem F.

References

[ACGH17] G. Arzhantseva, C. Cashen, D. Gruber, and D. Hume, *Characterizations of Morse quasi-geodesics via superlinear divergence and sublinear contraction*, Documenta Mathematica 22 (2017), 1193–1224.

[Beh17] J. Behrstock, *A counterexample to questions about boundaries, stability, and commensurability*, arXiv:1705.03984.

[BC12] J. Behrstock and R. Charney, *Divergence and quasimorphisms of right-angled Artin groups*, Mathematische Annalen 352 (2012), 339–356.

[BHS17] J. Behrstock, M. Hagen and A. Sisto, *Hierarchically hyperbolic spaces I: curve complexes for cubical groups*, Geometry & Topology 21 (2017), 1731–1804.

[BK02] N. Benakli and I. Kapovich, *Boundaries of hyperbolic groups*, Combinatorial and Geometric Group Theory (R.Gilman, et al, editors) Contemporary Mathematics 296 (2002), 39–94.

[Bou98] N. Bourbaki, *General topology. Chapters 1-4*, Translated from the French. Reprint of the 1989 English translation. Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1998. vii+437.

[BH99] M. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 319. Springer-Verlag, Berlin, 1999. xxii+643. ISBN 3-540-64324-9.

[Cas16] C. Cashen, *Quasi-isometries need not induce homeomorphisms of contracting boundaries with the Gromov product topology*, Analysis and Geometry in Metric Spaces 4 (2016), no. 1, 278–281.

[CM19] C. Cashen and J. Mackay, *A metrizable topology on the contracting boundary of a group*, Transactions of the American Mathematical Society 372 (2019), no. 3, 1555–1600.

[Cha07] R. Charney, *An introduction to right-angled Artin groups*, Geometriae Dedicata, 25 (2007), no. 1, 141–158 .

[CS15] R. Charney and H. Sultan, *Contracting boundaries of* CAT(0) *spaces*, J. Topology 8 (2015), no. 1, 93–117.

[CP01] J. Crisp and L. Paris, The solution to a conjecture of Tits on the subgroup generated by the squares of the generators of an Artin group, *Invent. Math.* 145 (2001), 19–36.

[CW04] J. Crisp and B. Wiest, *Embeddings of graph braid and surface groups in right-angled Artin groups and braid groups*, Algebr. Geom. Topol. 4 (2004), 439–472.

[CK00] C. Croke and B. Kleiner, *Spaces with nonpositive curvature and their ideal boundaries*, *Topology* 39 (2000), no. 3, 549–556.

[Fri37] A. Frink, *Distance functions and the metrization problem*, Bull. Amer. Math. Soc. 43 (1937), no. 2, 133–142.

[Gro87] M. Gromov, *Hyperbolic groups*, In Gersten, Steve M. (ed.). Essays in group theory, Mathematical Sciences Research Institute Publications. 8. New York: Springer. 75–263.

[Hag14] M. Hagen, *Weak hyperbolicity of cube complexes and quasi-arboreal groups*, Journal of Topology 7 (2014), no. 2, 385–418.

[HW08] F. Haglund and D. Wise, *Special Cube Complexes*, Geom. Funct. Anal. 17 (2008), no. 5, 1551–1620.

[Ka96] V. Kaimanovich, *Boundaries of invariant Markov operators: The identification problem*, in M. Pollicott and K. Schmidt (Eds.), *Ergodic Theory and* $\mathbb{Z}^d$ *Actions* (London Mathematical Society Lecture Note Series, pp. 127-176). Cambridge: Cambridge University Press.

[KM99] A. Karlsson and G. Margulis, *A multiplicative ergodic theorem and nonpositively curved spaces*, Comm. Math. Phys. 208 (1999), 107–123.

[Mah12] J. Maher, *Exponential decay in the mapping class group* Journal of the London Mathematics Society 86 (2012), no. 2, 366–386.

[NS13] A. Nevo and M. Sageev, *The Poisson boundary of* CAT(0) *cube complex groups.* Groups Geom. Dyn. 7 (2013), no. 3, 653–695.

[Qin16] Y. Qing, *Geometry of Right-Angled Coxeter Groups on the Croke-Kleiner Spaces* Geometriae Dedicata 183 (2016), no. 1, 113–122.

[QT18] Y. Qing and G. Tiozzo, *Excursions of generic geodesics in right-angled Artin groups and graph products*, arXiv:1810.07891.

[Sis17] A. Sisto, *Tracking rates of random walks* Israel Journal of Mathematics 220 (2017), 1–28.

[ST18] A. Sisto and S. Taylor, *Largest projections for random walks and shortest curves for random mapping tori.* to appear in Mathematical Research Letters.