RANK ONE ISOMETRIES IN SUBLINEARLY MORSE BOUNDARIES OF CAT(0) GROUPS

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Abstract. Given a sublinear function κ, the κ-Morse boundary $\partial_\kappa G$ of a CAT(0) group was introduced by Qing and Rafi and shown to be a quasi-isometry invariant and a metrizable space. In this paper, we prove several properties of the κ-Morse boundary that further generalize useful properties of the Gromov boundary. We first show that if X is a proper CAT(0) space, then $\partial_\kappa X$ is a strong visibility space. We also show that any CAT(0) group G with nonempty κ-boundary contains a rank one isometry; furthermore, the subspace consisting of strongly contracting rays is a dense subspace of $\partial_\kappa G$. These results generate several applications: any CAT(0) group with nonempty κ-Morse boundary is an acylindrically hyperbolic group; the collection of rank one isometries G contains is exponentially generic. Furthermore, we show that a homeomorphism $f\colon \partial_\kappa G \to \partial_\kappa G'$ comes from a quasi-isometry if and only if f is Morse quasi-möbius and stable. Lastly, we characterize exactly when the sublinearly Morse boundary is a compact space.

1. Introduction

Much of the geometric group theory originates from the studying of hyperbolic groups and hyperbolic spaces. Hyperbolic groups have solvable word problem and strong dynamical properties. One fundamental technique in the study of hyperbolic groups is by constructing boundaries for these groups. Gromov took the collection of all infinite geodesic rays (up to fellow traveling) in the associated Cayley graph, equipped this set with cone topology on these geodesics, and defined the space to be the boundary ∂G of the hyperbolic group G. The boundary ∂G is independent of the choice of a generating set and has rich topological, dynamical, metric, quasi-conformal, measure-theoretic and algebraic structures (see for example the survey by Kapovich and Benakli [KB02]).

CAT(0) spaces enjoy locally non-negative curvature and are contractible. Extension of the boundary theory to CAT(0) spaces and groups has been developing in the past decades. In this setting, the space of all geodesic rays together with the cone topology is called the visual boundary and is denoted by $\partial_v X$. It is shown by Croke and Kleiner that the visual boundary of a CAT(0) space is not in general a quasi-isometry invariant [CK00]. In [CS15], Charney and Sultan constructed the first quasi-isometrically invariant boundary for CAT(0) spaces called the contracting boundary. One consequence of being in the contracting boundary is that a given geodesic ray spends uniformly finite amount of time in each product region. In [Qin16], it was shown that, in the Croke-Kleiner example, failure to obtain quasi-isometry invariance comes from geodesic rays that spend linear amount of time (with respect to total time travelled) in each product region.

Hence, one can consider geodesic rays that spend a sublinear amount of time in each product region. In [QR19], Qing and Rafi introduce the sublinearly Morse boundary $\partial_\kappa X$ of a CAT(0) metric space X and show that $\partial_\kappa X$ is quasi-isometry invariant and metrizable. Qing and Tiozzo show that, for a right-angled Artin group G, $\partial_\kappa G$ is a model for Poisson boundaries associated to a random walk (G, μ). Intuitively, a (quasi-)geodesic ray is sublinearly Morse if it spends a sublinear amount of time in each maximal product region, with respect to total time travelled when it enters that product region.

In this paper we show that $\partial_\kappa X$ enjoys a variety of hyperbolic-like properties. Let X be a CAT(0) space, a subset $\mathcal{S}$ of the visual boundary $\partial_v X$ is said to be a *strong visibility space* if given an element $\alpha(\infty) \in \mathcal{S}$ and an arbitrary element $\beta(\infty) \in \partial_v X$, there exists a geodesic line γ such that γ is asymptotic in one direction to $\alpha(\infty)$ and to $\beta(\infty)$ in the other. We first prove:

Theorem A. *Let X be a proper CAT(0) space and let κ be a sublinear function. The κ-Morse boundary $\partial_\kappa X$ is a strong visibility space.*

Aside from serving as an evidence of a "hyperbolic-like" boundary, strong visibility gives rise to rank one isometries. A *rank one isometry* is a hyperbolic isometry such that none of whose axes bounds a flat half-plane. We remark that a rank one isometry is in general hard to establish directly, while the nonemptiness of $\partial_\kappa X$ for any κ serves as an alternative approach.

We say that a geodesic ray is *strongly contracting* if there exists a constant D such that all balls disjoint from the geodesic ray projects to sets of diameter at most D. We prove the following:

Theorem B. *Suppose G is a group that acts geometrically on a CAT (0) space X. If there exists a sublinear function κ such that $\partial_\kappa X \neq \emptyset$, then:*

1. *The group G contains a rank one isometry.*
2. *The subspace consisting of all strongly contracting geodesics is a dense subspace of $\partial_\kappa X$.*
3. *For any point $\mathsf{b} \in \partial_\kappa X$ representing a strongly contracting geodesic, the orbit $G\mathsf{b}$ is dense in $\partial_\kappa X$.*

We find necessary and sufficient conditions for $\partial_\kappa G$ to be compact:

Theorem C. *Suppose a group G acts geometrically on a proper CAT(0) space X such that $\partial_\kappa X \neq \emptyset$, then the following are equivalent:*

1. *Every geodesic ray in X is κ-contracting.*
2. *Every geodesic ray in X is strongly contracting.*
3. *$\partial_\kappa X$ is compact.*
4. *The space X is hyperbolic.*

We have several applications of Theorem A and Theorem B. First we can detect with weaker assumption when a group is *acylindrically hyperbolic.* Roughly speaking, a group is acylindrically hyperbolic if its action on hyperbolic spaces is less than properly continuously but still discrete (See Section 2.5 for definition). Acylindrically hyperbolic groups includes mapping class groups.

Corollary D. *Let G be a CAT(0) group which is not virtually cyclic. If $\partial_\kappa G \neq \emptyset$, then G is acylindrically hyperbolic.*

This theorem provides a new technique to prove that a CAT(0) group is acylindrically hyperbolic. This includes large classes of Artin groups. An example of an

Artin group of *large type* (See Definition 2.21) is given in Example 5.2. In general, it is open whether large type Artin groups are acylindrically hyperbolic.

Secondly, we use Theorem 1.9 in [Yan18] by Yang to provide a qualification for when rank one isometries of CAT(0) spaces are exponentially generic. For definitions, see Section 2.7.

Corollary E. *Let G is a CAT(0) group with $\partial_\kappa G \neq \emptyset$, then the elements that are rank one isometries are exponentially generic in G.*

The last application concerns quasi-isometry rigidity of CAT(0) groups. In 1996, Paulin gives the following characterization [Pau96]: if $f : \partial X \to \partial Y$ is a homeomorphism between the boundaries of two proper, cocompact hyperbolic spaces, then the following are equivalent

(1) f is induced by a quasi-isometry $h : X \to Y$.
(2) f is quasi-möbius.

Quasi-möbius maps are maps such that changes in the cross ratio are controlled by a continuous function. As an application of Theorem B, we give a similar characterization for sublinearly Morse boundaries. We note that *Morse quasi-möbius* is different from quasi-möbius as it is a condition that needs to be checked for infinitely many subspaces of $\partial_\kappa X$.

Corollary F. *Let X, Y be proper cocompact CAT(0) spaces with at least 3 points in their sublinear boundaries. A homeomorphism $f : \partial_\kappa X \to \partial_\kappa Y$ is induced by a quasi-isometry $h : X \to Y$ if and only if f is stable and Morse quasi-möbius.*

History. The sublinearly Morse boundary is preceded by various other approaches to construct a space at infinity for CAT(0) spaces and proper metric spaces in general. In [CS15] Charney and Sultan introduced the Morse boundary of a CAT(0) space which consists of geodesic rays such that the projection of disjoint balls to them are bounded uniformly. The Morse boundary of a CAT(0) space is a quasi-isometry invariant. This idea is generalized to proper geodesic metric space by Cordes [Cor17]. However, the Morse boundaries in general is not first countable as shown by Murray [Mur19] and is too small to be a model for Poisson boundaries (G, μ) for random walks on general right-angled Artin groups G. Building on [ACGH17], Cashen and Mackay put a metrizable topology on Morse boundary of proper geodesic space in [CM19]. The Morse boundary and the Cashen-Mackay boundary are shown to have many properties analogous to boundaries of hyperbolic spaces, and much of the work in this paper is inspired by the methods in [CM17] [CM19], [Liu19], [Mur19] and [Zal18].

Outline of the paper. In Section 2, we give all necessarily definitions and background. In Section 3, we prove Theorem A and Theorem B. In Section 4 we characterize when $\partial_\kappa X$ is a compact space and prove Theorem C. The last section contains three applications: acylindrically hyperbolic groups, exponential genericity in counting measure of CAT(0) groups and the connection between homeomorphisms on $\partial_\kappa X$ and quasi-isometries on X.

Acknowledgement. We thank Ruth Charney, Mathew Cordes, Jinying Huang, Curt Kent and Kasra Rafi for helpful conversations.

2. Preliminaries

2.1. Quasi-isometry and quasi-isometric embeddings.

Definition 2.1 (Quasi Isometric embedding)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. For constants $\mathsf{k} \geq 1$ and $\mathsf{K} \geq 0$, we say a map $f\colon X \to Y$ is a (k, K)–*quasi-isometric embedding* if, for all points $x_1, x_2 \in X$

$$\frac{1}{\mathsf{k}} d_X(x_1, x_2) - \mathsf{K} \leq d_Y\big(f(x_1), f(x_2)\big) \leq \mathsf{k}\, d_X(x_1, x_2) + \mathsf{K}.$$

If, in addition, every point in Y lies in the K–neighbourhood of the image of f, then f is called a (k, K)–quasi-isometry. When such a map exists, X and Y are said to be *quasi-isometric*.

A quasi-isometric embedding $f^{-1}\colon Y \to X$ is called a *quasi-inverse* of f if for every $x \in X$, $d_X(x, f^{-1}f(x))$ is uniformly bounded above. In fact, after replacing k and K with larger constants, we assume that f^{-1} is also a (k, K)–quasi-isometric embedding,

$$\forall x \in X \quad d_X\big(x, f^{-1}f(x)\big) \leq \mathsf{K} \qquad \text{and} \qquad \forall y \in Y \quad d_Y\big(y, f\, f^{-1}(x)\big) \leq \mathsf{K}.$$

A *geodesic ray* in X is an isometric embedding $\beta\colon [0, \infty) \to X$. We fix a base-point $\mathfrak{o} \in X$ and always assume that $\beta(0) = \mathfrak{o}$, that is, a geodesic ray is always assumed to start from this fixed base-point.

Definition 2.2 (Quasi-geodesics)**.** In this paper, a *quasi-geodesic ray* is a continuous quasi-isometric embedding $\beta\colon [0, \infty) \to X$ starting from the basepoint $\mathfrak{o}$.

The additional assumption that quasi-geodesics are continuous is not necessary for the results in this paper to hold, but it is added for convenience and to make the exposition simpler.

If $\beta\colon [0, \infty) \to X$ is a (q, Q)–quasi-isometric embedding, and $f\colon X \to Y$ is a (k, K)–quasi-isometry then the composition $f \circ \beta\colon [t_1, t_2] \to Y$ is a quasi-isometric embedding, but it may not be continuous. However, one can adjust the map slightly to make it continuous (see Lemma III.1.11 [BH09]). Abusing notation, we denote the new map again by $f \circ \beta$. Following Lemma III.1.11 [BH09], we have that $f \circ \beta$ is a $(\mathsf{kq}, 2(\mathsf{kq} + \mathsf{kQ} + \mathsf{K}))$–quasi-geodesic.

Similar to above, a *geodesic segment* is an isometric embedding $\beta\colon [t_1, t_2] \to X$ and a *quasi-geodesic segment* is a continuous quasi-isometric embedding

$$\beta\colon [t_1, t_2] \to X.$$

Notation. In this paper we will use α, β... to denote quasi-geodesic rays. If the quasi-geodesic constants are $(1, 0)$, we use $\alpha_0, \beta_0, ...$ to signify that they are in fact geodesic rays. Furthermore, let α be a (quasi-)geodesic ray $\alpha\colon [0, \infty) \to X$. We use $\alpha[s_1, s_2]$ to denote the segment of α between $\alpha(s_1)$ and $\alpha(s_2)$. On the other hand, if x_1, x_2 are points on α, then the segment of α between x_1 and x_2 is denoted $[x_1, x_2]_\alpha$. If a segment is presented without subscript, for example $[y_1, y_2]$, then it is a geodesic segment between the two points. Let β be a quasi-geodesic ray. For $\mathsf{r} > 0$, let t_r be the first time where $\|\beta(t)\| = \mathsf{r}$ and define:

$$\beta_\mathsf{r} := \beta(t_\mathsf{r}) \qquad \text{and} \qquad \beta|_\mathsf{r} := \beta[0, t_\mathsf{r}] = [\beta(0), \beta_\mathsf{r}]_\beta \tag{1}$$

which are points and segments in X, respectively.

2.2. **CAT(0) spaces and their boundaries.** A geodesic metric space (X, d_X) is CAT(0) if geodesic triangles in X are at least as thin as triangles in Euclidean space with the same triple of side-lengths. To be precise, for any given geodesic triangle $\triangle pqr$, consider the unique triangle $\triangle \overline{pqr}$ in the Euclidean plane with the same side-lengths. For any pair of points x, y on edges $[p, q]$ and $[p, r]$ of the triangle $\triangle pqr$, if we choose points $\overline{x}$ and $\overline{y}$ on edges $[\overline{p}, \overline{q}]$ and $[\overline{p}, \overline{r}]$ of the triangle $\triangle \overline{pqr}$ so that $d_X(p, x) = d(\overline{p}, \overline{x})$ and $d_X(p, y) = d(\overline{p}, \overline{y})$ then,

$$d_X(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y}).$$

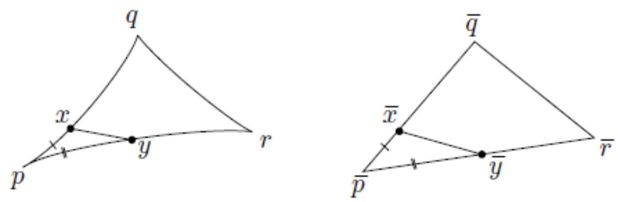

FIGURE 1. CAT(0) geometry

For the remainder of the paper, we assume X is a proper CAT(0) space. A metric space X is *proper* if closed metric balls are compact. a CAT(0) space has the following basic properties:

Lemma 2.3. *A proper CAT(0) space X has the following properties:*

(1) *For any two points x, y in X, there exists exactly one geodesic connecting them. Consequently, X is contractible via geodesic retraction to a base point in the space.*
(2) *The nearest point projection from a point x to a geodesic line β_0 is a unique point denoted $\pi_{\beta_0}(x)$, or simply x_{β_0}. In fact, the closest point projection map to a geodesic*
$$\pi_{\beta_0} \colon X \to \beta_0$$
is Lipschitz with respect to distances. The nearest point projection from a point x to a qausi-geodesic line β exists and is not necessarily unique. We denote the whole projection set $\pi_\beta(x)$.
(3) *For any $x \in X$, the distance function $d_X(x, \cdot)$ is convex. In other words, for any given any geodesic $[x_0, x_1]$ and $t \in [0, 1]$, if x_t satisfies $d_X(x_0, x_t) = td(x_0, x_1)$ then we must have*
$$d_X(x, x_t) \leq (1 - t)d_X(x, x_0) + td_X(x, x_1).$$

In addition, we need the following geometric properties of CAT(0) spaces:

Lemma 2.4 ([QR19])**.** *Consider a (q, Q)–quasi-geodesic segment β connecting a point $z \in X$ to a point $w \in X$. Let $x \in X$ and let y be a point in x_β, and let γ be the concatenation of the geodesic segment $[x, y]$ and the quasi-geodesic segment $[y, z]_\beta \subset \beta$. Then $\gamma = [x, y] \cup [y, z]_\beta$ is a $(3\mathsf{q}, \mathsf{Q})$–quasi-geodesic.*

Theorem 2.5 (The flat plane theorem [BH09])**.** *Suppose that X is a cocompact CAT(0) space. If X is not hyperbolic, then X contains an isometrically embedded copy of $\mathbb{E}^2$.*

Now we define the *visual boundary* of a CAT(0) space.

Definition 2.6 (visual boundary)**.** Let X be a CAT(0) space. The *visual boundary* of X, denoted $\partial_v X$, is the collection of equivalence classes of infinite geodesic rays, where α and β are in the same equivalence class, if and only if there exists some $C \geq 0$ such that $d(\alpha(t), \beta(t)) \leq C$ for all $t \in [0, \infty)$. The equivalence class of α in $\partial_v X$ we denote $\alpha(\infty)$.

Notice that by Proposition I. 8.2 in [BH09], for each α representing an element of ∂X, and for each $x' \in X$, there is a unique geodesic ray α' starting at x' with $\alpha(\infty) = \alpha'(\infty)$.

We describe the topology of the visual boundary by a neighbourhood basis: fix a base point $\mathfrak{o}$ and let α be a geodesic ray starting at $\mathfrak{o}$. A neighborhood basis for α is given by sets of the form:

$$\mathcal{U}_v\big(\alpha(\infty), r, \epsilon\big) := \{\beta(\infty) \in \partial X \,|\, \beta(0) = \mathfrak{o} \text{ and } \ d(\alpha(t), \beta(t)) < \epsilon \text{ for all } \ t < r\}.$$

In other words, two geodesic rays are close if they have geodesic representatives that start at the same point and stay close (are at most ϵ apart) for a long time (at least r). Notice that the above definition of the topology on $\partial_v X$ references a base-point $\mathfrak{o}$. Nonetheless, Proposition I. 8.8 in [BH09] proves that the topology of the visual boundary is base-point invariant.

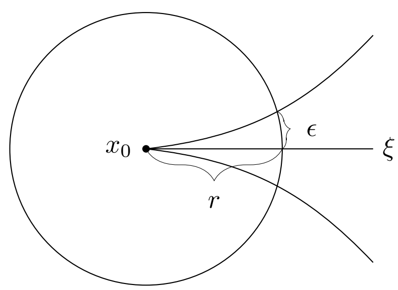

FIGURE 2. A basis for open sets

2.3. **Tits metric on ∂X.** In order to prove strong visibility we need the notion of angles. In this section we assume X to be a complete CAT(0) space. To begin with, the notion of an angle is well defined in CAT(0) space. Let α, α' denote two infinite geodesic rays emanating from $\mathfrak{o}$ and such that

$$\alpha \in \alpha(\infty), \alpha' \in \alpha'(\infty).$$

We define the *local angle* to be

$$\angle_{\mathfrak{o}}(\alpha(\infty), \alpha'(\infty)) := \angle_{\mathfrak{o}}(\alpha, \alpha') := \lim_{t \to 0} 2 \arcsin \frac{1}{2t} d_X(\alpha(t), \alpha'(t')).$$

The *angle* between two points $\alpha(\infty), \beta(\infty) \in \partial X$ is defined to be:

$$\angle(\alpha(\infty), \beta(\infty)) := \sup_{x \in X} \angle_x(\alpha^x, \beta^x),$$

where α^x, β^x are representatives of $\alpha(\infty), \beta(\infty)$ that originate from x. The *Tits metric*, denoted by d_T or $\angle_T$ is defined to be the length metric associated with $\angle(\cdot, \cdot)$.

Lemma 2.7. *let $\alpha(\infty), \beta(\infty) \in \partial X$ be where X is a complete CAT(0) spaces. Their angle falls into one of the following cases:*

- *If $d_T(\alpha(\infty), \beta(\infty)) < \pi$, then for any point $x \in X$, the geodesic rays α_x, β_x bounds a flat sector (Flat Sector Theorem, II.9.9 [BH09]).*
- *If $d_T(\alpha(\infty), \beta(\infty)) = \pi$, then by(II.9.20, [BH09]), there exists $\eta \in \partial X$ such that $\angle_T(\eta, \alpha(\infty)) = \angle_T(\eta, \beta(\infty)) = \pi/2$.*
- *If $d_T(\alpha(\infty), \beta(\infty)) > \pi$, then there is a geodesic ray connecting $\alpha(\infty)$ and $\beta(\infty)$.(II.9.21 (1) [BH09])*

Now we are ready to define *rank one isometry.* Formally, an isometry of X is *rank one* if there exists an axis of this isometry which does not bound any half flat. In [BB08], it is shown that one can detect rank one isometry from distance in Tits metric, which is the approach in this paper.

Proposition 2.8. *(Proposition 1.10 [BB08].) Suppose G is a group that acts geometrically on a CAT(0) space X, then the following are equivalent.*

- *The group G contains a rank one isometry.*
- *For any $\zeta \in \partial X$, there exists $\eta \in \partial X$ with $d_T(\zeta, \eta) > \pi$.*

2.4. Sublinearly Morse boundaries of CAT(0) spaces.

2.4.1. *sublinear functions.* Let $\kappa\colon [0, \infty) \to [1, \infty)$ be a sublinear function that is monotone increasing and concave. That is

$$\lim_{t\to\infty} \frac{\kappa(t)}{t} = 0.$$

The assumption that κ is increasing and concave makes certain arguments cleaner, otherwise they are not really needed. One can always replace any sub-linear function κ, with another sub-linear function $\overline{\kappa}$ so that

$$\kappa(t) \le \overline{\kappa}(t) \le \mathsf{C}\,\kappa(t)$$

for some constant C and $\overline{\kappa}$ is monotone increasing and concave. For example, define

$$\overline{\kappa}(t) = \sup\Big\{ \lambda\kappa(u) + (1-\lambda)\kappa(v) \;\Big|\; 0 \le \lambda \le 1,\ u, v > 0, \text{ and } \lambda u + (1-\lambda)v = t \Big\}.$$

The requirement $\kappa(t) \ge 1$ is there to remove additive errors in the definition of κ–contracting geodesics(See Definition 2.9).

2.4.2. *κ–Morse geodesic rays.* The boundary of interest in this paper consists of points in ∂X that are in the "hyperbolic-like". In proper CAT(0) spaces, they can be characterized in two equivalence ways.

Definition 2.9 (κ–contracting sets)**.** For $x \in X$, define $\|x\| = d_X(\mathfrak{o}, x)$. For a closed subspace Z of X, we say Z is *κ–contracting* if there is a constant c_Z so that, for every $x, y \in X$

$$d_X(x, y) \le d_X(x, Z) \quad \Longrightarrow \quad diam_X\big(x_Z \cup y_Z\big) \le \mathsf{c}_Z \cdot \kappa(\|x\|).$$

In fact, to simplify notation, we often drop $\|\cdot\|$. That is, for $x \in X$, we define

$$\kappa(x) := \kappa(\|x\|).$$

Definition 2.10 (κ–neighborhood)**.** For a closed set Z and a constant n define the (κ, n)–neighbourhood of Z to be

$$\mathcal{N}_\kappa(Z, \mathsf{n}) = \Big\{ x \in X \;\Big|\; d_X(x, Z) \le \mathsf{n} \cdot \kappa(x) \Big\}.$$

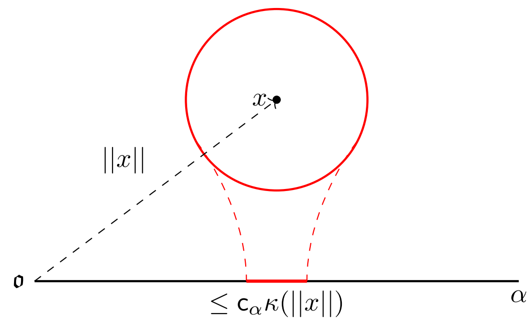

FIGURE 3. A κ–contracting geodesic ray.

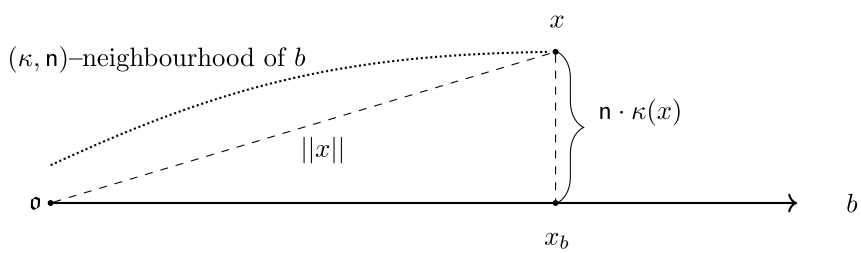

FIGURE 4. A κ-neighbourhood of a geodesic ray b with multiplicative constant n.

Definition 2.11 (κ–Morse sets)**.** We say a closed subset Z of X is κ*–Morse* if there is a function

$$\mathsf{m}_Z \colon \mathbb{R}_+^2 \to \mathbb{R}_+$$

so that if $\beta \colon [s,t] \to X$ is a (q, Q)–quasi-geodesic with end points on Z then

$$\beta[s,t] \subset \mathcal{N}_\kappa\big(Z, \mathsf{m}_Z(\mathsf{q}, \mathsf{Q})\big).$$

We refer to m_Z as the *Morse gauge* for Z. We always assume

$$\mathsf{m}_Z(\mathsf{q}, \mathsf{Q}) \geq \max(\mathsf{q}, \mathsf{Q}). \tag{2}$$

A geodesic is κ–contracting if and only if it is κ–Morse ([QR19]). In fact, the following technical tool is used several times in this paper. It states that if a geodesic ray Z is κ-contracting, then it is κ*-strongly Morse*. κ–Morse is defined in Definition 2.11, while κ-strongly Morse only requires the endpoints of the quasi-geodesic segment to be sublinearly close to Z, instead of on Z.

Theorem 2.12 ([QR19])**.** *Let X be a proper CAT(0) space. A geodesic ray α is κ–contracting if and only if it is κ–Morse. Specifically, let Z be a closed subspace that is κ–contracting. Then, there is a function $\mathsf{m}_Z \colon \mathbb{R}^2 \to \mathbb{R}$ such that, for every constants $\mathsf{r} > 0$, $\mathsf{n} > 0$ and every sublinear function κ', there is an $\mathsf{R} = \mathsf{R}(Z, \mathsf{r}, \mathsf{n}, \kappa') > 0$ where the following holds: Let $\eta \colon [0, \infty) \to X$ be a (q, Q)–quasi-geodesic ray so that $\mathsf{m}_Z(\mathsf{q}, \mathsf{Q})$ is small compared to r, let t_r be the first time $\|\eta(t_\mathsf{r})\| = \mathsf{r}$ and let t_R be the first time $\|\eta(t_\mathsf{R})\| = \mathsf{R}$. Then*

$$d_X\big(\eta(t_\mathsf{R}), Z\big) \leq \mathsf{n} \cdot \kappa'(\mathsf{R}) \quad \Longrightarrow \quad \eta[0, t_\mathsf{r}] \subset \mathcal{N}_\kappa\big(Z, \mathsf{m}_Z(\mathsf{q}, \mathsf{Q})\big).$$

Next, we observe that a κ–Morse geodesic ray does not bound a sector or a half flat.

Corollary 2.13. *A κ-contracting geodesic ray does not bound a sector or a half flat.*

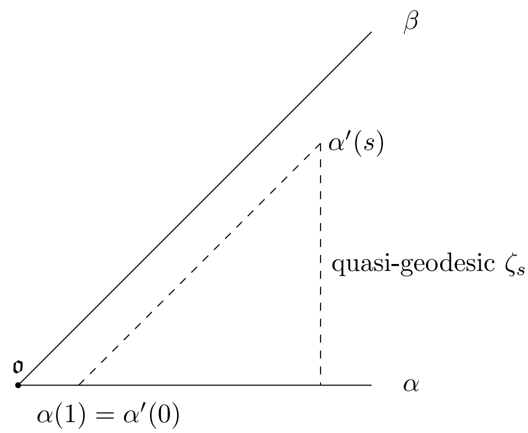

FIGURE 5. Since α and β bounds a sector, α cannot be κ–Morse. We show that ζ_s is a one-parameter family of quasi-geodesics, such that they all have bounded constants, and for any (κ, n)–neighbourhood, there exists an s such that the associated ζ_s does not stay in the (κ, n)–neighbourhood.

Proof. Suppose α is κ–Morse and α, β bounds a flat sector, suppose they intersect at angle u. At $\alpha(1)$, there exists a segment lying in the sector and is parallel to β, which we denote α'. The concatenation

$$\zeta_s := [\alpha'(0), \alpha'(s)] \cup [\alpha'(s), \alpha'(s)_\alpha]$$

is a quasi-geodesic of bounded constant, since it is a concatenation of geodesics intersecting at angle $\frac{1}{2}\pi - u$. This quasi-geodesic exists for any $s > 1$ and for every κ-neighbourhood there exists s such that this quasi-geodesic ζ_s is not contained in the neighbourhood (See Figure 5). Therefore α cannot be κ–Morse. □

Quasi-geodesic rays in X are grouped into equivalence classes to form $\partial_\kappa X$.

Definition 2.14 (κ–equivalence classes in $\partial_\kappa X$)**.** Let β and γ be two quasi-geodesic rays in X. If β is in some κ–neighbourhood of γ and γ is in some κ–neighbourhood of β, we say that β and γ *κ–fellow travel* each other. This defines an equivalence relation on the set of quasi-geodesic rays in X (to obtain transitivity, one needs to change n of the associated (κ, n)–neighbourhood).

We denote the equivalence class that contains β by $[\beta]$:

Definition 2.15 (Sublinearly Morse boundary)**.** Let κ be a sublinear function as specified in Section 2.4.1 and let X be a CAT(0) space.

$$\partial_\kappa X := \{ \text{ all } \kappa\text{-Morse quasi-geodesics } \}/\kappa\text{-fellow travelling}$$

We will discuss the topology of $\partial_\kappa X$ in Section 2.4.3.

We also use $\mathbf{a}, \mathbf{b}$ to denote κ-equivalence classes in $\partial_\kappa X$. It is shown that in CAT(0) spaces there is a unique geodesic ray in each equivalence class.

Lemma 2.16 (Lemma 3.5, [QR19])**.** *Let X be a CAT(0) space. Let $b\colon [0,\infty) \to X$ be a geodesic ray in X. Then b is the unique geodesic ray in any (κ, n)–neighbourhood of b for any n. That is to say, distinct geodesic rays do not κ–fellow travel each other.*

Lemma 2.17. *There is an 1-1 embedding of the set of points in $\partial_\kappa X$ into the points of $\partial_v X$.*

Proof. For each element $\mathbf{a} \in \partial_\kappa X$, consider its unique geodesic ray α. The associated $\alpha(\infty)$ is a element of $\partial_v X$. By Lemma 2.16 each equivalence class contains a unique geodesic ray. This map is well defined. Meanwhile, if two elements $\mathbf{a}, \mathbf{b} \in \partial_\kappa X$ contain the same geodesic ray, they are in fact the same set of quasi-geodesics, therefore this map is well-defined. Therefore we have an embedding of the set of points in $\partial_\kappa X$ into the points of $\partial_v X$. □

2.4.3. *Coarse visual topology on $\partial_\kappa X$.* We equip $\partial_\kappa X$ with a topology which is a coarse version of the visual topology. In visual topology, if two geodesic rays fellow travel for a long time, then they are "close". In this coarse version, if two geodesic rays and all the quasi-geodesic rays in their respect equivalence classes remain close for a long time, then they are close. Now we define it formally. First, we say a quantity D *is small compared to a radius* $\mathsf{r} > 0$ if

$$\mathsf{D} \leq \frac{\mathsf{r}}{2\kappa(\mathsf{r})}. \tag{3}$$

Recall that given a κ–Morse quasi-geodesic ray β, we denotes its associated Morse gauges functions $\mathsf{m}_\beta(\mathsf{q}, \mathsf{Q})$. These are multiplicative constants that give the heights of the κ–neighbourhoods.

Definition 2.18 (topology on $\partial_\kappa X$)**.** Let $\mathbf{a} \in \partial_\kappa X$ and $\alpha_0 \in \mathbf{a}$ be the unique geodesic in the class a. Define $\mathcal{U}_\kappa(\mathbf{a}, \mathsf{r})$ to be the set of points $\mathbf{b}$ such that for any (q, Q)-quasi-geodesic of $\mathbf{b}$, denoted β, such that $\mathsf{m}_\beta(\mathsf{q}, \mathsf{Q})$ is small compared to r, satisfies

$$\beta|_{\mathsf{r}} \subset \mathcal{N}_\kappa\big(\alpha_0, \mathsf{m}_{\alpha_0}(\mathsf{q}, \mathsf{Q})\big).$$

Let the topology of $\partial_\kappa X$ be the topology induced by this neighbourhood system.

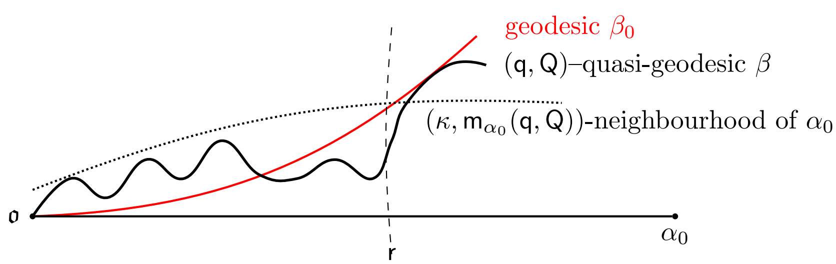

Figure 6. $\mathbf{b} \in \mathcal{U}_\kappa(\mathbf{a}, \mathsf{r})$ because the quasi-geodesics of $\mathbf{b}$ such as β, β_0 stay inside the associated $(\kappa, \mathsf{m}_{\alpha_0}(\mathsf{q}, \mathsf{Q}))$-neighborhood of α_0 (as in Definition 2.10), up to distance r.

Theorem 2.19 ([QR19]). *Let X be a proper CAT(0) space and let κ be a sublinear function. The κ–boundary of X, denoted $\partial_\kappa X$, is a metrizable space. Furthermore, $\partial_\kappa X$ is quasi-isometrically invariant.*

2.5. **Acylindrically hyperbolic groups.** Let G be a group with an isometric action on some geodesic hyperbolic metric space X. This action is called *acylindrical* if for every $R > 0$ there exists $N > 0, L > 0$ such that for every $x, y \in X$ with $d(x, y) > L$ one has

$$\#\{g \in G | d(x, gx) \leq R, d(y, gy) \leq R)\} \leq N.$$

Acylindricity is a weaker substitute for a proper action. It is shown that relative hyperbolic groups and mapping class groups are acylindrically hyperbolic. We have the following key result to decide when a CAT(0) group is acylindrically hyperbolic:

Theorem 2.20 ([Osi16]). *If a group G acts properly on a proper CAT(0) space and contains an element that is a rank one isometry, then G is either virtually cyclic or acylindrically hyperbolic.*

In particular, consider the class of *Artin groups*. Artin groups are natural combinatorial generalizations braid groups. For every finite simple graph Γ with vertex set S and with edges labeled by some integer in $\{2, 3, ...\}$, one associates the Artin-Tits group $A(\Gamma)$ with the following presentation:

$$A(\Gamma) = \langle S | \forall \{s, t\} \in \Gamma(1), w_m(s, t) = w_m(t, s) \text{ if the edge } \{s, t\} \text{ is labeled } m. \rangle,$$

where $w_m(s, t)$ is the word $stst...$ of length m. Note that when $m = 2$, then s and t commute, and when $m = 3$, then s and t satisfy the classical braid relation $sts = tst$. Also note that when adding the relation $s^2 = 1$ for every $s \in S$, one obtains the Coxeter group $W(\Gamma)$ associated to Γ.

Definition 2.21. An Artin group can further be classified as follows. The Artin group $A(\Gamma)$ is called:

- large type if all labels are greater or equal to 3,
- extra large type if all labels are greater or equal to 4,
- right-angled if all labels are equal to 2,
- spherical if $W(\Gamma)$ is finite, and
- type FC if every complete subgraph of Γ spans a spherical Artin subgroup.

Theorem 2.22 ([BM00]). *Let $A(\Gamma)$ be an Artin group, if $|S| = 3$ and all labels are greater or equal to 3, Then $A(\Gamma)$ is a CAT(0) group.*

It is in general an open question whether large type Artin group are acylindrically hyperbolic, and we give a positive answer to one example using $\partial_\kappa G$ in Example 5.2.

2.6. **Morse boundary and Morse quasi-mobius maps.** Recall a geodesic γ is *strongly contracting* if it is in the sublinearly Morse boundary whose associated sublinear function $\kappa = 1$: $\partial_1 X$. This implies the existence of a constant D such that all disjoint balls project onto γ to a set of diameter at most D, in which case we say γ is D-*strongly contracting*. Consider the set of all D-strongly-contracting geodesic rays emanating from $\mathfrak{o}$. We can think of this set as a subspace of the various boundaries we study in this paper: we use $\partial_v^{\mathsf{D}} X$ to denote the set of all D-contracting geodesic rays emanating from $\mathfrak{o}$ when equipped with the subspace topology of the visual boundary , and use $\partial_\kappa^{\mathsf{D}} X$ when equipped with the subspace topology of the κ-boundary.

Denote by $\partial_\kappa^{(n,\mathsf{D})} X$ the collection of all n-tuples $(a_1, a_2, ..., a_n)$ of distinct points $a_i \in \partial_\kappa X$ such that every bi-infinite geodesic connecting a_i to a_j is D-strongly contracting.

Definition 2.23. Let X, Y be proper geodesic CAT(0) space.

- A map $f : \partial_\kappa X \to \partial_\kappa Y$ is said to be 1-*stable* if for every D, there exists D' such that $f(\partial_\kappa^{\mathsf{D}} X) \subseteq \partial_\kappa^{\mathsf{D}'} Y$.
- A map $f : \partial_\kappa X \to \partial_\kappa Y$ is said to be 2-*stable* if for every D, there exists D' such that
$$f(\partial_\kappa^{(2,\mathsf{D})} X) \subseteq \partial_\kappa^{(2,\mathsf{D}')} Y.$$

Notice that it follows from the above definition that a 2-stable map f maps $\partial_\kappa^{(n,\mathsf{D})} X$ to $\partial_\kappa^{(n,\mathsf{D}')} X$ for all $n \geq 2$. Hence, it makes sense to make the following definition.

A map $f : \partial_\kappa X \to \partial_\kappa Y$ is said to be *stable* if it is both 1 and 2 stable.

Definition 2.24. The cross-ratio of a four-tuple $(a, b, c, d) \in \partial_\kappa^{(4,\mathsf{D})} X$ is defined to be $[a, b, c, d] = \pm \sup_{\alpha \in (a,c)} d(\pi_\alpha(b), \pi_\alpha(d))$, where the sign is positive if the orientation of the geodesic $(\pi_\alpha(b), \pi_\alpha(d))$ agrees with that of (a, c) and is negative otherwise.

Definition 2.25. A stable map $f : \partial_\kappa X \to \partial_\kappa Y$ is said to be D-quasi mobius if for every D, there exists a continuous map $\psi_{\mathsf{D}} : [0, \infty) \to [0, \infty)$ such that for all 4-tuples $(a, b, c, d) \in \partial_\kappa^{(4,\mathsf{D})} X$, we have $[f(a), f(b), f(c), f(d)] \leq \psi_{\mathsf{D}}(|[a, b, c, d]|)$.

Definition 2.26. Let $X_1 \subset X_2 \subset X_3 \subset ...$ be a nested sequence of topological spaces. The *direct limit* of $\{X_i\}$, denoted by $\varinjlim X_i$, is the space consisting of the union of all X_i given the following topology: A subset U is open in $\varinjlim X_i$ if $U \cap X_i$ is open in X_i for each i.

The following is a standard way to establish continuous maps between two nested sequences and the proof is left as an exercise for interested readers.

Lemma 2.27. *Let $\{X_i\}$, and $\{Y_j\}$ be two sequences of nested topological spaces. Let*
$$X = \varinjlim X_i \text{ and } Y = \varinjlim Y_i$$
be the direct limit of $\{X_i\}$ and $\{Y_i\}$ respectively. If $f : X \to Y$ is a map such that:

- *For each i there exists some j with $f(X_i) \subseteq Y_j$.*
- *$f|_{X_i} : X_i \to Y_j$ is continuous.*

Then f is continuous.

Consider the topological spaces $\partial_v^{\mathsf{D}} X$. The *Morse boundary* $\partial_\star X$ is the direct limit of the topological spaces $\partial_v^{\mathsf{D}} X$ where $\mathsf{D} \in \mathbb{N}$. In other words
$$\partial_\star X = \varinjlim \partial_v^{\mathsf{D}} X$$
Hence, a set U is open in $\partial_\star X$ if and only if $U \cap \partial_v^{\mathsf{D}} X$ is open for each D.

2.7. **Genericity of elements in countable groups.** Suppose that a countable group G admits a geometric action on a proper geodesic metric space (X, d). Fix a basepoint $\mathfrak{o} \in X$. Denote

$$N(n) := \{g \in G : d(\mathfrak{o}, g\mathfrak{o}) \leq n\}.$$

A subset W of G is said to be *generic* in the counting measure if as $n \to \infty$,

$$\frac{|W \cap N(n)|}{|N(n)|} \to 1.$$

We say that W is *exponentially generic* if the rate of convergence happen exponentially fast, that is to say:

$$1 - \frac{|W \cap N(n)|}{|N(n)|} < \alpha^n$$

for some $\alpha \in (0, 1)$.

3. STRONG VISIBILITY AND RANK ONE ISOMETRIES

In this section, we prove that a sublinearly Morse boundary of a CAT(0) space is a strong visibility space. We also prove that if G is a CAT(0) group, then $\partial_\kappa G \neq \emptyset$ implies there is a rank one isometry in G. We also prove that in the latter case, the subspace consisting of strongly contracting geodesics is a dense subspace of the κ-boundaries. To begin with, a subset $\mathcal{S}$ of $\partial_v X$ is a *strong visibility space* if given an element of $\mathcal{S}$ and an element of $\partial_v X$, there exists a bi-infinite geodesic connecting these two elements.

Theorem 3.1. *Let $(X, \mathfrak{o})$ be a proper CAT(0) space and let κ be a sublinear function. The κ-Morse boundary $\partial_\kappa X$ is a strong visibility space. That is to say, given an element $\boldsymbol{a} \in \partial_\kappa X$, and a point $\beta_0(\infty) \in \partial_v X$, there exists a bi-infinite geodesic line $\gamma : (-\infty, \infty) \to X$ such that $\gamma[0, -\infty) \in \boldsymbol{a}$ and $\gamma[0, \infty) \in \beta_0(\infty)$.*

Proof. Fix a basepoint $\mathfrak{o}$, let α_0 be the unique geodesic representative of $\mathbf{a}$ emanating from *go* and let β_0 denote the unique geodesic ray in $\beta_0(\infty)$ emanating from $\mathfrak{o}$. Consider $d_T(\alpha_0, \beta_0)$ as defined in Section 2.3. By Lemma 2.7 (1), (2) and Corollary 2.13, α_0 does not bound a half flat or a sector, and thus $d_T(\alpha_0, \beta_0) > \pi$.

By Lemma 2.7, there exists a geodesic ray connecting $\alpha_0(\infty)$ and $\beta_0(\infty)$. By Lemma 2.17, $\alpha_0(\infty)$ is associated with $\mathbf{a}$. Therefore there exists a geodesic ray connecting $\mathbf{a}, \beta_0(\infty)$. Since this is true for any element of $\partial_\kappa X$, $\partial_\kappa X$ is a strong visibility space.

□

From this observation we obtain the existence of rank one isometry:

Corollary 3.2. *Let G be a CAT(0) group acting geometrically on a CAT(0) space X. Suppose there exists a sublinear function κ such that $\partial_\kappa G \neq \emptyset$. Then G contains a rank one isometry of X.*

Proof. Since $\partial_\kappa G$ is nonempty, there is at least one element of $\partial_\kappa G$, which we denote $\mathbf{a}$ with a geodesic ray $\alpha \in \mathbf{a}$. Consider an element $\beta \in \partial_v X$. by Theorem 3.1, there exists a bi-infinite geodesic line connecting α and β. That is to say, $d_T(\alpha, \beta) \geq \pi$. However since $\alpha \in \partial_\kappa X$, by Corollary 2.13 it does not bound a half flat or a sector. That is to say, $d_T(\alpha, \beta) \gneq \pi$. By Lemma 2.17, $\alpha \in \partial_v X$. Thus for any point of $\beta \in \partial_v X$, we use α as a point of $\partial_v X$ to obtain $d_T(\alpha, \beta) \gneq \pi$. By Proposition 2.8, there exists a rank one isometry. □

Next we want to show that if X is a space with geometric action by a group, then rank-one isometries are in fact dense in $\partial_\kappa X$.

Theorem 3.3. *Let X be a proper CAT(0) space and let G act on X geometrically. Suppose κ is not a constant function. Then $\partial_1 X$ is a dense subset of $\partial_\kappa X$.*

Proof. The statement is trivial if $\partial_\kappa X$ is empty. If $\partial_\kappa X$ is non-empty, then by Corollary 3.2, there exists at least one rank one isometry $g \in G$. The axis of g defines two points in the κ-boundary, say g^+ and g^-. Let α_g be a geodesic line connecting g^+ to g^-, this is possible using Theorem 3.1. Notice that since g is rank one, the line α_g is a bi-infinite 1-Morse geodesic line. We may assume that the base point $\mathfrak{o}$ lives on the axis. Given any element $\mathbf{b} \in \partial_\kappa X$, consider the unique geodesic representative $\beta_0 \in \mathbf{b}$ that emanates from $\mathfrak{o}$. Since the action is cocompact, there exists a positive number C and a sequence $\{g_i\}$ such that each $g_i\mathfrak{o}$ is within C of β_0 and $d(g_i\mathfrak{o}, \mathfrak{o}) \geq i$. Furthermore, since the actions are isometries of X, the translates $g_i \cdot \alpha_g$ are bi-infinite, 1-Morse geodesic lines that passes through the points $g_i\mathfrak{o}$. Consider the segment $[\mathfrak{o}, g_i\mathfrak{o}]$. This segment forms two angles with two ends of the Morse geodesic $g_i \cdot \alpha_g$ and one of the angles is greater than or equal to $\pi/2$. Denote that end $(+)$ and consider the concatenation

$$[\mathfrak{o}, g_i\mathfrak{o}] \cup g_i \cdot \alpha_g[g_i\mathfrak{o}, (+)).$$

Since this is a concatenation of two geodesic segments at angle greater than or equal to $\pi/2$, this is a (2, 0)-quasi-geodesic.

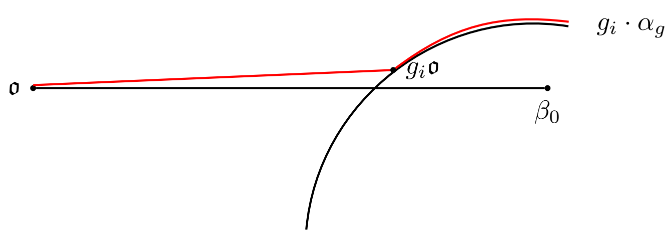

FIGURE 7. The concatenation $[\mathfrak{o}, g_i\mathfrak{o}] \cup g_i \cdot \alpha_g[g_i\mathfrak{o}, (+))$, marked red, is a $(2, 0)$–quasi-geodesic.

Let $\{\ell_i\}$ be the sequence of $(2, 0)$–quasi-geodesic rays constructed this way. Each ℓ_i is in a C-neighbourhood of β_0 for distance i. The equivalence classes of ℓ_i in $\partial_\kappa X$ we denote $[\ell_i]$. Given any ℓ_i, let the first segment $[\mathfrak{o}, g_i\mathfrak{o}]$ be denoted L_1:

$$L_1 := [\mathfrak{o}, g_i\mathfrak{o}],$$

and the rest of ℓ_i be denoted L_2. Consider a (q, Q)–quasi-geodesic $\eta \in [\ell_i]$. Without loss of generality, we assume η to be a connected path. Rename $p_i := g_i\mathfrak{o}$ and project the point p_i to η, and consider a point from the projection set and denote it q_i (see Figure 8):

$$q_i \in \pi_\eta(p_i).$$

Let Q_1 denote the concatenation

$$Q_1 = \eta[0, q_i] \cup [q_i, p_i].$$

By Lemma 2.4, Q_1 is a $(3\mathsf{q}, \mathsf{Q})$–quasi-geodesic. Similarly the concatenation

$$Q_2 = [p_i, q_i] \cup [q_i, \infty)_\eta$$

is also a $(3\mathsf{q}, \mathsf{Q})$-quasi-geodesic.

Since α_g is Morse, Q_2 is in a bounded neighbourhood of L_2. Let $\mathsf{m}_{L_2}(\cdot, \cdot)$ denote the Morse gauge of L_2. We have

$$d(q_i, L_2) \leq \mathsf{m}_{L_2}(3\mathsf{q}, \mathsf{Q}). \tag{4}$$

In particular, that implies $d(p_i, q_i) \leq \mathsf{m}_{L_2}(3\mathsf{q}, \mathsf{Q})$. By construction, we now have:

$$\|q_i\| \geq i - \mathsf{C} - \mathsf{m}_{L_2}(3\mathsf{q}, \mathsf{Q}) \tag{5}$$

Applying Lemma 2.4 to conclude also that Q_1 is a $(3\mathsf{q}, \mathsf{Q})$–quasi-geodesic with endpoints on $[\mathfrak{o}, p_i]$. But p_i is distance at most C from β_0. This means Q_1 can be replaced with a $(3\mathsf{q}, \mathsf{Q} + \mathsf{C})$–quasi-geodesic with endpoints on β_0. Therefore

$$d(q_i, \beta_0) \leq \mathsf{m}_{\beta_0}(3\mathsf{q}, \mathsf{Q} + \mathsf{C})\kappa(q_i). \tag{6}$$

By Theorem 2.12, for a given pair (q, Q) and any given r such that $\mathsf{m}_{\beta_0}(\mathsf{q}, \mathsf{Q})$ is small enough compared to r, there exists

$$\mathsf{R} = \mathsf{R}(\beta_0, \mathsf{r}, \mathsf{m}_{\beta_0}(3\mathsf{q}, \mathsf{Q} + \mathsf{C}), \kappa_{\beta_0})$$

such that a (q, Q)–quasi-geodesic $\eta \in [\ell_i]$ satisfies

$$d(\eta(t_R), \beta_0) \leq \mathsf{m}_{\beta_0}(3\mathsf{q}, \mathsf{Q} + \mathsf{C})\kappa_{\beta_0}(R) \tag{7}$$

implies

$$\eta|_{\mathsf{r}} \subset \mathcal{N}_\kappa(\beta_0, \mathsf{m}_{\beta_0}(\mathsf{q}, \mathsf{Q})).$$

Equation 5 tells us we can find i large enough such that $\|q_i\| \geq R$. Therefore by the continuity of η, there exists a point $\eta(t_R)$ satisfying Equation 7. Hence we conclude, there exists larger and larger r such that

$$[\ell_i] \in \mathcal{U}_\kappa(\beta_0, \mathsf{r}).$$

Since one can find a subsequence of $\{[\ell_i]\}$ for which this is true for arbitrarily large r, this subsequence converges to $\beta_0 \in \mathbf{b}$. Since each $[\ell_i] \in \partial_1 X$ and this holds for any $\mathbf{b}$, we conclude that $\partial_1 X$ is a dense subset of $\partial_\kappa X$.

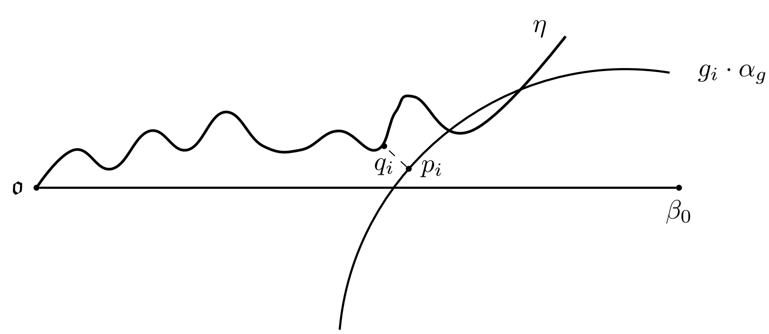

FIGURE 8. A (q, Q)–quasi-geodesic η can be decomposed and modified into two quasi-geodesics each of which has endpoints on β_0 and $g_i \cdot \alpha_g$, respectively.

$\square$

4. Compactness of κ-boundaries characterizes hyperbolicity

In [QR19], it is shown that if $G = \mathbb{Z}^2 \star \mathbb{Z}$, then $\partial_\kappa G$ is not compact. In this section we show that the κ-boundary is compact if and only if the underlying space is hyperbolic.

Consider the subset of all D-strongly contracting geodesic rays emanating from $\mathfrak{o}$ in a CAT(0) space X. The follwoing lemma states that equipping this subset with the subspace topology of the visual boundary or the subspace topology of the κ-boundary yields homeomorphic spaces. The intuitive reason for this is the following: since quasi-geodesics stay uniformly close to D-strongly contracting geodesics, the topology of fellow travelling of geodesics (the visual topology) and the topology of fellow travelling of quasi-geodesics (the topology of the κ-boundary) coincide.

Recall that we use $\partial_v^{\mathsf{D}} X$ to denote the set of all D-contracting geodesic rays emanating from $\mathfrak{o}$ when equipped with the subspace topology of the visual boundary , and use $\partial_\kappa^{\mathsf{D}} X$ when equipped with the subspace topology of the κ-boundary.

Lemma 4.1. *The identity map $id : \partial_v^{\mathsf{D}} X \to \partial_\kappa^{\mathsf{D}} X$ is a homeomorphism.*

Proof. We need to show that the map $id : \partial_v^{\mathsf{D}} X \to \partial_\kappa^{\mathsf{D}} X$ is a homeomorphism. Since $\partial_v^{\mathsf{D}} X$ is closed (Lemma 3.2 in [CS15]) and X is proper, $\partial_v^{\mathsf{D}} X$ must be compact. Also, the space $\partial_\kappa^{\mathsf{D}} X$ is metrizable by Theorem D in [QR19]. Hence, it suffices to show that the map id is a continuous map. Notice that since every geodesic ray in $\partial_v^{\mathsf{D}} X$ is D-strongly contracting for the same D, applying Theorem 2.12, we get an associated Morse function such that every geodesic ray β_0 is m-Morse, where m depends only on D and satisfies the following: For every constants $\mathsf{r} > 0$, $\mathsf{n} > 0$ and every sublinear function κ', there is an $\mathsf{R} = \mathsf{R}(\beta_0, \mathsf{r}, \mathsf{n}, \kappa') > 0$ where the following holds: Let $\eta \colon [0, \infty) \to X$ be a (q, Q)–quasi-geodesic ray so that $\mathsf{m}_{\beta_0}(\mathsf{q}, \mathsf{Q})$ is small compared to r, let t_r be the first time $\|\eta(t_\mathsf{r})\| = \mathsf{r}$ and let t_R be the first time $\|\eta(t_\mathsf{R})\| = \mathsf{R}$. Then

$$d_X\big(\eta(t_\mathsf{R}), \beta_0\big) \leq \mathsf{n} \cdot \kappa'(\mathsf{R}) \quad \Longrightarrow \quad \eta[0, t_\mathsf{r}] \subset \mathcal{N}_1\big(\beta_0, \mathsf{m}_{\beta_0}(\mathsf{q}, \mathsf{Q})\big) \subset \mathcal{N}_\kappa\big(\beta_0, \mathsf{m}_{\beta_0}(\mathsf{q}, \mathsf{Q})\big).$$

.

We claim the following:

Claim. Given $\mathbf{b} \in \partial_\kappa^{\mathsf{D}} X$, each neighbourhood of $\mathbf{b}$, denoted $\mathcal{U}_\kappa\big(\mathbf{b}, \mathsf{r}\big)$, must contain a visual neighbourhood basis of β_0, the unique geodesic ray in the class of $\mathbf{b}$.

Proof. To see this, let $\beta_0 \in \mathbf{b}$ be the unique geodesic ray starting at $\mathfrak{o}$. We wish to show that for any $\mathsf{r} > 0$, there exists r' and ϵ such that $\mathcal{U}_v\big(\mathbf{b}, \mathsf{r}', \epsilon\big) \subseteq \mathcal{U}_\kappa\big(\mathbf{b}, \mathsf{r}\big)$

In other words, we want to show that for any $\mathsf{r} > 0$, there exists r' and ϵ if a geodesic ray $\alpha_0 \in \mathbf{a}$ with $\alpha_0(0) = \mathfrak{o}$ satisfies $d(\alpha_0(t), \beta_0(t)) < \epsilon$ for $t \leq \mathsf{r}'$, then, any (q, Q)-quasi-geodesic representative α of a with $\mathsf{m}_{\beta_0}(\mathsf{q}, \mathsf{Q})$ small compared to r, we have

$$\alpha|_\mathsf{r} \subset \mathcal{N}_\kappa(\zeta, \mathsf{m}_{\beta_0}(\mathsf{q}, \mathsf{Q})).$$

Remember that $\alpha|_\mathsf{r} = \alpha([0, t_\mathsf{r}])$ where t_r is the first time where $||\alpha(t)|| = \mathsf{r}$.

Let r be given and let

$$n = \max\{\mathsf{m}_{\beta_0}(\mathsf{q}, \mathsf{Q}) + 1 | \mathsf{q}, \mathsf{Q} \leq \mathsf{r}\}.$$

By Theorem 2.12, with $Z = \beta_0$, there exists an $R = R(\mathsf{r}, n)$ such that any (q, Q)-quasi-geodesic representative β of a with $\mathsf{m}_\beta(\mathsf{q}, \mathsf{Q})$ small compared to r, we have

$$d(\beta(t_R), b) < n \Rightarrow \beta_\mathsf{r} \subset \mathcal{N}_1(\beta_0, \mathsf{m}_\beta(\mathsf{q}, \mathsf{Q})).$$

Choose $\mathsf{r}' = \mathsf{r} + R$ and $\epsilon = 1$. Hence, we want to show that if $d(a(t), b(t)) < 1$ for $t \leq \mathsf{r} + R$, then $\beta_{\mathsf{r}} \subset \mathcal{N}_\kappa(b, \mathsf{m}_{\beta_0}(\mathsf{q}, \mathsf{Q}))$ for β defined above. Since a is 1-Morse with gauge m, the Hausdorff distance between a and β is at most $\mathsf{m}(\mathsf{q}, \mathsf{Q})$. This implies that for any $0 < t \leq \mathsf{r} + R$, we have

$$d(a(t)), \beta(i_t)) < \mathsf{m}_{\beta_0}(\mathsf{q}, \mathsf{Q}),$$

for some i_t.

Therefore, if t_R is the first time with $\|\beta(t_R)\| = R$, we must have

$$d(a, \beta(t_R)) < \mathsf{m}_{\beta_0}(\mathsf{q}, \mathsf{Q}).$$

Now, since $d(a(t), b(t)) < 1$ for all $t < \mathsf{r} + R$ and as $d(a, \beta(t_R)) < \mathsf{m}_{\beta_0}(\mathsf{q}, \mathsf{Q})$, the triangle inequality gives

$$d(b, \beta(t_R)) \leq d(b, a) + d(a, \beta(t_R)) \leq 1 + \mathsf{m}_{\beta_0}(\mathsf{q}, \mathsf{Q}),$$

which by Theorem 2.12 implies that $\beta_r \subset \mathcal{N}_1(b, \mathsf{m}_{\beta_0}(\mathsf{q}, \mathsf{Q})) \subset \mathcal{N}_\kappa(b, \mathsf{m}_{\beta_0}(\mathsf{q}, \mathsf{Q}))$ which proves the claim. ☐

Now we are left to show that the map id is continuous. Let $\{\mathsf{c}_n\}, \mathsf{c} \in \partial_v^{\mathsf{D}} X$ with $\mathsf{c}_n \to \mathsf{c}$. Assume that $\mathsf{c}_n \to \mathsf{c}$ in $\partial_v^{\mathsf{D}} X$, we want to show that $\mathsf{c}_n \to \mathsf{c}$ in $\partial_\kappa^{\mathsf{D}} X$. Using the above claim, since each neighbourhood of c in $\partial_\kappa^{\mathsf{D}} X$ contains an open neighborhood of $\partial_v^{\mathsf{D}} X$, the statement is immediate.

☐

Corollary 4.2. *For a CAT(0) space X, the natural map $i : \partial_\star X \hookrightarrow \partial_\kappa X$ is continuous.*

Proof. Since $\partial_\star X = \varinjlim \partial_v^{\mathsf{D}} X$, we need only to show that $i_{\mathsf{D}} : \partial_v^{\mathsf{D}} X \hookrightarrow \partial_\kappa X$ is continuous for each D, but that is Lemma 4.1. ☐

Corollary 4.3 (Weak minimality)**.** *Suppose G acts geometrically on a CAT(0) space X with $\partial_\kappa X \neq \emptyset$. There exists a point $\mathsf{b} \in \partial_\kappa X$ such that the orbit $G\mathsf{b}$ is dense in $\partial_\kappa X$.*

Proof. Using Theorem B, $\partial_\kappa X$ contains an element b such that the unique geodesic ray α_0 representing b is strongly contracting. We claim that $G\mathsf{b}$ is dense in $\partial_\kappa X$. To see that, let $\mathsf{c} \in \partial_\kappa X$ and let β_0 be the unique geodesic ray representing c. Suppose $\mathcal{U}_\kappa(\mathsf{c}, \mathsf{r})$ is an open neighborhood of c. Using Theorem B, the open set $\mathcal{U}_\kappa(\mathsf{c}, \mathsf{r})$ must contain an element d whose unique geodesic representative $[\gamma_0]$ is strongly contracting. Now, notice that $\mathcal{U}_\kappa(\mathsf{c}, \mathsf{r}) \cap \partial_1 X$ is open in $\partial_1 X$. Hence, by Corollary 4.2, $i^{-1}(\mathcal{U}_\kappa(\mathsf{c}, \mathsf{r}) \cap \partial_1 X)$ is an open set in $\partial_\star X$ containing $[\gamma_0]$. Using Theorem 4.1 in [Mur19], the open set $i^{-1}(\mathcal{U}_\kappa(\mathsf{c}, \mathsf{r}) \cap \partial_1 X)$ must contain some orbit point $g[\alpha_0]$. Hence, the open set $\mathcal{U}_\kappa(\mathsf{c}, \mathsf{r})$ contains the orbit point $g\mathsf{b}$. ☐

Notice that the argument above proves the following stronger statement.

Corollary 4.4. *For any point $\mathsf{b} \in \partial_\kappa X$ representing a strongly contracting geodesic, the orbit $G\mathsf{b}$ is dense in $\partial_\kappa X$.*

Corollary 4.5. *Let X be a hyperbolic CAT(0) space, then $\partial_\kappa X$ is compact.*

Proof. If X is a hyperbolic space, then by Lemma 1.7 in [BH09] III.H every geodesic ray is 1-Morse, and hence every geodesic ray must be D-strongly contracting (Theorem 2.12)for a uniform D. This implies that the subspace $\partial_v^{\mathsf{D}} X$ defined above is the entire visual boundary, in other words, we have $\partial_v^{\mathsf{D}} X = \partial_v X$. Also, since every geodesic ray is D-strongly contracting, the subspace $\partial_\kappa^{\mathsf{D}} X$ defined above is the entire κ-boundary. That is to say, $\partial_\kappa^{\mathsf{D}} X = \partial_\kappa X$. Lemma 4.1 then yields a homeomorphism between the visual boundary of X, $\partial_v X$ and the κ-boundary of X, $\partial_\kappa X$. Therefore, as the visual boundary $\partial_v X$ of a hyperbolic space X is compact, the κ-boundary $\partial_\kappa X$ must also compact. $\square$

Theorem 4.6. *Suppose a group G acts geometrically on a CAT(0) space X such that $\partial_\kappa X \neq \emptyset$, then the following are equivalent:*

(1) *Every geodesic ray in X is κ–contracting.*
(2) *Every geodesic ray in X is strongly contracting.*
(3) *$\partial_\kappa X$ is compact.*
(4) *The space X is hyperbolic.*

Proof. We start by showing (3) implies (1). The statement is trivially true if $\partial_\kappa X$ is empty. If $\partial_\kappa X$ is non-empty, then by Corollary 3.2, there exists some rank one isometry g. This yields the existence of a geodesic line α_g which is 1-Morse. Let $\mathfrak{o}$ be a point on the line α_g and let α_0 be an arbitrary geodesic ray emanating from $\mathfrak{o}$. We will show now that α_0 is κ-contracting. Since the action of G on X is cocompact, there exists a $\mathsf{C} \geq 0$ and a sequence of group elements $\{g_i\} \subseteq G$ such that $d(\alpha_0(i), g_i\mathfrak{o}) \leq \mathsf{C}$ for each $i \in \mathbb{N}$ (the black dots in Figure 9). Now, consider the geodesic lines given by $g_i\alpha_g$ centered at $g_i\mathfrak{o}$ near α_0. We use $[\cdot,\cdot]$ to denote a geodesic segment between two points. By CAT(0) geometry, the concatenation of two geodesics at angle bounded below by $\pi/2$ form a (2,0)-quasi-geodesic. For each i, consider the concatenation $[\mathfrak{o}, g_i\mathfrak{o}]+[g_i\mathfrak{o}, \alpha_g(\infty)]$ and $[\mathfrak{o}, g_i\mathfrak{o}]+[g_i\mathfrak{o}, \alpha_g(-\infty)]$. It follows from construction that one of these two concatenations is a (2,0)-quasi-geodesic ray starting at $\mathfrak{o}$. Let y_i be the sequence of (2,0)–quasi-geodesic rays defined by concatenating $[\mathfrak{o}, g_i\mathfrak{o}]$ with either $[g_i\mathfrak{o}, \alpha_g(\infty)]$ or $[g_i\mathfrak{o}, \alpha_g(-\infty)]$ to form a sequence of (2,0)–quasi-geodesic rays, as given in the figure (the blue quasi-geodesic). Since $\partial_\kappa X$ is compact, up to passing to a subsequence, $[y_i]$ converges to some $\mathsf{a} \in \partial_\kappa X$. Hence, for each $\mathsf{r} > 0$, there exists k, such that if $i \geq k$, the sequence y_i satisfies

$$y_i|_{\mathsf{r}} \subset \mathcal{N}_\kappa\big(a_0, \mathsf{m}_{a_0}(2,0)\big),$$

where a_0 is the unique geodesic ray representing a. This implies that all initial segments of α_0 are in $\mathcal{N}_\kappa\big(a_0, \mathsf{C} + \mathsf{m}_{a_0}(2,0)\big)$, and hence

$$\alpha_0 \in \mathcal{N}_\kappa\big(a_0, \mathsf{C} + \mathsf{m}_{a_0}(2,0)\big).$$

Lemma 2.16 then implies that $\alpha_0 = a_0$ which finishes the proof.

Next we show that (1) implies (4). If every geodesic ray is κ-contracting, then X doesn't contain an isometric copy of $\mathbb{E}^2$, and hence, using Theorem 2.5, the space X must be hyperbolic. The implication (4) $\Rightarrow$ (3) is Corollary 4.5.

Lastly, we prove the equivalence between (2) and (4). For (4) $\Rightarrow$ (2) notice that Lemma 1.7 in [BH09] III.H states that if X is hyperbolic, then every geodesic ray is N-Morse for the same N. Now, by Theorem 2.12, every geodesic ray must be D-strongly contracting for the same D.

On the other hand, for (2) $\Rightarrow$ (4), by way of contradiction, suppose X is not a hyperbolic space, then it must isometrically contain a copy of $\mathbb{E}^2$ by the Flat Plane Theorem (Theorem 2.5). Let $\mathfrak{o} \in \mathbb{E}^2$ and the geodesic rays that stays entirely in the is not D–strongly contracting for any D. Therefore, (2) $\Rightarrow$ (4). $\square$

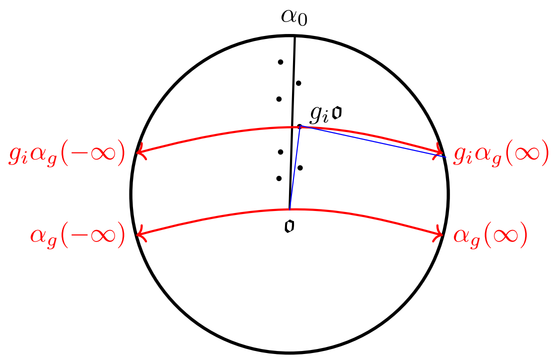

FIGURE 9. Translates of α_g by $\{g_i\}$ along α_0

5. Applications

In this section, we present three applications of the main theorems. Direct consequence of the existence of rank one isometry in κ–boundary result in Corollary 5.1 and Corollary 5.3. The fact that Morse geodesics are dense in $\partial_\kappa G$ plays a key part in the proof of Theorem 5.4.

5.1. **Acylindrically hyperbolic groups.** Theorem B gives a new way of showing that a certain CAT(0) group is acylindrically hyperbolic.

Corollary 5.1. *If G is a CAT(0) group such that $\partial G_\kappa \neq \emptyset$, then G must be acylindrically hyperbolic.*

Proof. If G is a CAT(0) group with $\partial G_\kappa \neq \emptyset$, then by Theorem B, G must contain a rank one isometry. Hence, by Theorem [Osi16], the group G must be acylindrically hyperbolic.

□

Example 5.2. Consider the Arin group defined by the following graph:

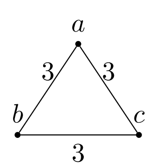

FIGURE 10. The defining graph Γ of Artin group $A(\Gamma)$

The universal cover of the presentation complex, endowed with word metric, consists of countably many copies of coarse "tree cross $\mathbb{R}$", which we call *blocks*. One can define the *intersection graph* of the blocks: In this intersection graph $\mathcal{I}_\Gamma$, every vertex is associated to a block. Two vertices in $\mathcal{I}_\Gamma$ are adjacent if and only if the two associated blocks intersect at a plane. The intersection graph of $A(\Gamma)$ is hyperbolic [JH17]. Furthermore, the projection (in word metric) of a block to another one whose distance is far enough is a point. Since the blocks are convex sets and their intersections are convex sets, a geodesic ray enters and leaves a (possibly infinite) sequence of them. Consider a unit-speed geodesic γ that spends $\kappa(t)$ amount of time in each block, where t is the time the geodesic enters the block. Proposition A.12 in [QR19] can be similarly applied to show that γ is κ-contracting, and hence a $\partial_\kappa X$ is nonempty. Since by [BM00] we know $A(\Gamma)$ is a CAT(0) group, Theorem 2.5 tells us that it is in fact an acylindrically hyperbolic group.

5.2. **Generic isometries in a CAT(0) group.** Work of [Yan18] shows that if a CAT(0) group contains a rank one isometry, then the subset of rank one isometries in G is exponentially generic. Therefore we obtain:

Corollary 5.3. *If G is a CAT(0) group such that $\partial_\kappa G \neq \emptyset$, then the collection of rank one isometries are exponentially generic in G.*

5.3. **Morse quasi-möbius homeomophisms on the κ-boundaries.** In [Pau96], the author characterizes homeomorphisms between boundaries of cocompact hyperbolic spaces that are induced by quasi-isometries. They characterize such homeomorphisms as the ones that are *quasi-möbius*. In this section, as an application of Theorem B and using work of [CM17], [CCM19], we prove a weaker version of this characterization:

Theorem 5.4. *Let X, Y be proper cocompact CAT(0) spaces with at least 3 points in their sublinear boundaries. A homeomorphism $f : \partial_\kappa X \to \partial_\kappa Y$ is induced by a quasi-isometry $h : X \to Y$ if and only if f is stable and Morse quasi-möbius.*

Corollary 5.5. *Let G and H be CAT(0) groups. Then G is quasi-isometric to H if and only if there exists a homeomorphism $f : \partial_\kappa G \to \partial_\kappa H$ which is Morse quasi-möbius and stable.*

Roughly speaking, the above corollary says that by understanding the κ–boundary of two CAT(0) groups G and H, we can tell if they belong to the same quasi-isometry class or not.

Lemma 5.6. *A (k, K)-quasi-isometry $h : X \to Y$ induces a stable homeomorphism*

$$\partial_\kappa h : \partial_\kappa X \to \partial_\kappa Y.$$

Proof. Fix $\mathfrak{o} \in X$ and let $\mathfrak{o}' = h(\mathfrak{o})$. Qing and Rafi show that a quasi-isometry h induces a homeomorphism ∂h on their respective κ-boundaries. If γ is a D-strongly contracting geodesic ray, then by Theorem 5.1 in [QR19], the unique geodesic ray starting at $h(\mathfrak{o})$ and representing $[f(\gamma)]$ must be D'-strongly contracting where D' depends on D, k and K. This implies that $\partial_\kappa h$ is 1-stable. Now, Theorem 5.8 of states that the map induced by h on the Morse boundary is 2-stable. Hence, we deduce that $\partial_\kappa h$ is stable.

□

Lemma 5.7. *Any homeomorphism $f : \partial_\kappa X \to \partial_\kappa Y$ such that f, f^{-1} are 1-stable induces a homeomorphism $g : \partial_\star X \to \partial_\star Y$ on their Morse boundaries, with $g(x) = f(x)$ for all $x \in \partial_\star X$.*

Proof. Let $f : \partial_\kappa X \to \partial_\kappa Y$ be a homeomorphism such that f and f^{-1} are 1-stable. Notice that by Theorem E in [QR19], we have that if $\kappa' < \kappa$, then the inclusion map

$$i : \partial_{\kappa'} X \to \partial_\kappa X$$

is continuous. Taking $\kappa' = 1$, yields that $i : \partial_1 X \to \partial_\kappa X$ is continuous. Hence, since both f and f^{-1} are 1-stable, the restriction of f to $\partial_1 X$ induces a homeomorphism $\overline{f} : \partial_1 X \to \partial_1 Y$, with $\overline{f} = f|_{\partial_1 X}$ where $\partial_1 X$ and $\partial_1 Y$ are given the subspace topology of the κ-boundary. Meanwhile,

$$\partial_1 X = \bigcup_{\mathsf{D}=1}^{\infty} \partial_1^{\mathsf{D}} X \text{ and } \partial_1 Y = \bigcup_{\mathsf{D}=1}^{\infty} \partial_1^{\mathsf{D}} Y$$

. Since $\partial_1^{\mathsf{D}} X$ is equipped with the subspace topology of $\partial_1 X$, the inclusion map

$$i^{\mathsf{D}} : \partial_1^{\mathsf{D}} X \hookrightarrow \partial_1 X$$

is continuous. Using Lemma 4.1, we get that

$$i^{\mathsf{D}} : \partial_v^{\mathsf{D}} X \hookrightarrow \partial_1 X$$

is continuous for every D, where $\partial_v^{\mathsf{D}} X$ is given the subspace topology of the visual boundary. Furthermore, since f is 1-stable, we have $\overline{f} \circ i^{\mathsf{D}} : \partial_v^{\mathsf{D}} X \hookrightarrow \partial_1^{\mathsf{D}'} Y$ for some D' where $\overline{f} \circ i^{\mathsf{D}}$ is continuous. Using Lemma 4.1, we obtain a continuous map

$$\overline{f} \circ i^{\mathsf{D}} : \partial_v^{\mathsf{D}} X \hookrightarrow \partial_v^{\mathsf{D}'} Y$$

for each D. Hence, by Lemma 2.27, we get a continuous map $g : \partial_\star X \to \partial_\star Y$. Applying the same argument above to f^{-1} yields a continuous map $g' : \partial_\star Y \to \partial_\star X$ with

$$g \circ g' = id_{\partial_\star X} \text{ and } g' \circ g = id_{\partial_\star Y},$$

which finishes the proof.

□

Regarding the homeomorphisms on Morse boundaries, the following is a characterization of when a homeomorphism comes from quasi-isometry:

Theorem 5.8 ([CCM19]). *Let X, Y be proper cocompact CAT(0) spaces with at least 3 points in their Morse boundaries. A homeomorphism $f : \partial_\star X \to \partial_\star Y$ is induced by a quasi-isometry $h : X \to Y$ if and only if f is is 2-stable and Morse quasi-möbius.*

5.3.1. *Proof of Theorem 5.4.*

Proof. ($\Rightarrow$) If h is a quasi-isometry, then $f := \partial h$ is stable by Lemma 5.6 . Also, f is Morse quasi-möbius by Theorem 5.8.

($\Leftarrow$) Using Lemma 5.7 any stable homeomorphism $f : \partial_\kappa X \to \partial_\kappa Y$ induces a homeomorphism $g : \partial_\star X \to \partial_\star Y$ on their Morse boundaries, with $g(x) = f(x)$ for all $x \in \partial_\star X$. Since f is Morse quasi-möbius and $g(x) = f(x)$ for $x \in \partial_\star X$, Theorem 5.8 implies the existence of a quasi-isometry $h : X \to Y$ such that $\partial h = g : \partial_\star X \to \partial_\star Y$. We wish to show that the induced map

$$\partial_\kappa h : \partial_\kappa X \to \partial_\kappa Y$$

agrees with f. Notice that as a set $\partial_\star X = \partial_1 X$, where $\partial_1 X$ is the subset of $\partial_\kappa X$ consisting of equivalence classes having a strongly contracting representative. Hence, we have $\partial_\kappa h(x) = \partial h(x)$ for all $x \in \partial_1 X \subseteq \partial_\kappa X$. Now, since $\partial h = g$, and $g(x) = f(x)$ on $\partial_1 X$, we get that

$$\partial_\kappa h(x) = \partial h(x) = g(x) = f(x)$$

for all $x \in \partial_1 X$. Therefore, $\partial_\kappa h(x) = f(x)$ for all $x \in \partial_1 X \subseteq \partial_\kappa X$. It remains to show that $\partial_\kappa h(x') = f(x')$ for all $x' \in \partial_\kappa X$. Let $x' \in \partial_\kappa X$, by Theorem 3.3, there exists a sequence $x_n \in \partial_1 X$ that converges to x'

$$x_n \to x'$$

in $\partial_\kappa X$. Since f is continuous on ∂X_κ, we have convergence

$$f(x_n) = \partial_\kappa h(x_n) \to f(x').$$

Also, since $\partial_\kappa h$ is continuous on $\partial_\kappa X$, we get that

$$\partial_\kappa h(x_n) \to \partial_\kappa h(x').$$

As $\partial_\kappa Y$ is Hausdorff, we obtain $\partial_\kappa h(x') = f(x')$. □

We remark that this theorem is far from satisfying, since Morse quasi-möbius requires one to check the quasi-möbius condition for every D, it is a much stronger condition than quasi-möbius.

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