Convexity Properties in the Outer Space: Balls are Weakly Convex

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Motivation

Question: Are balls in CV_n convex ?

Results:

- Out-going balls are weakly convex.
- In-coming balls in general are not convex.

Theorem

Theorem 1. *Given points $x, y \in \mathrm{CV}_n$, there exists a geodesic $[x,y]_{\mathrm{bf}}$ from x to y so that, for every loop α, and every time t,*

$$|\alpha|_t \leq \max\big(|\alpha|_x, |\alpha|_y\big).$$

Theorem 2. *Given a point $x \in \mathrm{CV}_n$, a radius $R > 0$ and points $y, z \in B_{\mathrm{out}}(x,R)$,*

$$[y,z]_{\mathrm{bf}} \subset B_{\mathrm{out}}(x,R).$$

where

$$B_{\mathrm{out}}(x,R) = \big\{y \in \mathrm{CV}_n \,\big|\, d(x,y) \leq R\big\}.$$

That is, the ball $B_{\mathrm{out}}(x,R)$ is weakly convex.

Set-up

$\mathrm{Out}(\mathbb{F}_n)$: the outer automorphism group of $\mathbb{F}_n$.

Outer Space CV_n: the space of all *marked metric graphs* of total length 1.

Lipschitz metric: let $x, y \in \mathrm{CV}_n$. A map $\phi\colon x \to y$ is a *difference of markings* map if $\phi \circ f_x \simeq f_y$. We only consider Lipschitz maps and we denote by L_ϕ the Lipschitz constant of ϕ. The Lipschitz metric on CV_n is defined to be:

$$d(x,y) := \inf_{\phi} \log L_\phi$$

where the infimum is taken over all differences of markings maps. Equivalently:

$$d(x,y) = \sup_{\alpha} \log \frac{|\alpha|_y}{|\alpha|_x}, \tag{0.1}$$

where α is an immersed loop, or Equivalently a conjugacy class in $\mathbb{F}_n$.

A geodesic in CV_n is a map $\gamma\colon [a,b] \to \mathrm{CV}_n$ so that, for $a \leq t \leq b$, we have

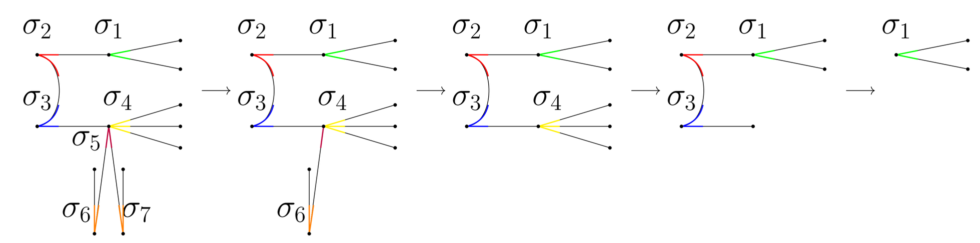
$$d\big(x, \gamma(t)\big) + d\big(\gamma(t), y\big) = d(x,y).$$

The difference of marking map $\phi\colon x \to y$ defines a gate structure on x. *Folding paths* with respect to the gate structure yields a large class of geodesics.

Key Question

How much length loss does each sub-gate account for?

Answer:



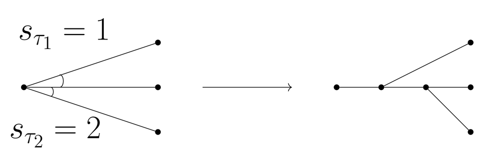
Combinatorial length loss: $c(\sigma_6,p) = 1, c(\sigma_7,p) = 1, c(\sigma_5,p) = 0, c(\sigma_4,p) = 3-1 = 2.$
$c(\sigma_2,p) = c(\sigma_3,p) = \frac{1}{2}, c(\sigma_1,p) = 1.$
$\sum c(\sigma,p) = 1+1+0+2+\frac{1}{2}+\frac{1}{2}+1 = 6 = |\operatorname{Pre}(p)| - 1$

Balanced Folding Path

Fold each sub-gate with respect to its contribution to the lengths loss at the destination point.

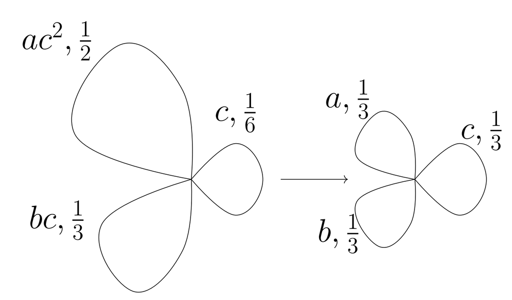
- Metric lengths loss: $\ell_\sigma = \int_{T_{\bar{y}}} c(\sigma,p)\, dp$
- Equivariant speed assignment: $s_\tau = \sum_{\hat{\tau} \supseteq \tau} \frac{\ell_{\hat{\tau}}}{|\hat{\tau}|-1}$

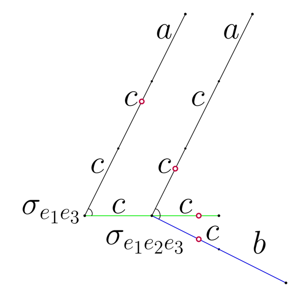
Example:



Example

$\mathbb{F} = \langle a, b, c\rangle$. The labels of edges of the rose indicates the associated difference of markings map.





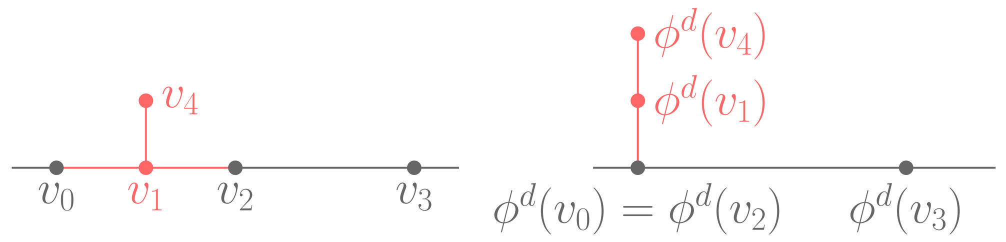
$$s_{e_1e_3} = l_{e_1e_3} + \frac{1}{2}\ell_{e_1e_2e_3} = \frac{1}{6} + \frac{1}{2} \times \frac{1}{3} = \frac{1}{3}$$

$$s_{e_1e_2} = s_{e_2e_3} = \frac{1}{2}\ell_{e_1e_2e_3} = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

That is to say, since ac^2 wraps over c twice while bc wraps over c once, infinitesimally, the folding associated to the former is twice as fast.

Decorated Difference of Markings Map

Modify the graphs x and y such that the tension graph after modification is all of x.

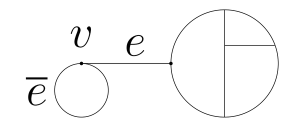


Obstructions to stronger properties

- Lengths cannot be made convex.
- Liberal folding paths do not stay in the ball.
$$y, z \in B_{\mathrm{out}}(x,2) \qquad \text{and} \qquad [y,z]_{\mathrm{ng}} \not\subset B_{\mathrm{out}}(x,R).$$
- Standard geodesics do not stay in the ball or the quasi-ball.
$$y, z \in B_{\mathrm{out}}(x,R) \qquad \text{and} \qquad [y,z]_{\mathrm{std}} \not\subset B_{\mathrm{out}}(x,2R-c).$$
- Greedy folding paths do not stay in the ball.
$$y, z \in B_{\mathrm{out}}(x,R) \qquad \text{but} \qquad [y,z]_{\mathrm{gf}} \not\subset B_{\mathrm{out}}(x,R).$$

Uniqueness of Geodesics

Theorem 3. *For points $x, y \in \mathrm{CV}_n$, the geodesic from x and y is unique if and only if there exists a rigid folding path connecting x to y.*



In-coming Balls

In-coming balls are:

$$B_{\mathrm{in}}(x,R) = \big\{y \in \mathrm{CV}_n \,\big|\, d(y,x) \leq R\big\}.$$

Theorem 4. *For any constant $R > 0$, there are points $x, y, z \in \mathrm{CV}_n$ such that, $y, z \in B_{\mathrm{in}}(x,2)$ but, for any geodesic $[y,z]$ connecting y to z,*

$$[y,z] \not\subset B_{\mathrm{in}}(x,R).$$