Subspace intersection graphs

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A R T I C L E   I N F O

Article history:
Received 15 August 2006
Received in revised form 24 June 2010
Accepted 25 June 2010
Available online 16 September 2010

Keywords:
Intersection graphs
Affine dimension

A B S T R A C T

Given a set $R$ of affine subspaces in $\mathbb{R}^d$ of dimension $e$, its intersection graph $G$ has a vertex
each subspace intersects. For each pair of positive integers $d$ and $e$ we obtain the class of $(d, e)$-
subspace intersection graphs. We classify the classes of $(d, e)$-subspace intersection graphs by containment,
either $e = 1$ or $e = d − 1$ or $d ≤ 4$.

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1. Introduction and preliminary results

Given a family of sets $S_1, S_2, \ldots, S_n$, one can form the intersection graph $G$ of the family by associating a vertex of $G$ with
each set and joining two distinct vertices with an edge if their corresponding sets have a nonempty intersection. Conversely,
finite graph can be viewed as the intersection graph of a family of sets in many different ways. If each set is a line segment
in the plane, then $G$ is called a segment intersection graph [1]. The most interesting problem for segment intersection graphs
has been Scheinerman’s conjecture, which asks whether all planar graphs are segment intersection graphs [2]. Hartman
et al. [5] and deCastro et al. [4] answered this question in the affirmative in the cases of bipartite and triangle-free graphs,
respectively. Recently Chalopin and Gonçalves have announced a proof that all planar graphs are segment intersection
graphs [3].

If each set $S_j$ is an affine subspace of $k^d$ for some field $k$, then $G$ has an affine representation in $k^d$. For a given $k$, the smallest
d for which $G$ has an affine representation is the affine dimension of $G$. If each set is a subspace of $k^d$, and two vertices are
adjacent if and only if their corresponding subspaces have a nontrivial intersection, then $G$ has a projective representation
in $k^d$ and the smallest such $d$ is the projective dimension of $G$ [6,7]. Pudlák and Rödl investigated the affine dimension and
projective dimension of bipartite graphs arising from Boolean functions, and gave asymptotic bounds on these dimensions. They
left the explicit construction of a graph with large affine or projective dimension as an open problem [6].

In this paper we consider graphs representable by $e$-dimensional affine subspaces of $\mathbb{R}^d$, where $e < d$. Though similar to
both segment intersection graphs and graphs with an affine representation, these graphs have not been previously studied.

Formally, we say that a graph $G$ is a $(d, e)$-subspace intersection graph or $(d, e)$-SI graph if there exists a set of $e$-dimensional
affine subspaces $R$ in $\mathbb{R}^d$ and a one-to-one correspondence between vertices in $G$ and subspaces in $R$, such that two vertices
$v$ and $w$ in $G$ are adjacent if and only if their corresponding subspaces intersect. Note that since $R$ is a set, the subspaces
are required to be distinct. For a given graph $G$, if such a set of subspaces $R$ exists, $G$ is called a $(d, e)$-subspace intersection
representation or $(d, e)$-SI representation of $G$. For ease of reference, if $G$ is a $(d, e)$-SI graph with $(d, e)$-SI representation $R$,
we denote the vertices of $G$ using lower-case letters, and the corresponding subspaces in $R$ using upper-case letters. For example, if $a$ and $b$ are vertices of $G$, then we denote their corresponding subspaces by $A$ and $B$. 
In this paper, we seek to order the classes of \((d, e)\)-SI graphs by set containment. Fig. 2 shows a partial order which summarizes our results. In this figure, edges represent set containment, and the graphs labeling the edges are separating examples.

**Lemma 1.** Every \((d, e)\)-SI graph is a \((d + k, e + j)\)-SI graph, for all \(k \geq j \geq 0\).

**Proof.** Given a \((d, e)\)-SI representation of a graph \(G\), we can consider it in \((d + k)\)-space, and use \(j\) of these new dimensions to increase the dimension of each of the affine subspaces to \((e + j)\). \(\square\)

Recall that a complete multipartite graph is a graph whose vertices can be partitioned into sets so that \(u \sim v\) if and only if \(u\) and \(v\) belong to different sets of the partition.

**Proposition 2.** A graph is a \((d, d - 1)\)-SI graph if and only if it is a complete multipartite graph, for all \(d \geq 2\).

**Proof.** If \(G\) is a \((d, d - 1)\)-SI graph, then \(G\) has a representation with \((d - 1)\)-dimensional hyperplanes in \(d\)-space. Note that two hyperplanes are disjoint if and only if they are not parallel. Hence \(G\) is a complete multipartite graph, with sets of vertices represented by parallel classes of hyperplanes forming the partite sets. Similarly, given a complete multipartite graph \(G\), we can form a \((d, d - 1)\)-SI representation of \(G\) by taking parallel classes of hyperplanes in \(d\)-space. \(\square\)

**Corollary 3.** A graph is a \((d, d - 1)\)-SI graph if and only if it is a \((2, 1)\)-SI graph. \(\square\)

Let \(G_1\) be the graph with vertices \(a, b,\) and \(c,\) and the single edge \([a, b]\). Note that \(G_1\) cannot be represented as the intersection graph of lines in the plane, since line \(C\) cannot be parallel with both line \(A\) and line \(B\). However, \(G_1\) can be easily represented as the intersection graph of lines in \(3\)-space. Hence we have the following proposition.

**Proposition 4.** The graph \(G_1\) is a \((3, 1)\)-SI graph but not a \((2, 1)\)-SI graph.

2. Main results

Let \(G_2\) be the graph formed by adding a single edge to the complete bipartite graph \(K_{3,3}\). We label the vertices of \(G_2\) as shown in Fig. 1.

**Theorem 5.** The graph \(G_2\) is a \((4, 2)\)-SI graph but not a \((d, 1)\)-SI graph for any \(d \geq 2\).

**Proof.** We present a \((4, 2)\)-SI representation of \(G_2\) by giving equations for the six planes in the representation. Using the convention that the four coordinates of \(\mathbb{R}^d\) are \(x, y, z,\) and \(w,\) define these six planes to be a \(A : z = 1, w = 0, B : z = 2, w = 0, C : x = 1, y = z, J : x = 1, y = −z,\) and \(K : x = 0, w = 0.\)

Now we prove that \(G_2\) is not a \((d, 1)\)-SI graph for any \(d\). Suppose by way of contradiction that \(R\) is a \((d, 1)\)-SI representation of \(G\) with lines \(A, B, C, I, J,\) and \(K\). Since \(I\) and \(J\) intersect, they determine a plane \(P\). Since \(A, B,\) and \(C\) are mutually disjoint, at most one of them contains the intersection of \(I\) and \(J\). Hence two of them (without loss of generality \(A\) and \(B\)) are both also contained in \(P\). So \(A\) and \(B\) are parallel. Since \(K\) intersects both \(A\) and \(B\), \(K\) is also in \(P\). Since the induced subgraph on the vertices \(i, j,\) and \(k\) is isomorphic to \(G_1\), this contradicts Proposition 4. Therefore \(G_2\) is not a \((d, 1)\)-SI graph for any \(d \geq 2\). \(\square\)

**Lemma 6.** A graph \(G\) is a \((d, e)\)-SI graph if and only if it is a \((d - 1, e - 1)\)-SI graph for all \(d \geq 2e + 1\).

**Proof.** By Lemma 1, every \((d - 1, e)\)-SI graph is also a \((d, e)\)-SI graph. Conversely, suppose that \(G\) is a \((d, e)\)-SI graph with \(n\) vertices, and \(R\) is a \((d, e)\)-SI representation of \(G\) consisting of the set of affine subspaces \(S_1, \ldots, S_n\) in \(\mathbb{R}^d\). We construct a \((d - 1, e - 1)\)-SI representation \(R'\) of \(G\) by projecting \(R\) onto a \((d - 1)\)-dimensional subspace \(V\) of \(\mathbb{R}^d\). We prove that we may choose \(V\) so that it has the following properties:

1. \(\dim(\text{proj}_V(S_i)) = \dim(S_i)\) for all \(1 \leq i \leq n\).
2. For any two subspaces \(S_i\) and \(S_j\) in \(R,\) \(\text{proj}_V(S_i)\) and \(\text{proj}_V(S_j)\) intersect if and only if \(S_i\) and \(S_j\) intersect.

If we have found such a subspace \(V,\) then the new set of subspaces \(\{\text{proj}_V(S_i) \mid S_i \in R\}\) will be a \((d - 1, e - 1)\)-SI representation of \(G,\) and the proof will be complete.

Recall that, for a subspace \(V\) in \(\mathbb{R}^d,\) the orthogonal complement \(V^\perp\) is the subspace \(\{v \in \mathbb{R}^d \mid v \perp V\}\). Every \((d - 1)\)-dimensional subspace \(V\) in \(\mathbb{R}^d\) corresponds to a unique line through the origin \(V^\perp\). We say that a projection onto \(V\) is a projection along \(V^\perp\). Also, given an affine subspace \(S_i\) in \(R,\) let \(S'_i\) be the unique subspace in \(\mathbb{R}^d\) parallel with \(S_i\) and passing through the origin. Note that \(\dim(\text{proj}_V(S_i)) = \dim(S_i)\) if and only if \(V^\perp\) is not contained in \(S'_i\).
For two affine subspaces $S_i$ and $S_j$ in $R$, let $p_i$ and $p_j$ be points in $S_i$ and $S_j$, respectively. Given a $(d - 1)$-dimensional subspace $V$ of $R^d$, the projection onto $V$ maps $p_i$ and $p_j$ onto the same point in $V$ if and only if the projection is along the line spanned by the vector $p_i - p_j$. Given a pair of non-intersecting affine subspaces $S_i$ and $S_j$, consider the span of all such vectors, $A_{ij}$. Note that $A_{ij}$ is also the span of all vectors in $S'_i$ and $S'_j$ and a single vector of the form $p_i - p_j$. Since $\dim(S_i) = \dim(S_j) = e$, $\dim(A_{ij}) \leq 2e + 1$.

Since $R$ has a finite number of subspaces, and $d > 2e + 1$ by hypothesis, the union $\delta$ of all the sets $A_{ij}$ and all the sets $S'_i$ cannot be all of $R^d$. So there exists some other line $l$ through the origin, not contained in $\delta$. Therefore the $(d - 1)$-dimensional subspace $l^1$ is the required subspace. $\Box$

The following theorem follows directly from Lemma 6.

**Theorem 7.** A graph $G$ is a $(d, e)$-$SI$ graph if and only if it is a $(2e + 1, e)$-$SI$ graph for all $d > 2e + 1$. In particular, $G$ is a $(d, 1)$-$SI$ graph if and only if it is a $(3, 1)$-$SI$ graph for all $d \geq 3$. $\Box$

We conjecture that Theorem 7 can be strengthened as follows.

**Conjecture 1.** A graph $G$ is an $(e + k, e)$-$SI$ graph if and only if $G$ is an $(e + 2, e)$-$SI$ graph, for $k \geq 2$. In particular, we conjecture that if $G$ is a $(5, 2)$-$SI$ graph then $G$ is a $(4, 2)$-$SI$ graph.

**Theorem 8.** Given a finite graph $G$, there exist positive integers $d$ and $e$ such that $G$ is a $(d, e)$-$SI$ graph.

**Proof.** Suppose $G$ is a finite graph with vertices $v_1, \ldots, v_n$. Let $E = \{e_1, \ldots, e_k\}$ be the set of all unordered pairs of vertices of $G$ (so $k = \binom{n}{2}$). Note that $E(G) \subseteq E$.

We define the affine subspace $V_i \subseteq R^k$ in the following way. Let $P_i$ be the set of positive integers $p$ such that $e_p$ contains $v_i$ and is not an edge of $G$. Then $V_i = \{x \mid x$ has an $i$ in coordinate $p$ for all $p \in P_i\}$. We modify the subspaces $V_i$ so that they all have the same dimension. So let $D$ be the largest dimension of any $V_i$, and $d$ be the smallest dimension of any $V_i$. For each $1 \leq i \leq n$ let $U_i$ be any affine subspace of $R^{D-d}$ of dimension $D - \dim(V_i)$, and let $V'_i = V_i \times U_i$. We claim that the set $E = \{V'_i\}$ is a $(k + D - d, D)$-$SI$ representation of $G$.

We must prove that two vertices $v_a$ and $v_b$ of $G$ are adjacent if and only if their corresponding subspaces $V'_a$ and $V'_b$ intersect. So first, suppose that $v_a \neq v_b$ in $G$, and $e_p = \{v_a, v_b\}$ in $E$. Then the $p$th coordinate of every point in $V'_a$ is $a$ and the $p$th coordinate of every point in $V'_b$ is $b$. Thus $V'_a \cap V'_b = \emptyset$.

On the other hand, suppose that $v_a \sim v_b$ in $G$, and $p$ is a coordinate in which every point in $V'_a$ has the same value $x$. Note that $x$ must be either $a$ or $0$ by the definition of $V'_a$. If $x = a$, then $e_p$ is a non-edge containing $v_a$. Since $v_a \sim v_b$, $e_p$ does not contain $v_b$, and so points in $V'_a$ can take on any value in their $p$th coordinate. If $x = 0$, then $p > k$, and so points in $V'_b$ can take on 0 as their $p$th coordinate. Hence we can find a point in both $V'_a$ and $V'_b$, so $V'_a \cap V'_b \neq \emptyset$. $\Box$

It is likely that the $(d, e)$-$SI$ representation of $G$ constructed in the proof of Theorem 8 is not the smallest such representation with respect to either $d$ or $e$.

By Corollary 3, the classes of $(2, 1)$-$SI$ graphs, $(3, 2)$-$SI$ graphs, etc., are all the same class of graphs, and by Lemma 1, this class of graphs is contained in all other classes of $(d, e)$-$SI$ graphs. Similarly by Theorem 7, the classes of $(3, 1)$-$SI$ graphs, $(4, 1)$-$SI$ graphs, etc., are all the same class of graphs, and by Lemma 1, this class of graphs is contained in all classes of graphs.
(d, e)-SI graphs other than the (2, 1)-SI graphs. Furthermore, if Conjecture 1 is true, then the class of (2, 1)-SI graphs and the classes of (e + 2, e)-SI graphs are all of the distinct classes of (d, e)-SI graphs. Fig. 2 shows the first few classes of this hierarchy, which summarizes the theorems in this paper.

Recall that the affine dimension of  \( G \) is the smallest value of \( d \) such that \( G \) is representable as the intersection graph of affine subspaces of \( d \)-space. Since there is only one choice for \( e \) if \( d = 2 \), the graphs with affine dimension 2 are precisely the complete multipartite graphs by Proposition 2. Pudlák and Rödl ask for explicit constructions of graphs with large affine dimension [6]. We have been unable to find a graph which is not a (4, 2)-SI graph, so we do not know a graph with affine dimension greater than 4.

**Open Question 1.** Find a graph which is not a (4, 2)-SI graph. More generally, find a graph with large affine dimension.

Acknowledgements

The authors thank Wojciech Kosek, Paul Humke, and Colin Starr for helpful conversations, and the reviewers for many helpful suggestions.

References


