

# Uniform Fellow Travelling Between Surgery Paths in the Sphere Graph

Yulan Qing

joint work with Matt Clay and Kasra Rafi

*yulan.qing@utoronto.ca*

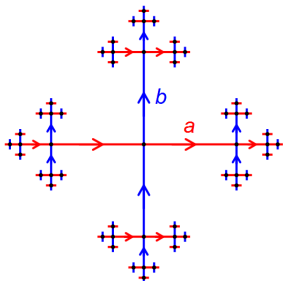
March 2017

**Group:**  $Out(\mathbb{F}_n)$

Let  $\mathbb{F}_n = \langle a_1, a_2, \dots, a_n \rangle$  be the free group of rank  $n$ ,

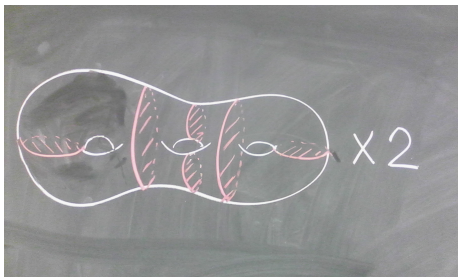
$$Out(\mathbb{F}_n) = Aut(\mathbb{F}_n)/Inn(\mathbb{F}_n)$$

- Finitely presented
- Maps to  $GL_n(\mathbb{Z})$
- Contains mapping class groups of punctured surfaces



**Space:** sphere complex

Let  $\underline{M}$  be the connected sum of  $n$  copies of  $S^1 \times S^2$  and fix an identification of  $\pi_1(\underline{M})$  with  $\mathbb{F}_n$ . A *sphere system*  $\underline{S} \subset \underline{M}$  is a finite union of disjoint essential (does not bound a 3-ball) embedded 2-spheres in  $\underline{M}$ . We specifically allow for the possibility that a sphere system contains parallel, i.e., isotopic, spheres. A sphere system is *filling* if the complementary regions  $\underline{M} - \underline{S}$  are simply-connected. A sphere complex is the simplicial complex of all such systems.



**Question:** Is the sphere complex uniformly hyperbolic?

**Answer:** we dont know.

**What we know:** the sphere complex is  $\delta$ -hyperbolic.(Handel-Mosher, Horbez-Hillion)

**What we can show:** Given any two points in the sphere complex, the forward and backward paths of surgery sequences have bounded Hausdorff distance.

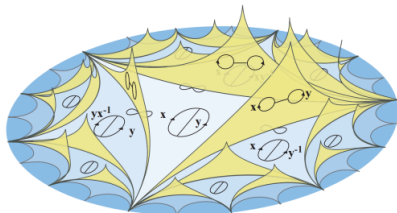


Figure 2: Outer space in rank 2.

**Figure:** Karen Vogtmann

## Theorem (Main Theorem)

Let  $\underline{S}$  and  $\underline{\Sigma}$  be two sphere systems and let

$$\underline{S} = \underline{S}_1, \underline{S}_2, \dots, \underline{S}_m = \underline{\Sigma}$$

be a surgery sequence starting from  $\underline{S}$  towards  $\underline{\Sigma}$  and

$$\underline{\Sigma} = \underline{\Sigma}_1, \underline{\Sigma}_2, \dots, \underline{\Sigma}_\mu = \underline{S}$$

be a surgery sequence in the opposite direction. Then, the Hausdorff distance between these paths as subsets of the sphere graph or free splitting graph is at most 2.

Why are we interested in uniform hyperbolicity?

- understanding geometry governing the family of spaces  $CV_n$ , for all  $n$
- group algorithm constants

What other spaces are uniformly hyperbolic?

- Curve complex
- Arc complex



What could fellow travel imply?

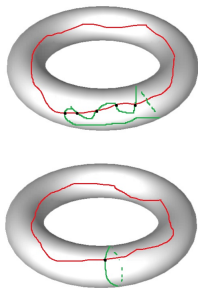
- In order to combine with *thin bigon condition*[Papasoglu, Pomroy] to obtain uniform hyperbolicity, we need to control all quasi-geodesics, and surgery paths are only a subset of quasi-geodesics.

### Theorem (Pomroy)

Let  $(X, d)$  be a geodesic metric space. If there is  $\rho, \epsilon$  so that  $\rho$ -bigons are uniformly  $\epsilon$ -thin, then  $X$  is hyperbolic.

To study the splitting complex we need to learn to track the relative position of two points in the complex:

- edge metric
- intersection number
- Guirardel Core: geometric model for intersection



**Figure:** Yen Duong

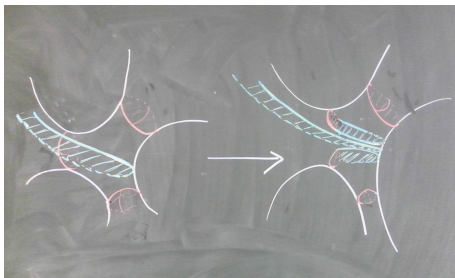


# What is the Guirardel Core?

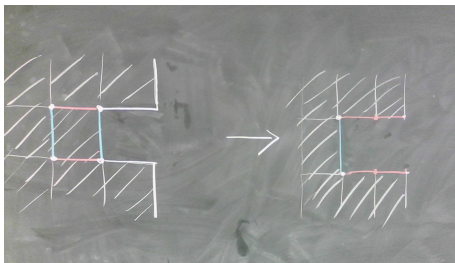
Suppose that  $S$  and  $\Sigma$  are filling sphere systems in normal form. The *core* of  $S$  and  $\Sigma$ , denoted  $Core(S, \Sigma)$ , is the square complex defined as follows.

- Vertices: components of  $M - (S \cup \Sigma)$ . Each such region corresponds to an intersection of a component  $P \subset M - S$  and a component  $\Pi \subset M - \Sigma$ .
- Edges: There is an edge between two vertices when the closures of the corresponding components of  $M - (S \cup \Sigma)$  have non-trivial intersection.
- Suppose that  $s \in S$  and  $\sigma \in \Sigma$  have non-empty intersection. Let  $P_1, P_2 \subset M - S$  be the components whose boundary contains  $s$  and let  $\Pi_1, \Pi_2 \subset M - \Sigma$  be the components whose boundary contains  $\sigma$ . Then four edges  $(s, \Pi_1)$ ,  $(P_1, \sigma)$ ,  $(s, \Pi_2)$  and  $(P_2, \sigma)$  form the boundary of a square with vertices  $(P_1, \Pi_1)$ ,  $(P_2, \Pi_1)$ ,  $(P_2, \Pi_2)$  and  $(P_1, \Pi_2)$  which is then filled in.

# Surgery Paths



**Figure:** A surgery move



## Lemma

*There exists a naturally defined non-surjective square map*

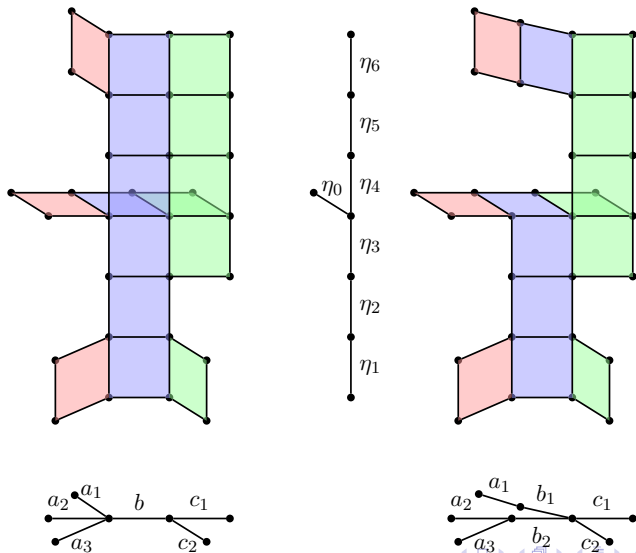
$$k_{i,i+1} : \text{Core}(S_{i+1}, \Sigma) \rightarrow \text{Core}(S_i, \Sigma).$$

## Lemma

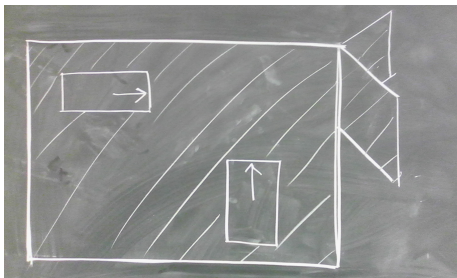
*If  $k_i(\text{Core}(S_i, \Sigma)) \cup \kappa_j(\text{Core}(S, \Sigma_j)) = \text{Core}(S, \Sigma)$ , then  $S_i$  and  $\Sigma_j$  are in normal form. In particular there exists a square isomorphism*

$$\Phi : \text{Core}(S_i, \Sigma_j) \rightarrow k_i(\text{Core}(S_i, \Sigma)) \cap \kappa_j(\text{Core}(S, \Sigma_j))$$

These two lemmas allow us to see the core of a pair of points along the forward and backward surgery paths *embedded* in  $Core(S, \Sigma)$  and observe how the embedding changes after each surgery.



Along a forward surgery path we perform Rips-like moves horizontally, while along a backward surgery path we perform a Rips-like moves vertically. Before the boundary pieces touch,  $\text{Core}(S_i, \Sigma_j)$  embeds in  $\text{Core}(S, \Sigma)$ ; If the boundary pieces touch, both  $S_i$  and  $\Sigma_j$  are both bounded distance to another sphere system.



Future questions.

- Are surgery triangles uniformly thin?
- Is there a normal form of three systems of spheres?
- Understanding of other geometric aspects of the core.
- How does other paths in the sphere complex change the core and what can we deduce from there?



Vincent Guirardel

Core and intersection number for group actions on trees  
*Ann. Sci. Ecole Norm. Sup. (4)*, 38 (2005), pp. 847888.



Jason Behrstock, Mladen Bestvina and Matt Clay

Growth of intersection numbers for free group automorphisms  
*Journal of Topology* 3 (2010), 280310.



Arnaud Hilion, Camille Horbez (2015)

The hyperbolicity of the sphere complex via surgery paths  
*Journal fr die reine und angewandte Mathematik*

Thank you!