Boundary of CAT(0) Groups With Right Angles

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This dissertation is dedicated to my mom, who would have been very proud.

Abstract

In [CK00], Croke and Kleiner present a torus complex whose universal cover has a non-locally connected visual boundary. They show that changing the intersection angle of the gluing loops in the middle torus changes the topological type of the visual boundary. In this thesis we study the effect on the topology of the boundary if the angle is fixed at $\pi/2$ but the lengths of the π_1-generating loops are changed. In particular, we investigate the topology of the set of geodesic rays with infinite itineraries. We identify specific infinite-itinerary geodesics whose corresponding subsets of the visual boundary change their topological type under the length change in the space. This is a constructive and explicit proof of a result contained in Croke and Kleiner's more general theorem in [CK02]. The construction and view point of this proof is crucial to proving the next result about Tits boundary: we show that the Tits boundaries of Croke Kleiner spaces are homeomorphic under lengths variation. Whether this is true for the general set of $CAT(0)$ 2-complexes is still open. We also study the invariant subsets of the set of geodesic rays with infinite itineraries.

In the next chapter, we consider the geometry of actions of right-angled Coxeter groups on the Croke-Kleiner space. We require the group act cocompactly, properly discontinuously and by isometries and determine that the resulting gluing angles of the loops must be $\frac{\pi}{2}$. Together with the first result we show that right-angled Coxeter groups does not have unique G-equivariant visual boundaries. This study aims to contribute to the investigation of whether right-angled Coxeter groups have unique boundaries.

Lastly, we begin the study of uniqueness of right-angled Coxeter groups that acts geometrically on a given CAT(0) space. We start the project by letting right-angled Coxeter groups at on a regular, infinite, 4-valence tree and provide a geometrical proof that if a right-angled Coxeter group acts geometrically on Tr_4, then it is an amalgamated product of finite copies of groups of the form $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. The last the result is the beginning of the project that aims to determine all the right-angled Coxeter groups that can act geometrically on the Croke-Kleiner space.

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Chapter 1

Introduction

1.1 History

Geometric group theory studies groups by observing and creating the connections between geometry and group theory. The leading idea in geometric group is: suppose a group G acts geometrically on a metric space X, and ask how do the geometric properties of X determine group theoretical properties of G, and conversely, how do the group theoretical properties of G determine the geometric properties of X? An interesting set of problems is to come up with suitable spaces for the groups to act upon. The first of these spaces to be considered are the Cayley graphs of groups. The Cayley graphs in this context are metric spaces with, in addition to the graph structure, the word metric. However, Cayley graphs of a group G with different generating sets have different local geometry. To resolve this issue there is the seminal concept of "quasi-isometry", which is meant to capture the large scale geometry of a metric space. Then the central problem in geometric group theory is to study and classify finitely generated groups according to quasi-isometry class. Therefore, any space or property that is associated with a group is hoped to be well-defined with respect to the whole quasi-isometric class of groups or spaces. The space we study in this document is the boundary of a group. We show that, with respect to a particular type of quasi-isometries, some $CAT(0)$ groups' boundary are not well-defined.

$CAT(0)$ spaces are metric spaces in which every pair of points can be joined by an arc isometric to a compact interval of the real line and in which every triangle satisfies the $CAT(0)$ inequality. This inequality captures the essence of non-positive curvature without requiring smoothness. Spaces which satisfy this condition display properties of both hyperbolic spaces as well as those of flat spaces.

Associated to any complete $CAT(0)$ space is a boundary at infinity ∂X, which can be constructed as the set of equivalence classes of geodesic rays in X. Two rays are equivalent if their images are a bounded Hausdorff distance apart. There is a natural topology on $X = X \cup \partial X$ called the *cone topology.* If X is a Riemannian manifold, ∂X is a sphere and $X \cup \partial X$ is X is homeomorphic to a closed ball, but for more general CAT(0) spaces the topology of ∂X can be rather complex. We use $\partial_\infty X$ to denote ∂X together with the cone topology.

What motivated the investigation of $CAT(0)$ groups is what is known about hyperbolic groups. A space is δ-hyperbolic if for all geodesic triangle in this space, any edge of the triangle lies in the δ-neighborhood of the other two edges. A group is *hyperbolic* if its Cayley graph is δ-hyperbolic. The visual boundary of a hyperbolic space is independent of the space's local geometry [Gro87]:

"If a finitely generated group G acts discretely, cocompactly and by isometries on two Gromov hyperbolic metric spaces X_1, X_2, then there is a G-equivariant homeomorphism $\partial_\infty X_1 \to \partial_\infty X_2$."

Gromov asked in [Gro93] whether this is still the case when the hyperbolicity assumption is dropped. In [CK00], Croke and Kleiner show that the answer is no. They give a construction of a family of $CAT(0)$ spaces $\{X_\alpha : 0 < \alpha \leqslant \frac{\pi}{2}\}$ each admitting a geometric action by the same group G. They showed that $\partial_\infty X_\alpha \neq \partial_\infty X_{\frac{\pi}{2}}$ for any $0 < \alpha < \frac{\pi}{2}$. We will call their construction the *Croke-Kleiner Space (α)*, where α is an angle in the construction that preserves the quasi-isometry type of the space. In [Wil05], Wilson shows that in fact $\partial_\infty X_\alpha \neq \partial_\infty X_\beta$ for all $\alpha \neq \beta$, so that G is a $CAT(0)$ group with uncountably many non-homeomorphic visual boundaries.

So far the focus of the geometric changes that impact the topology of the boundary concentrate on the angle changes. We are motivated by a different question. We want to know if we change the geometry in a different way, will the spaces be homeomorphic under the equivariant map, or if not, are they homeomorphic as topological spaces. We start with the original space X with gluing angle $\pi/2$ and instead vary the *lengths* of the gluing loops. This can be viewed as taking the space X - which is a $CAT(0)$ cube complex when the angle is $\pi/2$ - and changing the lengths of the sides of the squares that form a fundamental domain for the action of G. The resulting space, denoted X_l, has a rectangular structure instead of a cubical structure. Since both X_l and X admit a geometric group action by G, they are quasi-isometric as metric spaces. The following are natural questions to ask:

Question 1: Is $\partial_\infty X$ G-equivariantly homeomorphic to $\partial_\infty X_l$?

Question 2: If the answer to Question 1 is negative, then is the core of $\partial_\infty X$ homeomorphic to the core of $\partial_\infty X_l$? What about their Tits boundaries?

Question 3: If the answer to Question 1 is negative, then is $\partial_\infty X$ homeomorphic to $\partial_\infty X_l$ as topological spaces?

Independently, we want to study further the Tits boundary of the space. The Tits metric on the boundary induces a very different topology called the Tits topology, with the same underlying point set. Tits boundaries of δ-hyperbolic spaces are of little interest because they are all discrete sets regardless of the geometric dimension of the space. Since $CAT(0)$ spaces can contain hyperbolic sections as well as flat sections, its Tits boundary is more interesting topologically and group theoretically. In [Xie05], Xie shows that away from the endpoints, a geodesic segment in the Tits boundary is the visual boundary of an isometrically embedded Euclidean sector. Xie also shows that if two $CAT(0)$ 2-complexes are quasi-isometric, then the cores of their Tits boundaries are bi-Lipschitz. It is open whether the whole Tits boundaries are homeomorphic under quasi-isometry. In this document we prove that for the Croke-Kleiner example there is a homeomorphism between the cores in the Tits boundaries and in fact the whole Tits boundaries are homeomorphic as well.

Another open question that motivated part of this thesis is the boundary of a right-angled Coxeter group. It is unknown whether right-angled Coxeter groups have unique boundaries. We began an investigation of whether the Croke-Kleiner examples could be used as counter-examples for this class of CAT(0) groups. We show that this question is related to the larger question of whether a lengths change changes the homeomorphism type of the visual boundary of the space. O'Brien [O08] studies carefully in his thesis the existence and uniqueness of strict fundamental domains of right-angled Coxeter group acting on a $CAT(0)$ space. The work in the final chapter of this document is largely applications of his main results.

1.2 Right-angled Artin and Coxeter Groups

The fundamental group of the Croke-Kleiner construction is an example of a right-angled Artin group G. Therefore it is our goal to understand how the actions of this group G on these different $CAT(0)$ geometries interact with the topology of the ideal boundaries of these spaces. Our main theorem shows that certain length variations preserve the core components of the visual boundary and Tits boundary. However, if we change the translation distance of the actions, the G-equivariant map between the corresponding ideal boundaries does not extend to a homeomorphism. This latter result is proved by explicitly identifying a dense subset of components in the "irrational" part of the boundary we call the *dust*, whose homeomorphism type changes. We borrow terminology and ideas from Croke and

Kleiner [CK02]. In that paper, they study the question in a much more generalized setting of fundamental groups of 3-dimensional graph manifolds, of which of our result is a specific case. However, our proof and view point here is crucial to proving Theorem 1.4 about the Tits boundary.

Theorem 1.1. *On the components of ∂X other than the safe-path component, the G-equivariant action does not extend to a homeomorphism under the visual topology between ∂X and ∂X_l, where X_l has a nontrivial different set of length data.*

We explicitly identify a subset of the boundary whose homeomorphism type is changed under the $G-$equivariant homeomorphism. We also describes all length data changes that can be detected by these subset of the boundaries. These sets are termed *$n-$self-similar templates*, by Croke and Kleiner.

The next natural question is to decide whether there is any homeomorphism between the two boundaries. If the answer is affirmative, then we have a case of two spaces whose boundaries are not G-equivariantly homeomorphic via the natural extension, but are homeomorphic nonetheless, i.e. the group that acts geometrically on the spaces does not prescribe the homeomorphism type of their boundaries. If the answer is negative, we can see that the G-equivariant homeomorphisms indeed capture the topology of the spaces sufficiently.

Meanwhile we take a look at the core component of the visual boundary prove the following:

Theorem 1.2. *Let X be the Croke-Kleiner space with length data (1,1,1,1), and let X_l be the Croke-Kleiner space with length data (a, b, c, d), where at least one of a, b, c, d is not 1. There is an isometry between Tits boundaries:*

$$Core(\partial_T X) \sim Core(\partial_T X_l)$$

Theorem 1.3. *Let X be the Croke-Kleiner space with length data (1,1,1,1), and let X_l be the Croke-Kleiner space with length data (a, b, c, d), where at least one of a, b, c, d is not 1. The*

G-equivariant map between the $Core(\partial_\infty X)$ and $Core(\partial_\infty X_l)$ gives rise to homeomorphism.

It is worth noting that the G-equivariant map between the spaces X and X_l does not extend to a homeomorphism on the visual boundary, as pointed out to me by Professor Mooney and Professor Ancel. In Chapter 3 we include an example to show that the G-equivariant map between $Core(\partial_\infty X)$ and $Core(\partial_\infty X_l)$ is not by itself a homeomorphism.

In contrast to the conclusion we make about visual boundary, the study of the irrational rays allow us to prove the following new result:

Theorem 1.4. *Croke-Kleiner spaces of different lengths data have homeomorphic Tits boundaries.*

In contrast, for Right-angled Coxeter groups, it is still an open question as to whether a RACG can have more than one boundary up to homeomorphism. Thus it is natural to ask whether the Croke-Kleiner examples can be adjusted to give a RACG example with a non-unique boundary. There is an obvious RACG that acts geometrically on the three-tori space with $\frac{\pi}{2}$ angle, but we can show there is more than one group with that property. More illuminatingly, Theorem 0.2 states that a Right-angled Coxeter group can only act geometrically on spaces with translation distance variation. This result together with the result from RAAG motivates the conjecture that for a specific class of torus complexes, right-angled Coxeter groups act without unique boundary.

Right-angled Coxeter groups are generated by order-two elements whose two-element subsets either commute or have no relation at all. We assume here that a RACG acts geometrically on the three-torus example, and ask what we can conclude about the action and the group. The main results is the following:

Theorem 1.5. *Let G, an essential right-angled Coxeter group, acts geometrically on X, the universal cover of the three-torus complex, then the intersecting angle on the middle torus must be a right angle.*

1.3 Quasi-isometry Class of Right-angled Coxeter Groups

While working on the project described in the previous section, a question that arises naturally is: what are all the right-angled Coxeter groups that are quasi-isometric to the Croke-Kleiner space; in other words, how much of the geometry of the space determines the group. One typical theorem in this direction is "any group that is quasi-isometric to a tree is virtually free". Since Croke-Kleiner space is a concrete space and we add the assumption that the group should be a right-angled Coxeter group to begin with, we expect the resulting groups fit a narrow description. To begin answering the question, we start by asking:

Question 4: What are all the right-angled Coxeter groups that can act geometrically on an infinite, regular, 4-valence tree?
The main result is the following:

Theorem 1.6. *If a right-angled Coxeter group acts geometrically on an infinite, regular, 4-valence tree, then it is an amalgamated product of finite copies of groups of the form* $\mathbb{Z}_2$ *or* $\mathbb{Z}_2 \times \mathbb{Z}_2$.

The next step is to ask the same question about the space "$\mathscr{T} \times \mathbb{R}$". Since the boundary of "$tree \times \mathbb{R}$" contains exactly two points whose neighborhood is a cone over a Cantor set, which is distinguishable from other points in the boundary, we invoke the following theorem:

Theorem 1.7. *Suppose a group* G *acts geometrically on a CAT(0) space* X. *Then a group element* $g \in G$ *is virtually central if and only if the induced action of* g *on the visual boundary* $\partial_\infty X$ *is an identity action.*

At the end of chapter 4 we give an outline of our work in progress to answer this question.

1.4 Outline of the Thesis

Chapter 2 serves as a review of basics in geometric group theory that is relevant to this document. Readers can skip or reference this section as needed. Chapter 3 discusses the main results that pertain to the Croke-Kleiner space. Chapter 4 discusses the results about the local geometry when a right-angled Coxeter group acts on the Croke-Kleiner space. Chapter 5 starts a investigation of the isometry class of right-angled Coxeter groups of a given $CAT(0)$ space.

Chapter 2

Preliminaries

In this chapter , we collect all basic definitions and facts used in this thesis. This section loosely follows the notation of [BH99] and can be skipped by readers who are experienced with $CAT(0)$ geometry and $CAT(0)$ groups.

2.1 Group Actions

A group with the presentation $\langle S|R\rangle$ is defined to be the quotient of the free on on the generating set S by the normal closure of the set of relations R. A group is **finitely generated** if there exists a presentation where S is finite. In the rest of the thesis we are interested mainly in the following two classes of finitely generated groups:

Definition 2.1. An *Artin group* A is a group with presentation of the form

$$A = \langle s_1, s_2.., s_n|(s_i s_j)^{m_{ij}} = (s_j s_i)^{m_{ji}} \text{ for all } i \neq j\rangle$$

where $m_{ij} = m_{ji} \in \{2, 3, ...\infty\}$.
$(s_i s_j)^{m_{ij}}$ denotes an alternating product of s_i and s_j of length m^{ij}, beginning with s_i. If $m_{ij} = \infty$, then there is (by convention) no relation for s_i and s_i.
A **right-angled Artin group** [Cha07] is one in which $m_{ij} \in \{2, \infty\}$ for all i, j. In other

words, in the presentation for the Artin group, all relations are commutator relations:

$$s_i s_j = s_j s_i$$

The easiest way to specify a presentation for a right-angled Artin group is by means of a defining graph. This is a graph whose vertices are labeled by the generators $S = \{s_1, ..., s_n\}$ and whose edges connect pairs of vertices s_i, s_j if and only if $m_{ij} = 2$. Note that any finite, simplicial graph Γ is the defining graph for a right-angled Artin group.

Definition 2.2. Formally, a *Coxeter group* can be defined as a group with a presentation of the following form:

$$\{s_1, s_2, ...s_n | (s_i)^2 = 1, (s_i s_j)^{m_{ij}} = 1, \text{ where } m_{ij} \in \{2, 3, 4, ...\infty\}\}$$

A **right-angled** Coxeter Group [MRT07] is where $m_{ij} \in \{2, \infty\}$

Here are more basic facts of right-angled Coxeter Groups [GP08]:

- If s_i is not adjacent to s_j, then the order of $s_i s_j$ is infinite
- A right-angled Coxeter group is abelian if and only if it is finite.
- If w has finite order, then $w^2 = 1$. Right-angled Coxeter groups are distinguished from other Coxeter groups by this fact. That is, if every finite order element of a Coxeter group is two, then the Coxeter group is right-angled.
- A right-angled Coxeter group has a non-trivial center if and only if it can be written as $W \times \mathbb{Z}_2$

A **left group action** of G on a set X is a map $G \times X \to X$ such that:

1. $ex = x \; \forall x$
2. $g(hx) = (gh)x$

1. An action of a group G is *transitive* if for some $x \in X, G.x = X$.

2. An action of a group G is *faithful* if $\ker(\rho) = 1_G$.

3. An action of a group G is *free* if for every $x \in X$, stabilizer $(x) = 1_G$

Definition 2.3. An action of G on X is *cocompact* if X/G is compact in the topology induced by (1.3).

A group action is *properly discontinuous* if for every bounded open set B, the set

$$|\{g \in G | g.B \cap B = \phi\}|$$

is finite.

Definition 2.4. An *isometry* from one metric space (X, d) to anther (X', d') is a bijection

$$f : X \to X'$$

such that $d'(f(x), f(y)) = d(x, y)$ for all x, y. If such a map exists then (X, d) and (X', d') are *isometric.*

A group G acts *geometrically* on a metric space X if it acts properly discontinuously, cocompactly, and by isometries.

The group of all isometries from a metric space (X, d) to itself will be denoted *Isom(X).* If $G \subset Isom(X)$, then we say that G acts on X by isometries.

Definition 2.5. The *fixed point set* of $S \subset G$ on the space X is the set

$$\{x \in X | g.x = x \text{ for all } g \in S\}$$

2.2 CAT(0) Spaces and their Boundaries

Let X be a metric space. For any $x \in X$ and any $r > 0$, $B(x, r) = \{x' \in X : d(x, x') < 1\}$ and $\bar{B}(x, r) = \{x' \in X : d(x, x') \leqslant 1\}$ are respectively the open and closed metric balls with center x and radius r. For any subset $A \subset X$ and any $\epsilon > 0$, the ϵ-neighborhood of A is $N_\epsilon(A) = \{x \in X : d(x, a) \leqslant \epsilon$ for some $a \in A\}$. For any two subsets $A, B \subset X$, the Hausdorff distance between A and B is $d_H(A, B) = \inf\{\epsilon : A \subset N_\epsilon(B), B \subset N_\epsilon(A)\}$; $d_H(A, B)$ is defined to be ∞ if there is no $\epsilon > 0$ with $A \subset N_\epsilon(B)$ and $B \subset N_\epsilon(A)$.

A function is *proper* if the inverse image of every compact set is compact. A metric space X is said to be *proper* if the distance function is proper.

The **Euclidean cone** over a metric space X is the metric space $C(X)$ defined as follows. As a set $C(X) = X \times [0, \infty)/X \times \{0\}$. We use tx to denote the image for (x, t). We define

$$d(t_1x_1, t_2x_2) = \sqrt{t_1^2 + t_2^2 - 2t_1t_2cos(d(x_1, x_2))}$$

if $d(x_1, x_2) \leqslant \pi$ and

$$d(t_1x_1, t_2x_2) = t_1 + t_2$$

if $d(x_1, x_2) \geqslant \pi$. The point $O = X \times \{0\}$ is called the cone point of $C(X)$.

Let X be a metric space. A *geodesic* joining $x, y \in X$ is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$ and $d(c(t), c(t') = |t - t'|)$ for all $t, t' \in [0, l]$. (X, d) is a *geodesic metric space* if every two points are joined by a geodesic.

A *triangle* in a metric space X is the union of three geodesic segments where $c_1(b_1) = c_2(a_2), c_2(b_2) = c_3(a_3)$ and $c_3(b_3) = c_1(a_1)$. For any real number κ, let M_κ^2 stand for the 2-dimensional $c_i : [a_i, b_i] \to X(i = 1, 2, 3)$ simply connected complete Riemannian manifold with constant sectional curvature κ, and $D(\kappa)$ denote the diameter of $M_\kappa^2(D(\kappa) = \infty$ if $\kappa \leqslant 0)$. Given a triangle $\Delta = c_1 \bigcup c_2 \bigcup c_3$ in X where $c_i : [a_i, b_i] \to X(i = 1, 2, 3)$, a triangle Δ' in M_κ^2 is a *comparison triangle* for Δ if tehy have the same edge lengths, that is , if $\Delta' = c_1' \bigcup c_2' \bigcup c_3'$ and $c_i' : [a_i, b_i] \to X(i = 1, 2, 3)$. A point $x' \in \Delta'$ corresponds to a point $x \in \Delta$ if ther is some i and some $t_i \in [a_i, b_i]$ with $x' = c_i'(t_i)$ and $x = c_i(t_i)$. We notice if the perimeter of a triangle $\Delta = c_1 \bigcup c_2 \bigcup c_3$ in X is less than $2D(\kappa)$, that is, if length(c_1)+ length (c_2)+length$(c_3) < 2D(\kappa)$, then there is a unique comparison triangle (up to isometry) in M_κ^2 for Δ.

Definition 2.6. Let $\kappa \in \mathbb{R}$. A complete metric space X is called a CAT(κ) space if:

1. Every two points $x, y \in X$ with $d(x, y) < D(\kappa)$ are connected by a minimal geodesic segment;

2. For any triangle Δ in X with perimeter less than $2D(\kappa)$ and any two points $x, y \in \Delta$, the inequality $d(x, y) = d'(x', y')$ holds, where x' and y' are the points on a comparison triangle for Δ corresponding to x and y respectively.

Definition 2.7. A geodesic metric space X is a **CAT(0)** space if for all geodesic triangles $\triangle(p, q, r) \subset X$ and points $x \in [p, q]$, $y \in [p, r]$, $d(x, y) \leqslant d(\bar{x}, \bar{y})$ for comparison points on $\triangle(\bar{p}, \bar{q}, \bar{r}) \subset \mathbb{E}^2$.

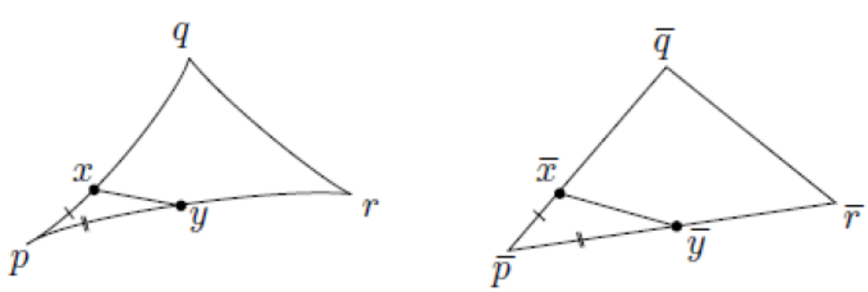

Figure 2.1: $CAT(0)$ geometry

Examples of $CAT(0)$ spaces

- $\mathbb{E}^n$, in this case all geodesic triangles are identical to their comparison triangles in Euclidean spaces.

- Trees. Trees are 1-dimensional $CAT(0)$ spaces and 0-hyperbolic space.

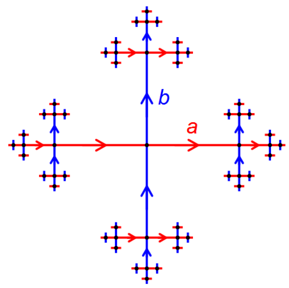

Figure 2.2: infinite, 4-valence tree

- Universal covers of compact Reimannian manifolds of non-positive curvature are $CAT(0)$ spaces, as $CAT(0)$ spaces can be alternatively characterized as locally $CAT(0)$, contractible spaces.

- Convex subspaces of $CAT(0)$ spaces are $CAT(0)$

- If X and Y are two $CAT(0)$ spaces, then with the product metric, $X \times Y$ is also $CAT(0)$.

- If two spaces X, XY are $CAT(0)$, then the new space obtained from gluing X and Y along a complete, convex subspace is also $CAT(0)$.

Follows immediately from the definition is the fact that $CAT(0)$ spaces are uniquely geodesic. Otherwise, consider the geodesic triangle whose vertices are a pair of points with two geodesics between them and a third point on one of the geodesic. Applying the $CAT(0)$ inequality to this triangle shows that it is in fact a degenerative triangle. Therefore $CAT(0)$ spaces are uniquely geodesic. Therefore, $CAT(0)$ spaces are contractible (having the homotopy type of a point) via the unique geodesic ray emanating from a base point to any point in the space.

Specifically, a *geodesic ray* in X is a geodesic $c : [0, \infty) \to X$. Consider the set of geodesic rays in X. Two geodesic rays c_1 and c_2 are said to be *asymptotic* if $f(t) := d(c_1(t), c_2(t))$ is a bounded function. It is easy to check that this defines an equivalence relation. The set of equivalence classes is denoted by ∂X and called the *boundary* of X. If $\xi \in \partial X$ and c is a geodesic ray belonging to ξ, we write $c(\infty) = \xi$. For any $\xi \in \partial X$ and any $x \in X$, there is a unique geodesic ray $c_{x\xi} : [0, \infty) \to X$ with $c_{x\xi}(0) = x$ and $c_{x\xi}(\infty) = \xi$. The image of $c_{x\xi}$ is denoted by $x\xi$.

There are two topologies we can put on ∂X. Set $\bar{X} = X \bigcup \partial_\infty X$. The cone topology on $\bar{X}$ has as a basis the open sets of X together with the sets

$$U(x, \xi, R, \epsilon) = \{z \in barX | x \notin B(x, R), d(c_{xz}(R), c_{x\xi}(R)) < \epsilon\}$$

where $x \in X$, $\xi \in \partial_\infty X$ and $R > 0, \epsilon > 0$. The topology on X induced by the cone topology coincides with the metric topology on X.

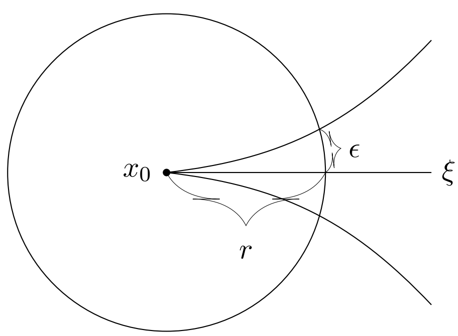

Figure 2.3: A basis for open sets

The set ∂X together with the cone topology is called the *visual boundary* of X, denoted $\partial_\infty X$.

There is another metric on ∂X. Let $c_1, c_2 : [0, \infty) \to X$ be two geodesic ray with $c_1(0) = c_2(0) = x$. For $t_1, t_2 \in (0, \infty)$, consider the comparison angle $\overline{\angle_x}(c_1(t_1), c_2(t_2))$. Let ξ_1 and ξ_2 be points of ∂X represented by c_1 and c_2. We define the *Alexandrov angle* between ξ_1

Table 2.1: Visual Boundary vs. Tits Boundary

X	$\partial_\infty X$	$\partial_T X$
$\mathbb{E}^n$	S^n	S^n
Regular-valence, infinite tree	Cantor set	discrete set
$\mathbb{H}^n$	S^n	discrete set

and ξ_2 at point x to be

$$\angle_x(\xi_1, \xi_2) := lim_{t_i \to 0}\overline{\angle_x}(c_1(t_1), c_2(t_2))$$

We define the *Tits angle* between ξ_1 and ξ_2 to be

$$\angle_T(\xi_1, \xi_2) := \sup_{x \in X} \angle_x(\xi_1, \xi_2)$$

The *Tits metric* d_T on $\partial_\infty X$ is the path metric induced by $\angle_T$. We denote $\partial_T X := (\partial_\infty X, d_T)$. From the definition we see that

$$\angle_x(\xi_1, \xi_2) \leqslant \angle_T(\xi_1, \xi_2)$$

and it can be shown that if X is a $CAT(0)$ space, and $\xi_1, \xi_2 \in \partial_T X$, then $\partial_T X$ is a $CAT(1)$ space.

The set ∂X together with the topology arising from d_T is called the *Tits boundary* of X, denoted $\partial_\infty X$.

In general the visual boundary of the space is different from the Tits boundary of the space. The following table list the ideal boundaries and the Tits boundaries of three basic spaces.

Consider $\mathbb{H}^2$. In the disk model, ∂X can be identified with the unit circle. For any two points ξ, η on the boundary, there is a geodesic line in $\mathbb{H}^2$ joining ξ and η. Therefore $\angle(\xi, \eta) = \pi$ for any two distinct points one the boundary, hence $\partial_T \mathbb{H}^2$ is a discrete set.

2.3 Quasi-isometry

The first examples of space on which the group acts are Cayley graphs. Cayley graphs of the same group with different generating set "look" different. To establish a well-defined correspondence between a group and the set of spaces on which it acts nicely, quasi-isometry is a needed concept that captures the "large-scale" geometry of spaces.

Theorem 2.8. *If X is isometric to X', then their boundaries are homeomorphic, i.e. $\partial_\infty X \simeq \partial_\infty X'$ and $\partial_T X \simeq \partial_T X'$.*

Definition 2.9. Let (X_1, d_1) and (X_2, d_2) be metric spaces. A (not necessarily continuous) map $f : X_1 \to X_2$ is called a (λ, ϵ)-*quasi-isometric embedding* if there exist constants $\lambda \geqslant 1$ and $\epsilon \geqslant 0$ such that for all $x, y \in X_1$

$$\frac{1}{\lambda} d_1(x, y) - \epsilon \leqslant d_2(f(x), f(y)) \leqslant \lambda d_1(x, y) + \epsilon$$

If, in addition, there exists a constant $C \geqslant 0$ such that every point of X_2 lies in the Cneighborhood of the image of f , then f is called a (λ, ϵ)-quasi-isometry. When such a map exists, X_1 and X_2 are said to be *quasi-isometric.*

If a finitely generated group G acts cocompactly, properly discontinuously and by isometries on a metric space X, then we say that G is *quasi-isometric* to X.

If X, Y are metric spaces and $f : X \to Y$, then, f is an isometric embedding if $\forall x, x' \in X$, $d_X(x, x') = d_Y(f(x), f(x'))$ If X, Y are metric spaces and $f : X \to Y$ then f is a quasi-isometric embedding if $\exists C \geqslant 1 s.t. \forall x, x'$

$$\frac{1}{C}d_X(x,x') - C \leqslant d_Y(f(x), f(x')) \leqslant Cd_X(x,x') + C$$

f is not necessarily continuous.

Theorem 2.10. *(Švarc-Milnor lemma) If X is a complete, locally compact geodesic metric space and G acts geomet- rically on X, then G is finitely generated and X is quasi-isometric to every Cayley graph for G.*

2.4 Bass-Serre Theory and Group Amalgamation

In this section we present a basic introduction to the Bass-Serre theory according to [Ser80]. The Bass-Serre theory is a fundamental structure theorem in group theory that relates geometric and combinatorial aspects of a group. Generally speaking, a group that acts on a tree decomposes into smaller pieces encoded in the stabilizer data via iterated applications of HNN extensions and free products with amalgamation. We will only be concerned with amalgamated products in this thesis.

We begin by constructing the notion of graphs. A graph is a pair $A = (V, E)$ comprising a set V of vertices together with a set E of unoriented edges, which are 2-element subsets of V. If and edge e is the pair (v_1, v_2), then we say the edge e is *incident* to the vertex v_1 (resp. v_2), and the v_1 is *adjacent* to v_2. This type of graph may be described precisely as undirected and simple.

A graph A' is a *subgraph* of A if it forms a graph in itself and $V(A') \subset V(A)$, $E(A') \subset E(A)$. An *edge path* is a sequence of edges $e_1, e_2, ...e_n$ such that each adjacent pair of edges in the sequence shares a vertex, and non-adjacent edges do not share any vertex. A *cycle* is an edge path except for that e_0 and e_n shares a vertex and $n \geqslant 3$.
A graph is *connected* if there is a path between any two distinct vertices. A *tree* is an undirected simple graph that is connected and has no cycles.

Definition 2.11. A *graph of groups* $\Gamma = (\Gamma, \mathscr{G})$ is a connected graph Γ equipped with the following

- to each vertex $v \in V(\Gamma)$, an assignment of a group G_v (called the *vertex group of* v)
- to each edge $e \in E(\Gamma)$, an assignment of a group G_e (called the *edge group of* e).
- to each edge $e = (v_i, v_t) \in E(\Gamma)$, an assignment of an injective group homomorphisms h_i, h_t where:

$$h_i : G_e \to G_{v_i}$$
$$h_t : G_e \to G_{v_t}$$

Now we define Serre's fundamental groups of a graph of groups. Let Γ be a simple graph. The *fundamental group* of the graph of groups $\pi_1(\Gamma, \mathscr{G})$ is the quotient of the free product of groups G_v and G_e by the normal subgroup generated by elements

$$\{h_{i_\alpha}(g)h_{t_\alpha}(g)^{-1} | g \in G_{e_\alpha}, e_\alpha \in E(\Gamma)\}$$

Suppose the graph Γ is a single edge with vertex groups Γ_1, Γ_2, edge group H and injective homomorphisms ϕ_1, ϕ_2, then the *amalgamated free product* of Γ_1 and Γ_2 along H is the fundamental group the graph of groups, i.e. it is the quotient of the free product $\Gamma_1 * \Gamma_2$ by the normal subgroup generated by the conjugates of the elements

$$\{\phi_1(h)\phi_2(h)^{-1} | h \in H\}$$

We write the amalgamated product as $\Gamma_1 *_H \Gamma_2$

Definition 2.12. The *Bass-Serre tree* T associated to an amalgamated free product $\Gamma = \Gamma_1 *_H \Gamma_2$ is constructed as follows: take disjoint union of $\Gamma \times [0,1]$, i.e. copies of $[0,1]$ indexed by elements of Γ, then quotient by the equivalence relation generated by

$$(\gamma\gamma_1, 0) \sim (\gamma, 0)$$

$$(\gamma\gamma_2, 1) \sim (\gamma, 1)$$
$$(\gamma h, t) \sim (\gamma, t)$$

for all $\gamma \in \Gamma$, $\gamma_1 \in \Gamma_1$, $\gamma_2 \in \Gamma_2$ $h \in H$ and $t \in [0, 1]$

The action of Γ by left translation on the index set of the disjoint union permutes the edges and is compatible with the equivalence relation, therefore it induces an action Γ on T by isometries. The quotient of the tree T by this action is an interval $[0, 1]$; the subgroup $H \subset \Gamma$ is the stabilizer of an edge in T and the stabilizers of the vertices of this edge are the subgroups $\Gamma_1 \subset \Gamma$ and $\Gamma_2 \subset \Gamma$.

Theorem 2.13. *If a group Γ has a amalgamated product decomposition, then Γ acts on the corresponding Bass-Serre tree with quotient an edge.*

2.5 Strict Fundamental Domain

One of the key ideas in geometric group theory is to view groups are geometric objects. A strict fundamental domain and all its translates can be thought of as geometric representations of each group element in the space. In this section, we suppose that G is a group acting on a metric space X by isometries.
For each point $x \in X$, the *orbit* of x is the set

$$\{y \in X | y = gx \text{ for some } g \in G\}$$

Definition 2.14. Let K be a closed subset of X. K is a *strict fundamental domain* of G on X if every orbit meets K exactly once.

According to Bass-Serre Theory, in the case of a group acting on a tree, the strict fundamental domain determines an amalgamated product decomposition of the group with the amalgamating groups finite. Conversely, if the original group has this decomposition, then there is a tree in which the group acts geometrically. The critical tools to our study come from the thesis of O'Brien.

Proposition 2.15. *[O08]*

Suppose G acts geometrically on a uniquely geodesic space X with a strict fundamental domain K whose translates are locally finite. Then

1. *K is convex, and*

2. *the the quotient X/G is isometric to K*

The full strength of O'Brien's result generalize the Bass-Serre theory from trees to uniquely geodesic spaces. Once we obtain a strict fundamental domain K, we can study the stabilizer groups of the topological boundaries of K and recover the amalgamated product decomposition of the group. Specific to our result, we let a right-angled Coxeter group act on the nerve tree of the Croke-Kleiner space and use the following theorem:

Theorem 2.16. *[O08] If a G acts geometrically on a tree, then there is a fundamental domain that is a finite sub-tree.*

In Chapter 3 and Chapter 4, we will apply specific construction of the strict fundamental domain to obtain the desired results.

Chapter 3

The Croke Kleiner Space

In this chapter we examine a concrete counterexample of the conjecture that if a finitely generated group G acts properly discontinuously, cocompactly and by isometries on two CAT(0) spaces X_1, X_2, then there is a G-equivariant homeomorphism between their ideal boundaries $\partial_\infty X_1 \to \partial_\infty X_2$.

The example is first constructed by Croke and Kleiner. In their original construction of a class of spaces CK_θ, they prove that the ideal boundaries $\partial_\infty CK_\theta$ is different for $\theta = \pi/2$ from that of $\theta \neq \pi/2$. They show that the two boundaries with different values of θ are not homeomorphic as topological spaces. We follow the construction and let θ be fixed at $\pi/2$ and vary other geometric data of the space to show that there are cases when two boundaries agree on a dense subset, but the G-equivariant homeomorphism types differ on the whole space. The main theorem we prove in this chapter states that with all above assumptions the two boundaries are not G-equivariantly homeomorphic. It remains an open question whether the two boundaries as topological spaces are homeomorphic or not.

It is worth noting that a more general construction of this class of space is studied in [CK02]. In [CK02], the authors study discrete, cocompact, isometric actions of groups on a class of graph manifolds and the induced actions on ideal boundaries. In [CK02], the Bass-Serre tree of the space is isomorphic to that of the Croke-Kleiner space. However, in the generalization, each vertex in the Bass-Serre tree is a hyperbolic-by $\mathbb{R}$ space and each edge is $\mathbb{E}^2$. The fundamental group is a graph of groups where each vertex group is $F \times Z$ where F is a non-elementary hyperbolic group, and each edge group is $\mathbb{Z}^2$. With a group that acts discretely, cocompactly and by isometries on the space, they specify "geometric data" MLS_v and "τ_v". In their notation, the geometric data we consider in this space all have $\tau_v = 0$. [CK02] proves that the geometric data determines the equivariant homeomorphism type of the boundary by studying templates and the method of "shadowing". In this document we will be only considering the case where $\tau_v = 0$ and we will be using template as well, but we argue straight from the $CAT(0)$ geometry of the template. While Theorem 1.1 is a special case of the general result in [CK02], the technique in the proof of Theorem 1.1 is crucial to proving Theorem 1.4.

In the last section we study the set of points in the boundary with infinite itineraries, which we call the "dust". We show that the set of isolated points and interval components are both uncountable. We also give examples of subsets of the dust that are invariant under the G-equivariant map, in the sense that points are mapped to points, and intervals are mapped to intervals.

Before we prove the theorem about the equivariant homeomorphism type of the boundary, we also study the core component of the boundary, the union of all circles, more carefully and we prove

1. The G-equivariant map on the Croke-Kleiner space with length variation is an isometry of the cores with the Tits metric.

2. There is an homeomorphism on the cores (of the boundary) with visual topology.

3.1 Construction

The space X constructed by Croke and Kleiner in [CK00] is the universal cover of a torus complex Y. Start with a flat torus T_2 with the property that a pair b, c of unoriented, π_1-generating simple closed curves in T_2 meets at a single point at an angle α, $0 < \alpha \leqslant \frac{\pi}{2}$. Let b, c have length 1. Let T_1, T_3 be flat tori containing simple closed essential loops, a, b_1 and c_1, d, respectively, such that $length(b_1) = length(b)$, $length(c_1) = length(c)$. Let a, d also have length 1. Let Y be the union of T_1, T_2, T_3 with b_1 identified isometrically with b and c_1 with c. Let X be the universal cover of Y. Let $Y_1 = T_1 \cup T_2$, and let $Y_2 = T_3 \cup T_2$, X_i be the universal cover of Y_i in X.

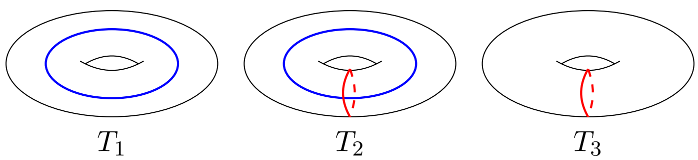

Figure 3.1: Tori Complex

Definition 3.1. A *barrier* is a maximal connected component of the universal cover of T_2 in X. A *block*, B, is a maximal, connected component of the universal cover of Y_i in X, which we denoted X_i.

Each block, as well as each barrier is a closed, connected and locally convex subset of X. Let $\mathscr{B}$ and $\mathscr{W}$ denote respectively the collection of all blocks and barriers. We will prove later that $\mathscr{B}$ and $\mathscr{W}$ are countably infinite sets.

Let $\mathscr{T}_4$ be the regular 4-valence, infinite tree that is isomorphic to the Cayley graph of $F_2 = \langle a, b \rangle$. A block is isometric to the metric product of T_4 with the real line $\mathbb{R}$. The intersection of two blocks can be either an empty set or a barrier. Two blocks are *adjacent* if and only if their intersection is a barrier.

Definition 3.2. The *geometric data* associated with the Croke-Kleiner space consists of three intersecting angles and four translation distances. The three intersection angles are

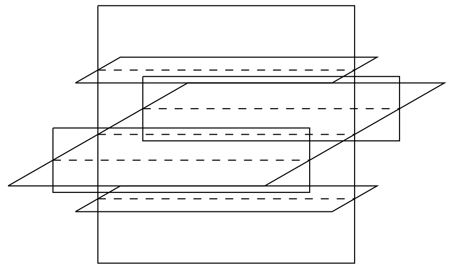

Figure 3.2: A Block

that of the intersecting angle of the three pairs of π_1-generating, simple closed curves on the three tori, which we denote $\theta_1, \theta_2, \theta_3$. In this chapter we fix $\theta_1 = \theta_2 = \theta_3 = \pi/2$ unless otherwise specified. The four lengths are the translation distance of a, b, c, d. Since the θ_i are right angles, we can use $|a|, |b|, |c|, |d|$ to denote the translation distance, or by abuse of notation a, b, c, d. It can be easily checked that length variation is a quasi-isometry but not an isometry on the space.

3.1.1 The boundaries of X

Let $\partial_\infty X$ and $\partial_T X$ denote respectively the visual boundary and the Tits boundary of the space X. $\partial_\infty B$ is homeomorphic to the suspension of a Cantor set and $\partial_T B$ is the suspension of an uncountable discrete set with each suspension arc having length $\pi/2$.

Definition 3.3. The two suspension points of $\partial_\infty B$, called *poles* of the block, are the equivalence classes of geodesics correspond with the pair $\{b^n(x), b^{-n}(x)\}$, or the pair $\{c^n(x), c^{-n}(x)\}$, as $n \to \infty$.

Definition 3.4. A *longitude* of the block is an arc in $\partial_\infty B$ joining the two poles. It can also be thought of as the suspension of a point in the Cantor set.

We say that a geodesic ray ξ enters a plane V if there are values $r < R$ in the domain of ξ such that $\xi([r, R]) \subset V$. ξ enters a block if it enters a non-barrier plane of the block.

Definition 3.5. An *itinerary* of a geodesic ray, $It(\xi)$, is the sequence of blocks that the geodesic ray enters in order. An itinerary can be either finite or infinite.

We say that $\xi \in \partial_\infty X$ is a *vertex* if there is a neighborhood U of ξ such that the path component of ξ in U is homeomorphic to the cone over a Cantor set, with ξ corresponding to the vertex of the cone.

A path $c : [0, 1] \to \partial_\infty X$ is *safe* if $c(t)$ is a vertex for only finitely many $t \in [0, 1]$. Since the property of being joinable by a safe path is an equivalence relation on pairs of points, and since $\partial_\infty B_1 \cup \partial_\infty B_2$ is safe path connected when B_1 is adjacent to B_2, it follows that $\bigcup_{B \in \mathscr{B}} \partial_\infty B$ is safe path connected. It is shown in [CK00] that $\bigcup_{B \in \mathscr{B}} \partial_\infty B$ is a *safe-path component* of $\partial_\infty X$.

3.1.2 The Right-Angled Artin Group

One way to specify a presentation for a right-angled Artin group is by means of a defining graph. Given a presentation:

$$G = \langle s_1, s_2, ...s_n | s_i s_j = s_j s_i \text{ for some pairs of } i, j \in \{1, 2, ...n\} \rangle$$

A defining graph is a simple graph whose vertices are labeled by the generators $S = \{s_1, ..., s_n\}$ and whose edges are pairs of vertices (s_i, s_j) if and only if $s_i s_j = s_j s_i$. Note that any finite, simplicial graph Γ is a defining graph for some right-angled Artin group.

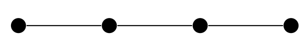

Figure 3.3: Defining graph of an RAAG

The group that acts properly discontinuously, cocompactly, and by isometries on the Croke-Kleiner space is among the smallest non-trivial right-Angled Artin groups. We define this group as:

$$G = \langle a, b, c, d | aba^{-1}b^{-1}, bcb^{-1}c^{-1}, cdc^{-1}d^{-1} \rangle$$

G is the fundamental group of the space X. The covering space action of the group satisfies the assumption of a geometric action. Alternatively, we can see that the group G is an amalgamated product of groups that act geometrically on spaces that are glued equivariantly along subspaces that are acted on geometrically by a common subgroup that embeds into both amalgams. Therefore, by Theorem II 11.18 of [BH99], G acts geometrically on the space constructed. The amalgamated product decomposition is not unique. Let

$$G_1 = G_2 = (\mathbb{Z} \times \mathbb{Z}) *_{\mathbb{Z}} (\mathbb{Z} \times \mathbb{Z})$$

then let $H = \mathbb{Z} \times \mathbb{Z}$, G can be expressed as

$$G = G_1 *_H G_2$$

The elements of H act geometrically on the barrier, the elements of G_1 and G_2 act geometrically on the respective $\mathscr{T} \times \mathbb{R}$ where $\mathscr{T}$ is an infinite, 4-valence tree. The amalgamated product induces a rooted Bass-Serre tree with the vertices corresponding to cosets of G_i and edges corresponding to cosets of H [Ser80].

Recall that an *itinerary* of a geodesic ray, $It(\xi)$, is the sequence of blocks that the geodesic ray enters in order. An infinite itinerary geodesic ray corresponds to an infinite path in the Bass-Serre tree starting at the root x_0.

A finer decomposition of G is the amalgamated product of three copies of $\mathbb{Z} \times \mathbb{Z}$:

$$G = (\mathbb{Z} \times \mathbb{Z}) *_{\mathbb{Z}} (\mathbb{Z} \times \mathbb{Z}) *_{\mathbb{Z}} (\mathbb{Z} \times \mathbb{Z})$$

The corresponding induced tree we will define later as the *nerve*.

3.1.3 The Bass-Serre Trees

The Bass-Serre tree is a simplicial tree which when quotiented by an amalgamated product results in an edges whose vertex stabilizers are the amalgams. In the context of the Croke Kleiner space, the first tree described above records the blocks and their intersections. Let

ρ be the following projection from the space X to a graph:

- Each block is projected onto a vertex
- two vertices are adjacent if and only if two blocks are adjacent

We call the image of ρ the *Bass Serre covering tree*, or simply the *Bass-Serre tree* with respect to the amalgamated product decomposition $G = G_1 *_H G_2$. A group or a space does not necessarily have a unique Bass-Serre tree. We study another Bass-Serre tree of the space in the next section.

Let x_0 be the base point in X. Consider the block containing x_0 and all the adjacent blocks. There is a bijection between the set of adjacent blocks and the cosets of H in G_1 (or G_2). The vertices are indexed by $G/G_i, i = 1, 2$, the edges are indexed by G/H. By this construction the Bass-Serre tree is a locally countably infinite tree.

Using the Bass-Serre tree we make the following observation:

Proposition 3.6. *Given any two blocks B and B', $\partial_\infty B \cap \partial_\infty B'$ is either ϕ, S^0 or S^1.*

Proof. If two blocks are adjacent in the Bass-Serre tree, then they share a common barrier W, their boundaries meet exactly in $\partial_\infty W \sim S_1$. If they are at distance two in the Bass-Serre tree, then the vertex in between the two vertices that represents the two blocks represents a block in which there are two disjoint barriers which are shared respectively with the two blocks. The boundary of barriers are circles and these two circles intersects at exactly two points, which is S_0. If the two blocks are at distance 3 or more in the Bass-Serre tree then their intersection is ϕ.

□

3.1.4 The Nerve

There is another Bass-Serre tree of the space that corresponds to the second amalgamated product decomposition (In the original paper [CK00], however, the term "nerve" refers to the tree associated with the first decomposition). By abuse of notation, let T_i denote the fundamental groups of the tori.

$$T_i = \mathbb{Z} \times \mathbb{Z}, i = 1, 2, 3$$

$$L_j = \mathbb{Z}, j = 1, 2$$

$$G = T_1 *_{J_1} T_2 *_{J_2} T_3$$

The *nerve* is a simplicial tree that records the intersections of planes that are lifts of the tori. Each vertex in the *nerve* represent a plane that is a lift of T_1, T_2 or T_3, two vertices are adjacent if and only if two planes intersect along a line. The nerve is also a regular, infinite and locally infinite tree. Each vertex is adjacent to countably many other vertices. The vertices representing lifts of T_1 (and respectively T_3) are adjacent to countably infinite copies of vertices representing lifts of T_2, while the lifts of T_2 are adjacent to both countably many T_1 and countably many T_3.

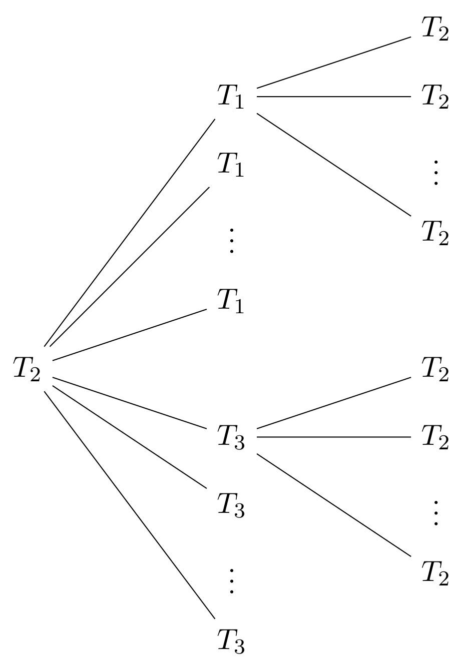

Figure 3.4: The Nerve

Table 3.1: Path component, safe-path component and the core component

	Visual boundary	Tits boundary
core component	union of all circles	union of all circles
safe-path component	union of all circles	union of all circles
path component	union of all circles and the points in the dust	union of all circles

3.2 The Core Component

In this $CAT(0)$ 2-complex X, let

$$Core(\partial_\infty X) = \cup c \text{ where } c \text{ varies over all the topological circles in } \partial_\infty X$$
$$Core(\partial_T X) = \cup c \text{ where } c \text{ varies over all the topological circles in } \partial_T X\ .$$

In this section we will study the Tits boundary of the space X under length variation and its core; we will also study the core of the visual boundary. Before we proceed, we first compare the core to a couple of other concepts.

Two points x, y in a topological space X are said to be in the same path component if there exists a path from x to y in X. The equivalence classes of X under this equivalence relation are called the *path components* of X.

Two points x, y in the Croke-Kleiner space are said to be in the same safe-path component if there exists a path from x to y in X that passes only finitely many poles. The equivalence classes of X under this equivalence relation are called the *safe-path components* of X.

Let d_c be the induced path metric of ∂_T on Core$(\partial_T X)$. For $\xi, \mu \in Core(\partial_T X)$, $d_c(\xi, \mu) < \infty$ if and only if ξ, μ lie in the same path component of Core$(\partial_T X)$ and in this case there is a minimal Tits geodesic contained in Core$(\partial_T X)$ that connects ξ and μ. In particular,

$d_c(\xi, \mu) = d_T(\xi, \mu)$ if $d_c(\xi, \mu) < \infty$.

Xie [Xie05] shows that if two CAT(0) 2-complexes are quasi-isometric then the cores of their Tits boundaries are bi-Lipschitz. We show that in the case of this particular example, even stronger is true: the Tits boundaries of X and X_l are homeomorphic, and the cores of the ideal boundaries are homeomorphic. Recall by X_l, we mean a Croke-Kleiner space with non-trivial lengths data (a, b, c, d) and where the angles $\theta_1 = \theta_2 = \theta_3 = \pi/2$

Theorem 3.7. *Let X be the Croke-Kleiner space with length data (1,1,1,1), and let X_l be the Croke-Kleiner space with length data $l = (a, b, c, d)$, where at least one of a, b, c, d is not 1. There is an isometry between Tits boundaries:*

$$Core(\partial_T X) \backsimeq Core(\partial_T X_l)$$

Proof. We define the isometry map as follows. On the core component, each pole is the pole of a unique block, which is indexed by G/G_i. Each circle is indexed by G/T_i. We first map poles and circles one-to-one G-equivariantly. Each circle either has two poles or four poles. On the circles with two poles, each half-circle between two pole points has Tits length π. The arc is between two pole with representatives $\{\lim_{n\to\infty} b^n, \lim_{n\to\infty} b^{-n}\}$ or $\{\lim_{n\to\infty} c^n, \lim_{n\to\infty} c^{-n}\}$. We parametrize the arc from the positive representative to the negative representative, by its Tits distance d_t from the positive pole $d_T \in [0, \pi]$. For the circles with four poles, we parametrize the arcs from the b-type poles to the c-type poles, by its Tits distance d_T from the b-type poles, $d_T \in [0, \frac{\pi}{2}]$.

Therefore, each ξ is denoted by a 2-tuple coordinate

$$(g, d_T)$$

where $g \in G/T_i$ and $d_T \in [0, \pi]$

The map between $Core(\partial_T X)$ and $Core(\partial_T X_l)$ maps bijectively an element of index $(g_i, (d_T)_i)$ to the element of the same index. This is a homeomorphism since an open set away from the pole is an open subinterval on a single arc.

□

Next we construct a homeomorphism between the ideal boundaries.

Theorem 3.8. *Let X be the Croke-Kleiner space with length data (1,1,1,1), and let X_l be the Croke-Kleiner space with length data (a, b, c, d), where at least one of a, b, c, d is not 1. The G-equivariant map between the $Core(\partial_\infty X)$ and $Core(\partial_\infty X_l)$ is a homeomorphism.*

Proof. We map the circles and poles G-equivariantly as in the previous proof. On each circle, each point represents an equivalence class of infinite geodesics in that plane that are parallel. Let ξ be an equivalence class in the plane indexed by $\{g_i\}$. The plane is stabilized by a conjugate of one of the following pairs of elements:

$$\{a, b\}, \{b, c\}, \{c, d\}$$

Let b^∞ and c^∞, respectively, denote the tails of the principal axes in the respective planes. Suppose θ is the angle between ξ and the principle axis, $\theta \in [0, \pi]$. Suppose $\tan(\theta) \in \mathbb{Q}$, then

$$tan(\theta) = \frac{p}{q}, p, q \in \mathbb{N}$$

in lowest terms. If the plane is stabilized by a conjugate of $\{a, b\}$, consider the sequence of group elements $\{g_i = a^{ip} b^{iq}, i \in \mathbb{N}\}$. Conjugate the sequence to the plane. The orbit of a point of choice in the plane, x, under this sequence, is $x, g_1(x), g_2(x), g_3(x)$.... This sequence necessarily has a limit that is a point in the visual boundary of this plane, denoted ξ_l. Let the map $f : Core(\partial_\infty X) \longrightarrow Core(\partial_\infty X_l)$ be defined equivariantly as

$$f(\xi) = \xi_l$$

Since $\mathbb{Q}$ is a dense subset of $\mathbb{R}$, the map extends to the whole core component. Consider any sequence of points in $\partial_\infty X$ with a limit in $\partial_\infty X$. An Arzelà-Ascoli argument shows that this map respect limits in the visual boundary by construction.

$\square$

Remark. The G-equivariant map, between X and X_l is not a homeomorphism that extends from the space to the core. For example, let x_0 be the base-point in the block spanned by generators b, c, d, where the length data is $(b, c, d) = (1, 1, 1)$. Let $(bc)^\infty$ denoted the infinite word

$$bcbcbcbcbc...$$

and $[w]$, with w an infinite word, denotes the geodesic in the equivalence class of the infinite word w.

Consider the following sequence of geodesics

$$\xi_i = [(cd)^i(bc)^\infty]$$

Each geodesic in this sequence is at Tits distance $arctan(1)$ from one of the poles of the block. The limit of the sequence:

$$\lim_{i\to\infty} \xi_i = \lim_{i\to\infty} [(cd)^i(bc)^\infty] = [(cd)^\infty]$$

which is also at Tits distance $arctan(1)$ from one of the poles.

However, if we let the lengths data be $(b, c, d) = (1, 2, 1)$ and apply the G-equivariant identity map f, then

$$[f(\xi)]$$

is a sequence of geodesics at Tits distance $arctan(1)$ 1 from the pole, while

$$[f((cd)^\infty)]$$

is a geodesic at Tits distance $arctan(1/2)$ from the pole. Since the Tits distance function is continuous on the visual boundary, this shows that the G-equivariant map is not an homeomorphism on the visual boundary of the core component.

Remark. Given the example above, we can conjecture that any map that comes from a group automorphism cannot be extended to a homeomorphism on the visual boundary. It remains open whether there is a homeomorphism between the visual boundaries of the core components that comes from a map of the space.

Conjecture. Any G-equivariant map between the spaces X and X_l does not extend to a homeomorphism on the visual boundary.

3.3 Proof of the Theorem 1.1

Recall **Theorem 1.1**:

On the components of ∂X other than the safe-path component, the G-equvariant action does not extend to a homeomorphism under the visual topology between ∂X and ∂X_l, where X_l has a nontrivial different set of length data.

In this section we present the complete proof of Theorem 1.1.

Since blocks are convex, a geodesic cannot revisit any block which it left. Therefore a geodesic visits a finite, or infinite set of blocks, sequentially. If the whole geodesic visits a finite set of blocks, then it must stabilize in the last block of that sequence. Let ξ be a geodesic that does not stabilize in any block. We project the geodesic on to the first Bass-Serre tree T_{BS}. The projection is well defined because the space is simply connected and contractible. The image of the projection of ξ onto the Bass-Serre tree is an half-infinite path with one end point at the base vertex and the other end correspond to a end of the tree. On the other hand, if the whole geodesic visits only a finite set of blocks, then it projects onto a finite-length path in the Bass-Serre tree. This gives a map

$$f : \partial X \longrightarrow \{\text{finite segments in } T_{BS}\ \} \cup \text{ends of } T_{BS}$$

Similarly each geodesic can be projected onto the nerve tree and the image is a finite or infinite path in the nerve. Each vertex on this path represents a plane and each edge an intersection between two planes. The planes and the intersections are respectively indexed by G/T_i and G/L_j.

Definition 3.9. The union of all the planes on the path we define as the *pre-template*.
Two adjacent planes intersect in an $\mathbb{R}$-line, corresponding to a coset of $\mathbb{Z}$. Consider the sequence of all $\mathbb{R}$-lines. Two consecutive $\mathbb{R}$-lines lie in the plane "between" them, in which they either intersect or are parallel. It is straight forward to see that the sequence of $\mathbb{R}$-lines can be grouped into a new sequence of collections of $\mathbb{R}$-lines if we group all the maximal subsequences of consecutive parallel $\mathbb{R}$-lines together. Take the subset of the pre-template that's bounded by each pair of consecutive $\mathbb{R}$-lines. It should be noted that if two consecutive $\mathbb{R}$-lines are perpendicular to one another, then they cut the plane into four quadrants. Of the four quadrants, there is exactly one that is disjoint from the $\mathbb{R}$-lines immediately before and after this pair, namely the portion "between" this pair of $\mathbb{R}$-lines. The union of all $\mathbb{R}$-lines and the subset of the pre-template bounded between them we call the *template*.

Consider all the $\mathbb{R}$-lines that are perpendicular to its preceding line. These are the lines via which the geodesic ray exits the preceding block and enters a new block. The intersecting point with its preceding line divides the line into two halves. Out of these two halves, there is exactly one that lies in the same half-plane as the next intersecting point. That half-infinite line, together withe the intersecting point, we will define as a *half-exit*.

Given any infinite-itinerary geodesic ray, we can collect the following sets:

- points $\{x_i : i = 0, 1, 2, ...\}$

- strips: $\{S_i : i = 1, 2, ...\}$We combine all the parallel strips in succession into one bigger parallel strip. By doing this, the two borders of the big strip necessarily intersect the previous entrance and next exit perpendicularly. The two perpendicular pairs yields two points that are intersections of entrances and exits. The second intersection point picks out one of the two quarter-planes in the first perpendicular pair that ensures the local geodesity of the geodesics. Since the path is infinite, at every perpendicular pair, there is a unique quarter plane that is traversed by the geodesic. The template can be reduced to the alternating sequence of quarter-planes and strips that are spliced successively along entrances and exits.

3.3.1 Coordinates and Adjusted Coordinates

Let $\{x_i : i = 0, 1, 2...\}$ denote the sequence of intersecting points in the template, with x_0 being the base-point. We let x_0 have coordinate $(0, 0)$. l_1 and ϵ_1 measures the dimensions of the unique rectangle whose diagonal corners are x_0 and x_1. One side of the rectangle is path traced by a power of b (respectively a power of c) and let l_1 be the power, i.e. one side of the rectangle is a path traced by the word b^{l_1}. Likewise, the other side of the rectangle is a word in $\langle a, c\rangle$ (respectively, $\langle b, d\rangle$). a and c are not commutative, but we sum up the powers of a and the powers of c and $\epsilon_1 = (\epsilon_{11}, \epsilon_{12})$ is an ordered pair where ϵ_{11} is the sum of the power of a and ϵ_{12} is the sum of power of c. In general, $\epsilon_i = (\epsilon_{i1}, \epsilon_{i2})$ is an ordered pair where ϵ_{i1} is the sum of the power of a (resp. d) and ϵ_{i2} is the sum of power of c (resp. b).

Similarly, (l_i, ϵ_i) measures the coordinate of x_i with respect to x_{i-1}. The collection $\{(l_i, \epsilon_i) : i = 1, 2, 3...\}$ we call the *coordinates* of the template.

With coordinates we can calculate the *adjusted coordinates* $\{(a_i, b_i : i = 0, 1, 2, ...)\}$ as follows:

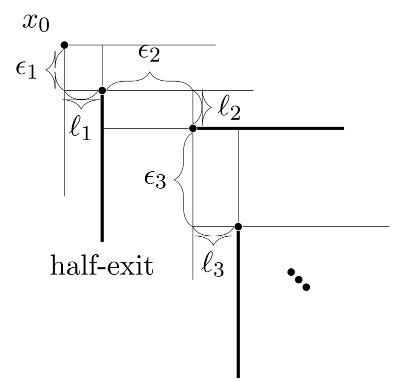

Figure 3.5: Template

$$
\begin{aligned}
& a_1 = l_1, b_1 = \epsilon_1 \\
& a_2 = l_1 + \epsilon_2, b_2 = \epsilon_1 + l_2 \\
& a_3 = l_1 + \epsilon_2 + l_3, b_3 = \epsilon_1 + l_2 + \epsilon_3 \\
& a_4 = l_1 + \epsilon_2 + l_3 + \epsilon_4, b_4 = \epsilon_1 + l_2 + \epsilon_3 + l_4 \\
& \vdots
\end{aligned}
$$

This notation is not ideal in the following sense: when adding the coordinates together, we simply add the powers of the same generators together. thus, a_i and b_i are essentially 4-tuples recording the corresponding accumulative power of a, b, c, and d.

Having established templates and adjusted coordinates for each infinite itinerary geodesic ξ, we show that given an infinite itinerary $\{B_{g_i}\}$, there exists a template such that all geodesic rays with that itinerary travel on this template. More specifically we will show that there is a unique half-exit between two consecutive blocks in an infinite itinerary.

Proposition 3.10. *Given any four consecutive blocks in the itinerary $B_{g_i}, B_{g_{i+1}}, B_{g_{i+2}}, B_{g_{i+3}}$, there is a unique half-exit between $B_{g_i}, B_{g_{i+1}}$, i.e. all geodesics following the itinerary $B_{g_i}, B_{g_{i+1}}, B_{g_{i+2}}$ intersect that half-exit as they exit B_{g_i} and enter $B_{g_{i+1}}$.*

Proof. Let W_i be the barrier that separates separating $B_{g_i}, B_{g_{i+1}}$. The intersection:

$$cl(B_{g_{i+1}}/W_i) \cap W_i$$

consists of a countable set of infinite lines that any geodesic leaving B_{g_i} and entering $B_{g_{i+1}}$ have to intersect. Similarly, let W_{i+1} be the barrier that separates separating $B_{g_{i+1}}, B_{g_{i+2}}$.consider the intersection:

$$cl(B_{g_{i+1}}/W_{i+1}) \cap W_{i+1}$$

consists of a unique infinite line.

The block $B_{g_{i+1}}$, like any other block, is a product $\mathscr{T}_4 \times \mathbb{R}$ which corresponds to to the algebraic decomposition $F_2 \times \mathbb{Z}$. The two sets of infinite lines corresponds to copies of $\mathbb{R}$ indexed by vertices of the T_4. Since the countable set of infinite lines lies on a path in the tree, which the second set that is made up of a single infinite line is at a point in the tree not on this path. By the structure of T_4 there is a unique infinite line from the first set that is at the shortest distance (in the tree) from the singleton in the second set. That unique line we identify as the *exit* between B_{g_i} and $B_{g_{i+1}}$. Similarly we identify the unique exit between $B_{g_{i+1}}$ and $B_{g_{i+2}}$, call these two exits E_i, E_{i+1}. Consider the two intersections x_i, x_{i+1}

$$(cl(B_i/W_i) \cap W_i) \cap E_i$$

$$(cl(B_{i+1}/W_{i+1}) \cap W_{i+1}) \cap E_{i+1}$$

The projection of x_{i+1} onto E_i, i.e. the point on E_i that is closest in the $CAT(0)$ geometry to x_{i+1} lies on one of the two half infinite lines of E_i cut by x_i. That unique half of E_i we call the *half-exit* between B_{g_i} and $B_{g_{i+1}}$. □

3.3.2 n-Self similar templates

In this section, we construct a countable set of geodesic rays, each with infinite itinerary $\mathscr{S}$ such that the G-equivariant homomorphism on each element of $\mathscr{S}$ is not homeomorphic when the lengths (a, b, c, d) are not $(1, 1, 1, 1)$.

We define the *n-self-similar template* to be the templates such that for some $n \in \mathbb{N}$ and all $i, j \in \mathbb{N}$, we have

$$l_{i+2j} = n^j l_i,$$
$$\epsilon_{i+2j} = n^j \epsilon_i$$

Therefore we have: $l_1 = l_1, l_3 = nl_1, l_5 = n^2 l_1, l_7 = n^3 l_1 ...$
$\epsilon_1 = \epsilon_1, \epsilon_3 = n\epsilon_1, \epsilon_5 = n^2\epsilon_1, \epsilon_7 = n^3\epsilon_1 ...$
The template is specified by the data $(n, l_1, \epsilon_1, l_2, \epsilon_2)$ and the adjusted coordinates are:

$$x_0 : (0, 0)$$
$$x_1 : (l_1, \epsilon_1)$$
$$x_2 : (l_1 + \epsilon_2, \epsilon_1 + l_2)$$
$$x_3 : ((n + 1)l_1 + \epsilon_2, (n + 1)\epsilon_1 + l_2)$$
$$x_4 : ((n + 1)l_1 + (n + 1)\epsilon_2, (n + 1)\epsilon_1 + (n + 1)l_2)$$
$$\vdots$$
$$x_i : (\frac{1 - n^k}{1 - n} l_1 + \frac{1 - n^{k-1}}{1 - n}\epsilon_2, \frac{1 - n^k}{1 - n}\epsilon_1 + \frac{1 - n^{k-1}}{1 - n} l_2), \text{if } i = 2k + 1$$
$$x_i : (\frac{1 - n^k}{1 - n} l_1 + \frac{1 - n^k}{1 - n}\epsilon_2, \frac{1 - n^k}{1 - n}\epsilon_1 + \frac{1 - n^k}{1 - n} l_2), \text{if } i = 2k$$

Limit of the tangent of the angles of geodesic rays passing through x_{even} is

$$\lim_{k\to\infty} \frac{\frac{1-n^k}{1-n}\epsilon_1 + \frac{1-n^k}{1-n} l_2}{\frac{1-n^k}{1-n} l_1 + \frac{1-n^k}{1-n}\epsilon_2}$$

$$= \lim_{k\to\infty} (\frac{1-n^k}{1-n}) \frac{\epsilon_1 + l_2}{l_1 + \epsilon_2}$$

$$= \frac{\epsilon_1 + l_2}{l_1 + \epsilon_2}$$

Limit of the tangent of the angles of geodesic rays passing through x_{odd} is

$$\lim_{k\to\infty} \frac{\frac{1-n^k}{1-n}\epsilon_1 + \frac{1-n^{k-1}}{1-n} l_2}{\frac{1-n^k}{1-n} l_1 + \frac{1-n^{k-1}}{1-n}\epsilon_2}$$

$$= \lim_{k\to\infty} \frac{(\frac{1}{n^{k-1}} - n)\epsilon_1 + (\frac{1}{n^{k-1}} - 1) l_2}{(\frac{1}{n^{k-1}} - n) l_1 + (\frac{1}{n^{k-1}} - 1)\epsilon_2}$$

$$= \lim_{k\to\infty} \frac{n\epsilon_1 + l_2}{n l_1 + \epsilon_2}$$

$$= \frac{n\epsilon_1 + l_2}{n l_1 + \epsilon_2}$$

If the two above limits overlap, then the itinerary corresponds to an arc, otherwise, the itinerary corresponds to a point. That is to say, given $n \in \mathcal{N}$, if

$$\frac{\epsilon_1 + l_2}{l_1 + \epsilon_2} > \frac{n\epsilon_1 + l_2}{n l_1 + \epsilon_2}$$

then the collection of geodesic rays forms an interval at infinity; otherwise, if

$$\frac{\epsilon_1 + l_2}{l_1 + \epsilon_2} \leqslant \frac{n\epsilon_1 + l_2}{n l_1 + \epsilon_2}$$

the boundary component consists of a point.

Meanwhile, compute from the data $\frac{\epsilon_1}{l_1}$ and $\frac{\epsilon_1 + l_2}{l_1 + \epsilon_2}$ also shows if the two tangent lines overlap in the first two strips. Therefore it suffices to point out that

$$\frac{\epsilon_1 + l_2}{l_1 + \epsilon_2} > \frac{n\epsilon_1 + l_2}{nl_1 + \epsilon_2} \text{ if and only if } \frac{\epsilon_1}{l_1} < \frac{\epsilon_1 + l_2}{l_1 + \epsilon_2}$$

Therefore we have the following main lemma:

Lemma 3.11. *An n-self-similar template $(l_1, \epsilon_1, l_2, \epsilon_2)$ corresponds to a point if and only if the set of geodesics allowed by the first two strips is a point.*

Example Let the self-similar data be (2,1,2,1) and let $n = 2$, the first strip has sides a, b^2, the second strip has sides d, c^2.

The adjusted coordinates of the x_{even} are (3,3), (9,9), (21,21)... The adjusted coordinates of x_{odd} are (2,1), (7,5), (17, 13), (37, 25), ...$(2^{k+1} + 2^{k-1} - 3, 2^{k+1} - 3)$, if $n = 2k - 1$

The tangent of angle of the half-exits originated from x_{even} is $\frac{1}{1} = \frac{3}{3} = \frac{9}{9}... = 1$; The tangent of angle of the half-exits originated from x_{odd} is:

$$\begin{aligned}
&\lim_{n\to\infty} \frac{2^{k+1} - 3}{2^{k+1} + 2^{k-1} - 3} \\
=&\lim_{n\to\infty} \frac{2^{k+1} + 2^{k-1} - 3 - 2^{k-1}}{2^{k+1} + 2^{k-1} - 3} \\
=&1 + \lim_{n\to\infty} -\frac{2^{k-1}}{2^{k+1} + 2^{k-1} - 3} \\
=&1 - \frac{1}{4} \\
=&0.75
\end{aligned}$$

Since $1 - 0.75 > 0$, the image at infinity is an interval.

In X_l let the geometric data be $(|a| = 2, |b| = 1, |c| = 1, |d| = 2)$, then

$$\frac{\epsilon_1 + l_2}{l_1 + \epsilon_2} = 1 = \frac{2\epsilon_1 + l_2}{2l_1 + \epsilon_2}$$

Hence the image is a point under the G-equivariant map.

Proof of Theorem 1.1:

Proof. Given a non-trivial length change, it is straightforward to check that there exist choices of the first two strips such that the set of geodesic allowed by them changes its cardinality from uncountably infinite to one, or one to uncountably infinite. From Lemma 3.12, we see that the n-self-similar templates constructed from those chosen first two strips will have visual boundaries whose homeomorphism type change with the corresponding lengths change. Since a group equivariant map acts on itineraries, i.e. block sequences, by identity, then it will map the corresponding visual boundaries of the templates to their counterparts in X_l. Since the cardinality of the interval is different from the cardinality of a single point, the map cannot be homeomorphic. □

We make specific what are the sets of trivial length data.

Proposition 3.12. *Two length data* (a, b, c, d) *and* $(a', b', c', d,)$ *are equivalent if either of the following is true:*

- $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \frac{d}{d'} = r$*, for some* $r \in \mathbb{R}^+$*, or*
- $\frac{b}{a} = \frac{b}{c} = \frac{d}{c} = r$ *for some* $r \in \mathbb{R}^+$ *and* $\frac{b'}{a'} = \frac{b'}{c'} = \frac{d'}{c'} = r'$ *for some* $r' \in \mathbb{R}^+$*.*

Proof. If we view the template as entirely embedded in a rectangle as in Figure 3.5, the first equivalence changes the dimension of the template by multiplying both sides by the same fraction. In the second equivalence relation, the entire (infinite) rectangle is stretched by a factor of r or r', in either case, stretching the whole template does not change the homeomorphism type of its boundary. It can be observed from the form of the adjusted coordinates that

$$x_i : (n_1|b| + n_2|d|, n_3|a| + n_4|c|), n_1, n_2, n_3, n_4 \in \mathbb{N}$$

are the vertices of the half-exits, therefore $\frac{b}{a} = \frac{b}{c} = \frac{d}{c} = r$ for some $r \in \mathbb{R}^+$ and $\frac{b'}{a'} = \frac{b'}{c'} = \frac{d'}{c'} = r'$ for some $r' \in \mathbb{R}^+$, does not change whether the limits is an arc or a point.

$\square$

3.4 Dust

We refer to the set of points in $\partial_\infty X$ whose itineraries are infinite as the *dust* and denote this set by $\mathscr{D}$. $\mathscr{D}$ is the complement of the safe-path component in $\partial_\infty X$. The set $\mathscr{D}$ with its topology being the subspace topology is of independent interest, see Section 3.7. As a subset of $\partial_\infty X$ we are interested in $\mathscr{D}$ for the following reasons:

- As previously shown if we vary the length data, then the G-equivariant map is homeomorphic on the core but not homeomorphic on the $\mathscr{D}$.
- $\mathscr{D}$ is a dense subset of $\partial_\infty X$ and exhibits Cantor-set-like properties.
- A *poisson boundary* is [KM99] the ergodic components of the time shifts of the random paths space. Nevo and Sageev define that an ultra-filter which has the property that no descending collection of half-spaces terminates is called *nonterminating*, let $U_{NT}(X)$ denote the collection of nonterminating ultra-filters, and let $B(X)$ denote the closure of $U_{NT}(X)$ in the Tychonoff topology on the set of all ultra-filters. It can be conjectured that the Poisson boundary of X has its support on $\mathscr{D}$.

We are now ready to fully describe the visual boundary: any given point in $\partial_\infty X$ is an equivalence class of geodesics ξ with either a finite itinerary or an infinite itinerary. If ξ has a finite itinerary, then it is in the core, if ξ has an infinite itinerary then it is in the dust. Therefore

$$\partial_\infty X = \text{Core} \cup \mathscr{D}$$

Next consider the G-equivariant map on the Tits boundaries,

Theorem 3.13. *If we vary length data (a, b, c, d), the G-equivariant map on the Tits boundaries of the X takes the core isometrically to the core, and when restricted to the dust is a homotopy equivalence.*

Proof. The first half of the statement is true by section 3.2. To see the map on the dust, consider the G-equivariant map built in the Theorem 1.1, the map takes connected components to connected components. A priori, some of the points are mapped to intervals or intervals to points. So it is a homotopy equivalence away from the dust, the map maps poles to poles , blocks to blocks. □

Consider all the connected components of $\mathscr{D}$.

Proposition 3.14. *There are two types of set in* $\mathscr{D}$*,*

- *An interval, as the limit of arcs with pole points as endpoints. The intervals are closed. There is no path in* $\partial_\infty X$ *between points of one of the intervals and the core. For an explicit proof of this, see [CMT06].*
- *A point, as the shared limit of two sequences of pole points.*

Proof. Let $\mathcal{S}_i \subset \partial_\infty X$ be the set of all geodesic rays that reaches the strip S_i and the half-exit between S_i and S_{i+1}, and stabilize in the barrier that contains the half-exit. Since geodesic rays exit each strip via a half-exit, the set of all geodesic rays that reaches the half-exit and stays in the barrier spans an arc in the visual boundary whose Tits length is $\pi/2$. Since the arc is in the boundary of a barrier, and one of its endpoint is a pole, i.e. the end that consists of a geodesic that is parallel to the half-exit, then the other end of the $\pi/2$ arc also is a pole point. Consider the sequence of all arcs constructed in this manner for each half-exit. By the visual topology, the limit of the sequence is the boundary of this template, and each arc has Tits length $\pi/2$.

Let x_0 be the base point, consider the angle *at* x_0 of the geodesics in each of the arcs in the sequence described above. The angles are closed sub-intervals of $[0, \pi/2]$ of the form $[0, \alpha]$ or $[\beta, \pi/2]$. The Tits boundary of the template is an infinite intersection of closed intervals of the form $[0, \alpha]$ or $[\beta, \pi/2]$ and hence is closed. Since we are taken intersections of closed sub-intervals of the interval $[0, \pi/2]$, the intersection can be either a point or a closed interval. Therefore, both the points in the dust and the intervals in the dust are limit of arcs with pole points as endpoints. Since the Tits boundary and visual boundary shares the same underlying points, the corresponding visual boundary is also a closed subinterval or a point, and hence a closed set.

□

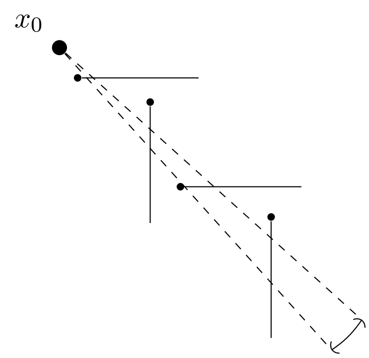

Figure 3.6: An isolated interval in $\mathscr{D}$

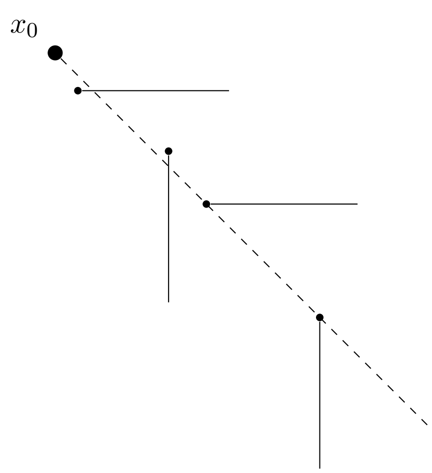

Figure 3.7: An isolated point in $\mathscr{D}$

Remark. The path component of $\partial_\infty X$ contains the safe-path component and all the points in $\mathscr{D}$.

Furthermore, the set $\mathscr{D}$ has "Cantor-set-like" properties, i.e. The set of intervals are dense in $\mathscr{D}$; the set of points are dense in $\mathscr{D}$ as well. This is because whether an infinite-itinerary corresponds to a point or an intervals depends on its "tail" behavior.

The set of self-similar templates constructed in the previous section is a countably infinite set. In the set $\mathscr{D}$ each component is in one-to-one correspondence with an infinite-itinerary. To specify an infinite-itinerary, we have an infinite sequence of blocks, each block is chosen from a countably infinite set. Therefore, $\mathscr{D}$ corresponds to the ends of the Bass-Serre tree, or equivalently, the set of all infinite sequences of natural numbers, which is uncountable. Let's study the cardinality of $\mathscr{D}$.

Proposition 3.15. *Let X be the universal cover of the three-tori complex studied in this paper with any geometric data, there are uncountably many isolated points and uncountably many isolated intervals in $\mathscr{D}$.*

Proof. To show that there are uncountably many intervals, we prove that there is a surjective function from a subset of all intervals to the real numbers $(1, \infty)$, i.e. for every real number $r \in (1, \infty)$, there is an interval in the dust whose Tits length is $\frac{\pi}{4} - tan^{-1}(r)$. To produce an itinerary that corresponds to an arc of Tits length $\frac{\pi}{4} - tan^{-1}(r)$, we first suppose r is rational, $r = \frac{p}{q}, p > q$. Suppose the geometric data are $|a| = |b| = |c| = |d| = 1$. Given any finite sequence of strips and half-exits, $\{S_i\}_{i=1,2,...n}$, in the last exits S_n, the start of the next half-exit, x_{n+1} can have any rational adjusted coordinates (a, b) where a, b are positive integers, which means in particular it can have its adjusted coordinates to be an integer multiple of (p, q), or an integer multiple of $(1, 1)$. Therefore, let the adjusted coordinates of the alternating strips be the ones with (ip, iq) and (j, j), $i \in \mathbb{N}$ and $j \in \mathbb{N}$. The length of this interval is

$$tan^{-1}(p/q) - tan^{-1}(1)$$

Since we are taking an countably infinite intersection, the resulting interval is a closed interval and its length is exactly $tan^{-1}(p/q) - tan^{-1}(1)$ for any rational number $r = p/q > 1$.

Suppose $r \in (1, \infty)$ an irrational number, let $\{\frac{p_i}{q_i}\}_{i \in \mathbb{N}}$ be a sequence of rationals converges uniformly to r. Similarly let the half of strips with adjusted coordinates lie on the $x = y$ line in the template and the other alternating half of strips have adjusted coordinates $(p_i/q_i), i \in \mathbb{N}$. The lengths of the interval will again be

$$\lim_{i \to \infty} tan^{-1}(\frac{p_i}{q_i}) - tan^{-1}(1) = tan^{-1}(p/q) - tan^{-1}(r)$$

for any irrational number $r = p/q > 1$. Since $\mathscr{D}$ assumes the cardinality of the real, and we showed by construction that a subset of it surjects onto the real interval $(1, \infty)$, therefore, the set of all intervals has the cardinality of the real.

Likewise, To show that there are uncountably many isolated points, we again shows that a subset of it surjects onto the real interval $(0, \infty)$, for each rational $\frac{p}{q}$, let the two alternating sets of strips have adjusted coordinates that lie on the $\frac{p}{q}$-line or a sequence of rationals $(p_i/q_i), i \in \mathbb{N}$, the rest of the argument is identical.

Suppose the length data is any given (a, b, c, d), where the numbers$|a|, |b|, |c|, |d|$ can be rational or irrational. Then let the term "rational" denote rational in terms of sum of exponents of each side, i.e. the number of squares (or rectangles) on each side of the strips and the rest of the argument is identical. $\square$

The corollary from the proof above is the following:

Corollary 3.16. *The cardinality of the set of isolated points in the dust is $|\mathbb{R}|$; the cardinality of the set of isolated intervals in the dust is $|\mathbb{R}|$.*

We will use this corollary to prove the homeomorphism of Tits boundary in Section 3.6.

Proposition 3.17. *The dense subsets in $\mathscr{D}$ and in $\partial_\infty X$ are the following:*

- *The set of all points in $\mathscr{D}$ is a dense subset of $\mathscr{D}$; the set of all intervals in $\mathscr{D}$ is a dense subset of $\mathscr{D}$.*
- *$\mathscr{D}$ is a dense subset of $\partial_\infty X$*

Proof. By the construction of template the distinction between a point and an interval depends on the "tail behavior" of an itinerary. Since visual topology is characterized by "fellow-travel", we can form a sequence of points $\{\xi_i\}$ in $\mathscr{D}$ that limit to an interval ζ in $\mathscr{D}$ by taking the ξ_i to have the first i blocks in the itinerary of ζ and then have the itinerary of a point as its "tail". Thus the set of points in $\mathscr{D}$ is dense in $\mathscr{D}$. Likewise for any $\zeta \in \mathscr{D}$, let $\{\xi_i\}$ be an infinite sequence of intervals such chat the first i blocks of ξ_i is the first i blocks in the itinerary of ζ, and the rest of the itinerary of ξ is that of an interval.

To show that $\mathscr{D}$ is dense in $\partial_\infty X$. consider a finite-itinerary ray ξ in $\partial_\infty X$. Let $\{w_i\}$ be an infinite sequence of reduced, finite words that limit to ξ. The words w_i have the same, finite itinerary for all $i \geqslant N$ for some N. Take $\{w_j\}, j = 1, 2, 3...$ where $w_j = w_{i-N}$ such that the sequence $\{w_j\}$ share the same, finite itinerary and limits of ξ. To find a sequence of points in the dust that limits to ξ, consider the word $w_1(ad)^\infty$. The itinerary of each word is infinite and the sequence limits to ξ.

$\square$

We also identify some subsets of $\mathscr{D}$ that does not change under certain length variation. Specifically we exhibit a set of itineraries that always corresponds to points and a set of itineraries that always corresponds to intervals.

3.5 Invariant Subsets of $\mathscr{D}$

Having studied a specific subset of $\mathscr{D}$ whose G-equivariant topological type changes as the lengths data change. We now provide some sufficient conditions for subsets of $\mathscr{D}$ to have invariant G-equivariant topological type under lengths change. We begin with the definitions related to the angle metric.

Definition 3.18. Let X be a complete CAT(0) metric space and let $c : [0, a] \to X$ and $c' : [0, a'] \to X$ be two geodesic paths with $c(0) = c'(0)$. Given $t \in (0, a]$ and $t' \in (0, a']$, we consider the comparison triangle $\overline{\Delta}(c(0), c(t), c'(t')$ and the comparison angle $\overline{\angle}_{c(0)}(c(t), c'(t'))$. The *(Alexandrov) angle* between the geodesic paths c and c' is the number $\angle_{c,c'} \in [0, \pi]$ defined by:

$$\angle(c, c') := \limsup_{t,t' \to 0} \overline{\angle}(c(t), c'(t')) = \lim_{\epsilon \to 0} \sup_{0<t,t'<\epsilon} \overline{\angle}(c(t), c'(t'))$$

One can express $\angle(c, c')$ purely in terms of the distance function by noting that

$$\cos(\overline{\angle}_{c(0)}(c(t), c'(t'))) = \frac{1}{2tt'}(t^2 + t'^2 - d(c(t), c'(t'))^2)$$

Definition 3.19. Let X be a complete CAT(0) space. The angle between two geodesic segments which have a common end point is defined to be the angle between the unique geodesics which issue from this point and whose images are the given segments. If X is uniquely geodesic, $p \neq x$ and $p \neq y$, then the angle between the geodesic segments $[p, x]$ and $[p, y]$ may be denoted $\angle_p(x, y)$. The angle $\angle(\xi, \eta)$ between $\xi, \eta \in \partial_\infty X$ is defined to be:

$$\angle(\xi, \eta) = \sup_{x \in X} \angle_x(\xi, \eta)$$

Definition 3.20. Let X be a complete CAT(0) space. and let ∂X be spaces of all equivalence class of geodesic rays. The *Tits metric*, d_T is defined as follows:

1. If ξ and η are points of ∂X which cannot be joined by a path which is rectifiable in the angular metric, then $d_T(\xi, \eta) = \infty$.
2. If ξ and η are at distance less than π apart are joined by a unique geodesic in $(\partial X, \angle)$, so for $\angle(\xi, \eta) < \pi$, then $d_T(\xi, \eta) = \angle(\xi, \eta)$

Proposition 3.21. *If for some $g \in G$, the path that traces the infinite word g^∞ traces out the template of an infinite itinerary, then the boundary of the itinerary is always a point in $\mathscr{D}$ under any lengths data.*

Proof. By tracing out a template we mean the path from x_0 to x_1 along the two sides of the rectangle S_1 followed by the path from x_1 to x_2 along the two sides of the rectangle S_2, etc. Recall that the coordinates of an infinite itinerary is the sequence $\{(l_i, \epsilon_i) : i = 1, 2, 3...\}$ where (l_i, ϵ_i) measures the coordinate of x_i with respect to x_{i-1}.
l_i measures the length of the side that is labeled by $\langle b \rangle$ (resp. $\langle c \rangle$) and ϵ_i measures the length of the sides labeled by $\langle a, c \rangle$ (resp. $\langle b, d \rangle$).
If an infinite itinerary is traced by g^∞ for some $g \in G$, then

$$l_i = l_1, \text{ if } i \text{ is odd;}$$
$$l_i = l_2, \text{ if } i \text{ is even;}$$
$$\epsilon_i = \epsilon_1, \text{ if } i \text{ is odd;}$$
$$\epsilon_i = \epsilon_2, \text{ if } i \text{ is even.}$$

Therefore the adjusted coordinates are:

$$\begin{aligned}
&x_0 : (0,0)\\
&x_1 : (l_1, \epsilon_1)\\
&x_2 : (l_1 + \epsilon_2, \epsilon_1 + l_2)\\
&x_3 : (2l_1 + \epsilon_2, 2\epsilon_1 + l_2)\\
&x_4 : (2l_1 + 2\epsilon_2, 2\epsilon_1 + 2l_2)\\
&\vdots\\
&x_i : ((k+1)l_1 + k\epsilon_2, (k+1)\epsilon_1 + kl_2), \text{if } i = 2k+1\\
&x_i : (kl_1 + k\epsilon_2, k\epsilon_1 + kl_2), \text{if } i = 2k
\end{aligned}$$

Limit of the tangent of the angles of geodesic rays passing through x_{even} is

$$\begin{aligned}
&\lim_{k\to\infty} \frac{k\epsilon_1 + kl_2}{kl_1 + k\epsilon_2}\\
=&\frac{\epsilon_1 + l_2}{l_1 + \epsilon_2}
\end{aligned}$$

Limit of the tangent of the angles of geodesic rays passing through x_{odd} is

$$\begin{aligned}&\lim_{k\to\infty}\frac{(k+1)\epsilon_1+kl_2}{(k+1)l_1+k\epsilon_2}\\=&\frac{\epsilon_1+l_2}{l_1+\epsilon_2}\end{aligned}$$

Since the two limits are equal, the visual boundary component corresponds to this itinerary is a point in $\mathscr{D}$. When the lengths data change from (a, b, c, d) to (a', b', c', d'), the value of $l_1, \epsilon_1, l_2, \epsilon_2$ change, while the limits are still equal and therefore the boundary component remains a point for all lengths data.

$\square$

We can further strengthen this proposition into the following statement:

Proposition 3.22. *If an infinite itinerary has bounded coordinates, then its visual boundary is always a point in $\mathscr{D}$.*

Proof. The coordinates of an infinite itinerary is defined as above. If an infinite itinerary has bounded coordinates, then there is a positive real number $R \in \mathbb{R}^+$ such that

$$l_i \leqslant R, \text{ for all } i$$
$$\epsilon_i \leqslant R, \text{ for all } i$$

Consider the diameter, in Tits metric, of the connected components of $\mathscr{D}$. For the rest of this chapter, let

$$\angle_{x_0}(p, q)$$

denote the angle at the point x_0 of two geodesics, one emanates from x_0 and reaches p, the other eminates from x_0 and reaches q. Given that the Croke-Kleiner space X is a uniquely geodesic space, the angle $\angle_{x_0}(p, q)$ is well-defined. Another

The Euclidean geometry of the template forces that, for each template T and the sequence of strips S_i

$$diam_T(\partial_\infty T) < \angle_{x_0}(x_i, x_{i+1}), \text{ for all } i$$

The angle $\angle_{x_0}(x_i, x_{i+1})$ is less than the "visual angle" of the diameter of each strip at base point x_0.

$$diam_T(\partial_\infty T) < \angle_{x_0}(x_i, x_{i+1}) < \angle_{x_0}(diam(S_i)), \text{ for all } i$$

The "visual angle" of an object is the diameter of the object divided by the distance traveled to reach the project. The diameter of each strip is bounded by:

$$\sqrt{R^2 + R^2} = \sqrt{2}R$$

The distance traveled to get to S_i is $i/2$ times a scalar multiple of $diam(S_1) + diam(S_2)$. Therefore, it can be verified that

$$\lim_{i \to \infty} \frac{\sqrt{2}R}{i/2(diam(S_1) + diam(S_2))} = 0$$

As we change the lengths data, $(diam(S_1) + diam(S_2))$ changes by a scalar factor, but the above limit still goes to zero. Therefore for bounded strips the $diam_T(\partial_\infty T) = 0$ for any lengths data and hence the boundary is invariantly a point. Since there is a natural 1-1 map from the points of Tits boundary to the points of visual boundary, the visual boundary of such template is also a point.

$\square$

Next we want to discuss further what other growth of the function $(l(i), \epsilon(i)$ will result in a point in the visual boundary. If there is a constant $c > 0$ such that

$$f(x) \leqslant cg(cx + c) + c$$

for all x in the domain, then we say $f \preceq g$. If $f \preceq g$ and $g \preceq f$, then as functions f and g are *equivalent*. A function grows linearly if it is equivalent to a linear function, and a function grows polynomially if it is equivalent to a polynomial function.

Proposition 3.23. *Any itinerary whose coordinate grows linearly corresponds to a point and is invariant.*

Proof. Similar to the previous proof, The dimension of the strips l_i, ϵ_i can be thought of as functions of their indices, $l(i), \epsilon(i)$ If the dimension of the strips $l(i), \epsilon(i)$ grows linearly as a function of i, then the distance traveled to S_i is the partial sum of the strips that come before it. the partial sum grows quadratically,

$$\lim_{n\to\infty} \frac{\sqrt{2}n}{n^2} = 0$$

Therefore the Tits boundary is a point. If we change the length data, the numerator and denominator in the limit both change by a scalar and the limit remains zero. Again we invoke the bijection between the point set of Tits boundary and the point set of visual boundary and conclude that the visual boundary is also a point.

□

Proposition 3.24. *Any itinerary whose coordinate grows polynomially corresponds to a point and is invariant.*

Proof. As is with the previous proof we use the fact:

$$diam_T(\partial_\infty T) < \angle_{x_0}(diam(S_i)), \text{ for all } i$$

Since

$$
\begin{aligned}
& diam_T(\text{diagonal}(S_i)) \\
= & \lim_{i\to\infty} \frac{\text{diagonal}(S_i)}{\sqrt{(l_1 + l_2 + ... l_{i-1})^2 + (\epsilon_1 + \epsilon_2 + ... \epsilon_{i-1})^2}} \\
= & \lim_{i\to\infty} \frac{\sqrt{l_i^2 + \epsilon_i^2}}{\sqrt{(l_1 + l_2 + ... l_{i-1})^2 + (\epsilon_1 + \epsilon_2 + ... \epsilon_{i-1})^2}}
\end{aligned}
$$

Definition 3.25. We say that a template or itinerary grows *polynomially* if and only if $l_i = f(i)$ and $\epsilon_i = g(i)$ where both $f(i)$ and $g(i)$ are polynomials of the same degree.

If a sequence grows polynomially of degree k, then its partial sum grows polynomially of degree $k + 1$ by discrete calculus, therefore the limit

$$
\begin{aligned}
& \lim_{n\to\infty} \frac{\alpha n^k}{\beta n^{k+1}} \text{ for some rational numbers } \alpha, \beta \\
& = \lim_{n\to\infty} \frac{\alpha}{\beta n} \\
& = 0
\end{aligned}
$$

□

In the proof of the main theorem, for a n-self-similar template, both $f(i)$ and $g(i)$ grows exponentially, and the function itself over its partial sum does not vanish as $i \to \infty$, therefore the boundary is dependent upon $(l_1, \epsilon_1, l_2, \epsilon_2)$. We understand that the proposition is not a complete characterization. That is to say, not all exponentially growing template vary the topological type under length change, i.e. there exists itinerary of "weak" exponential growth with G-equivariant homeomorphism type which is however invariant under lengths data change. In the example below, $f(i)$ grows exponentially while $g(i)$ is constant.

Example Consider the two strips have sides $a^{\epsilon_1}, b^{l_1}, d^{\epsilon_2}, c^{l_2}$. for some $n \in \mathbb{N}$ we have

$$k = 2m + 1 : l_k = n^m l_1, \epsilon_k = \epsilon_1$$

$$k = 2m + 2 : l_k = l_2, \epsilon_k = n^m \epsilon_2$$

Therefore we have:

$$l_1 = l_1, l_3 = nl_1, l_5 = n^2 l_1, l_7 = n^3 l_1 ...$$

$$\epsilon_1 = \epsilon_3 = \epsilon_5 = \epsilon_7 ...$$

$$l_2 = l_4 = l_6 = l_8 ...$$

$$\epsilon_4 = n\epsilon_2, \epsilon_6 = n^2 \epsilon_2, \epsilon_8 = n^3 \epsilon_2 ...$$

The adjusted coordinates are:

$$
\begin{aligned}
&x_0 : (0,0)\\
&x_1 : (l_1, \epsilon_1)\\
&x_2 : (l_1 + \epsilon_2, \epsilon_1 + l_2)\\
&x_3 : (2l_1 + \epsilon_2, (n+1)\epsilon_1 + l_2)\\
&x_4 : (2l_1 + (n+1)\epsilon_2, (n+1)\epsilon_1 + 2l_2)\\
&x_5 : (3l_1 + (n+1)\epsilon_2, (n^2+n+1)\epsilon_1 + 2l_2)\\
&x_6 : (3l_1 + (n^2+n+1)\epsilon_2, (n^2+n+1)\epsilon_1 + 3l_2)\\
&x_7 : (4l_1 + (n^2+n+1)\epsilon_2, (n^3+n^2+n+1)\epsilon_1 + 3l_2)\\
&x_8 : (4l_1 + (n^3+n^2+n+1)\epsilon_2, (n^3+n^2+n+1)\epsilon_1 + 4l_2)\\
&\vdots\\
&x_i : (kl_1 + \frac{1-n^{k-1}}{1-n}\epsilon_2, \frac{1-n^{k-1}}{1-n}\epsilon_1 + kl_2), \text{if } i = 2k\\
&x_i : ((k+1)l_1 + \frac{1-n^{k-1}}{1-n}\epsilon_2, \frac{1-n^{k}}{1-n}\epsilon_1 + kl_2), \text{if } i = 2k+1
\end{aligned}
$$

Limit of the tangent of the angles of geodesic rays passing through x_{even} is

$$
\begin{aligned}
&\lim_{k\to\infty} \frac{\frac{1-n^{k-1}}{1-n}\epsilon_1 + kl_2}{kl_1 + \frac{1-n^{k-1}}{1-n}\epsilon_2}\\
=&\frac{\epsilon_1}{\epsilon_2}
\end{aligned}
$$

Limit of the tangent of the angles of geodesic rays passing through x_{odd} is

$$\lim_{k\to\infty} \frac{\frac{1-n^k}{1-n}\epsilon_1 + kl_2}{(k+1)l_1 + \frac{1-n^{k-1}}{1-n}\epsilon_2}$$
$$= n\frac{\epsilon_1}{\epsilon_2}$$

Compare the two limits,

$$\frac{\epsilon_1}{\epsilon_2} < n\frac{\epsilon_1}{\epsilon_2}$$

The inequality holds when ϵ_1 and ϵ_2 change lengths. If the original template corresponds to point, it will always a point; if the original template corresponds to an interval, it will always be an interval.

Remark. By construction there is at least a countable, dense subset of components whose homeomorphism types change under the G-equivariant map; meanwhile there is at least a countable, dense subset of components whose homeomorphism types are invariant under the G-equivariant map. The complete characterization of the invariant sets in $\mathscr{D}$, their group theoretical significance and what roles they play in other related boundaries such as the Poisson boundary [AN11] and the contracting boundary, are possible directions for future research that is highly interesting.

3.6 Revisit the Tits Boundary

Consider the Tits boundaries of the whole space $\partial_T X$, $\partial_T X_l$. The cores are isometric as shown in 3.2, the $\mathscr{D}$ in Tits topology is the union of an uncountable set of intervals and an uncountable set of points. If lengths variation does not change the cardinality of the uncountable sets, then the Tits boundaries are homeomorphic. We know that $\mathscr{D}$ is in bijection with the set of all infinite sequence of the natural numbers, which is the cardinality of the real numbers. We also know that the set of points in $\mathscr{D}$ and the set of all intervals in *dust* are both uncountable sets. The following two propositions would be sufficient in order to prove that the Tits boundaries are homeomorphic:

Proposition 3.26. *There is a bijection between the connected components of $\mathscr{D}$ in the following sense:*

1. *There is a bijection between the sets of all intervals in dust of X and X_l.*

2. *There is a bijection between the sets of all points in dust of X and X_l.*

Proof. Since the set $\mathscr{D}$, by construction in Lemma 3.15, is in bijection with the set of all infinite sequence of natural numbers, each number in the sequence denoting a block, $\mathscr{D}$ has cardinality that of $\mathbb{R}$. By the proof of Proposition 3.11 and by Corollary 3.12, the set of all points as well as the set of all intervals surject on to the set of all real numbers. A subset of the real numbers that surjects on to the set of real numbers is necessarily in bijection with the set reals. Therefore, two such sets are in bijection to one another. □

Hence we can conclude the following:

Theorem 3.27. *The Tits boundaries of X and X_l are homeomorphic.*

Proof. The Tits boundary of a Croke-Kleiner space necessarily consists of the core and the dust $\mathscr{D}$. The core components are isometric to one another. The dusts consist of isolated points and isolated intervals. we showed in the previous proposition that there is a bijection between the sets of isolated points and a bijection between the sets of isolated intervals. Therefore the Tits boundaries are homeomorphic. □

3.7 Open Questions and Conjectures

Question 1: Is $\partial_\infty X$ homeomorphic to $\partial_\infty X_l$ as topological spaces?

This was the original motivation of the project. In [CK00] with angle change, the spaces are shown to be different topological spaces, which settles the question of unique boundary. However, changing angle changes the space greatly. for instance, X_θ is no longer a CAT(0) *cube* space. If we respect the local, rectangular structure, then we are only able to show unique boundary up to group equivariance. For geometers and topologists it remains highly interesting whether two spaces whose only difference are the dimensions of the rectangles have a topologically different "space at infinity".

Question 2: Complete characterization of $\mathscr{D}$ as topological spaces. Are $\mathscr{D}$s of Croke-Kleiner spaces of different lengths data homeomorphic?

We know so far that $\mathscr{D}$ consists of an uncountable set of connected components. Each connected component is either a point or an interval. Each connected component is the limit of a sequence of connect components. The set of all point-sets is a dense subset. The set 0f all interval-sets is also a dense subset.

In the course of studying the dust we considered the concept of "cantorval"[Nit13]. Formally, A *symmetric Cantorval* is a nonempty compact subset S of the real line such that

1. S is the closure of its interior (i.e., the nontrivial components are dense)
2. Both endpoints of any nontrivial component of S are accumulation points of trivial (i.e., one-point) components of S.

The remarks above establish a full topological classification of subsum sets for summable positive sequences, proven by Guthrie and Nymann:

Theorem 3.28. *(Guthrie-Nymann) The subsum set of a positive summable sequence is one of the following:*

1. *a finite union of (disjoint) closed intervals;*
2. *a Cantor set;*
3. *a symmetric Cantorval.*

A cantorval demonstrates a lot of the same topological properties that we observe in dust however, we were never able to show that the dust embeds in $\mathbb{R}$. If $\mathscr{D}$ were a cantorval, then Question 2 would be answered because all cantorvals are homeomorphic.

Question 3: Complete characterization of invariant subspaces of *dust* under lengths variation.
We provide some necessarily conditions for a subset of *dust* to be invariant. In general it remains open how to characterize a template that is invariant under length variation.

Question 4: Is there a map between the X and X_l that extends to a homeomorphism between $\partial_\infty Core(X)$ and $\partial_\infty Core(X_l)$?
We conjecture that there is not a G-equivariant map between the spaces X and X_l that extends to homeomorphisms on the visual boundary of the cores. If the conjecture is true, then the next question to consider is that if we view X and X_l only as topological spaces, is there any maps on the space that extends ”nicely” to the boundary.

Question 5: We prove in Section 3.6 that for this specific example of CAT(0) space, the Tits boundaries of X_l and X are homeomorphic to one another. Since the angles in the geometric data are $\pi/2$, for all the rational lengths change, the space X_l can be given a new cubing such that it is a new CAT(0) cube complex. Therefore we can ask the question: does a rank-1 CAT(0) *cube* complex have unique Tits boundary?

Chapter 4

Right-angled Coxeter Groups

The previous chapter provides an example of a right-angled Artin group which does not have a unique equivariant boundary. It is still an open question whether the class of right-angled Coxeter groups have unique boundary, or whether they have unique equivariant boundary. To investigate the question of unique boundary, we consider the action of right-angled Coxeter groups on the space constructed by Croke-Kleiner. Suppose the action is geometric, we want to know that given one such right-angled Coxeter group, does varying all the geometric data $(\theta_1, \theta_2, \theta_3, a, b, c, d)$ changes the homeomorphism of its equivariant boundary. The main result is that the right-angled Coxeter group that acts geometrically on the Croke-Kleiner space does not have an equivariantly unique boundary. Therefore we answer the question about the uniqueness of equivariant visual boundaries.

Specifically, we prove that unlike the case of right-angled Artin groups, the Croke-Kleiner space does not support a geometric action by a right-angled Coxeter group if the three angles on the tori $\theta_1, \theta_2, \theta_3$ are not fixed at $\pi/2$.

Take X as defined in Chapter 3. There are many embedded flats in X. Among them are the flats that are universal covers of the tori T_i, which we call *special flats*. In this chapter we prove that:

Theorem 4.1. *Let W, a right-angled Coxeter group, act geometrically on the Croke-Kleiner* *complex and preserve special flats, then the angles $\theta_1, \theta_2, \theta_3$ must all be right angles.*

This implies that the "right-angled" in the terminology "right-angled Coxeter group" turns out to be literal and is consistent with the "geometric" property of the group. Furthermore, given the fact that if we fix the gluing angle of the Croke-Kleiner space at $\pi/2$ and change the side lengths of the tori, the resulting boundaries are not equivariantly homeomorphic to each other, we conclude the following:

Corollary 4.2. *There exists right-angled Coxeter groups that does not have unique equivariant visual boundary.*

Aside from answering the open question, this close examination of the interplay between right-angled Coxeter actions and the $CAT(0)$ geometry of the Croke-Kleiner space shows that right-angled Coxeter groups can be more "geometrically rigid" than its counterpart in the class of right-angled Artin groups.

4.1 Proof of Theorem 1.5

In this section we restate and proof the main result of this chapter, Theorem 1.5:

Let G, an essential right-angled Coxeter group, acts geometrically on X, the universal cover of the three-torus complex, then the intersecting angle on the middle torus must be a right angle.

The proof makes use of the Bass-Serre tree defined as the *nerve* in the last chapter: let each lift of T_i be projected down to a vertex. Two vertices are adjacent if and only if two corresponding planes intersect along a line. The nerve is an infinite tree, $\mathscr{T}_N$ in which every vertex has countably infinite adjacent vertices. Each lift of T_2 is adjacent to countably infinite lifts of both T_1 and T_3; lifts of T_1 or T_3 are only adjacent to countably infinite lifts of T_3.

A *strict fundamental domain* is a closed set in which there is exactly one point in every orbit. By O'Brien[O08], the action of a Coxeter group W on a space X has a strict fundamental domain if and only if for every $x \in X$ and $w \in W$, every path from x to $w.x$ meets the

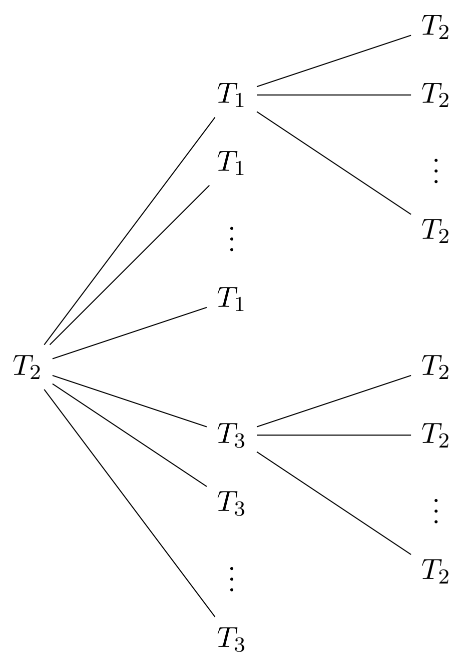

Figure 4.1: The Nerve

fixed point set of X. Since the action is on a tree, this condition(which is called *generalized reflection*) is satisfied, therefore there exists a strict fundamental domain of a right-angled Coxeter group acting on its nerve.

4.2 The Gluing Theorem

To begin with, let the group be presented as following with a minimal generating set:

$$W = \{s_1, s_2, ...s_n | s_i^2 \text{ for all } i, [s_i, s_j] \text{ for some pairs } i, j\}$$

The main theorem of this chapter is the following:

Theorem 4.3. *Let W, a right-angled Coxeter group, act geometrically on the Croke-Kleiner space and preserve special flats, then the intersection angle on the middle torus must be $\frac{\pi}{2}$.*

Recall a *special flat* is a flat that is a lift of T_i. For the remainder of the paper, we use "planes" to refer to such flats. In order to prove the theorem, we start with lemmas about the action of the generators and the stabilizers of each plane.

Lemma 4.4. *For the given complex, each generator of the right-angled Coxeter group acts on the nerve tree without inversion.*

Proof. In the nerve, we have vertices that are labeled by the torus of which they denote the universal cover, by abuse of notation, the edges in the nerve are also labeled by either of the following pairs:

$$\{T_2, T_1\}, \{T_2, T_3\}$$

Isometry in the space induces a homeomorphism on the boundary. The boundary of the planes $\widetilde{T_2}$ is a circle with four poles, the boundary of the planes $\widetilde{T_1}$ and $\widetilde{T_3}$ are circles with two poles. Since an isometry of the space induces a homeomorphism on the boundary that takes circles to circles and poles to poles, an isometry cannot invert edges. $\square$

In the following definitions we lay out O'Brien's construction[O08] of a strict fundamental domain:

Definition 4.5. For a group element $w \in W$, let X^w be the fixed point set of w. Let $\mathscr{T}_w(Y)$ be the set of components of X/X^w. If $w = s_i$, we simply write $\mathscr{T}_i$.
Let

$$\mathscr{T} := \bigcup_{i \in I} \mathscr{T}_i$$

Let T denote a connected component of $\mathscr{T}_i$, and let

$$\tilde{\mathscr{T}_i} := \{T \cup X^{s_i} | T \in \mathscr{T}\}$$

The elements of $\tilde{\mathscr{T}_i}$ we denote as $\tilde{T} := T \cup X^{s_i}$.
We know that if s_i and s_j does not commute, then $X^{s_i} \cap X^{s_j} = \phi$. Moreover, since X^{s_j} is connected, there exists a unique component in $\mathscr{T}_i$ that contains X_{s_j}, which we denote T^y_{ij}, a *yes*-component, and a *no*-component is defined as $T^n_{ij} := s_i.T^y_{ij}$.
Let

$$\tilde{T^y} := \bigcap_{i \in I} \bigcup_{j \in \underline{kl(i)}} (T^y_{ij} \cup X^{s_i})$$

By this construction we have the following corollary: Let us take a generator and mark its fixed point set in the Bass-Serre tree. Then there is exactly a strict fundamental domain whose vertices are labeled by all the generators.

Lemma 4.6. *Suppose a right-angled Coxeter group acts geometrically on the Croke-Kleiner space and takes planes to planes. If a group element w fixes a plane $\widetilde{T_i}$ set-wise, suppose $w = s_k s_{k-1} ... s_2 s_1$, then each s_i fixes $\widetilde{T_i}$ set-wise.*

Proof. Without loss of generality, let s_1 not fix the plane T_i, otherwise let $w = s_k s_{k-1} ... s_2$. Let j be the smallest number such that the subword $s_j s_{j-1} ... s_2 s_1$ fixes the T_i. Consider generators s_1 and s_j. In $\mathscr{T}_0$, s_j and s_1 each label a vertex, v_{s_j} and v_{s_1}. $\mathscr{T}_0$ also contains a lift of T_i, label it v_0. Since $\mathscr{T}_0$ is a tree, there are unique paths (v_0, v_{s_j}) and (v_0, v_{s_1}). The word $s_{j-1} s_{j-2} ... s_2$ takes the edges (v_0, v_{s_j}) to the edges (v_0, v_{s_1}). This contradicts the assumption that T_2 is a strict fundamental domain. Therefore, each s_i fixes $\widetilde{T_i}$ set-wise. $\square$

Now we can analyze the stabilizer subgroups of each T_i.

Proposition 4.7. *Given the universal cover of T_i, denoted $\widetilde{T_i}$ consider the stabilizer subgroup $Stab(\widetilde{T_i})$, then $Stab(\widetilde{T_i})$ is generated by a (conjugate) of a subset of the generating set $\{s_1, s_2, ... s_n\}$, respectively.*

Proof. Each generator acts simplicially on the nerve tree of planes. Furthermore, let every edge has length 1, then each group element acts isometrically on the tree. Each generator is of order two. Therefore the fixed point set of each generator acting on the nerve tree is either an induced subgraph or the midpoint of an edge. Lemma 5 rules out the latter case.

Since W acts cocompactly on the space, it acts cocompactly on the nerve. By Bass-Serre theory, there exists a minimal finite tree that is the strict fundamental domain of W on the nerve, which we denote by $\mathscr{T}$. This tree is the fundamental domain of the group acting on this tree, therefore generators of W does not take points of $\mathscr{T}$ to points of $\mathscr{T}$.

Since each generator acts on the nerve tree without inversion, take a generator and label the set of vertices in the nerve tree that it fixes. O'Brien[O08] proves that there is a copy of $\mathscr{T}$ whose vertices are labeled by the generators, not their conjugates, of the group. Take this copy of T with the labeling and denote it $\mathscr{T}_2$. Consider the stabilizer subgroup of each vertex in $\mathscr{T}_2$, since each group element that stabilizes a vertex of $\mathscr{T}_2$ is generated by a subset of generators that stabilizes the vertex, we conclude that $Stab(\widetilde{T}_i)$ are **special subgroups**, i.e. they are generated by a subset of generators.

□

How does $Stab(\widetilde{T}_i)$ act on $\widetilde{T}_i$? We claim the group acts isometrically, cocompactly.

Proposition 4.8. *$Stab(\widetilde{T}_i)$acts cocompactly and by isometries on the plane.*

Proof. The group acts cocompactly on the the space. If K is a fundamental domain for $W \curvearrowright X$, then

$$K \cap \widetilde{T}_i$$

is the fundamental domain for the actions of $Stab(\widetilde{T}_i)$. Therefore $Stab(\widetilde{T}_i)$ acts cocompactly on the plane it stabilizes. □

Next we study a right-angled Coxeter group acting cocompactly and by isometries on a 2-dimensional Euclidean plane. For a presentation of a right-angled Coxeter group

$$W = \{s_1, s_2, ...s_n | s_i^2 \text{ for all } i, [s_i, s_j] \text{ for some pairs } i, j\}$$

We associate a **defining graph**, whose vertices are generators of W and edges are the two-element subsets of the generating set that commutes. First one can rule out the defining graphs on less than or equal to three vertices since they either have 0, 2, or infinite ends.

Recall the Svarc-Milnor Lemma of group acting geometrically on Euclidean flats:

Proposition 4.9. *(The Svarc-Milnor Lemma). Let X be a length space. If Γ acts properly, cocompactly by isometries on X, then Γ is finitely generated and for any choice of base-point $x_0 \in X$, the map $y \to yx_0$ is a quasi-isometry.*

Also recall Gromov's Theorem [BH99]:

Theorem 4.10. *If a finitely generated group is quasi-isometric to $\mathbb{Z}^n$ then it contains $\mathbb{Z}^n$ as a subgroup of finite index.*

Lemma 4.11 (Key Lemma)**.** *Suppose W is a right-angled Coxeter group acting cocompactly and by isometries on the plane $\mathbb{E}^2$. Then we claim that W must be the direct product of two copies of the infinite dihedral group.*

Proof. We know that the group W has at least four generators. Since W contains $\mathbb{Z}^2$ as a subgroup of finite index, it is not hyperbolic. By [Mou88], if Γ is the defining graph of W, then in Γ there exists induced subgraphs A, B such that $\langle A\rangle$, $\langle B\rangle$ are infinite and A-join-B is a subgraph of Γ. In particular, there exists two infinite order elements $\gamma_1' = s_1t_1, \gamma_2' = s_2t_2$ such that the subgraph on the vertices s_1, s_2, t_1, t_2 is a join of two pairs of non-adjacent vertices. The subgraph is a chordless 4-cycle, where s_1 is adjacent to s_2 and t_2, and t_1 is adjacent to s_2 and t_2.

The actions of s_1, t_1, s_2, t_2 are order-2 isometries of the plane, which are either reflecting across a straight line l, or rotate around a point p by π. Two such elements commute in the following cases:

- l_1 and l_2 intersecting at right angle
- $l \cap p \neq \phi$

An infinite order action must be a composition of these order-2 isometries as one of the following cases:

1. $l \cap p = \phi$
2. $l_1 \cap l_2 = \phi$

3. $p_1 \cap p_2 = \phi$

In the $\mathbb{Z}^2$ subgroup, there are two elements of infinite order, both generators in one of the three pairs of elements commutes with both generators of another one, not necessarily different, of the three pairs.
(1) and (1), impossible since a point cannot simultaneously coincide with another point off the line and be on the line, For the same reason, (1) and (3) is also impossible.
(1) and (2), impossible since there is only one straight line that passes perpendicularly through another line and a point off that line;
(2) and (3), impossible, since one point cannot be on two parallel lines;
(3) and (3), impossible since one point cannot coincides with two points.
Therefore the only possibility is (2) and (2): two pairs of parallel lines intersecting at right angle. The defining of this group consists of four vertices and four edges connecting up to a four-gon.
To have this group as a subgroup of finite index, by the Finite Index Lemma [MRT07] we must have in the defining graph a complete graph joined to the chord-less 4-cycle. This is to say the generators not in the chord-less 4-cycle commutes with the four reflections. By the previous argument, there cannot be order-2 symmetries of the plane that commutes with all four reflections. Therefore, if a right-angled Coxeter group acts geometrically on a plane, the actions of the group restricted to the plane is isomorphic to

$$G = D_\infty \times D_\infty =< a, b, c, d | a^2, b^2, c^2, d^2, [a, c], [a, d], [b, c], [b, d] >$$

□

A direct corollary is the following,

Corollary 4.12. *If a right-angled Coxeter group acts geometrically on a plane, then it is isomorphic to $D_\infty \times D_\infty$. The $D_\infty \times D_\infty$ acts on the plane like two pairs reflections cross lines. Each pair consists of two reflections whose fixed-point sets are parallel axes, and the* *two pairs of axes intersect at right angle.*

Next we study how the stabilizer subgroups piece together and determine the gluing angle of the complex.

Theorem 4.13. *If a right-angled Coxeter group acts geometrically on the Croke-Kleiner space and preserves special flats, then the angles $\theta_1, \theta_2, \theta_3$ must all be $\pi/2$.*

Proof. Consider a special flat that is a lift of either T_1 or T_3, without loss of generality, let it be T_1. Each of these flats is adjacent to countably lifts of T_2. The intersections we can label $l_{2,1}$. The $l_{2,1}$s are cosets of $\mathbb{Z}$ and therefore are parallel, bi-infinite geodesic rays. Since any nontrivial action, when restricted to a plane, is by reflection, there is necessarily an axis of reflection. If an axis of reflection is at an angle $\theta \neq 0, \pi/2$ with the $l_{2,1}$s, then the reflection takes a copy of $l_{2,1}$ to its image, which is not a copy of $l_{2,1}$. Since the boundary of $l_{2,1}$ is a pair of cone points, and isometry of the space induces homeomorphisms on the boundary, we arrive at a contradiction. therefore the axes of reflection is at angle 0 or $\pi/2$ with the $l_{2,1}$s. By Key Lemma, the reflection axes are two pairs of parallel lines, and one pair is perpendicular to the other. therefore the reflection axes in a plane labeled by T_1, T_3 is either parallel or perpendicular to the $l_{2,1}$s. Furthermore, by Proposition 4.7, the reflections are generators of the group W.

Now consider the special flats that are lifts of the middle torus T_2. In these flats, there are two sets of intersections with neighboring planes, labeled accordingly $l_{2,1}$ and $l_{2,3}$. All the $l_{2,1}$s are parallel to one another; all the $l_{2,3}$s are parallel to one another. Consider the angle θ between $l_{2,1}$ and $l_{2,3}$. Suppose $\theta \neq \pi/2$, then the only possibility for a set of four reflection, configured in the way specified in Key Lemma, can take intersections to intersections is to have them reflect across the diagonals of the unit parallelograms in the plane, as shown in Figure 4.2. In Figure 4.2, the solid lines are $l_{2,1}$ and $l_{2,3}$, the dashed lines are the axes of reflections.

In this case, it takes a two-letter word to reflect $l_{2,1}$ onto itself across a point. We argued

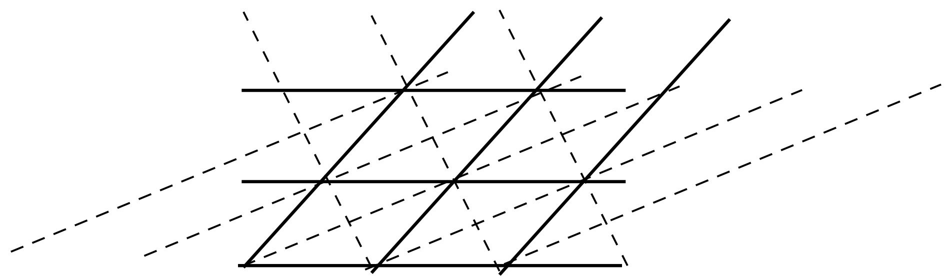

Figure 4.2: Gluing Theorem

in the first paragraph that there are generators that reflect $l_{2,1}$ to itself across a point. Since $l_{2,1}$ is also in the flat that is a lift of T_2, the same generator then act as reflection on the corresponding T_2 and its axis intersects $l_{2,1}$. However there are already reflection axes intersecting $l_{2,1}$ as established in the previous paragraph and neither of them reflect $l_{2,1}$ onto itself. Therefore we need to have a third reflection axis that is not parallel to the two existing axes. This configuration contradicts the Key Lemma. Therefore, it is not possible to have the intersection angle of $l_{2,1}$ and $l_{2,3}$ be $\theta \neq \pi/2$.

□

Remark. The main theorem states that if a right-angled Coxeter group acts geometrically on a Croke-Kleiner space and preserves special flats, then the angle must be fixed at $\frac{\pi}{2}$, therefore, the only parameters for the group action are the length data. We know that changing the length data changes the G-equivariant homeomorphism type of the boundary, it suffices to verify that changing the length data does not violates the requirement of a isometric, properly discontinuous, and cocompact action.

Since we can vary the distances between two parallel reflecting axes on each plane, we can indeed obtain actions of varied "translation lengths" and the action is still geometric, therefore, we can conclude from this and the previous chapter the following result:

Proposition 4.14. *If a right-angled Coxeter group acts geometrically on the Croke-Kleiner space $\widetilde{X}$, then changing the lengths data of the tori changes the equivariant type of it visual boundary. There are uncountably many equivariant visual boundaries of of the space.*

4.3 A Concrete Action

As discussed in the introduction of this chapter, we assumed without verifying that there does exist a right-angled Coxeter group that acts geometrically on the Croke-Kleiner space. In this section a specific RACG is given. In general there may be more than one right-angled Coxeter groups that is quasi-isometric to the Croke-Kleiner space. Let W be the right-angled Coxeter group defined by the following graph:

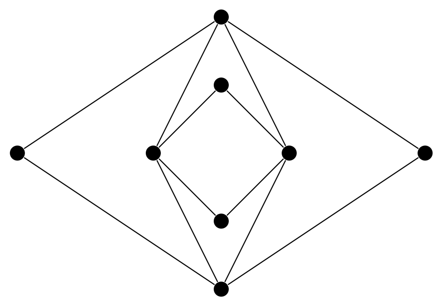

Figure 4.3: Defining graph of an RACG

Consider the Cayley graph of W with respect to this generating set. There are three "diamonds". Each "diamond" in the defining graph corresponds to an $H = D_\infty \times D_\infty$ whose Caley graph is a $\mathbb{Z} \times \mathbb{Z}$ lattice with each edge being replaced by a double-edge, i.e. a pair of edges that shares starting and ending vertices. Three diamonds generates three types of such double-edged lattices. These lattices are identified along double-edged $\mathbb{Z}-$lines according to the amalgamated product decomposition:

$$W = H *_{D_\infty} H *_{D_\infty} H$$

We observe that the Caley graph embeds into the Croke-Kleiner space as its dual: each vertex of the Cayley graph represent a unit square in the Croke Kleiner space and two vertices of the Cayley graph are connected by a pair of double-edges if and only if the corresponding two unit squares share a common edge in the Croke-Kleiner space. We observe that group W acts in its own Cayley graph by reflecting through the mid-point of the double edges, and it is easy to check that the induced action on the Croke-Kleiner space in which it embeds is indeed a geometric action.

In fact, we can observe by this construction that the Croke-Kleiner space is the Davis Complex of the group W. A Davis Complex can be constructed from the defining graph as follows:

1. Start with a a base vertex v_0

2. Construct the 1-skeleton of a unit, n-dimensional cube, where n is the number of vertices in the defining graph. We label all the edges by the vertices they corresponds to. For each vertex in the defining graph, there are 2^{n-1} edges with that label.

3. For every complete subgraph on k vertices in the defining graph, fill all the corresponding k-cubes in the lattice constructed previously. For each complete subgraph of k vertices, there are 2^{n-k} copies of cubes to fill in the lattice.

4. Take the universal cover of the previous cube complex.

We observe here that a pair of groups act on the same space X:

- a right-angled Artin group as its fundamental group

- a right-angled Coxeter group whose Davis complex is X

The defining graphs of them are as follows:

Γ_0 Γ_1

Figure 4.4: RAAG vs. RACG

In fact this phenomenon is true in general as observed in [DJ00] by Davis and Januszkiewicz. In this paper, for each right-angled Artin group, there is a corresponding right-angled Coxeter group which can be created in the following steps, let Γ be the defining graph of the right-angled Artin group:

1. Double the vertex set Γ, i.e. take two copies of the vertex set, label them with V_1, V_2; let vertices of Γ be *preimages* of vertices of V_1, V_2;

2. $v, v' \in V_1$ are adjacent if and only if their preimages are adjacent in Γ;

3. $v, v' \in V_2$ are adjacent if and only if their preimages are adjacent in Γ;

4. $v \in V_1, v' \in V_2$ are adjacent if and only if their preimages are adjacent in Γ.

The resulting graph Γ' determines uniquely[Rad03] a right-angled Coxeter group. [DJ00] proves that both groups always acts geometrically on the Davis complex of the latter.

Chapter 5

Uniqueness of Right-angled Coxeter Groups

An interesting project that occurred during the study documented in Chapter 4 is the following: what are all the right-angled Coxeter groups that acts geometrically on the given Croke-Kleiner space? We know that all such groups are quasi-isometric to one another by definition, but if we add the assumption that they are all right-angled Coxeter groups, can more things be said about them as a collection? Are they all a finite index subgroup of a certain group, or can they be constructed in a similar pattern, etc? The previous study points to some vague notion of "geometric rigidity", but in this chapter the question is open-ended and we start by investigating the action of right-angled Coxeter groups on an infinite, proper tree. A priori, an interested reader can ask the same question about any space of interest.

This is work in progress and the next natural step is to ask the question about the full Croke-kleiner space.

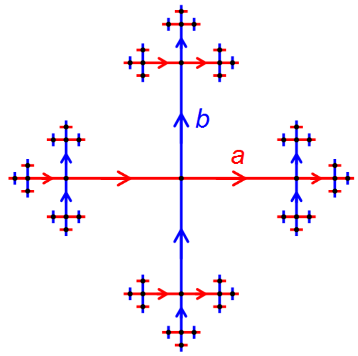

Figure 5.1: infinite, 4-valence tree

Start with the infinite tree:

The space is $CAT(0)$, 1-dimensional, 0-hyperbolic, and in many senses one of the first non-trivial $CAT(0)$ spaces to be act upon. We give a precise characterization of the RACGs that is quasi-isometric to this space and provide a geometric proof to the following theorem:

Theorem 5.1. *If a right-angled Coxeter group acts geometrically on $\mathscr{T}_4$, then it is an amalgamated product of finite copies of groups of the form $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

The statement is not true in reverse, i.e. there are groups that are amalgamated product of finite copies of groups of the form $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ but does not act geometrically on $\mathscr{T}_4$, for example:

$$(\mathbb{Z}_2 \times \mathbb{Z}_2) * (\mathbb{Z}_2 \times \mathbb{Z}_2)$$

The proof is a straight forward application of O'Brien's thesis [O08]. The thesis generalizes Bass-Serre theory of group acting on trees. Specifically O'Brien gives conditions for the existence of strict fundamental domain of right-angled Coxeter groups acting on a $CAT(0)$ space and constructs the strict fundamental domain if one does exists.

5.1 Proof

Suppose a right-angled Coxeter group W acts geometrically on $\mathscr{T}_4$. By O'Brien's thesis [O08], there exists a strict fundamental domain whose construction is laid out as follows.

Definition 5.2. For an element $w \in W$, let X^w be the fixed point set of w. Let $\mathscr{T}_w(Y)$ be the set of components of X/X^w. If $w = s_i$, we simply write $\mathscr{T}_i$.

Let

$$\mathscr{T} := \bigcup_{i \in I} \mathscr{T}_i$$

Let T denote a connected component of $\mathscr{T}_i$, and let

$$\tilde{\mathscr{T}_i} := \{T \cup X^{s_i} | T \in \mathscr{T}\}$$

The elements of $\tilde{\mathscr{T}_i}$ we denote as $\tilde{T} := T \cup X^{s_i}$.

We know that if s_i and s_j does not commute, then $X^{s_i} \cap X^{s_j} = \phi$. Moreover, since X^{s_j} is connected, there exists a unique component in $\mathscr{T}_i$ that contains X^{s_j}, which we denote T^y_{ij}, a *yes*-component, and a *no*-component is defined as $T^n_{ij} := s_i.T^y_{ij}$.

Let

$$\tilde{T^y} := \bigcap_{i \in I} \bigcup_{j \in \underline{kl(i)}} (T^y_{ij} \cup X^{s_i})$$

By Mark O'Brien's thesis there exist a finite subgraph T_0 that is the strict fundamental domain of the action of W on X. Moreover, if we mark out the fixed point sets of all generators on $\mathscr{T}_4$, then there exists a image, $g.T_0$ for some g, whose topological boundary in $\mathscr{T}_4$ contains all the fixed point sets of the generators of W.

Proposition 5.3. *The fixed point set of a an order-two action on $\mathscr{T}_4$ is one of the following:*

- *Mid-point of an edge*
- *a connected, spanning, finite subgraph of $\mathscr{T}_4$*
- *a connected, spanning, infinite subgraph of $\mathscr{T}_4$*

Proof. The fixed-point set of a group element is the set

$$\{x \in X | g.x = x\}$$

denoted by X^g. If X is a $CAT(0)$ space, then X^g is a closed, convex $CAT(0)$ subset of X, finite or infinite. Suppose X^g is a finite subset Γ that is not a subgraph, i.e. there exists in the topological boundary (with respect to $\mathscr{T}_4$) of X^g a point x that not a vertex of $\mathscr{T}_4$. Since $x \in X$ then x is an interior point of an edge e. However any isometry of $\mathscr{T}_4$ takes e to either to itself or to another edge. If the isometry takes e to itself, then x is the mid-point of an edge; if an isometry takes e to another edge then the interior $int(e)$ is either contained in X^g or disjoint from X^g, and hence X^g is a finite subgraph.
Similarly if X^g is an infinite subset, it has to be an infinite subgraph as well.

$\square$

Therefore T_0 by construction is a finite subset of $\mathscr{T}_4$ that consists of edges and half-edges, where *half-edges* denotes the segment between the mid-point of an edge together with the midpoint and one end point.

Furthermore, consider the degree of the vertices in T_0. We say an edge is *incident* to a vertex if the vertex is one of its end vertices. If a vertex $v \in T_0$ has degree 3, then by construction of T_0 there is exactly one edge e' incident to v that is not in the strict fundamental domain. However the image of e' under the generators whose fixed-point sets contribute to the inclusion of v in T_0 has to be one of the three edges, contradiction the definition of a strict fundamental domain.

If a vertex $v \in T_0$ has degree one, then v is the fixed point set of special subgroup of order 4. A special subgroup of a right-angled Coxeter group of order 4 is necessarily $\mathbb{Z}_2 \times \mathbb{Z}_2$. However, there is not an element of order 4 in $\mathbb{Z}_2 \times \mathbb{Z}_2$. which is another contradiction.

Therefore every vertex in T_0 is of degree 2 or 4, every mid-point in T_0 is of degree 1.
A mid-point necessarily has stabilizer subgroup $\mathbb{Z}_2$, and a degree-2 vertex has stabilizer subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$. A degree-4 vertex subgroup has trivial stabilizer subgroup.

The stabilizer subgroup of any half-edge is trivial since any isometry that fixes the half-edge also fixes the whole edge and is the identity on the mid-point. By Bass-Serre theory, the stabilizer subgroup of the half-edge embeds into the stabilizer subgroup of the mid-point. The only subgroup of $\mathbb{Z}_2$ that is identity on the half-edge is the trivial group. Therefore the stabilizer subgroup of each half-edge is the trivial group. The stabilizer subgroup of an edge is Z_2, which is an inversion of the edge.

Therefore, In T_0, every degree 1 vertex has stabilizer groups $\mathbb{Z}_2$, every degree 2 vertex has stabilizer group $\mathbb{Z}_2 \times \mathbb{Z}_2$, every degree 4 subgroups has trivial stabilizer group. Every half-edge has trivial stabilizer subgroup. Every full edge's stabilizer group is one of the two:

- If either of its end vertices is stabilized by the trivial group, then the edge's stabilizer group is trivial;
- If both its end vertices are stabilized by $\mathbb{Z}_2 \times \mathbb{Z}_2$, then the edge's stabilizer group is $\mathbb{Z}_2$

If a vertex v_0 and both of its neighboring vertices v_1, v_2 are all of degree 2, with the edges label $e_1 = (v_0, v_1), e_2 = (v_0, v_2)$, then the edges groups G_{e_1}, G_{e_2} embeds into different copies of $\mathbb{Z}_2$ in the stabilizer group of v_0 which is $\mathbb{Z}_2 \times z_2$.
Lastly, by Bass-Serre theory, the graph of groups of T_0 is W.

For the remainder of the chapter we see a few examples of the main theorem. In the examples we use $\circ$ to denote the mid-points and $\bullet$ to denote the vertices.

Example 1

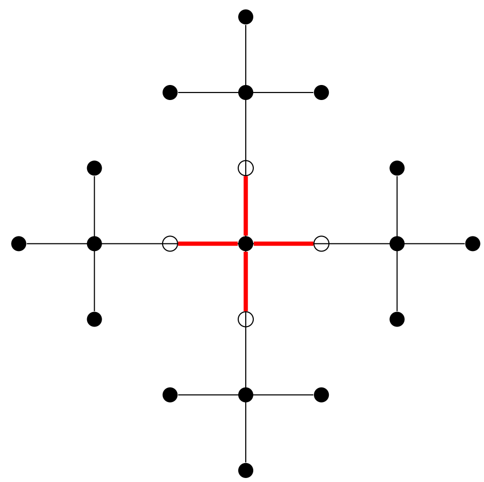

Figure 5.2: Example 1

Since the stabilizer subgroup of a mid-edge point is Z_2, and the stabilizer subgroup of a valence-4 vertex in the fundamental domain is trivial, the right-angled Coxeter group is:

$$
\begin{aligned}
G &= \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 \\
&= \mathop{\scalebox{1.5}{$*$}}_{i=1}^{4} (\mathbb{Z}_2)_i
\end{aligned}
$$

Example 2

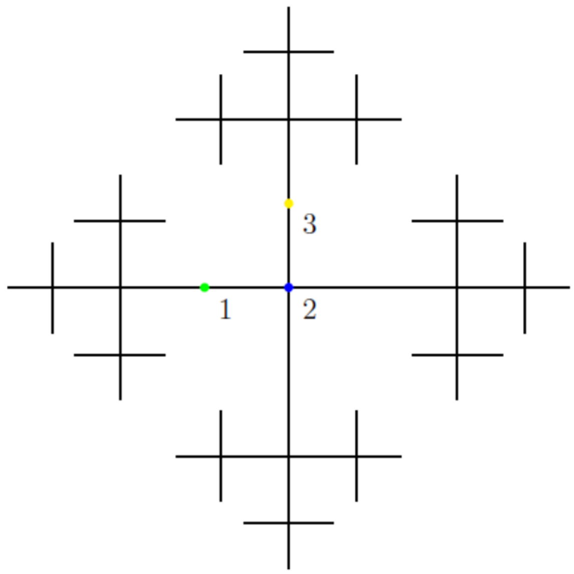

Figure 5.3: Example 2

The stabilizer groups of vertex 1 and 3 are $\mathbb{Z}_2$, the stabilizer of 2 is $\mathbb{Z}_2 times \mathbb{Z}_2$. The edge stabilizer groups embed into its corresponding vertex groups, hence the stabilizer groups of half-edges are necessarily trivial. Therefore the right-angled Coxeter groups associated with this strict fundamental domain is

$$G = \mathbb{Z}_2 * (\mathbb{Z}_2 \times \mathbb{Z}_2) * \mathbb{Z}_2$$

Example 3

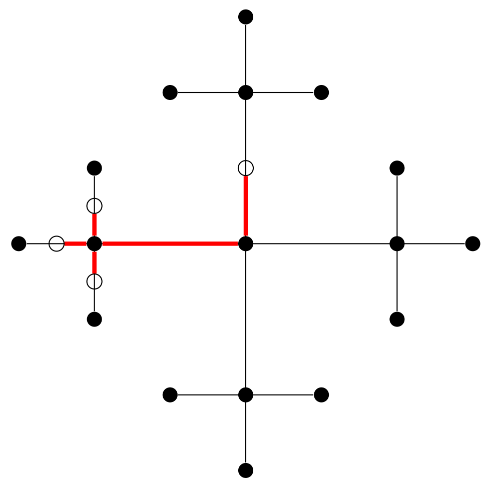

Figure 5.4: Example 3

The mid-edge vertex groups are $\mathbb{Z}_2$. The valence-4 vertex has trivial stabilizer group. The valence-2 vertex has stabilizer:

$$\mathbb{Z}_2 \times \mathbb{Z}_2$$

The edge groups embed into their corresponding vertex groups and hence are all trivial, therefore the right-angled Coxeter group associated with this strict fundamental domain is:

$$\begin{aligned} G &= \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * (\mathbb{Z}_2 \times \mathbb{Z}_2) \\ &= \mathop{\Huge *}_{i=1}^{4} (\mathbb{Z}_2)_i * (\mathbb{Z}_2 \times \mathbb{Z}_2) \end{aligned}$$

Example 4

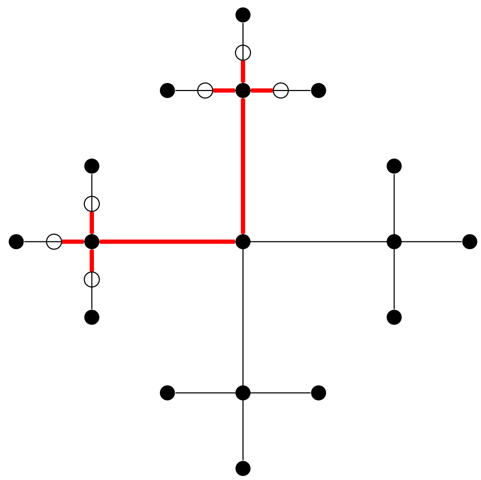

Figure 5.5: Example 4

Every mid-edge vertex is stabilized by $\mathbb{Z}_2$. The two valence-4 vertices are stabilized by trivial groups. The valence-2 vertex in the middle is stabilized by $\mathbb{Z}_2 \times \mathbb{Z}_2$. The two full edges' stabilizer groups should embed into their vertex groups and thus are both trivial. Therefore the right-angled Coxeter group associated with this strict fundamental domain is

$$\begin{aligned} G &= \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * (\mathbb{Z}_2 \times \mathbb{Z}_2) \\ &= \mathop{\ast}_{i=1}^{6} (\mathbb{Z}_2)_i * (\mathbb{Z}_2 \times \mathbb{Z}_2) \end{aligned}$$

Example 5

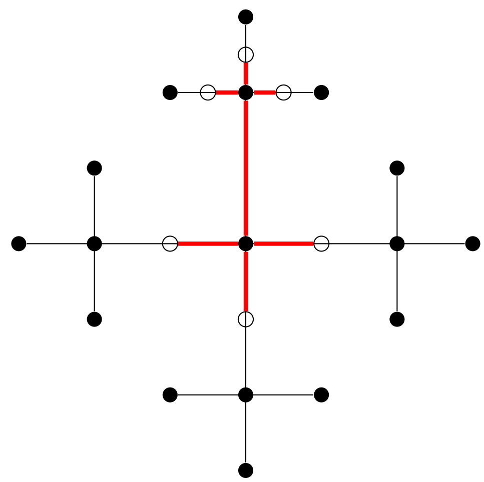

Figure 5.6: Example 5

Since every mid-edge point is stabilized by $\mathbb{Z}_2$, and every valence-4 vertex is stabilized by a trivial group, the right-angled Coxeter group associated with this strict fundamental domain is

$$\begin{aligned} G &= \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 \\ &= \mathop{\ast}_{i=1}^{6} (\mathbb{Z}_2)_i \end{aligned}$$

Notation Index

A partial list of mathematical symbols

$\mathscr{T}_4$	infinite, regular, 4-valence tree
$\mathscr{T}_N$	locally countably infinite, regular, infinite tree
∂X	the space at infinity for X
$\partial_\infty X$	visual boundary
$\partial_T X$	Tits boundary
X_l	the Croke-Kleiner space with nontrivial lengths data
$Core(X)$	union of all circles
$\mathscr{B}$	union of all blocks
$\mathscr{D}$	the dust
$\mathscr{W}$	union of all barriers
$It(\xi)$	the itinerary of a geodesic ξ
$\mathcal{S}_i \subset \partial_\infty X$	the set of all geodesic rays that reaches the strip S_i and the half-exit between S_i and S_{i+1}, and stabilize in the barrier that contains the half-exit
$\prod\limits_n{}^* G$	free product of n copies of the group G

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