Differential Posets and Dual Graded Graphs

by

Yulan Qing

B.A. in Mathematics and Physics
Colorado College, May 2006

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

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Abstract

In this thesis I study r-differential posets and dual graded graphs. Differential posets are partially ordered sets whose elements form the basis of a vector space that satisfies DU-UD=rI, where U and D are certain order-raising and order-lowering operators. New results are presented related to the growth and classification of differential posets. In particular, we prove that the rank sequence of an r-differential poset is bounded above by the Fibonacci sequence and that there is a unique poset with such a maximum rank sequence. We also prove that a 1-differential lattice is either Young's lattice or the Fibonacci lattice. In the second part of the thesis, we present a series of new examples of dual graded graphs that are not isomorphic to the ones presented in Fomin's original paper.

Thesis Supervisor: Richard Stanley
Title: Professor of Mathematics

Acknowledgments

My deepest gratitude goes to my advisor Richard Stanley for guiding me in my research with great knowledge and great compassion, to Tan Fucheng and Xiao Liang for countless insightful discussions and to Ruoxia for her laughter.

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Chapter 1

Introduction

Richard Stanley(1988) has defined and studied a class of partially ordered sets called "differential posets," In this thesis, we state and prove some basic properties of r-differential posets and focus on two specific constructions of r-differential posets, namely Y^r and $Z(r)$. We follow the notation of [3] and refer to [3] for basic definitions of partially ordered sets. In Chapter 2, we give one main result of this thesis that gives upper bound on the rank sequences of r-differential posets. In Chapter 3 we settle the conjecture that a 1-differential lattice is either Y or $Z(1)$. Richard Stanley studied properties of operators U and D which move up and down by one rank in the poset. In Chapter 4 we present some of these properties and introduce dual graded graphs. Dual graded graphs has been developed independently by Fomin(1992), which used a similar approach of functions of U and D, in a more general context. We provide newly found examples of dual graded graphs according to Fomin's definition. In Chapter 5, we will discuss in detail the future research methods related to higher dimensional differential lattices and lower bound on growth of rank sequences of 1-differential posets.

1.1 Differential Posets

Let r be a positive integer. A poset P is called *r-differential* if it satisfies the following three conditions [1]:

1. (D1) P is locally finite and graded and has a $\mathbf{0}$ element,

2. (D2) If $x \neq y$ in P and there are exactly k elements of P which are covered by both x and y, then there are exactly k elements of P which cover both x and y.

3. (D3) If $x \in P$ and x covers exactly k elements of P, then x is covered by exactly $k + r$ elements of P.

For x and y in a poset P, if $x \leq p$ and $y \leq p$, then p is an *upper bound* of x and y. If for all upper bounds of x and y, there exists an element z such that $z \leq p$ for all of them, then z is the *least upper bound* of x and y, denoted $x \vee y = z$.
An *interval [x, y]* in a poset is subposet consists of the set of points z satisfying $x \leq z \leq y$. A *chain* is a totally ordered set. A finite chain of cardinality $n + 1$ has *rank* n. A poset has *rank*(or *height*) n if the longest maximal chain is finite of rank n. The *rank* of an interval is its rank as a poset. A poset is graded if every maximal chain has the same finite rank.
If P is an r-differential poset for some r, then we call P a *differential poset.* If P has a unique minimal element $\mathbf{0}$, the *rank* of $x \in P$ is the maximal length of a saturated chain with largest element x. Thus, $\mathbf{0}$ is rank 0, every elements which covers an element of rank n has rank $(n-1)$. A rank function $\rho(x)$ maps each element to its rank.

1.2 Properties of r-Differential Posets

There are a number of properties concerning r-differential posets.

Proposition 1 (Stanley). *If P is a poset satisfying (D1) and (D2), then for $x \neq y$ in P the integer k of (D2) is equal to 0 or 1.*

Proof. Suppose the contrary. Let x and y be elements of minimal rank for which $k > 1$. Thus x and y both cover elements $x_1 \neq y_1$ of P. But then x_1 and y_1 are elements of smaller rank with $k > 1$, a contradiction. ☐

Proposition 2 (Stanley). *In any differential poset P, The number of closed paths of length $2n$ starting and ending at $\mathbf{0}$ is equal to $n!$. We will prove this in Chapter 4.*

Let's see some examples of differential posets. One type of 1-differential posets is the poset of partition of integers, ordered by inclusion of diagrams. Given a non-negative integer n, a *partition of n* is a finite, non increasing sequence of positive integers, $\lambda = (\lambda_1, \lambda_2, ...\lambda_k)$, such that$\lambda_1 + \lambda_2 + ...\lambda_k = n$. We denote this by $\lambda \vdash n$. For example, there are five partitions of 4: (4), (3,1), (2,2), (2,1,1) and (1,1,1,1). There is one partition of 1. We define an order on the partition as follows: given two partitions $\lambda = (\lambda_1, \lambda_2, ...\lambda_k)$ and $\mu = (\mu_1, \mu_2, ...\mu_l)$, $\lambda \leq \mu$ if and only if $k \leq l$ and $\lambda_i \leq \mu_i$ for $1 \leq i \leq k$. The poset whose elements are partition of integers, ordered by this relation, is called *Young's lattice* and denoted by $\mathbf{Y}$. Below is the first few levels of Young's lattice.

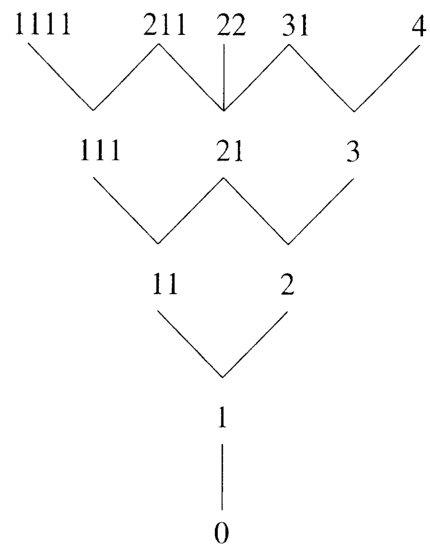

Figure 1-1: Young's lattice

Another type of 1-differential posets is denoted $Z(r)$ and is constructed as follows. Let A_r be an alphabet of $r+1$ letters. We will regard the letters as consisting of the number 1 with r different "colors", denoted $1_1, 1_2, ...1_r$ together with the number 2. Thus $A_r = 1_1, 1_2, ...1_r, 2$. Let $A*_r$ consists of all finite words $a_1a_2a_3...a_k$ (including the empty word $\oslash$) of elements of A_r.

Let $r \in \mathbf{N}$. Define a poset $Z(r)$ as follows. As a set, $Z(r)$ coincides with $A*_r$. If $\omega \in Z(r)$, then define ω' to be covered by ω in $Z(r)$ if either:

1. (a) ω' is obtained from ω by changing a 2 to some 1_i, provided that the only letters to the left of this 2 are also 2's, or

2. (b) ω' is obtained from ω by deleting the first letter of the form 1_i.

This defines the cover relations in $Z(r)$, and hence by transitivity a partial ordering of $Z(r)$. We call $Z(r)$ the *Fibonacci r-differential poset.* Below is the first few levels of the Fibonacci lattice.

A *lattice* is a non-empty poset L such that, for all $x, y \in L$, the least upper bound $x \vee y$ and the greatest lower bound $x \wedge y$ exists. A lattice is *modular* if, for all $x, y, z \in L$ such that $x \leq z$, $x \vee (y \wedge z) = (x \vee y) \wedge z$. If L has a minimal and a maximal element, denoted 0 and 1, it is *complemented* if, for all $x \in L$, there exists

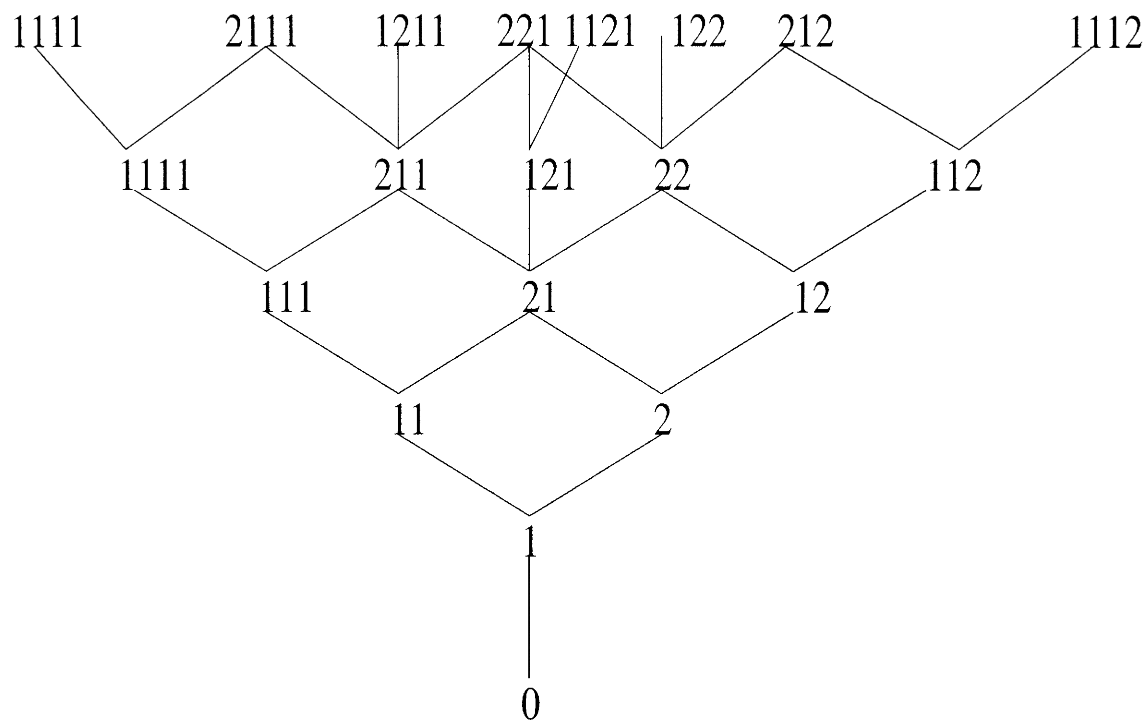

Figure 1-2: the Fibonacci lattice

$y \in L$ such that $x \vee y = 1$ and $x \wedge y = 0$.

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Chapter 2

Differential Posets

2.1 Statistics on the Growth of Differential Posets

This chapter focuses on the growth of rank numbers of differential posets. By definition, every 1-differential poset has a unique minimal element. It is shown in [1] that the number of elements of rank 0, 1, ,2... is non-decreasing. In fact, we prove in this chapter that it is strictly increasing. More specifically, the rank number p_i is the number of elements of rank i in P. The *rank sequence* is the infinite sequence $(p_0, p_1, p_2 ...)$. In this chapter we are going to show that the rank sequence is strictly increasing. We prove that the rank sequences are bounded above by the Fibonacci sequence. The proof also shows that there is a unique poset that realizes this tight upper bound. It is conjectured in [1] that the rank sequence is bounded below by the sequence of integer partitions. We will discuss this conjecture more in Chapter 5.

Let x be an element in the poset P. With P we associate a graph representation of the poset called Hasse Diagram. In a Hasse Diagram, each vertex represents an element of P. There is a vertical line segment between two vertices x and y, with x below y, if and only if $x < y$ in P, and there is no z such that $x < z < y$. In this case, we say y covers x, and denote $y \searrow x$ or $x \nearrow y$. Any such diagram (with labeled vertices) uniquely determines a partial order.

Moreover, in the Hasse Diagram, each vertex may have line segments between itself and elements that covers it , which are called up-edges. Likewise line segments between itself and elements covered by it are called down-edges. The number of up-edges of x is denoted $deg_{(u)}(x)$ and is called up-degree, the number of down-edges of x is denoted $deg_{(d)}(x)$ and is called down-degree. If a vertex in Hasse diagram has $deg_{(d)}(x) = 1$, we call the vertex, as well as the corresponding poset element, a *singleton*.

Lemma 1. *If any element of rank i is a singleton, then $p_i > p_{i-1}$, for $i \geq 1$.*

Proof. This is a straight forward observation. Suppose a is a singleton, $\rho(a) = i$. Then there exists a set of 6 elements such that ($a \searrow b \nearrow c \searrow d \nearrow e$ and $b \searrow f \nearrow d$) as shown below.

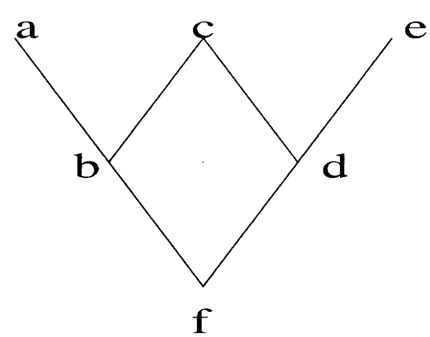


Figure 2-1: A unit of a differential poset

If the Hasse diagram above contains all the elements of rank i and $i-1$ in P, then $p_i > p_{i-1}$ and the proof is complete. Otherwise, other elements belong to one of the following cases:

1. x covers b or x covers d. If x is a singleton, then $p_i > p_{i-1}$. If x is not a singleton, x covers a new element that is different from b or d, or else we contradict Proposition 1.

2. x is covered by c or x is covered by e. If x is covered by c, then at least two other elements are force to cover x, See Figure 2-2. If x is covered by c, then at least one other element is forced to cover x, simply by definition.

We can "trace out" the rest of the elements in these two cases. Applying (D2)

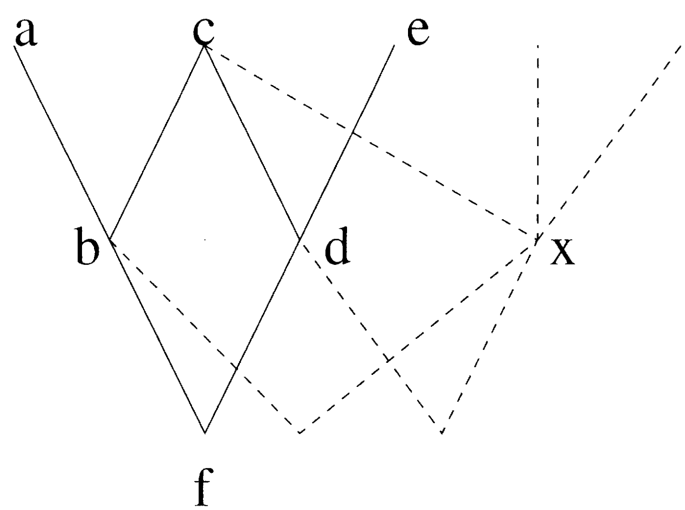


Figure 2-2: Case 2

and (D3) repeatedly, we get that each unit that we trace out add on at least as many elements in rank i as in rank $i-1$. $\square$

Lemma 2. *There exists at least one singleton of rank i for each $i \geq 2$.*

Proof. By the definition of r-differential posets, the first few ranks of a 1-differential poset are unique as shown:

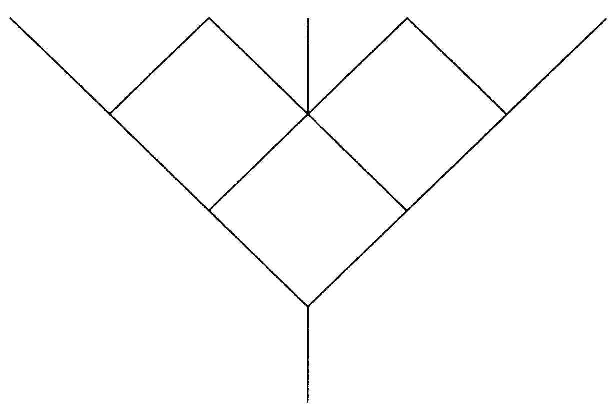


Figure 2-3: First 5 ranks of 1-differential posets

Therefore, there exist at least one singleton of rank 3. We observe that in any 1-differential poset, such an element is always covered by an element of the same property, i.e. a singleton that covers an element of up-degree 2, from which we conclude that there is always a singleton of each rank. $\square$

Lemma 1 and Lemma 2 together give rise to the following result:

Theorem 1. *The rank sequence of any 1-differential poset is strictly increasing, i.e.* $p_i < p_{i+1}$, *for* $i \geq 1$.

Next, we give a partial order between two rank sequences $\alpha = (a_1, a_2, a_3...), \beta = (b_1, b_2, b_3...)$ by $\alpha < \beta$ if any only if $a_i < b_i$ for all i. We show now that the maximum element of this poset of all rank sequences is the Fibonacci sequence, and $Z(1)$ is the poset uniquely corresponds with the sequence.

Theorem 2. *The rank sequence of any 1-differential poset cannot grow faster than the Fibonacci sequence, and there is only one poset with the rank sequence(1,1,2,3,5,8,...), namely, the Fibonacci lattice.*

Proof. Since the partial order is lexicographic order, i.e. for the smallest i such that $a_i > b_i, \beta = (b_1, b_2, b_3..., b_i, ...)$ is eliminated from the set of candidates for the maximum element, since β is either less than, or incomparable to α. Since we are searching for a maximum element, if it exists, we then disregard β.
Any 1-differential poset up to rank 5 is uniquely determined. At rank 6, there are two possibilities, shown as follows.

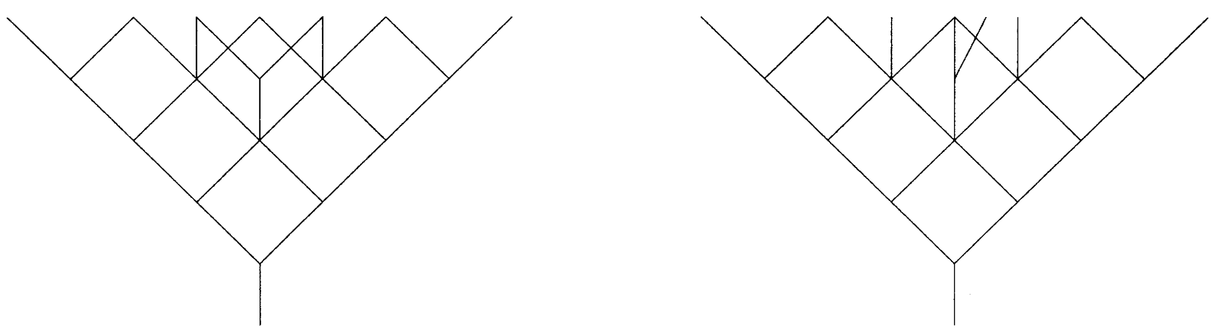


Figure 2-4: First 6 rank

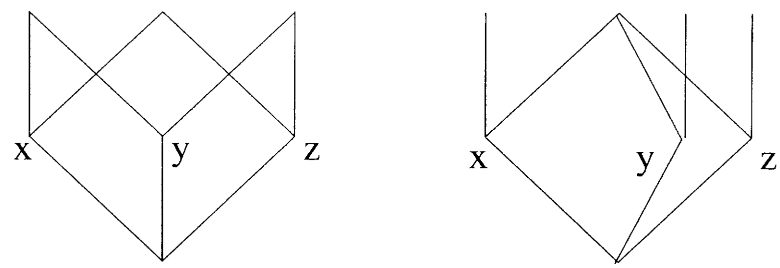
By the previous principle we disregard the choice on the left. To develop the poset upward to include as many elements as possible, we have the following three cases:

1. $deg_{(u)}(x) = 2$;

2. $deg_{(u)}(x) = 3$;

3. $deg_{(u)}(x) \geq 4$.

When $deg_{(u)}(x) \geq 4$, we refer to Lemma 6 in Chapter 3. The lemma shows that in this case x will be reflected as if in Z(1). The case of $deg_{(u)}(x) = 2$ is trivial, the element x will be reflected as well. Then we are left with the case of $deg_{(u)}(x) = 3$. When every other element is isomorphic to elements in Z(1), x will be a choice between the following two. Since we are asking for the highest number of elements in each rank, we choose Type II.



Type I **Type II**

Figure 2-5:

□

It is conjectured that the rank sequence of a differential poset grows at least as fast as the sequence of integer partition. Young's lattice is an example of a poset with such a rank sequence. It is unclear if there are other differential posets that also has the same rank sequence. We will discuss more on this topic in Chapter 5.

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Chapter 3

Differential Lattices

3.1 1-Differential Lattices

A *lattice* is a partially ordered set in which every pair of elements has a least upper bound, called their join, and a greatest lower bound, called their meet. The following fact is easy to verify.

Proposition 3 (Stanley). *The two types of* 1*-differential posets introduced in Chapter 1, Y and $Z(r)$, are both lattices.*

Proposition 4 (Stanley). *Let L be a lattice satisfying (D1) and (D3). Then L is r-differential if and only if L is modular.*

Proof. It is known that a locally finite lattice is modular if and only if the following condition is satisfied:

For all $x, y \in L$, x and y cover $x \wedge y$ if and only if $x \vee y$ covers x and y.

But this condition is equivalent to (D2). □

We would like to settle the following conjecture by Richard Stanley:

Theorem 3. *The only 1-differential lattices are Young's lattice Y and the Fibonacci lattice $Z(1)$.*

We start by defining two types of local orders of up-degree 3. Let x, y and z be three elements of a poset, $\rho(x) = \rho(y) = \rho(z) = i$ and x, y and z all covers a same fourth element of a rank $i-1$. There are two possibilities for the join of x, y and z of rank $i+1$. We will call them type I and II. In type I x_1 covers y, z, y_1 covers x, z z_1 covers x, y, and $x_1 \neq y_1 \neq z_1$. In type II, s covers x, y and z. See the following diagram:

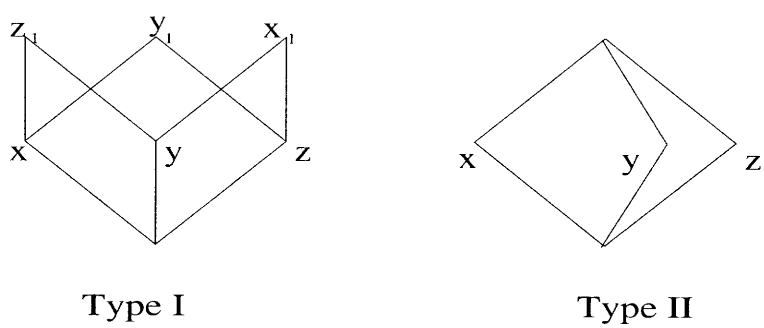


Figure 3-1: Type I and Type II

We can quickly establish the following necessary condition:

Lemma 3. *The three elements of type I has to have a common cover of the next rank.*

Proof. The elements pairwise has meets at the lower rank, therefore they have joins at the higher rank that is either one element covers all three of them or three element each cover a pair of them. If it were the latter then we have a Hasse diagram of the following case where element x_2, y_2 does not have a meet, contradicting the fact that P is a lattice. □

Next we again refer to the fact that there is only one 1-differential poset up to rank 5, and only two non-isomorphic 1-differential posets up to rank 6. One of them is isomorphic to Young's lattice up to rank 6, the other is isomorphic to the Young-Fibonacci lattice up to rank 6. Our main theorem is an inductive result of the following two lemmas.

Lemma 4. *If a 1-differential lattice is isomorphic to Young's lattice up till rank $n-1$, where $n \geq 6$, then it is isomorphic to Young's lattice up to rank n.*

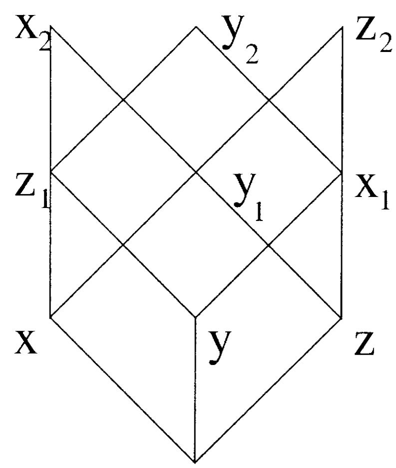


Figure 3-2: Not a lattice

Lemma 5. *If a 1-differential lattice is isomorphic to the Fibonacci lattice up till rank $n-1$, where $n \geq 6$, then it is isomorphic to the Fibonacci lattice up to rank n.*

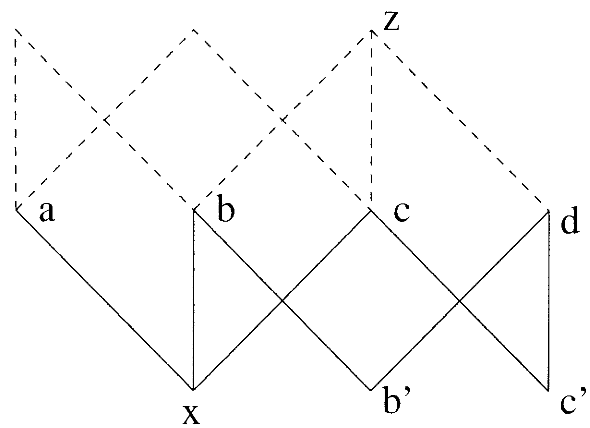
To prove Lemma 4, we need the following proposition.

Proposition 5. *If L is isomorphic to Young's lattice up to rank n, $n \geq 6$, then for the elements of rank n, if a, b, c covers the same element x, then a, b, c do NOT have a same cover in rank $n+1$. i.e. it is a type I.*

Proof. Since L is isomorphic to Young's lattice up to rank n, elements of rank up to n have a one-to-one bijection with elements of Young's lattice, thus we can label them with Young's diagram, and use terms such as outer corner and inner corner.

The following diagram illustrates the proof. The Hasse Diagram with filled line shows the lattice up till rank n, and the dashed line shows the lattice beyond rank n. Since x has up-degree at least 3, then it has at least two outer corner. Since $n > 5$, at least one of the the two outer corners does not belong to either the first or the last row of the Young's diagram. Then for b, c, they can each delete a same outer corner to get b', c' which has a cover d. By Lemma 4, b, c, d has a common cover z, but a and d does not cover an element by way of construction, therefore z cannot cover a, which proves the statement.

□



Since P is a differential poset, every triple of $y_1 \vee y_2 = x$, there is a unique corresponding $y_1 \wedge y_2 = z$.

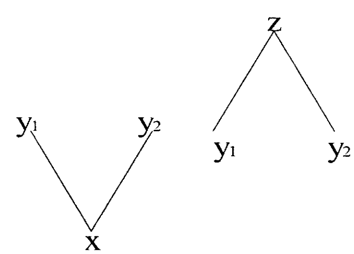


Figure 3-3:

Since P is same at the Young's lattice for ranks up to n, one of the following occurs if $P \neq Y$ at rank $n+1$:

1. Three pairwise different joint in Young's lattice versus a same joint in P.
   In this case, since y_1, y_2, y_3 must cover a same element on a lower level, but if they are covered by a same joint in P then they contradict Proposition 5.

2. A same joint of three elements in P versus three pairwise different joints in Young's lattice.
   In this case, They must have three different meets which gives a contradiction to Lemma 3.

Therefore, both cases are impossible which implies $P = Y$.

Next, we would like to prove Lemma 5. Assume that P is isomorphic to the Fibonacci lattice up to rank n. there is a bijection between the elements of P and elements of the Fibonacci lattice up to rank n. First, by way of construction, for any i, there does not exists a set of 3 vertices of rank i in the Hasse diagram of the Fibonacci lattice such that there are three different meet of rank $i-1$ for the three two-element subsets. See bellow Figure 3-3.

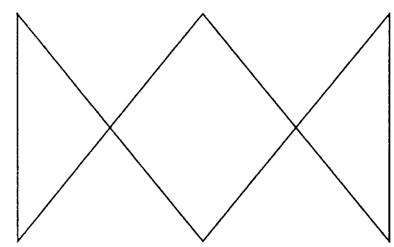


Figure 3-4:

Thus for any element $x \in Z(1)$, for every element covered by x, there is at least one distinct element that covers x to satisfy (D2). Therefore, for every edge in the up-degree of x, there is a unique edge in the down-degree of x that corresponds with it. Since (D3), there can be two up-edges that correspond with at most one down-edge. Therefore given an element x of rank $i-1$, let it have up-degree r,

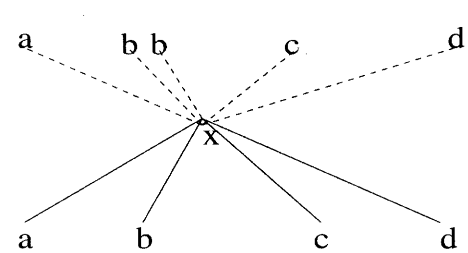


Figure 3-5: for each of a, b, c, d down edges, there can be at most 2 up edges that (D2)-correspondence with each.

There can be at most two up-edges from each of $\{x_1, x_2, x_3 ...\}$.

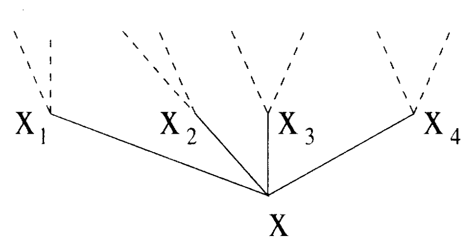


Figure 3-6:

Suppose x_1 has two up-edges that will be part of this part of the diagram, y_1 and y_2. The other elements $\{x_2, x_3, x_4 ...\}$ are partitioned into two sets, one is covered by y_1, the other set covered by y_2. Then an element that is covered by y_1, say x_2, has a distinct common cover with each of the elements in set 2. The covers has to be distinct to not contradict Proposition 1. Therefore we conclude there are at most 1 element in set 2, likewise there is at most 1 element in set 1. Therefore if the up-degree of x is greater than 3, we are force to have a reflection.

In fact what we just proved is not dependent upon the lattice condition. It can be stated formally as following:

Lemma 6. *If a 1-differential poset is isomorphic to Z(1) up till rank n where $n \geq 5$, then for an element x of rank n, if $deg_{(u)}(x) \geq 4$, then there is exactly one element that will cover the covers of x.*

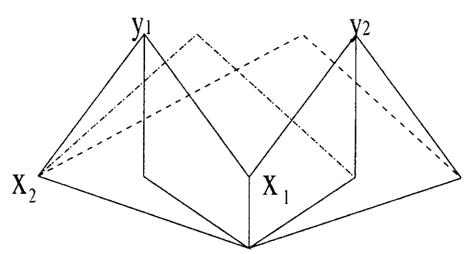


Figure 3-7:

3.2 A Singular Case

If there exists an element of rank n, whose up-degree is exactly 3, and the 1-differential lattice is the Fibonacci lattice up till rank n, it is impossible to continue to grow the poset into a lattice beyond rank $(n+5)$. This will be illustrated as follows.

Throughout the rest of the proof, we will use a_{ij} to denote elements of the i^{th} level. For convenience, we start with the element of up-degree 3, and label it a_{11}. We also assume that the 1-differential lattice is isomorphic to Z(1) for elements of label a_{2j} and all elements below

1. If $deg_{(u)}(a_{11}) = 3$, let a_{21}, a_{22} and a_{23} covers a_{11}. In the Fibonacci lattice, each element is covered by exactly one singleton. Thus one of a_{21}, a_{22} and a_{23} can have down degree 1. By construction of the Fibonacci lattice, at least one of a_{21}, a_{22} and a_{23} will have down degree 3, so let a_{23} covers a_{11}, a_{12} and a_{13}. By the structure of the Fibonacci lattice, a_{11}, a_{12} and a_{13} have one cover in common, thus again by definition only one of a_{11}, a_{12} and a_{13} can have up-degree 2, the others have up-degree at least 3. Therefore let a_{12} be covered by a_{23}, a_{24} and let a_{13} be covered by a_{23}, a_{25} and a_{26}. Likewise, we let a_{20} and a_{21} covers a_{10}

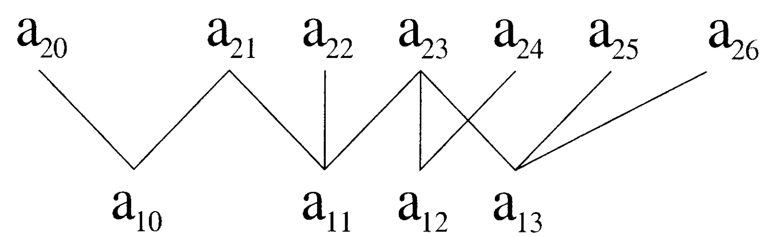


Figure 3-8:

2. Since we are interested in the case when a_{21}, a_{22} and a_{23} has three distinct covers. We apply Lemma 3 and Proposition 5 to uniquely determine the left side of the lattice

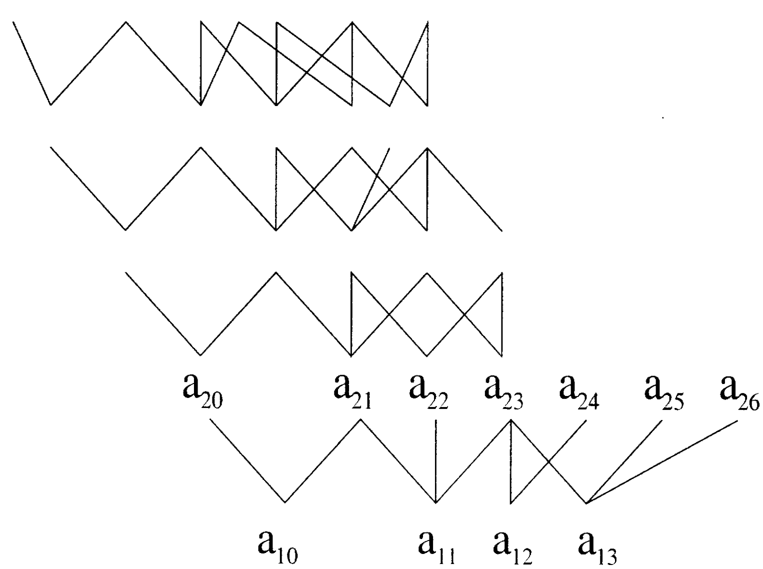


Figure 3-9:

3. Consider the up-degree of a_{23}. $deg_{(u)}(a_{23}) = deg_{(d)}(a_{23}) + 1 = 4$. There are

two elements a_{12} and a_{13} that needs elements to cover a_{23} correspondingly to satisfy (D2). Since $a_{12} \neq a_{25} \neq a_{26}$, the corresponding covers a_{35} and a_{37} are also distinct. a_{36}, a_{38} and a_{39} exist by (D3).

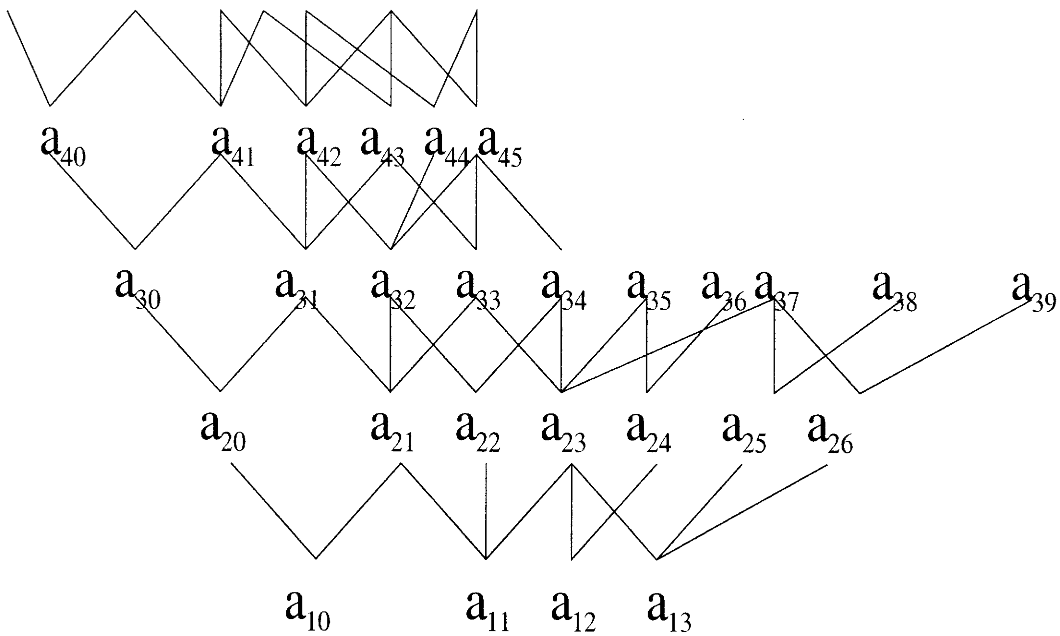


Figure 3-10:

4. Now consider the up-degree of a_{33}.

   First we complete the fifth rank. In the diagram following, we use different line styles for to distinguish between the different reasons that uniquely determines the corresponding edge.

   (a) The modularity of differential lattices. For cases that are relevant to this section, whenever there is a sublattice isomorphic to the figure below, the elements a, b, c, d have a common cover: dotted line.

   (b) Lemma 3: filled line

   (c) (D2) and (D3):dashed line

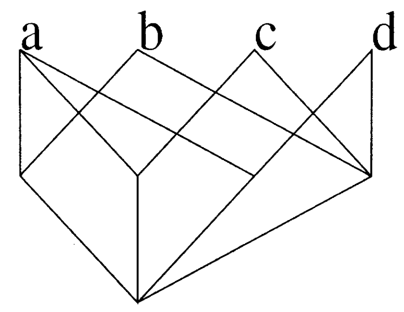


Figure 3-11:

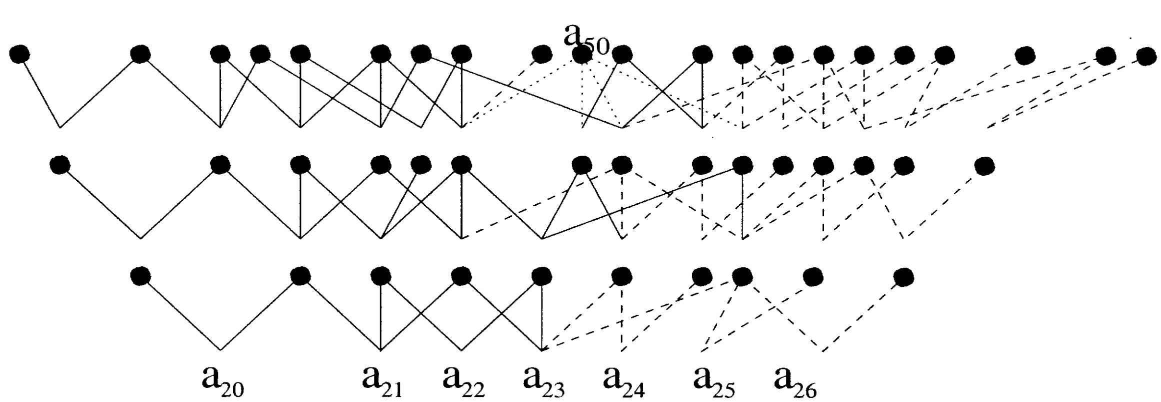


Figure 3-12:

5. We will show that the element a_{50} cannot have enough up-degrees to satisfy its differential lattice definitions.

   We start by complete some of the edges by Lemma 3.

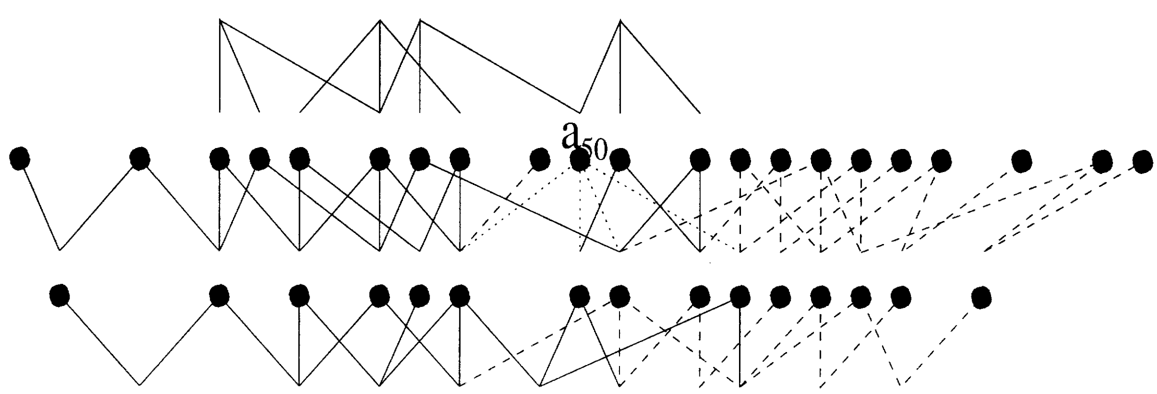


Figure 3-13:

Then add more edges according to (D2) and (D3)

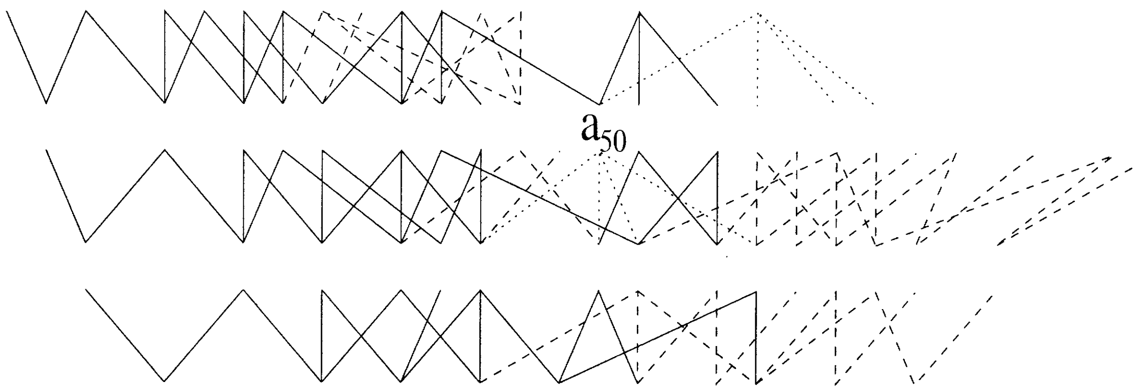


Figure 3-14:

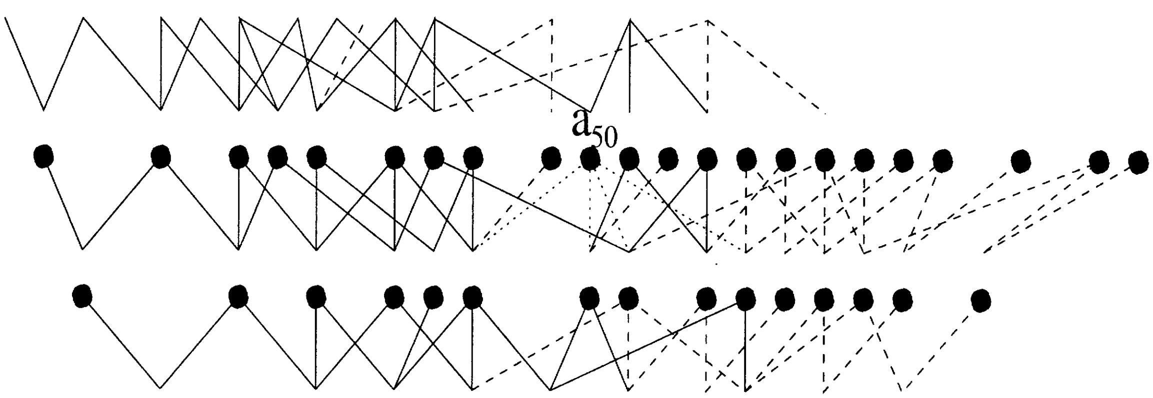


Figure 3-15:

We realize that even a_{50} already has maximal up-degree 5, it still does not have covers with a_{51} or a_{52}, hence the contradiction.

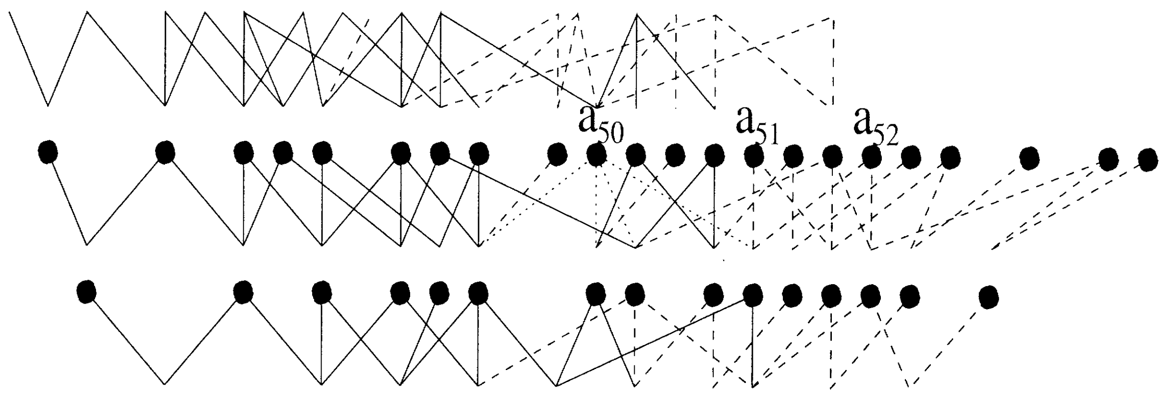


Figure 3-16:

3.3 Conclusion

We used enumerative methods to prove that a 1-differential lattice can only be either Young's lattice or the Fibonacci lattice. The one special case in last section does not have to be an induced sublattice. There can be other covering relations between these elements and other elements of the lattice. One can easily derive the higher dimensional generalization of Lemma 6, however, that approach can leave many special cases. In Chapter 5 we are going to discuss a different approach to study higher dimensional differential lattices that uses edge labeling.

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Chapter 4

Dual Graded Graphs

4.1 Linear Operators on r-Differential Posets and Dual Graded Graphs

Given a locally finite poset P and $x \in P$, define

$$C^-(x) = y \in P : x \text{ covers } y$$

$$C^+(x) = y \in P : y \text{ covers } x$$

Let P be a locally finite poset such that for all $x \in P$ the set $C^-(x), C^+(x)$ are finite. Define two linear transformations U, D by the condition that for $x \in P$:

$$Ux = \sum_{y \in C^+(x)} y,$$

$$Dx = \sum_{y \in C^-(x)} y,$$

We are going to state and prove two most important results regarding the pair of linear transformations:

Theorem 4 (Stanley). *Let P be a locally finite graded poset with $\mathbf{0}$, with finitely many elements of each rank. Let r be a positive integer. The following two conditions are equivalent:*

1. *P is r-differential.*

2. *$DU - UD = rI$*

Proof. Let $x \in P$. Then $DUx = \sum_y c_y y$, where $c_y = \#(C^+(x) \cap C^+(y))$. Moreover, $UDx = \sum_y d_y y$, where $d_y = \#(C^-(x) \cap C^-(y))$. Hence $DU - UD = rI$ if and only if for all $x, y \in P$,

$$\#(C^+(x) \cap C^+(y)) = \#(C^-(x) \cap C^-(y)), \text{ if } x \neq y$$

$$\#(C^+(x) \cap C^+(y)) = \#(C^-(x) \cap C^-(y)), \text{ if } x \neq y$$

and

$$\#(C^+(x) = r + \#(C^-(x)$$

.

which are precisely the conditions for P to be r-differential. □

Another fundamental result states:

Proposition 6 (Stanley). *For a differential poset P, we have $DU^n = nU^{n-1} + U^n D$ for all $n \geq 1$.*

Proof. The case $n = 1$ follows immediately. In general, if $DU^k = kU^{k-1} + U^kD$, then

$$\begin{aligned} DU^{k+1} &= (DU)U^k \\ &= (I + UD)U^k \\ &= U^k + U(DU^k) \\ &= U^k + U(kU^{k-1} + U^kD) \\ &= U^k + kU^k + U^{k+1}D \\ &= (k+1)U^k + U^{k+1}D \end{aligned}$$

□

Now we can state and prove the result mentioned in the last section.

Proposition 7. *In any differential poset P, The number of closed paths of length $2n$ starting and ending at $\mathbf{0}$ is equal to $n!$.*

Proof. The number of such closed paths exactly euqals to the coefficient of $\mathbf{0}$ in $D^nU^n\mathbf{0}$. We have trivially $D^0U^0\mathbf{0} = 0$, so the claim is valid for $n = 0$. Inductively,

$$D^{k+1}U^{k+1}\mathbf{0} = D^k(DU^{k+1}\mathbf{0} = (k+1)D^kU^k\mathbf{0})$$

.

And the result follows. □

There are many directions in which one can generalize the concept of a graded poset such that $DU - UD = I$. One way is to have the D and U operate on two different set of edges, which results in the study of *dual graded graphs*.

4.2 Dual Graded Graphs

Defined by Fomin in [5], an ***r**-dual graded graph* is a pair of graded graphs (G_1, G_2) such that

1. G_1 and G_2 have the same vertex set and rank function ρ

2. G_i has edge set E_i. Edges in E_1 are directed upward, i.e. in the direction of increasing rank, and G_2 edges downward. We introduce the U and D operator as before. and the relations hold that:

$$D_{n+1}U_n = U_{n-1}D_n + r_n I_n$$

where $\mathbf{r} = r_n$ is a sequence of constants, i.e. $r_n \in K$.

An r-differential poset is an r-dual graded graph where $E_1 = E_2$ and $\mathbf{r} = r, r, r, ...$ We refer to [5] [4] and [6] for studies on graded algebras. We will focus on enumerative properties of these dual grades graphs. We will present graphs that satisfy one of the following commutation relations [5]:

$$DU = \mathbf{q}UD + \mathbf{r}I$$

where $\mathbf{q} = \{q_n\}$ and $\mathbf{r} = \{r_n\}$ are elements of K^∞. Differential posets are dual graded graphs with $\mathbf{q} = \{1, 1, 1, ...\}$ and $\mathbf{r} = \{1, 1, 1, ...\}$ It is shown that if a pair of dual graded graphs satisfies the above relation, then we have the following path enumeration [5]:

Theorem 5. *Assume the up and down operators in an oriented graded graph G with zero satisfy the above relations. Then for any $x \in P_k$:*

$$\sum_{y \in P_{k+l}} e(\hat{0} \to y \to x) = e(x)\Pi_{s=k+1}^{k+l} \sum_{i=0}^{s-1} r_i \Pi_{j=i+1}^{s-1} q_j$$

Where $e(x \to y)$ denotes the number of paths between x and y. In a graph with zero, let $e(y) = e(\hat{0} \to y)$. The studies of enumerative properties of dual graded graphs give rise to combinatorial identities involving $e(x)$. A typical example is the Young-Frobenius identity

$$\sum_{x \in P_n} e(x)^2 = n!$$

We will present new examples of dual graded graphs that are not analyzed in [5].

4.2.1 Example 1: The Catalan Tree

Let the vertex set be number string of the form $x = a_1a_2a_3a_4...a_n$, with $1 = a_1 \leq a_2 \leq a_3 \leq ...a_n \leq n$ and $\rho(x) = n$

U-edges: $x \nearrow y$ if and only if y can be obtained from x by adding a number to the right end of x.

D-edges: $x \searrow y$ if and only if y can be obtained from $x =$ by deleting a_i from $a_1a_2a_3a_4...a_n$ only if $a_{i-1} = a_i$

These are the first few ranks:

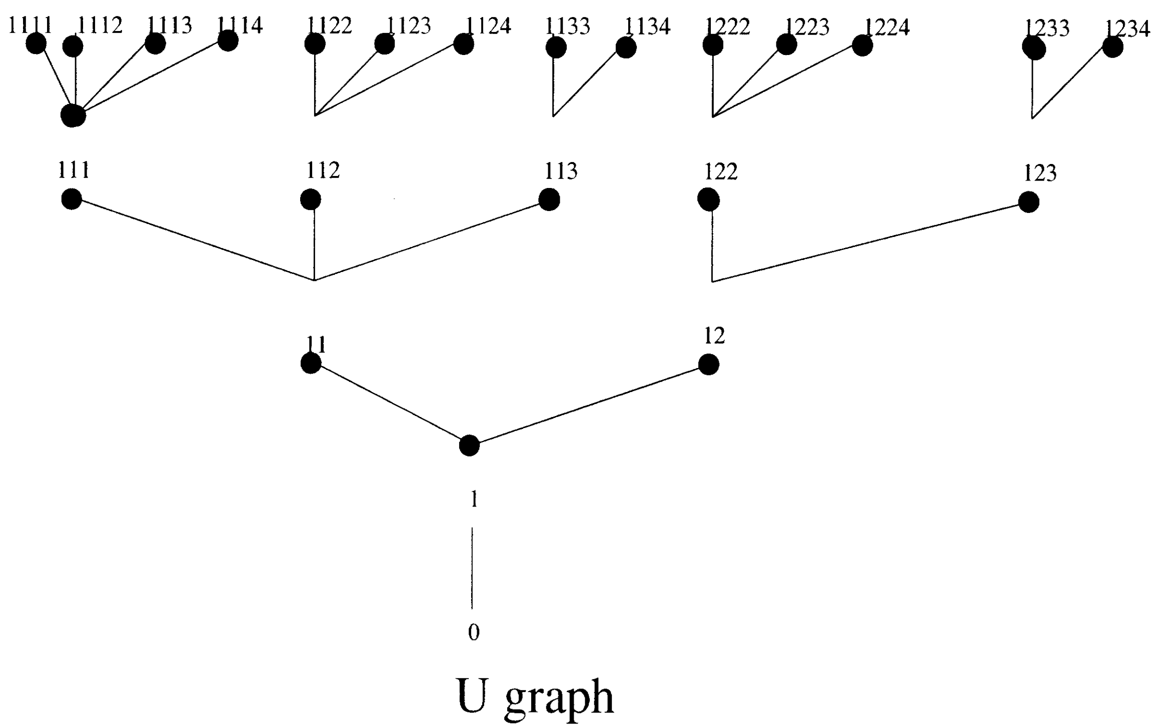

Figure 4-1: U graph

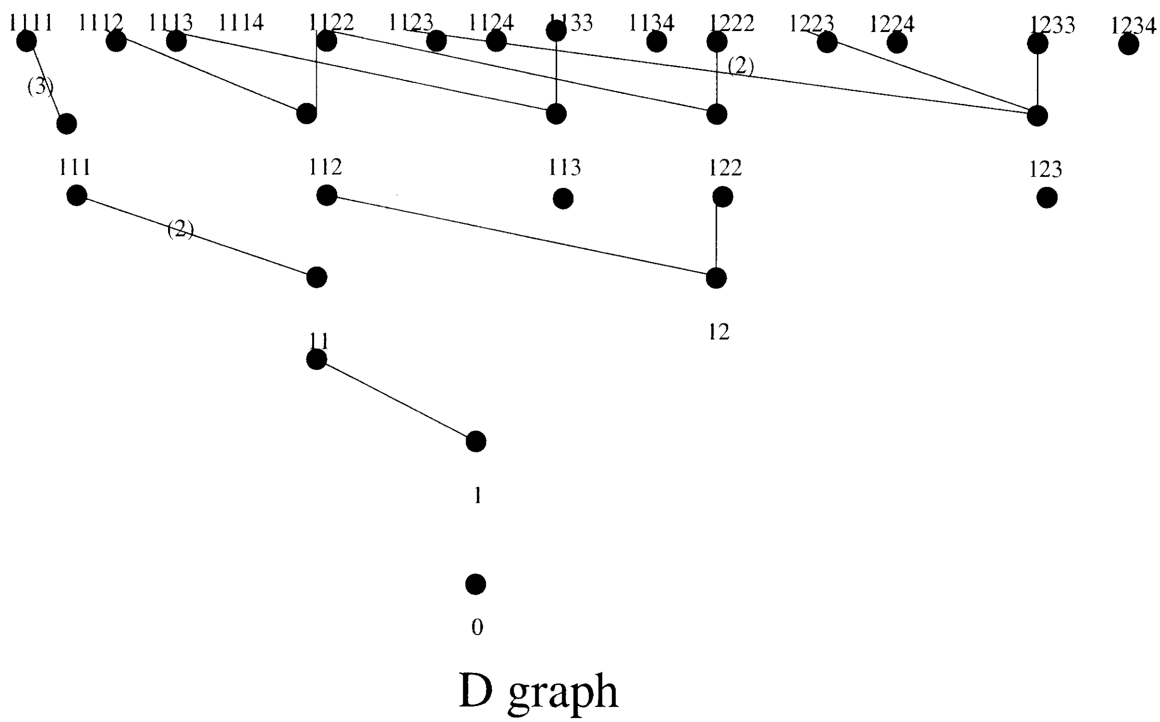

Figure 4-2: D graph

It can be checked that this is a pair of dual graded graphs with $DU - UD = I$. By Theorem 5,

$$\sum_{y \in P_{k+l}} e(\hat{0} \to y \to x) = e(x)\Pi_{s=k+1}^{k+l} \sum_{i=0}^{s-1} r_i \Pi_{j=i+1}^{s-1} q_j$$

Because in this case we have $\mathbf{q} = \{1, 1, 1, ...\}$ and $\mathbf{r} = \{1, 1, 1, ...\}$, we have

$$\sum_{y \in P_{k+l}} e(\hat{0} \to y \to x) = e(x)r^l(k+l)!/k!$$

In the case of a Catalan tree, $e(x) = 1$for all x, so we have for any $x \in P_k$:

$$\sum_{y \in P_{k+l}} e(\hat{0} \to y \to x) = (k+l)!/k!$$

Let $k = 1$, $\sum_{y \in P_{l+1}} e(\hat{0} \to y \to x) = \sum_{y \in P_{l+1}} e(\hat{0} \to y \to 0) = (l+1)!$.

4.2.2 Example 2: Binary Trees

Let the vertex set be a strings of 1s and 0s of finite length. The length of the string is the rank of the element.

U edges: $x \nearrow y$ if and only if y can be obtained from x by adding a number to the right end of x.

the **D** edges: $x \searrow y$ if and only if y can be obtained from x by the following process:

1. If x ends with 1, delete 1
2. If x ends with 0, find the right most 1, i.e. $x = a_1a_2...a_k100...0$, then y can be any vertex that starts with $a_1a_2...a_k$

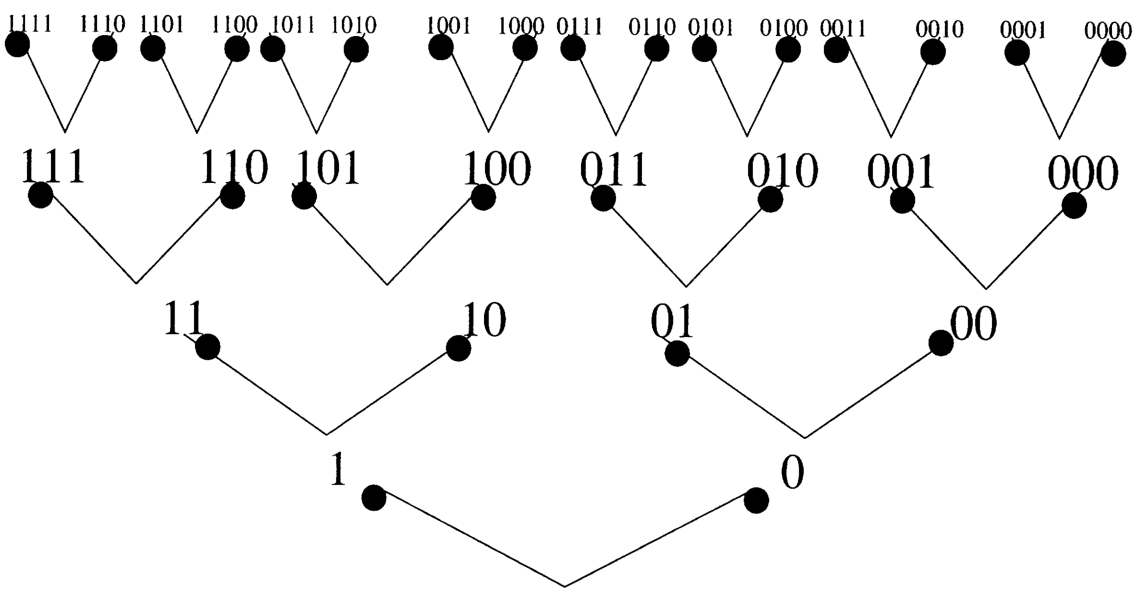

Figure 4-3: U graph

Proposition 8. *This is a pair of dual graded graphs with $DU - UD = I$*

$$\sum_{y \in P_{k+l}} e(\hat{0} \to y \to x) = e(x)r^l(k+l)!/k!$$

Again we have $e(x) = 1$ and $\sum_{y \in P_{l+1}} e(\hat{0} \to y \to x) = (l+1)!$.

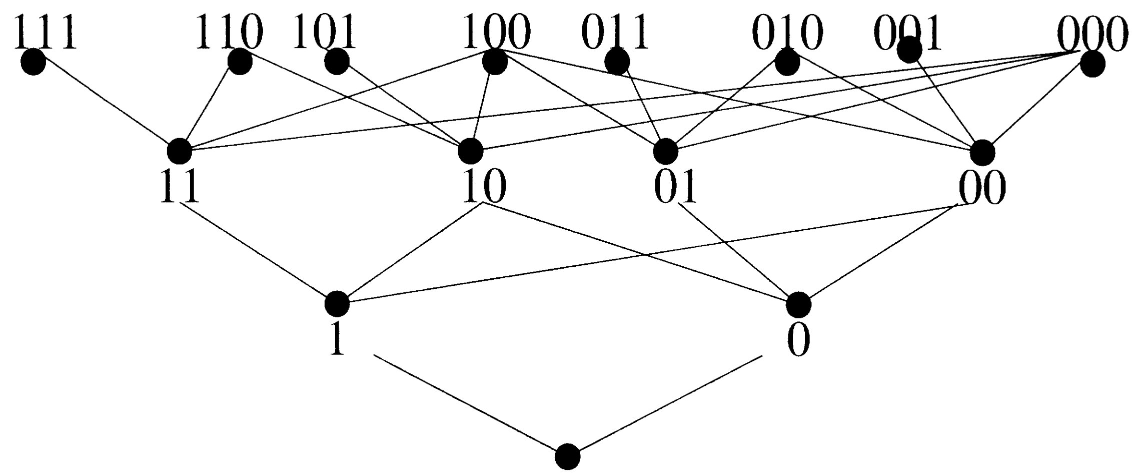

Figure 4-4: D graph

4.2.3 Example 3: Graph Coverings

Let G be a graph,define a *covering of* G to be a graded graph such that

1. At each level, the vertex set is the vertex set of G
2. $x \nearrow y$ if and only if x is adjacent to y in G
3. $x \searrow y$ if and only if x is adjacent to y in G

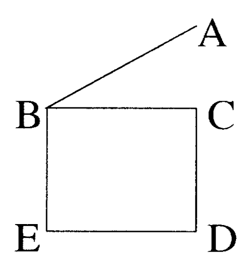

Figure 4-5: G

Proposition 9. *The covering of* G *is a self-dual graph with* $DU - UD = 0$*, where self-dual means* $G_1 = G_2$ *in the definition of dual graded graphs.*

This is an example of a pair of dual graded graphs that is infinite in both the positive and and the negative direction. Locally it satisfies the condition $DU - UD = 0$, and any interval of this subset is graded.

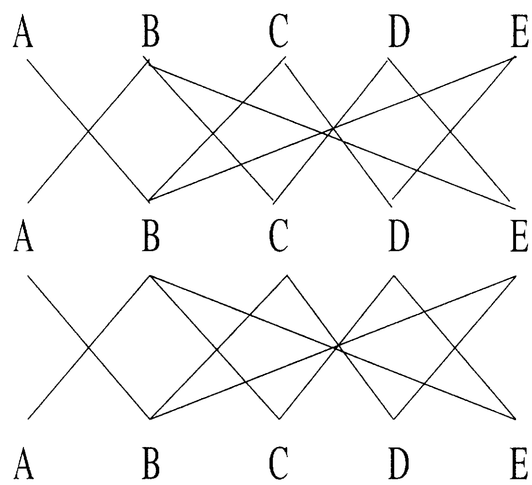

Figure 4-6: U graph and D graph

4.2.4 Example 4: Hex Variation

The Hex graph is presented in [5]. It is interesting because its $e(x)$ for ranks $2n$ and $2n+1$ are the ordinary binomial coefficients $\binom{n}{k}$, $1 \leq k \leq n$. One can substitute the "wall" of the hexagon for a finite chain of n elements for any n. Here is an example when $n = 1$:

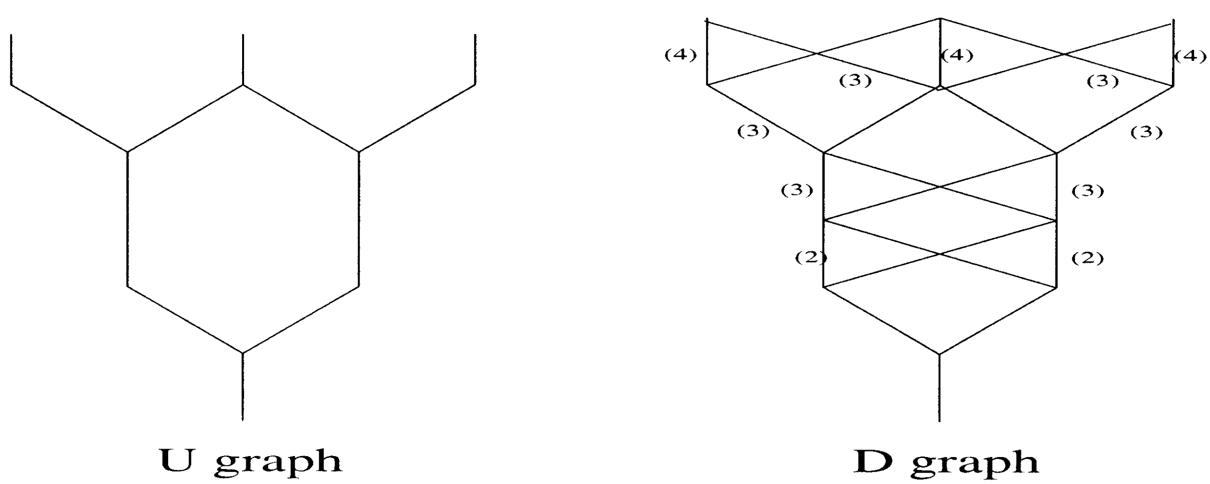

Figure 4-7: Hex Variation

4.2.5 Example 5: The Path-Sum

We learned that differential posets are dual graded graphs with $\mathbf{q} = \{1, 1, 1, ...\}$ and $\mathbf{r} = \{1, 1, 1, ...\}$. In every differential poset, there are exactly two paths in which every element has $deg_{(u)} = 2$. We can take any two differential posets, and identify such a path, and then path-sum them into a bigger dual graded graph. We take an example of the following two differential posets:

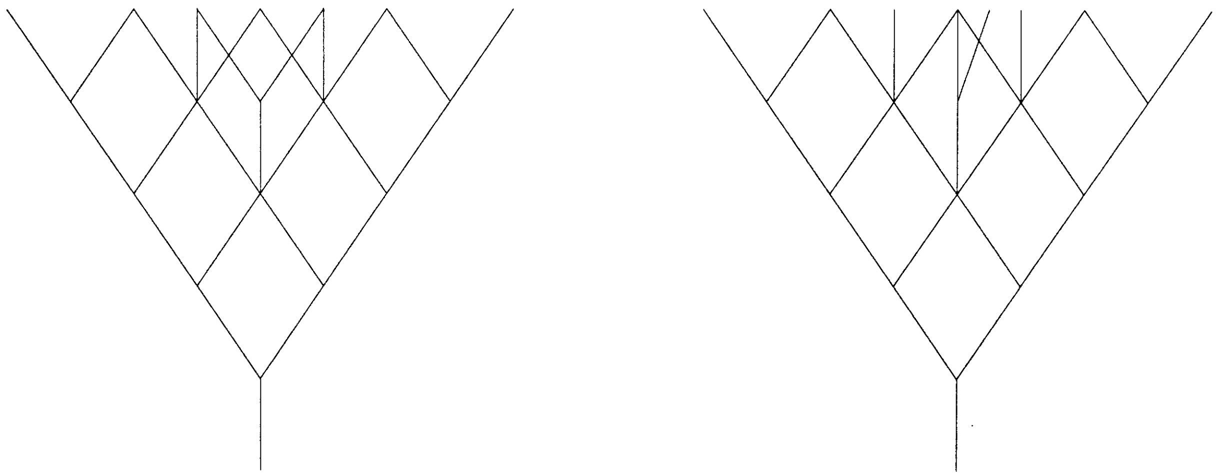

Figure 4-8: Examples of two posets

In the following figure, the filled lines are both up-edges and down edges, the dash lines are only down edges:

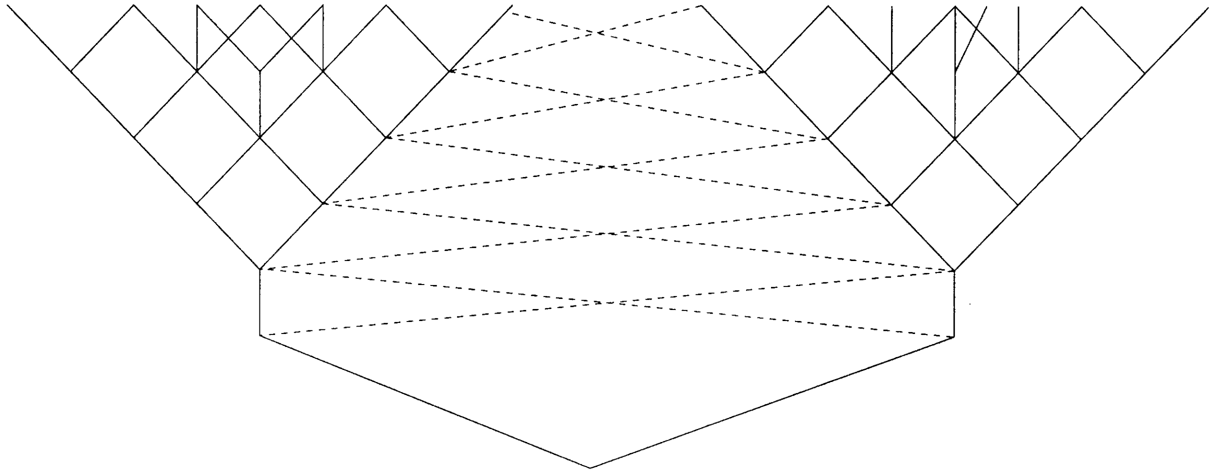

Figure 4-9: Path-Sum

4.3 Conclusion

The dual graded graphs are tools to study one of the most important algorithm in algebraic combinatorics, the Robinson-Schensted correspondence, as well as a variety of algebras. We mainly search for examples that either has interesting enumerative properties, or ones whose vertex set as the Weyl group of algebras. This chapter listed some of the non trial examples found, of which the Catalan tree is the most interesting and is worth further study.

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Chapter 5

Higher Dimensional Lattices and Lower bounds on Rank Sequences of Differential Posets

In this chapter we discuss the next step in this research. The directions of future research will concentrate on two topics: searching for a lower bound for rank sequence of 1-differential posets and generalizing the result of Chapter 3 to higher dimensional lattices.

5.1 A Lower Bound for Differential Posets

The minimal growth of a differential poset is conjectured to be the integer partition sequence. To studying this conjecture. Let's first relax (D3). In the definition of differential posets, if we only consider (D1), (D2) and develop the poset along the two side edges, together with Proposition 1, we get Figure 5-1.

Each time we add a singleton to satisfy (D3), we introduce another copy of Figure 5-1. As we start to add them into the diagram, some of the new elements coincide with others. The Fibonacci lattice is the arrangement in which such coincidence happens least. It will be helpful to show that Young's lattice is the arrangements in which the coincidence happens most.

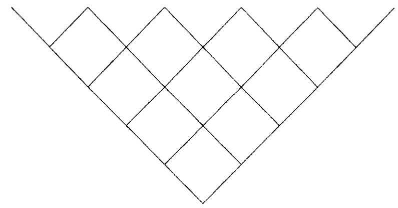

Figure 5-1:

5.2 Higher Dimensional Lattices

To study r-dimensional lattices where $r > 1$, we introduce the concept of complemented interval. let's remind us from Chapter 1 that an *interval [x, y]* in a poset is a subposet consists of the set of points z satisfying $x \leq z \leq y$. It contains at least the points x and y. A lattice L with $\hat{0}$ and $\hat{1}$ is *complemented* if for all $x \in L$ there is a $y \in L$ such taht $x \wedge y = \hat{0}$ and $x \vee y = \hat{1}$. If every interval [x,y] of L is itself complemented, then L is *relatively complemented.* We present two propositions on complemented intervals in differential lattices. The proofs are found in [1].

Proposition 10 (Stanley). *Let L be a lattice satisfying (D1) and (D3). Then L is r-differential if and only if L is modular.*

Proposition 11 (Stanley). *Let r be a positive integer. The following two conditions on a poset P are equivalent.*

1. *P is an r-differential lattice (necessarily modular) such that every complemented interval has length ≤ 2.*

2. *P is isomorphic to Z(r).*

If a 2-differential lattice is such that every complemented interval has length $\leq$ 2, then it is isomorphic to $Z(r)$. If not, then suppose P is a 2-differential lattice in which there is at least one complemented interval of length 3 or higher. Let's assign such complemented intervals a rank of its minimal element. Since there is at least one complemented interval of length 3 or higher, let α be such an interval with the smallest rank. By definition, there exists a sub-lattice of α, denoted β, that satisfies the following two properties:

1. $\hat{0}_{\in\alpha} = \hat{0}_{\in\beta}$

2. length$(\beta) = 3$

Thus β is isomorphic to:

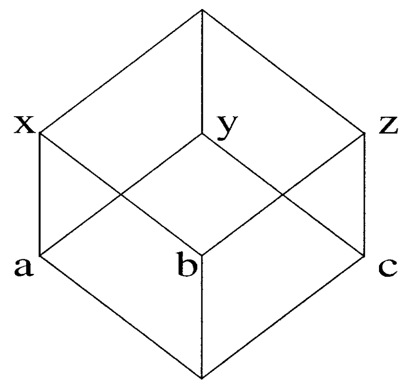

Figure 5-2: The smallest, lowest complemented interval

Given a 2-differential lattice, if two complemented interval base at rank 1 are both of length ≥ 3, we introduce an algorithm that labels the edges of the lattice r(red) or b(blue), such that the r-labeled edges consist a differential lattice of lower dimension, likewise for the b-labeled edges. Moreover, they are put together such that the lattice is a product of the two lattices of lower dimension. For higher dimensional lattices, we recursively apply the decomposition through labeling until the resulting lattices are of dimension 1, or one such that all its complemented intervals are of length ≤ 2.

5.3 A Lattice Decomposition Algorithm on a 2-differential Lattice

Let x and y that both cover $\hat{0}$. label the respective edges r and b. Since $x \wedge y = \hat{0}$, x and y are covered by a same element of rank 2, which will give a set of four elements with a diamond relations:

For every diamond relation with a pair of adjacent edges labeled. we have one of the two following ways of developing it:

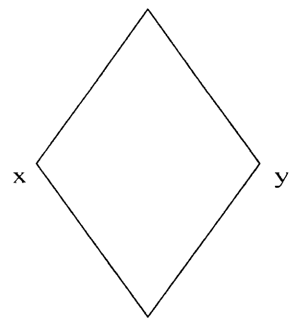

Figure 5-3:

1. If the labeled edges are of a same label, then label the rest two edges with the same label.

2. If the labeled edges are different labels, then label the opposite edges the same way.

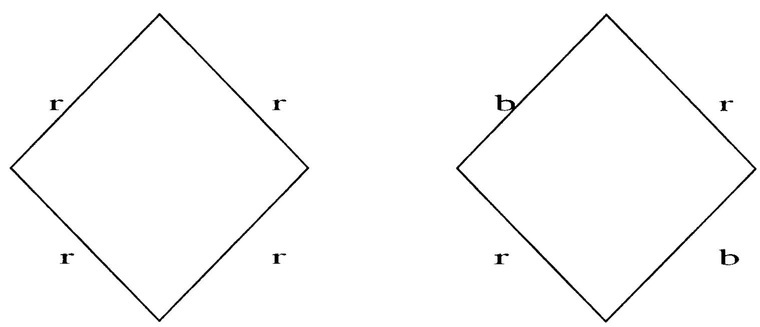

Figure 5-4:

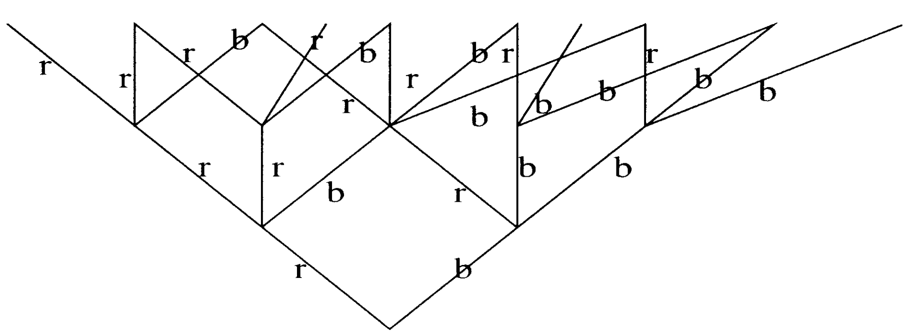

Figure 5-5:

In the next stage of this research, we aim at studying and refining such a labeling, so that it becomes well defined, i.e., each edge is assigned to exactly one kind of labeling, and can be generalized to other type of 2-differential lattices. More over, we hope to prove that the labeling has some if not all of the following characteristics:

$$deg_{(u,r)} = deg_{(d,r)} + 1$$
$$deg_{(u,b)} = deg_{(d,b)} + 1.$$

If such a labeling exists, consider the sublattice that consists of all r-edges and the set of elements $\mathbf{A}$ incident to these edges. For $x, y \in \mathbf{A}$, then it will be helpful to show:

$$x \wedge y \in \mathbf{A} \text{ if and only if } x \wedge y \in \mathbf{A}$$

To sum up, these kinds of labeling are aiming at to prove or disprove the following conjecture of Richard Stanley:

An r-differential lattice P where $r > 1$ is a product of Young's lattices and the Fibonacci lattices.

If P and Q are posets, then the *direct* (or *Cartesian*) *product* of P and Q is the poset $P \times Q$ on the set $\{(x, y) : x \in P, y \in Q\}$ such that $(x, y) \leq (x', y')$ in $P \times Q$ if $x \leq x'$ in P and $y \leq y'$ in Q.

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