Growth of hyperbolic groups

11 Sep 2020, GGTEA

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Joint work with Zlil Sela on the arxiv

G : f.g. gp, S : a generating set, Cayley(G,S).

$B_n(G,S) \subset$ Cayley : the ball of radius n at $1 \in G$.

- The exp growth rate of (G,S) is :

$$e(G,S) = \lim_{n\to\infty} |B_n(G,S)|^{\frac{1}{n}} \geq 1.$$

- G has exp growth if $\exists S$ ($\forall S$) s.t $e(G,S) > 1$.

$$e(G) = \inf_S e(G,S) \geq 1$$

- G has unif exp growth if $e(G) > 1$.

Q For which G, is there S s.t $e(G) = e(G,S)$? (de la Harpe)

$\exists G$ f.g. of exp growth, but not unif exp growth.

For such G, $\nexists S$ s.t $e(G) = e(G,S)$.

$$\begin{array}{cc} \shortparallel & \vee \\ 1 & 1 \end{array}$$

§ Some examples.

- $e(F_k) = 2k-1 = e(F_k, \{\text{The free gen.}\})$

  This is more or less only example s.t $e(G)$ is known & $e(G)$ is known to be realized by some S (and which S).

- $e(\pi_1(\Sigma_g)) \geq 4g-3$ de la Harpe

  $e(\pi_1(\Sigma_g), \{\text{Standard}\}) = 4g-1-\varepsilon_g \overset{?}{=} e(\pi_1(\Sigma_g))$

  $\uparrow$ 2g-elements

  $\varepsilon_g \to 0$ $(g \to \infty)$

§ Results

Define for f.g. G,

$$\xi(G) = \{ e(G,S) \mid \text{for all } S,\ |S|<\infty,\ \langle S\rangle = G \} \subset \mathbb{R}_{\geq 1}, \text{ countable}$$

$$e(G) = \inf \xi(G).$$

<u>Th1</u> (Sela–F)

If G is non-elem. hyp, then $\xi(G)$ is well-ordered.

In particular, $e(G)$ is realized by some S.

<u>Cor</u> (Sela '99, Reinfeldt–Weidmann '14)

If G is non-elem hyp gp, then G is Hopf, i.e. if $G \twoheadrightarrow G$, then is om.

(new pf) de la Harpe

Use <u>Th</u> (Arzhantseva–Lysenok) Growth tightness

If $f : (G_1, S) \twoheadrightarrow (G_2, f(S))$ is a surj between hyp gps G_1 & G_2, s.t. f is not isom, then $e(G_1, S) > e(G_2, f(S))$.

Now, take S s.t. $e(G) = e(G,S)$. Suppose $f : G \twoheadrightarrow G$ is given, not isom.

then $f : (G,S) \twoheadrightarrow (G, f(S))$, $\underset{\substack{\| \\ e(G)}}{e(G,S)} > \underline{\underline{e(G,f(S))}}$. ⨳ □

L4

Th2 ~~For~~ If G is hyp,
for each $r \in \xi(G)$, then at most finitely many S
s.t. $e(G,S) = r$, upto $\mathrm{Aut}(G)$ (finiteness)

- G f.g. gp.

Define $\Theta(G) = \{ e(H,S) \mid H < G,\ \langle S \rangle = H,\ |S| < \infty \} \subset \mathbb{R}_{\geq 1}$
$\cup$ H has exp growth
$\xi(G)$

Th3 If G is non-ele hyp. $\Theta(G)$ is well-ordered.
finiteness hold.

~~Th~~ $\xi(G)$ is well-ordered.

Define $\xi(G) =$ The ordinal of $\xi(G)$: growth ordinal

$\mathbb{N}$ its ordinal is ω.

Th4 If G is non-ele hyp, then $\xi(G) \geq \omega^{\omega}$

Conj $\xi(G) = \omega^{\omega}$.

true for free gps, surface gps.

Rem

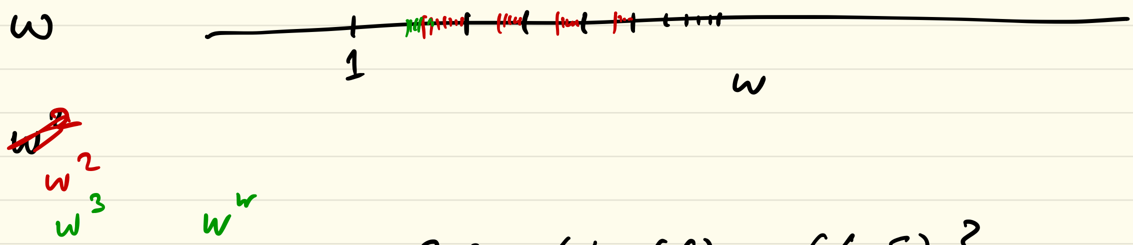
Similar to vol of hyp 3-mflds:

$\{\mathrm{vol}(M) \mid M^3$ is hyp 3-mfld of vol $< \infty\} \subset \mathbb{R}_{>0}$

hyp is well-ordered, its ordinal is ω^ω,

for each r, only finitely many M.

(Thurston, + Jorgensen)



Q. for which G, $\exists S$ s.t. $e(G) = e(G, S)$?

linear gps, MCG, $\mathrm{Out}(F_n)$ etc

Q. $G = F(a,b)$ $\quad e(G) = e(G, \{a, b\}) = 3$.

what is the second smallest $e(G, S)$, which S ?

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<u>Pf of Th 1</u> $\mathfrak{Z}(C)$ is well-ordered.

By contradiction: Suppose $\exists \{S_n\}$ seq of gen sets for G

s.t. $e(G, S_1) > e(G, S_2) > e(G, S_3) > \cdots$

It follows $\exists L$ s.t. $\forall n, \ |S_n| \le L$.

(Arzhantseva-Lysenok: $\exists C(G) > 0$ s.t. $\forall S, \ e(G,S) \geq C \cdot |S|$)

- Passing to a subseq, $|S_n| = \ell$, $\forall n$.

$$S_n = \{x_1^n, x_2^n, \ldots, x_\ell^n\}$$

- Take F_ℓ with $S = \{s_1, \ldots, s_\ell\}$

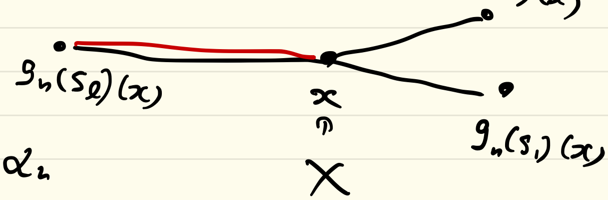
$$\exists g_n : \begin{array}{ccc} F_\ell & \twoheadrightarrow & G \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ s_i & \longmapsto & x_i^n \end{array} \qquad \text{Then } e(G, S_n) = e(G, \underbrace{g_n(S)}_{= S_n})$$

- Fix a Cayley graph X of G. $G \curvearrowright X$

Via $\{g_n\}_{n=1}^{\infty}$, $\exists$ seq of actions $\alpha_n : F_\ell \curvearrowright X$.

<u>Claim</u> $\{\alpha_n\}_{n=1}^{\infty}$ "diverges".

$g_n : (F_\ell, S) \to (G, S_n)$ $\quad S_n = \{ \quad \}$
$\qquad\qquad\qquad\qquad\quad \parallel$
$\qquad\qquad\qquad\qquad g_n(S).$



$X = \mathrm{Cayley}(G)$

$\alpha_n : F_\ell \curvearrowright (X, d)$

$\{\alpha_n\}$ diverges i.e., minimal displacement of α_n

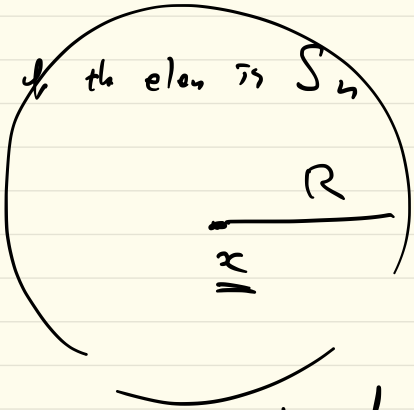
- $\rho_n = \min_{x \in X} \max_i |g_n(s_i)x - x|_X$.

$\forall x \in X$ is moved at least ρ_n by one of the elem in S_n

$\boxed{\text{Then } \rho_n \to \infty}$ ✓

why: otherwise, only finitely many choice for S_n, upto conj in G. ~~X~~

$e(G, S_n)$



- Now, we rescale (X, d) by $\frac{1}{\rho_n}$ to (X, d_n), $d_n = \frac{d}{\rho_n}$

Go to limit $\lim_{n \to \infty} (X, d_n) = Y$: $\mathbb{R}$-tree, $F_\ell \curvearrowright Y$

$$F_\ell \curvearrowright^{d_n} X = \mathrm{Cayley}(G) \rightsquigarrow F_\ell \curvearrowright Y = \lim \left(X, \frac{d}{\rho_n}\right)$$

- This action does not have a global fix pt
  since $\forall y \in Y$, $\exists s_i \in S$, $|s_i(y) - y| \geq 1$
- $F_\ell \curvearrowright Y$ is maybe not faithful, so divide it by the kernel of the action, K.

$L = F_\ell / K$. Then $L \curvearrowright Y$ faithful, by isom, no gl. fix pt.

L is called a limit gp (Sela) over G.

$\{g_n : F_\ell \to G\}$ (fixed G)

$\eta : F_\ell \twoheadrightarrow L$

Now, I quote one thm:

Th (Sela, R-W)

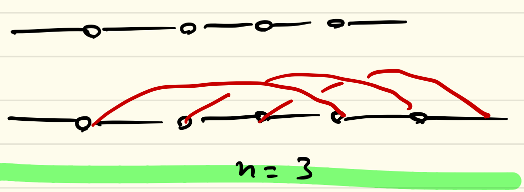
For large n, $\exists h_n : L \twoheadrightarrow G$ s.t. $g_n = h_n \circ \eta$.

$$\begin{tikzcd}
(F_\ell, S) \arrow[d, "\eta"'] \arrow[dr, "g_n"] & \\
(L, \eta(S)) \arrow[r, "\exists h_n"'] & (G, g_n(S))
\end{tikzcd}$$

$g_n(S) = S_n$

ex

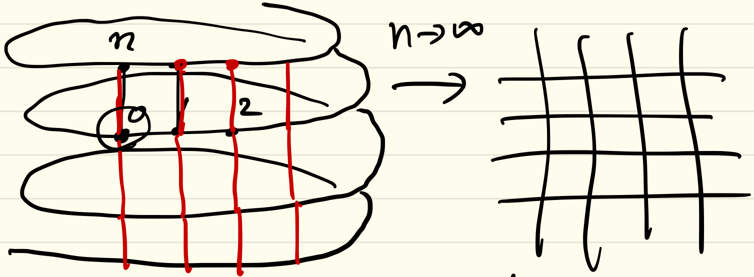
$$G = (\mathbb{Z}, \{\pm 1\}) = (\mathbb{Z}, \underbrace{\{1, n\}}_{S_n})$$



$$g_n : F_2 \to \mathbb{Z}$$

$$a \mapsto 1$$

$$b \mapsto n$$



$$L = \mathbb{Z}^2$$

$$\begin{tikzcd}
(F_2, \{a, b\}) \arrow[d, "\eta"'] \arrow[dr, "g_n"] & \\
(\mathbb{Z}^2, \{(1,0), (0,1)\}) \arrow[r, "\exists h_n"'] & (\mathbb{Z}, \{1, n\})
\end{tikzcd}$$

$$h_n : (1,0) \to 1$$

$$(0,1) \to n$$

$$\begin{tikzcd} (F_\ell, S) \arrow[d, "\eta"'] \arrow[dr, "g_n"] & S_n \\ (L, \eta(S)) \arrow[r, "h_n"'] & (G, g_n(S)) \arrow[u, phantom, "\shortparallel"] \end{tikzcd}$$

$g_n = h_n \circ \eta \Rightarrow e(G, g_n(S)) \leq e(L, \eta(S))$ easy

- "lim" $(G, g_n(S)) = (L, \eta(S))$

main <u>prop</u> $\lim_{n\to\infty} e(G, g_n(S)) = e(L, \eta(S))$ $\boxtimes$

Remember $\underline{e(L, \eta(S))} \geq e(G, g_1(S)) > e(G, g_2(S)) > e(G, g_3(S)) > \cdots$

prop $\cancel{\times}$ // Th 1.

want to show $e(L, \eta(S)) - \varepsilon \leq e(G, g_n(S))$

fix m large s.t. B_m^L in $(L, \eta(S))$ sees $e(L, \eta(S))$

for large enough n <u>B_m in $(G, g_n(S))$</u> looks like B_m^L.