Growth of hyperbolic groups
11 Sep 2020, GGTEA
Koji Fujiwara, joint with Zlil Sela
Joint work with Zlil Sela on the arxiv

\[ G : \text{f.g. gp}, \quad S : \text{a generating set}, \quad \text{Cayley}(G, S). \]

\[ B_n(G, S) \subset \text{Cayley} : \text{the ball of radius } n \text{ at } 1 \in G. \]

- The \underline{exp growth rate} of \((G, S)\) is:

\[ e(G, S) = \lim_{n \to \infty} \frac{1}{|B_n(G, S)|} \geq 1. \]

- \(G\) has \underline{exp growth} if \(\exists \) \((S) \text{ s.t. } e(G, S) > 1.\)

\[ e(G) = \inf_S e(G, S) \geq 1 \]

- \(G\) has \underline{unif exp growth} if \(e(G) > 1.\)

For which \(G\), is there \(S\) s.t. \(e(G) = e(G, S)?\) (de la Harpe)

\(\exists G \text{ f.g. of exp growth, but not unif exp growth.}\)

For such \(G\), \(\exists S\) s.t. \(e(G) = e(G, S).\)
Some examples.

- \( e(F_k) = 2^k - 1 = e(F_k, \{ \text{The free gen.} \} ) \)

This is more or less only example set \( e(C) \) is known & \( e(C) \) is known to be realized by some \( S \) (and which \( S \)).

- \( e(\pi_1(\Sigma_g)) \geq 4g - 3 \) de la Harpe

\[ e(\pi_1(\Sigma_g), \{ \text{standard} \} ) = 4g - 1 - \varepsilon_g \]

\[ \text{2g-elements} \quad \varepsilon_g \to 0 \\
\varepsilon_{g \to 100} \]
§ Result

Define for \( S, A, \)
\[ \exists (A) = \{ f \in (A, S) \mid \forall S, < S > = A \frac{3}{2} < \mathbb{R} \geq 1, \ \text{countable} \} \]
\[ |S| < \infty \]
\[ e(A) = \inf \exists (G). \]

Th 1 (Sela - F)
If \( G \) is non-elon, hyf, then \( \exists (A) \) is well-ordered.
In particular, \( e(A) \) is realized by some \( S \).

Cor (Sela '99, Reinfelde - Weidman '14)
If \( G \) is non-elon hyf 5p, then \( G \) is Hopf, i.e., if \( G \to G \), then \( G \) is on.

(new pt) de la Houppe
Use Th (Arasonhese - Lysak) Growth rigness
If \( f: (G_1, S) \to (G_2, f(S)) \) is a surj between hyf \( G_1, G_2 \),
so \( f \) is not Tson, then \( e(G_1, S) > e(G_2, f(S)). \)

Now, take \( S \) s.t \( e(G) = e(G, S) \). Suppose \( f: G \to G \) is given,
then \( f: (G, S) \to f(S), e(G, S) > e(G, f(S)). \)
\[ e(G) \]

\[ \leftarrow \]
Theorem 2.8. If \( G \) is bzp

for each \( R \leq G \), then at most finitely many \( S \)

such that \( e(G, S) = 0 \), up to \( Aut(G) \) (for reasons)

- \( R \leq S \).

Define \( \Theta(G) = \{ S \in G | H \leq G, S \leq S = H, |S| < \infty \} \leq |R| \geq 1 \)

such that \( H \) has exp growth.

Theorem 2.8. If \( G \) is non-elm bzp, \( \Theta(G) \) is well-orded.

Factors: hold.

\( \exists (C) \) is well-orded.

Define \( \exists(C) = \) The ordinal of \( \exists(C) \) : growth ordred

In its ordred is \( \omega \).

Theorem 2.8. If \( G \) is non-elm bzp, then \( \exists(C) \geq \omega \).

Corollary: \( \exists(C) = \omega \).

true for free gps, surfer gps.
\textbf{Rem}

\textit{Sturla to vol b hyp 3-wlds:}

\$\text{vol}(M) \mid M^3 \text{ is hyp 3-wld if vol} < \infty \ 3 < IR > 0$

\textit{hyp} is well-ordered, its ordinal is \( w^w \),

for each \( r \), only finitely many \( M \).

\textit{(Thuris, + Jørgensen)}

\[ w \quad 1 \quad \cdots \quad w \]

\[ w^2 \quad w^3 \quad w^w \]

\textbf{Q.} for which \( G \), \( 3 \leq e(G) \leq e(G, S) \)?

\textit{Linea gps, MCA, \( \omega \omega(F_4) \) etc}

\textbf{Q.} \( G = F(a, b) \quad e(G) = e(G, f^a, b^b) = 3 \).

\textit{What is the second smallest} \( e(G, S) \), \textit{which} \( S \)?
Proof of Thm 3. Let $\mathcal{C}$ be well-ordered.

By contradiction: Suppose $\exists S_n$ is a set of sets in $\mathcal{C}$

$$e(S_n) > e(S_{n+1}) > e(S_{n+2}) > \cdots$$

It follows $\exists L < \forall n, |S_n| \leq L.$

(Arabian Lemma: $\forall (C, G) > 0 \iff \forall S, e(C, S) \geq C \cdot |S| \leq L$)

- Pick any $S_n$, a subseq. $|S_n| = 2$, $\forall n$.

$$S_n = \{ x_1^n, x_2^n, \ldots, x_L^n \}$$

- Take $F \rightarrow S = \{ S_1, \ldots, S_L \}$

$$\exists g_n : F \rightarrow G, \quad \text{then } e(G, S_n) = e(G, g_n(S))$$

- Fix a Cayley graph $X \rightarrow G$. $G \rightarrow X$

$$\forall a \in \mathcal{A}, \exists \mathcal{A} \cdot a = \mathcal{A} \cdot a \cdot a$$

Claim $\{ d_n \}_{n=1}^{\infty}$ diverges.
\[ g_n : (\mathbb{F}_\ell, S) \to (\mathcal{C}, S_n) \]
\[ S_n = \{ \} \]
\[ g_n(s) \]

\[ X \in \text{Cayley}(\mathcal{C}) \]
\[ d_n : \mathbb{F}_\ell \to (X, d) \]

\[ g_n^3 \text{ diverges i.e., minimal displacement of } d_n \]

\[ s_n = \min_{x \in X} \max_{i} |g_n(s) x - x| \]

\[ X \subseteq X \text{ is moved at least } s_n \text{ by one of the elements in } S_n \]

\[ \text{Then } s_n \to \infty \]

\[ \text{Why: otherwise, only finitely many choices to } S_n, \text{ up to cong in } \mathcal{C}. \]

\[ e(\mathcal{C}, S_n) \]

Now, we rescale \((X, d)\) by \( \frac{1}{s_n} \) to \((X, d_n)\), \( d_n = \frac{d}{s_n} \)

Go to limit \( \lim_{n \to \infty} d_n (X, d_n) = Y : R - \text{tree, } \mathbb{F}_\ell \to Y \)
$Fe^X = Cayley(G)$ where $Fe^X Y = \log \left( X, \frac{d}{p_n} \right)$

- This action does not have a global fix pt

- $Fe^X Y$ is maybe not faithful, so divide it by the kernel of the action $K$.

$$L = Fe/K.$$ Then $L \simeq Y$ faithful, by 750 w, no gl. fix pt.

$L$ is called a limit gp over $G$.

\[ \begin{array}{c}
\{ \text{g}_n : Fe \to G \} \\
\end{array} \]

\[ \eta : Fe \to L \]

Now, I run out the:

\[ \text{Th (Sela, R-W)} \]

For large $n$, $\exists h_n : L \to G$

\[ \therefore g_n = h_n \circ \eta. \]
$G = (\mathbb{Z}, f_n)$

$= (\mathbb{Z}, \{1, n^3\})$

$n = 3$

$g_n : F_2 \rightarrow \mathbb{Z}$

$a \mapsto 1$

$b \mapsto n$

$F_2 = \{a, b\}$

$\{1, 0\} \hookrightarrow \mathbb{Z}$

$\mathbb{Z}^2 \hookrightarrow \mathbb{Z}^3$

$h_n : (1, 0) \mapsto 1$

$(0, 1) \mapsto n$
\[
(F_k, S) \\
\eta \downarrow \left\{ S_n, S_n \right\} \\
(L, \eta(S)) \xrightarrow{h_n} (G, g_n(s))
\]

\[g_n = h_n \circ \eta \Rightarrow e(G, g_n(s)) \leq e(L, \eta(s)) \quad \text{easy.}\]

\[
\lim (G, g_n(s)) = (L, \eta(s))
\]

Remark: \[e(L, \eta(s)) > e(G, s_1(s)) > e(G, s_2(s)) > \cdots\]

Want to show: \[e(L, \eta(s)) - \varepsilon \leq e(G, g_n(s))\]

Fix \( m \) large enough \( B_m \) in \((L, \eta(s))\) sees \( e(L, \eta(s)) \)

Fix large \( m \) \( B_m \) in \((G, g_n(s))\) looks like \( B_m \).