Sublinearly Morse Boundary

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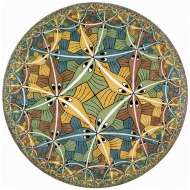
Gromov boundary of a δ–hyperbolic space

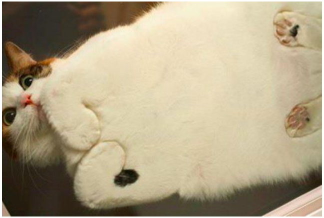
- A point in the boundary is a geodesic ray or a family of quasi-geodesic rays up to fellow traveling.
- cone topology



Gromov boundary of a hyperbolic space is QI-invariant.

Visual boundary of CAT(0) spaces

- geodesics, up to fellow travel.
- cone topology



–Croke-Kleiner: the visual boundary is not QI-invariant.

Key: geodesics are Morse in a Gromov hyperbolic space.

A quasi-geodesic ray γ is Morse if given any pair (q, Q), there exists constant $n(q, Q)$ such that all (q, Q)–quasi-geodesics whose endpoints are on γ stays inside the $n(q, Q)$-neighbourhood of γ.

Morse boundary(Charney-Sultan, Cordes, Cashen-Mackay): Morse geodesics.

–Not large enough from the point of view of random walk.

κ-Morse boundary

Space: $(X, \mathfrak{o})$ is a proper, geodesic space, with a fixed base-point $\mathfrak{o}$.

Points in the boundary: families of quasi-geodesic rays starting at $\mathfrak{o}$.

Fix a sublinear function $\kappa(t)$. Let $\|x\| = d(\mathfrak{o}, x)$. A κ–neighbourhood around a quasi-geodesic γ is a set of point x

$$\mathcal{N}_\kappa(\gamma, n) := \{x \mid d(x, \gamma) \leq n \cdot \kappa(\|x\|)\}$$

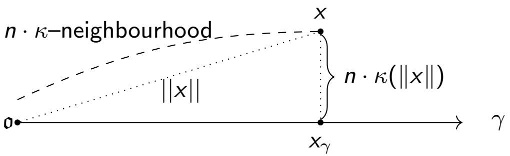


Figure: A κ–neighbourhood of γ

A quasi-geodesic ray γ is κ–Morse if there exists a proper function $m_\gamma : \mathbb{R}^2 \to \mathbb{R}$ such that for any sublinear function κ' and for any $r > 0$, there exists R such that for any (q, Q)–quasi-geodesic β with $m_\gamma(q, Q)$ small compared to r, if

$$d_X(\beta_R, \gamma) \le \kappa'(R) \qquad \text{then} \qquad \beta|_r \subset \mathcal{N}_\kappa\big(\gamma, m_\gamma(q, Q)\big)$$

The function m_γ will be called a Morse gauge of γ.

Equivalence class: given two quasi-geodesics α, β based at $\mathfrak{o}$, we say that $\beta \sim \alpha$ if they sublinearly track each other: i.e. if

$$\lim_{r \to \infty} \frac{d(\alpha_r, \beta_r)}{r} = 0.$$

Let $\partial_\kappa X$ denote the set of equivalence class of κ–Morse quasi-geodesic rays, equipped with coarse cone topology.

Theorem (Q-Rafi, Q-Rafi-Tiozzo)

Let X be a proper, geodesic metric space, then $\partial_\kappa X$ is a topological space that is quasi-isometrically invariant, and metrizable.

Theorem (Q-Rafi-Tiozzo)

For sublinear functions κ and κ' where $\kappa(t) \leq \kappa'(t)$ for any $t > 0$, we have $\partial_\kappa X \subset \partial_{\kappa'} X$ where the topology of $\partial_\kappa X$ is the subspace topology associated to the inclusion. Further, letting $\partial X = \cup_\kappa \partial_\kappa X$, we obtain a topological space that contains all $\partial_\kappa X$ as topological subspaces.

Examples:

- $\mathbb{Z}^2$
- $\mathbb{H}^2$
- $\mathbb{Z} \star \mathbb{Z}^2$

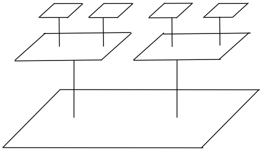


Figure: A tree of flats.

Random walk and Poisson boundaries

Let $\langle S \rangle$ be a symmetric generating set with a probability distribution μ. A *random walk* is a process on a group G where sample paths are $s_{r_1} s_{r_2} s_{r_3} \ldots$, $s_{r_i} \in \langle S \rangle$.

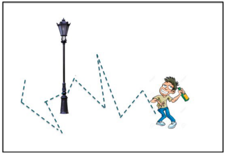


Figure: A random walk.

Definition

Given a finitely generated group and a probability measure μ with finite support, its *Poisson boundary* is the maximal measurable set to which almost all sample paths converge, with hitting measure ν arising from μ.

Kaimanovich: Let G be a hyperbolic group, then Gromov boundary is a model for it's associated Poisson boundary.

Theorem (Gekhtman-Q-Rafi)

Let X be a rank-1 CAT(0) space, and $G \curvearrowright X$ geometrically. Then there exists a κ such that the Poisson boundary can be identified with $\partial_\kappa G$.

Proof idea:

A unit speed, parametrized geodesic ray τ in X is said to be recurrent if there is a number $N > 0$ such that for each $R > 0$ and $\theta \in (0, 1)$ there is an $L_0 > 0$ such that for $L > L_0$ length θL subsegment of $\tau([0, L])$ contains N-(strongly) contracting subsegment of length at least R.

1. A generic sample path tracks a recurrent geodesic ray.
   - Stationary measure: follow the proof of Baik-Gekhtman-Hamenstädt.
   - Patterson Sullivan measure (defined by Ricks): Birkhoff ergodic theorem.
2. A recurrent geodesic ray is sublinearly Morse.

Mapping class groups

S: oriented surface of finite type

$Map(S) := \mathrm{Homeo}^+(S)/\mathrm{Isotopy}$.

Kaimanovich-Masur: The space of projective measured foliations with the corresponding harmonic measure can be identified with the Poisson boundary of random walks on the associated mapping class group.

Theorem (Q-Rafi-Tiozzo)

Let μ be a finitely supported, non-elementary probability measure on $Map(S)$. Then for an integer p depending only on the topology of S and $\kappa(t) = \log^p(t)$, we have

1. *Almost every sample path (w_n) converges to a point in $\partial_\kappa Map(S)$;*
2. *The κ–Morse boundary $(\partial_\kappa Map(S), \nu)$ is a model for the Poisson boundary of $(Map(S), \mu)$ where ν is the hitting measure associated to the random walk given by μ.*

We now consider the set of points in $\mathcal{EL}$ that have *logarithmically bounded projection* to all subsurfaces. Let θ be a fixed set of filling curves on S once and for all. Given a proper subsurface $Y \subsetneq S$, let ∂Y denote the multi-curve of boundary components of Y and define

$$\|Y\|_S := d_S(\theta, \partial Y).$$

Similarly, for $x \in Map(S)$, define

$$\|x\|_S := d_S\big(\theta, x(\theta)\big).$$

Definition

For a constant $c > 0$, let L_c be the set of points $\xi \in \mathcal{EL}$ such that

$$d_Y(\mathfrak{o}, \xi) \leq c \cdot \log\|Y\|_S \tag{1.1}$$

for every subsurface $Y \subsetneq S$.

Theorem (Q-Rafi-Tiozzo)

the Poisson boundary can be identified with $\partial_\kappa G$ *for the following groups.*

- *Right-angled Artin groups,* $\kappa(t) = \sqrt{t \log t}$.
- *Relative hyperbolic groups,* $\kappa(t) = \log t$
- *Mapping class groups,* $\kappa(t) = \log^p t$

Some properties of κ-Morse geodesic ray in $\mathrm{CAT}(0)$ spaces

Definition

Let b be a geodesic ray and fix some $t > 0, r > 0$. Let $\rho_\kappa(r,t)$ denote the infimum of the lengths of all paths from $b(t - r\kappa(t))$ to $b(t + r\kappa(t))$ which lie outside the open ball of radius $r\kappa(t)$ about $b(t)$. Given such a geodesic ray b, we define the κ-*lower divergence* of b to be growth rate of the function

$$div_\kappa(r) := \inf_{t > r\kappa(t)} \frac{\rho_\kappa(r,t)}{\kappa(t)}.$$

	b is Morse	$b \in \partial_{\log(t)} X$	$b \in \partial_{\sqrt{t}} X$
1- lower-divergence	superlinear	linear	linear
$\log(t)$-lower-divergence	superlinear	superlinear	linear
$\sqrt{t}$-lower-divergence	superlinear	superlinear	superlinear

Q-Murray-Zalloum: The κ-lower divergence of b is at least quadratic.

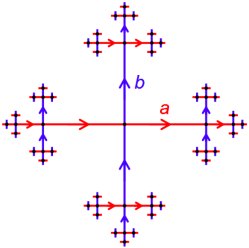
Alternative characterization of κ-Morse elements: hyperplane geometry

Two hyperplanes h_1, h_2 are:

Strongly separated: no hyperplane crossing both.

k-separated: the number of hyperplanes crossing both are bounded above by k.

k-well-separated: the number of crossing both and do not contain a *facing triple* are bounded above by k.



Theorem (Q-Murray-Zalloum)

Let X be a locally finite cube complex. A geodesic ray $b \in X$ is κ-contracting if and only if there exists $c > 0$ such that b crosses an infinite sequence of hyperplanes $h_1, h_2, \ldots$ at points $b(t_i)$ satisfying:

1) $d(t_i, t_{i+1}) \leq c\kappa(t_{i+1})$.
2) h_i, h_{i+1} *are* $c\kappa(t_{i+1})$*-well-separated.*

Corollary (Q-Murray-Zalloum)

κ-Morse geodesic rays project to infinite diameter sets in the contact graph of right-angled Artin groups.

κ-Morse vs. κ-contracting.

In $\mathrm{CAT}(0)$ space we use the *nearest-point* projection.

Definition

A set is *D-contracting* if there exists a constant D such that all disjoint ball projects to sets of diameter at most D on the set. A set is *contracting* if it is D-contracting for some D.

Definition

Similarly a set is κ-contracting if there exists a constant c such that each disjoint ball $B(x, r)$ is projected to sets of diameter at most $c \cdot \kappa(x)$.

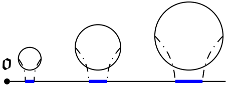


Figure: A sublinearly contracting geodesic ray

Example: tree of flats.

Theorem (Charney-Sultan)

In CAT(0) spaces, A geodesic ray is Morse if and only if it is contracting.

Theorem (Q-Rafi)

In CAT(0) spaces, κ-Morse is equivalent to κ-contracting.

Theorem (Q-Rafi-Tiozzo)

In proper geodesic spaces, sublinearly Morse is equivalent to sublinearly contracting, but the sublinear functions may differ.

However, in many groups/spaces, nearest-point projection is not well understood nor is it helpful to use. More generally,

Definition

Let (X, d_X) be a proper geodesic metric space and $Z \subseteq X$ a closed subset, and let κ be a sublinear function. A map $\pi_Z \colon X \to Z$ is a *κ-projection* if there exist constants D_1, D_2, depending only on Z and κ, such that for any points $x \in X$ and $z \in Z$,

$$\operatorname{diam}_X(\{z\} \cup \pi_Z(x)) \leq D_1 \cdot d_X(x, z) + D_2 \cdot \kappa(x).$$

A κ-projection differs from a nearest point projection by a uniform multiplicative error and a sublinear additive error.

- Nearest-point projections, the projection we use in mapping class groups (Duchin-Rafi) and in relatively hyperbolic group (Q-Rafi-Tiozzo) are examples of κ-projections.
- Since X is assumed to be proper, projections exist, not necessarily unique.

For a closed subspace Z of a metric space (X, d) and a κ-projection π onto Z, we say Z is κ–weakly contracting with respect to π if there are constants C_1, C_2, depending only on Z, such that, for every $x, y \in X$

$$d(x, y) \leq C_1 d(x, Z) \Rightarrow \mathrm{d}_X(\pi(x), \pi(y)) \leq C_2 \cdot \kappa(x).$$

- Axes of Pseudo-Anosov elements in mapping class groups are not known to be contracting but they are weakly contracting. (Masur-Minsky, Rafi-Verberne)

Theorem (Q-Rafi-Tiozzo)

Every κ–weakly contracting set, with respect to a κ-projection, is κ-Morse. Every κ-Morse set is κ'-weakly-contracting for some κ'.

Definition of Coarse cone Topology

We define the set $\mathcal{U}(\beta, r) \subseteq X \cup \partial_\kappa X$ as follows.

- An equivalence class $\mathbf{a} \in \partial_\kappa X$ belongs to $\mathcal{U}(\beta, r)$ if for any (q, Q)-quasi-geodesic $\alpha \in \mathbf{a}$, where $m_\beta(q, Q)$ is small compared to r, we have the inclusion
$$\alpha|_r \subseteq \mathcal{N}_\kappa(\beta, m_\beta(q, Q)).$$

Proof ideas for random walks

Sisto-Taylor: Projections systems.

- Relative hyperbolic groups
- Curve complex of subsurfaces in mapping class group.
- Hierarchically hyperbolic groups.

Let G be a group and let $(\mathcal{S}, Z_0, \{\pi_Z\}_{Z\in\mathcal{S}}, \pitchfork)$ be a projection system on G. Let (w_n) be a random walk on G. Then there exists $C \geq 1$ so that, as n goes to ∞,

$$\mathbb{P}\big(\sup_{Z\in\mathcal{S}} d_Z(1, w_n) \in [C^{-1}\log n, C\log n]\big) \to 1$$

2. Maximality: the tracking is sublinear. Sisto, Tiozzo, Maher-Tiozzo, Karlsson-Margulis, Q-Rafi-Tiozzo.

Question

- What are the "shapes" of $\partial_\kappa G$ for different G?
- Can $\partial_\kappa G$ be part of a compact space?
- When does a group G has a $\partial_\kappa G$ that can be identified with the Poisson boundary?

Thank you!