GEOMETRIC INEQUALITIES

Consequences of Besicovitch inequality:
questions related to Papasoglu's isoperimetric inequality on the two-sphere and Loewner's systolic inequality.

1. Let (S^2, g) be a Riemannian 2-sphere of area 1. Does there exists a curve γ of length ≤ 100, dividing (S^2, g) into two discs of area $\frac{1}{2}$?

2. In class we proved Papasoglu's isoperimetric inequality on the Riemannian two-sphere using Besicovitch inequality.

We considered some related results and problems, among them the following theorem of Balacheff and Sabourau:

Theorem For any Riemannian 2-sphere M there exists a Morse function $f : M \to \mathbb{R}$, such that $length(f^{-1}(x)) \leq 100\sqrt{Area(M)}$.

Conjecture (Calabi-Croke) The optimal constant in the theorem above is $\sqrt[4]{12}$.

Problem: Show that this is the constant you obtain when M is a 2-sphere obtained by gluing two equilateral triangles along their boundary.

(You want to show that in this case every continuous function from M to $\mathbb{R}$ will have a fiber of length at least twice the height of the triangle).

What is the constant that you get for the round sphere?

Let us define the waist of M, denoted by $W(M)$, as the infimum of

$$\max_{x \in \mathbb{R}} length(f^{-1}(x))$$

over all Morse functions $f : M \to \mathbb{R}$.

(Definitions similar to this one appear frequently! Take some time to process it. Is it clear to you that the Theorem above can be restated in the form of inequality $W(M) \leq 100\sqrt{Area(M)}$?).

The Calabi-Croke conjecture has been proved for certain classes of spheres, which are very close to the Calabi-Croke sphere (two equilateral triangles glued together).

3. **Open problem (Buser-Guth)** Does there exist a universal constant $C > 0$ with the following property:

For every 2-dimensional Riemannian surface M there exists a continuous map $F : M \to \Gamma$ into a graph Γ with $Length(F^{-1}(x)) \leq C\sqrt{Area(M)}$ for all $x \in \Gamma$?

4. Suppose $M = (T^3, g)$ is a Riemannian 3-torus of volume 1. Does there always exist a homologically non-trivial 2-torus $T \subset M$ of area less than 1000 in M?