GEOMETRIC INEQUALITIES

Open Problems

1. In class we considered the space ΛS^2 of closed curves on the two-sphere and the space ΠS^2 of simple closed curves on the two-sphere.

Given a Riemannian metric on the 2-sphere we defined widths

$$W_i^{\Lambda}(S^2, g) = \inf\{\max_t E(f(t)), f : P \to \Lambda S^2\}$$

$$W_i^{\Pi}(S^2, g) = \inf\{\max_t E(f(t)), f : P \to \Pi S^2\}$$

where the infimum is taken over maps f from some simplicial complex into ΛS^2 or ΠS^2 that represent a non-trivial element in the ith homology.

Open problem: Is it true that $W_i^{\Lambda}(S^2, g) = W_i^{\Pi}(S^2, g)$ for all $i = 1, 2, 3$ and every metric g?

Some progress (in the case i=1) in this direction was obtained in Chambers-Liokumovich1 and Chambers-Liokumovich2, where constructing families of simple closed curves from closed curves with self-intersections was introduced.

2. For every Riemannian 2-sphere (S^2, g) it is known that there exists a closed geodesic γ on (S^2, g) with $Length(\gamma) \leq 4\sqrt{2}\sqrt{Area(S^2, g)}$. (see Rotman, Croke).

Conjecture: There exist two distinct closed geodesics γ_1 and γ_2, such that

$$Length(\gamma_1)Length(\gamma_2) \leq 100 Area(S^2, g)$$

("Distinct" here means that they should have different images; you can not take the same geodesic traversed twice, for example)

3. In class we proved isoperimetric inequalities for lipschitz cycles of arbitrary codimension in $\mathbb{R}^n$.

Open problem: prove a parametric isoperimetric inequality.

More precisely, let $\{z_t\}_{t \in X}$ denote a family of Lipschitz cycles with integer coefficients in $\mathbb{R}^n$ that are continuous in the flat distance.

(The flat distance is defined as follows:

$$\mathcal{F}(z_1, z_2) = \inf\{Vol(\tau) : \partial\tau = z_1 - z_2\}$$

where the infimum is taken over Lipschitz $(k+1)$-chains τ. You can read about it in Fleming, W., Flat chains over a finite coefficient group, Trans. Amer. Math. Soc. 121 (1966), 160-86).

Does there exist a continuous family of $(k+1)$-chains $\{\tau_t\}_{t \in X}$, such that $\partial\tau_t = z_t$ and

$$\mathrm{Vol}(\tau_t) \leq C(n,k) \sup_t \mathrm{Vol}(z_t)^{\frac{k+1}{k}} ?$$