# GEOMETRIC INEQUALITIES

Problem set 1

Let $X$ be a metric space and $s > 0$. The $s$-dimensional Hausdorff content $\mathrm{HC}_s(U)$ of a subset $U \subset X$ is defined as the infimum of $\sum r_i^s$ over all coverings

$$U \subset \bigcup B(x_i, r_i) \subset X$$

Unlike the definition of the Hausdorff measure, in the definition of the Hausdorff content we do not take the limit with the radii of balls in the covering tending to zero. Using Hausdorff content instead of Hausdorff measure is sometimes advantageous, because Hausdorff content ignores "waviness" of the set on small scales. For example, take a very thin cylinder in $\mathbb{R}^3$. One can make the area of the cylinder very large by adding many small wrinkles to its surface. However, the 2-dimensional Hausdorff content will remain small, reflecting the fact that the surface is still close to something 1-dimensional.

1. Observe that for every compact set $K$ the Hausdorff content $\mathrm{HC}_s(K)$ is finite for all $s > 0$ (unlike the Hausdorff measure).

2. Give an example of two disjoint open sets $U_1, U_2 \subset \mathbb{R}^2$, such that $\mathrm{HC}_k(U_1 \cup U_2) \neq \mathrm{HC}_k(U_1) + \mathrm{HC}_k(U_2)$ for some $k$. (In particular, Hausdorff content is not a measure).

3. Show that

$$\mathrm{HC}_k(U)^{\frac{1}{k}} \geq \mathrm{HC}_m(U)^{\frac{1}{m}}$$

for $k \leq m$.

4. Consider $\mathbb{R}^n_\infty$, the Euclidean space with $l_\infty$ metric. Let $U \subset \mathbb{R}^n_\infty$ be an open set. Prove the isoperimetric inequality

$$\mathrm{HC}_n(U) \leq \mathrm{HC}_{n-1}(\partial U)^{\frac{n}{n-1}}$$

(Hint: use the prove of Loomis-Whitney inequality)

5. Show that this inequality is sharp.

**Open problem**

Let $U \subset \mathbb{R}^n_\infty$ be an open set and $m < n$ an integer. Prove the isoperimetric inequality

$$\mathrm{HC}_m(U) \leq \mathrm{HC}_{m-1}(\partial U)^{\frac{n}{n-1}}$$

This innocent looking question has important implications for a certain important problem in systolic geometry (which may have applications in manifold learning in AI research).

6. Let $(S^2, g)$ be a Riemannian 2-sphere of area 1. Does there exists a curve $\gamma$ of length $\leq 100$, dividing $(S^2, g)$ into two discs of area $\frac{1}{2}$?