RENORMALIZATION HORSESHOE FOR CRITICAL CIRCLE MAPS

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Abstract. We prove that the invariant horseshoe for the renormalization of critical circle maps is globally uniformly hyperbolic with one-dimensional unstable direction. This extends our earlier result on hyperbolicity of periodic orbits of the renormalization, and completes the proof of Lanford's Program.

With this paper we complete our description of the renormalization picture for critical circle maps, began in [Ya1], [Ya2], [Ya3], and [EY]. The renormalization theory of critical circle maps serves to explain the universality phenomena which are observed in the families of one-dimensional dynamical systems given by smooth homeomorphisms of the circle with a finite number of (non-flat) critical points. Historically, this is one of the two main examples of universality in one-dimensional dynamics, the other being the Feigenbaum universality. A renormalization operator for critical circle maps appeared in the papers of Ostlund *et al* [ORSS] and Feigenbaum *et al* [FKS]. The most general formulation of the main conjectures of the theory is due to Lanford [Lan1], [Lan2], and became known as the Lanford's Program.

The renormalization theory of critical circle maps has developed in parallel with the other one-dimensional renormalization theory, that of the unimodal maps of the interval. The recent spectacular progress in the latter began with the seminal work of Sullivan [Sul1], [Sul2], [MvS], who gave a conceptual explanation of the existence of the Feigenbaum fixed point using the methods of holomorphic dynamics and the Teichmüller theory. The results of Sullivan were adapted to the critical circle maps by de Faria [dF1], [dF2], who, in particular, introduced an appropriate analogue of the Douady-Hubbard quadratic-like maps. McMullen's approach to constructing the renormalization horseshoe for bounded type [McM2] has also been translated into the critical circle map setting by de Faria and de Melo [dFdM2]. In our works [Ya1], [Ya2] we have extended the above mentioned results to all combinatorial types, and constructed the global renormalization horseshoe for critical circle maps. The two one-dimensional renormalization theories diverged at the next stage: proving the hyperbolicity of the horseshoe presented different challenges in the two contexts. In the unimodal setting it has been shown by Lyubich [Lyu4], [Lyu5]. His proof roughly consists of two parts: endowing the space of quadratic-like maps with the structure of an analytic manifold, in which the stable sets of the renormalization operator

Date: September 9, 2002.

The author gratefully acknowledges the financial support of NSERC and the Connaught Foundation of the University of Toronto.

become submanifolds; and showing the existence of a single unstable direction for the renormalization. The second step, which is the more difficult of the two, and involves Lyubich's new Rigidity Theorem [Lyu3], for critical circle maps is elementary. The first step, on the other hand, can not be carried out for critical commuting pairs – which is the language in which the renormalization of critical circle maps has been defined. We have resolved this difficulty in [Ya3] by introducing a new, more natural, renormalization construction, the cylinder renormalization operator. The paper [Ya3] contains the proof of hyperbolicity of the periodic orbits of this transformation. In this paper we prove that the whole renormalization horseshoe is hyperbolic for the cylinder renormalization operator. The two novel ingredients are the construction of the stable lamination for the renormalization, made difficult in this case by the absense of the hybrid class – external map decomposition; and the study of the limiting case, the parabolic renormalization carried out in [EY].

Acknowledgement. The author would like to express his gratitude to Mikhail Lyubich for many stimulating discussions.

1. Preliminaries

Some notations. We will generally assume that the reader is familiar with the renormalization theory of critical circle maps, and will only recall its main points briefly. For a detailed discussion, we refer the reader to [Ya3].

We use dist and diam to denote the Euclidean distance and diameter in $\mathbb{C}$. We shall say that two real numbers A and B are K-commensurable for $K>1$ if $K^{-1}|A|\leq|B|\leq K|A|$. The notation $D_r(z)$ will stand for the Euclidean disk with the center at $z\in\mathbb{C}$ and radius r. The unit disk $D_1(0)$ will be denoted $\mathbb{D}$. The plane $(\mathbb{C}\setminus\mathbb{R})\cup J$ with the parts of the real axis not contained in the interval $J\subset\mathbb{R}$ removed will be denoted $\mathbb{C}_J$. By the circle $\mathbb{T}$ we understand the affine manifold $\mathbb{R}/\mathbb{Z}$, it is naturally identified with the unit circle $S^1=\partial\mathbb{D}$. The real translation $x\mapsto x+\theta$ projects to the rigid rotation by angle θ, $R_\theta:\mathbb{T}\to\mathbb{T}$. For two points a and b in the circle $\mathbb{T}$ which are not diametrically opposite, $[a,b]$ will denote the shorter of the two arcs connecting them. As usual, $|[a,b]|$ will denote the length of the arc. For two points $a,b\in\mathbb{R}$, $[a,b]$ will denote the closed interval with endpoints a, b without specifying their order. The cylinder in this paper, unless otherwise specified will mean the affine manifold $\mathbb{C}/\mathbb{Z}$. Its equator is the circle $\{\operatorname{Im} z=0\}/\mathbb{Z}\subset\mathbb{C}/\mathbb{Z}$. A topological annulus $A\subset\mathbb{C}/\mathbb{Z}$ will be called an equatorial annulus, or an equatorial neighborhood, if it has a smooth boundary and contains the equator. By "smooth" in this paper we will mean "of class C^∞", unless another degree of smoothness is specified. The notation "C^ω" will stand for "real-analytic".

Critical circle maps. A critical circle map is an orientation preserving automorphism of $\mathbb{T}$ of class C^3 with a single critical point c. A further assumption is made that the critical point is of cubic type. This means that for a lift $\bar{f}:\mathbb{R}\to\mathbb{R}$ of a critical circle map f with critical points at integer translates of $\bar{c}$,

$$\bar{f}(x) - \bar{f}(\bar{c}) = (x-\bar{c})^3(\text{const} + O(x-\bar{c})).$$

We note that all the renormalization results will hold true if in the above definition "3" as the order of smoothness and the order of the critical point is replaced by any other odd number. To fix our ideas, we will always place the critical point of f at $0 \in \mathbb{T}$.

Being a homeomorphism of the circle, a critical circle map f has a well-defined rotation number, denoted $\rho(f)$. It is useful to represent $\rho(f)$ as a contined fraction with positive terms

$$\rho(f) = \cfrac{1}{r_0 + \cfrac{1}{r_1 + \cfrac{1}{r_2 + \cdots}}} \tag{1.1}$$

Further on we will abbreviate this expression as $[r_0, r_1, r_2, \ldots]$ for typographical convenience. Note that the numbers r_i are determined uniquely if and only if $\rho(f)$ is irrational. In this case we shall say that $\rho(f)$ (or f itself) is of type bounded by B if $\sup r_i \leq B$; it is of a periodic type if the sequence $\{r_i\}$ is periodic. For a map f such that the continued fraction (1.1) contains at least $m+1$ terms, we denote $\{p_m/q_m\}$ the best rational approximation of $\rho(f)$ given as the truncated continued fraction $p_m/q_m = [r_0, r_1, \ldots, r_{m-1}]$, an set $I_m \equiv [0, f^{q_m}(0)]$. Recall that an iterate $f^k(0)$ is called a *closest return* of the critical point if the arc $[0, f^k(0)]$ contains no iterates $f^i(0)$ with $i < k$. One verifies then that the iterates $\{f^{q_m}(0)\}$ are closest returns of 0.

By a classical result of Poincaré every circle homeomorphism f with an irrational rotation number is semi-conjugate to the rigid rotation $R_{\rho(f)}$. Yoccoz [Yoc] has shown that in the case when f is a critical circle map, the semiconjugacy actually becomes a conjugacy.

1.1. **Some function spaces.** In this section we will introduce some spaces of functions which will play a role in what follows. Firstly, we recall the definition of the Epstein class, introduced by Eckmann and Epstein [EE]. An orientation preserving interval homeomorphism $g : I = [0, a] \to g(I) = J$ belongs to the *Epstein class* $\mathcal{E}$ if it extends to an analytic three-fold branched covering map of a topological disk $G \supset I$ onto the double-slit plane $\mathbb{C}_{\tilde{J}}$, where $\tilde{J} \supset \text{cl}\, J$. The importance of $\mathcal{E}$ will lie in its invariance under the renormalization of commuting pairs (see below). We refer the reader to [Ya3] for a detailed discussion of $\mathcal{E}$ and further definitions. We will equip it with the Carathéodory topology. For $s \in (0,1)$ we will denote $\mathcal{E}_s$ a sequentially compact subset of $\mathcal{E}$ consisting of maps with a geometric bound s.

Let us introduce some notation: suppose, B is a complex Banach space whose elements are functions of the complex variable. Let us say that the *real slice* of B is the real Banach space $B^{\mathbb{R}}$ consisting of the real-symmetric elements of B. If X is a Banach manifold modelled on B with the atlas $\{\Psi_\gamma\}$ we shall say that X is *real-symmetric* if $\Psi_{\gamma_1} \circ \Psi_{\gamma_2}^{-1}(B^{\mathbb{R}}) \subset B^{\mathbb{R}}$ for any pair of indices γ_1, γ_2. The *real slice of* X is then defined as the real Banach manifold $X^{\mathbb{R}} \subset X$ given by $\Psi_\gamma^{-1}(B^{\mathbb{R}})$ in a local chart Ψ_γ. An operator A defined on a subset of X is *real-symmetric* if $A(X^{\mathbb{R}}) \subset X^{\mathbb{R}}$.

Denote π the natural projection $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$. For an equatorial annulus $U \subset \mathbb{C}/\mathbb{Z}$ let $\mathbf{A}_U$ be the space of bounded analytic maps $\phi : U \to \mathbb{C}/\mathbb{Z}$, suc h that $\phi(\mathbb{T})$ is homotopic to $\mathbb{T}$, equipped with the uniform metric. We shall turn $\mathbf{A}_U$ into a real-symmetric complex Banach manifold as follows. Denote $\tilde{U}$ the lift $\pi^{-1}(U) \subset \mathbb{C}$. The space of functions $\tilde{\phi} : \tilde{U} \to \mathbb{C}$ which are analytic, continuous up to the boundary, and 1-periodic, $\tilde{\phi}(z+1) = \tilde{\phi}(z)$, becomes a Banach space when endowed with the sup norm. Denote that space $\tilde{\mathbf{A}}_U$. For a function $\phi : U \to \mathbb{C}/\mathbb{Z}$ denote $\check{\phi}$ an arbitrarily chosen lift $\pi(\check{\phi}(\pi^{-1}(z))) = \phi$. Observe that $\phi \in \mathbf{A}_U$ if and only if $\tilde{\phi} = \check{\phi} - \mathrm{Id} \in \tilde{\mathbf{A}}_U$. We use the local homeomorphism between $\tilde{\mathbf{A}}_U$ and $\mathbf{A}_U$ given by

$$\tilde{\phi} \mapsto \pi \circ (\tilde{\phi} + \mathrm{Id}) \circ \pi^{-1}$$

to define the atlas on $\mathbf{A}_U$. The coordinate change transformations are given by $\tilde{\phi}(z) \mapsto \tilde{\phi}(z+n)+m$ for $n, m \in \mathbb{Z}$, therefore with this atlas $\mathbf{A}_U$ is a real-symmetric complex Banach manifold.

Now let $f : U \to \mathbb{C}/\mathbb{Z}$ be an analytic critical circle map. By definition, there is a neighborhood of the equator in which 0 is the only critical point of f. Let $\tilde{f} : \tilde{U} \to \mathbb{C}$ be a lift of f. The Argument Principle implies that for $\epsilon > 0$ small enough, if $\tilde{g} \in \tilde{\mathbf{A}}_U$ is real-analytic, with $\tilde{g}'(0) = -1$, $\tilde{g}''(0) = 0$, and $||f - g|| < \epsilon$, then $g = \pi \circ (\tilde{g} + \mathrm{Id}) \circ \pi^{-1}$ is a critical circle map as well. Let $\epsilon(f)$ be the supremum of such ϵ, and set

$$\mathbf{C}_U = \cup_f \{\pi \circ (\tilde{g} + \mathrm{Id}) \circ \pi^{-1} |\ \tilde{g} \in \tilde{\mathbf{A}}_U, \tilde{g}'(0) = -1,\ \tilde{g}''(0) = 0,\ \text{and } ||\tilde{g} - \tilde{f}|| < \epsilon(f)\}.$$

As shown in [Ya3], the space $\mathbf{C}_U$ is a codimension 2 submanifold of $\mathbf{A}_U$. A tangent space to $\mathbf{C}_U$ may be identified with the codimension 2 Banach subspace of $\tilde{\mathbf{A}}_U$, given by

$$\mathbf{B}_U = \{\tilde{\phi} \in \tilde{\mathbf{A}}_U \text{ with } \tilde{\phi}'(0) = 0, \tilde{\phi}''(0) = 0\}.$$

We shall say that f is a *critical cylinder map* if $f \in \mathbf{C}_U$ for some $U \subset \mathbb{C}/\mathbb{Z}$. Let us denote by $\mathbf{C}_U^{\mathbb{R}}$ the real slice of $\mathbf{C}_U$:

$$\mathbf{C}_U^{\mathbb{R}} = \{f \in \mathbf{C}_U \text{ with } \bar{f}(z) = f(\bar{z})\} = \{f \in \mathbf{A}_U,\ \text{such that } f|_{\mathbb{T}} \text{ is a critical circle map}\}.$$

2. Universality and renormalization

2.1. **Definition of renormalization of critical circle maps using commuting pairs.**

The strong analogy with universality phenomena in statistical physics and with the already discovered Feigenbaum-Collett-Tresser universality in unimodal maps, led the authors of [FKS] and [ORSS] to explain the existence of the universal constants by introducing a renormalization operator acting on critical circle maps. The definition is by no means straightforward, we refer the reader to [Ya3] for the motivating discussion.

Definition 2.1. A *commuting pair* $\zeta = (\eta, \xi)$ consists of two C^3-smooth orientation preserving interval homeomorphisms $\eta : I_\eta \to \eta(I_\eta),\ \xi : I_\xi \to \xi(I_\xi)$, where

(I) $I_\eta = [0, \xi(0)],\ I_\xi = [\eta(0), 0]$;

(II) Both η and ξ have homeomorphic extensions to interval neighborhoods of their respective domains with the same degree of smoothness, which commute, $\eta\circ\xi = \xi\circ\eta$;
(III) $\xi \circ \eta(0) \in I_\eta$;
(IV) $\eta'(x) \neq 0 \neq \xi'(y)$, for all $x \in I_\eta \setminus \{0\}$, and all $y \in I_\xi \setminus \{0\}$.

The space of C^r-smooth commuting pairs considered modulo an affine conjugacy can be further equipped with a natural C^r distance (see [dFdM1]).
The commutation condition allows one to iterate the extensions of the maps of a commuting pair. We can also use it to perform the following glueing construction. We can regard the interval $I = [\eta(0), \xi \circ \eta(0)]$ as a circle, identifying $\eta(0)$ and $\xi \circ \eta(0)$ and define $f_\zeta : I \to I$ by

$$f_\zeta = \begin{cases} \xi \circ \eta(x) \text{ for } x \in [\eta(0), 0] \\ \eta(x) \text{ for } x \in [0, \xi \circ \eta(0)] \end{cases}$$

The mapping ξ extends to a C^3-diffeomorphism of open neighborhoods of $\eta(0)$ and $\xi \circ \eta(0)$. Using it as a local chart we turn the interval I into a closed one-dimensional manifold M. Condition (II) above implies that the mapping f_ζ projects to a well-defined C^3-smooth homeomorphism $F_\zeta : M \to M$. Identifying M with the circle by a diffeomorphism $\phi : M \to \mathbb{T}$ we recover a critical circle mapping $f^\phi = \phi \circ F_\zeta \circ \phi^{-1}$. The critical circle mappings corresponding to two different choices of ϕ are conjugated by a diffeomorphism, and thus we recovered a C^3-smooth conjugacy class of circle mappings from a critical commuting pair.

Let f be a critical circle mapping, whose rotation number ρ has a continued fraction expansion (1.1) with at least $m+1$ terms, and let $p_m/q_m = [r_0, \ldots, r_{m-1}]$. The pair of iterates $f^{q_{m+1}}$ and f^{q_m} restricted to the circle arcs I_m and I_{m+1} correspondingly can be viewed as a critical commuting pair in the following way. Let $\bar{f}$ be the lift of f to the real line satisfying $\bar{f}'(0) = 0$, and $0 < \bar{f}(0) < 1$. For each $m > 0$ let $\bar{I}_m \subset \mathbb{R}$ denote the closed interval adjacent to zero which projects down to the interval I_m. Let $\tau : \mathbb{R} \to \mathbb{R}$ denote the translation $x \mapsto x+1$. Let $\eta : \bar{I}_m \to \mathbb{R}$, $\xi : \bar{I}_{m+1} \to \mathbb{R}$ be given by $\eta \equiv \tau^{-p_{m+1}} \circ \bar{f}^{q_{m+1}}$, $\xi \equiv \tau^{-p_m} \circ \bar{f}^{q_m}$. Then the pair of maps $(\eta|\bar{I}_m, \xi|\bar{I}_{m+1})$ forms a critical commuting pair corresponding to $(f^{q_{m+1}}|I_m, f^{q_m}|I_{m+1})$. Henceforth we shall simply denote this commuting pair by

$$(f^{q_{m+1}}|I_m, f^{q_m}|I_{m+1}). \tag{2.1}$$

This allows us to readily identify the dynamics of the above commuting pair with that of the underlying circle map, at the cost of a minor abuse of notation.

Following [dFdM1], we say that the *height* $\chi(\zeta)$ of a critical commuting pair $\zeta = (\eta, \xi)$ is equal to r, if

$$0 \in [\eta^r(\xi(0)), \eta^{r+1}(\xi(0))].$$

If no such r exists, we set $\chi(\zeta) = \infty$, in this case the map $\eta|I_\eta$ has a fixed point. For a pair ζ with $\chi(\zeta) = r < \infty$ one verifies directly that the mappings $\eta|[0, \eta^r(\xi(0))]$ and $\eta^r \circ \xi|I_\xi$ again form a commuting pair. For a commuting pair $\zeta = (\eta, \xi)$ we will denote by $\widetilde{\zeta}$ the pair $(\widetilde{\eta}|\widetilde{I_\eta}, \widetilde{\xi}|\widetilde{I_\xi})$ where tilde means rescaling by the linear factor $\lambda = -\frac{1}{|I_\eta|}$.

Definition 2.2. The *renormalization* of a real commuting pair $\zeta = (\eta, \xi)$ is the commuting pair

$$\mathcal{R}\zeta = (\widetilde{\eta^r \circ \xi}|\widetilde{I_\xi},\ \widetilde{\eta}|[0, \widetilde{\eta^r(\xi(0))}]).$$

The non-rescaled pair $(\eta^r \circ \xi | I_\xi, \eta|[0, \eta^r(\xi(0))])$ will be referred to as the *pre-renormalization* $p\mathcal{R}\zeta$ of the commuting pair $\zeta = (\eta, \xi)$.

For a pair ζ we define its *rotation number* $\rho(\zeta) \in [0,1]$ to be equal to the continued fraction $[r_0, r_1, \ldots]$ where $r_i = \chi(\mathcal{R}^i\zeta)$. In this definition $1/\infty$ is understood as 0, hence a rotation number is rational if and only if only finitely many renormalizations of ζ are defined; if $\chi(\zeta) = \infty$, $\rho(\zeta) = 0$. Thus defined, the rotation number of a commuting pair can be viewed as a rotation number in the usual sense:

Proposition 2.1. *The rotation number of the mapping F_ζ is equal to $\rho(\zeta)$.*

For $\rho = [r_0, r_1, \ldots] \in [0,1]$ let us set

$$G(\rho) = [r_1, r_2, \ldots] = \left\{ \frac{1}{\rho} \right\},$$

where $\{x\}$ denotes the fractional part of a real number x (G is usually referred to as the *Gauss map*). As follows from the definition,

$$\rho(\mathcal{R}\zeta) = G(\rho(\zeta))$$

for a real commuting pair ζ with $\rho(\zeta) \neq 0$.

The renormalization of the real commuting pair (2.1), associated to some critical circle map f, is the rescaled pair $(\widetilde{f^{q_{m+2}}}|\widetilde{I_{m+1}}, \widetilde{f^{q_{m+1}}}|\widetilde{I_{m+2}})$. Thus for a given critical circle map f the renormalization operator recovers the (rescaled) sequence of the first return maps:

$$\{(\widetilde{f^{q_{i+1}}}|\widetilde{I_i}, \widetilde{f^{q_i}}|\widetilde{I_{i+1}})\}_{i=1}^{\infty}.$$

A *critical commuting pair* is a commuting pair (η, ξ) whose maps are real-analytic. We shall also impose a technical assumption that ξ analytically extends to an interval $(a,b) \ni 0$ with $\xi(a,b) \supset [\eta(0), \xi(0)]$, and has a single critical point 0 in this interval. The space of critical commuting pairs modulo affine conjugacy will be denoted by $\mathbf{C}$; its subset consisting of pairs ζ with $\chi(\zeta) = \infty$ will be denoted by $\mathbf{S}_\infty$. Renormalization is a transformation $\mathcal{R} : \mathbf{C} \setminus \mathbf{C}_\infty \to \mathbf{C}$. It is not difficult to show (see [Ya2]) that this transformation is injective:

Proposition 2.2. *The map $\mathcal{R} : \mathbf{C} \setminus \mathbf{S}_\infty \to \mathbf{C}$ is one-to-one.*

A commuting pair ζ lies in the Epstein class if both of its maps are. It is clear that the Epstein class $\mathcal{E}$ is invariant under the action of $\mathcal{R}$. It was shown by various people such as Sullivan (in the unimodal case), Świątek, Herman, and Yoccoz, that the renormalizations of *any* C^3-smooth commuting pair with an irrational rotation number converge to the Epstein class, at a geometric rate in the C^2-metric (see [dFdM1]). Moreover, there exists $\sigma \in (0,1)$ such that all limit points lie in $\mathcal{E}_\sigma$.

2.2. **Renormalization of maps of the cylinder.** The cylinder renormalization operator $\mathcal{R}_{cyl}$ was introduced in [Ya3] as a more natural way to renormalize analytic critical circle maps. It has two crucial advantages: the operator acts on maps themselves, rather than their conjugacy classes; and extends to an analytic operator in the Banach manifold $\mathbf{C}_U$ of analytic critical circle maps defined in some equatorial neighborhood $U \subset \mathbb{C}/\mathbb{Z}$. We briefly recall the definition of $\mathcal{R}_{cyl}$ below, the reader is directed to [Ya3] for the detailed exposition.

First note that in [Ya3] we constructed a closed equivalence relation denoted $\underset{\text{conf}}{\approx}$ between commuting pairs in the Epstein class $\mathcal{E}_\sigma$ with the property that if $\zeta_1 \underset{\text{conf}}{\approx} \zeta_2$, then the analytic extensions of the commuting pairs are conjugate in a specific neighborhood of the interval of definition by a conformal change of coordinates. In particular, $\rho(\zeta_1) = \rho(\zeta_2)$. Moreover, there exists a value $N \in \mathbb{N}$ such that if ζ_1 and ζ_2 are N-times renormalizable, then $\mathcal{R}^N \zeta_1 \underset{\text{conf}}{\approx} \mathcal{R}^N \zeta_2$. This last property implies that periodic orbits of $\mathcal{R}^N$ constructed in Theorem 3.1 project to the quotient space.

The cylinder renormalization operator defined in [Ya3] is a real-symmetric analytic operator $\mathcal{R}_{cyl}$ from an open subset $\mathbf{R}$ of $\mathbf{C}_U$ to $\mathbf{C}_U$ (thus $\mathcal{R}_{cyl}$ maps an open subset of $\mathbf{M} \equiv \mathbf{C}_U^{\mathbb{R}}$ into $\mathbf{M}$), such that the following holds. For every $f \in \mathbf{M}$ such that the continued fraction of $\rho(f)$ has at least r terms, $f \in \mathbf{R}$. For such maps the operator $\mathcal{R}_{cyl}$ changes the rotation number by shifting its continued fraction expansion r terms to the left. Moreover, for every $r \neq \infty$ there is a canonical immersion

$$\mathbf{S}_r \cap \mathcal{E}_\sigma / \underset{\text{conf}}{\approx} \xrightarrow{\iota_r} \mathbf{M}$$

such that $\rho(\iota_r(\zeta)) = \rho(\mathcal{R}(\zeta))$, and $\iota_r \circ \mathcal{R}^N \circ \iota_r^{-1} \equiv \mathcal{R}_{cyl}$.

Every map f in $\mathbf{R}$ posesses a domain C_f called a fundamental crescent, which is bounded by two simple curves l and $f^{q_N}(l)$ which meet at two endpoints p_1, p_2, which are repelling fixed points of the iterate f^{q_N}. Moreover, the quotient of $\overline{C_f \cup f^{q_N}(C_f)} \setminus \{p_1, p_2\}$ by the iterate f^{q_N} is conformally isomorphic to $\mathbb{C}/\mathbb{Z}$. The first return map R_f of C_f under this isomorphism becomes $\mathcal{R}_{cyl} f$.

2.3. **Explanation of universality: Lanford's Program.** The original golden-mean universality hyperbolicity conjecture was described in [ORSS] and FKS. It was later generalized by Lanford [Lan2], and in its most general form became known as the *Lanford's Program.* Since stating a conjecture allows for a degree of vagueness, we simply use $\mathcal{R}$ for the renormalization operator without specifying which one we have in mind; the original Program was stated for critical commuting pairs, but we will prove it for $\mathcal{R}_{cyl}$ acting on critical circle maps.

To outline the Lanford's Program, let us give an informal description of the action of renormalization on the space of critical circle maps.

Let $\rho = \phi(\theta) : \mathbb{T} \to \mathbb{T}$ be a monotone continuous function such that $\phi^{-1}(\rho)$ is a point for ρ irrational and an interval otherwise. Imagine the space of critical circle maps as an infinite-dimensional cylinder $\mathbf{C} = \mathbb{T} \times \mathbf{C}'$, where the rotation number of a commuting pair $\zeta(\theta, \cdot)$ with the equatorial coordinate $\theta \in \mathbb{T}$ is $\rho(\theta)$. The cylinder $\mathbf{C}$ is partitioned into

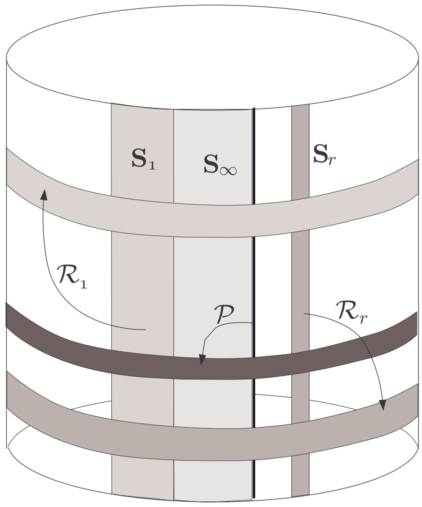

FIGURE 1

strips (cf. Figure 1)

$$\mathbf{S}_r = \{\zeta \in \mathbf{C} | \rho(\zeta) = [r, r_1, \ldots]\} \text{ for } r = 1, \ldots, \infty$$

A boundary component of the strip $\mathbf{S}_\infty$ is the hypersurface $\mathbf{P}_\infty \subset \mathbf{S}_\infty$ with the property that a pair $\zeta \in \mathbf{P}_\infty$ if and only if it has a single fixed point with unit eigenvalue (ζ is a "parabolic pair"). The sets $\mathbf{S}_r$ accumulate on $\mathbf{P}_\infty$ in clockwise direction.

It is natural to think of the transformation $\mathcal{R} : \mathbf{C} \setminus \mathbf{S}_\infty \to \mathbf{C}$ as being defined piecewise, given on each $\mathbf{S}_r$, $r \neq \infty$ by the formula:

$$\mathcal{R}_r : (\eta, \xi) \mapsto (\eta, \eta^r \circ \xi).$$

The operator $\mathcal{R}_r$ expands the strip $\mathbf{S}_r$ in the equatorial direction, mapping it onto a thin equatorial cylinder, which intersects all the vertical strips. We may close the domain of $\mathcal{R}$ by adjoining the hypersurface $\mathbf{P}_\infty$ to it and setting $\mathcal{R}$ on $\mathbf{P}_\infty$ to an appropriately defined (multivalued) parabolic renormalization transformation $\mathcal{P}$, with this convention the image of $\mathbf{P}_\infty$ is also an equatorial cylinder. Lanford conjectured that this picture can be appropriately endowed by a smooth metric, so that the operator $\mathcal{R}$ is uniformly expanding in the θ direction and uniformly contracting in the complementary directions. Thus $\mathcal{R}$ posesses an infinite-dimensional "horseshoe" structure. The "boxes"

$$\mathcal{R}_{r_{-1}}(\mathbf{S}_{r_{-1}} \cap \mathcal{R}_{r_{-2}}(\mathbf{S}_{r_{-2}} \cap \cdots (\mathcal{R}_{r_{-n}} \mathbf{S}_{r_{-n}}) \cdots) \cap \mathbf{S}_{r_0} \cap \mathcal{R}_{r_0}^{-1}(\mathbf{S}_{r_1} \cap \mathcal{R}_{r_1}^{-1}(\mathbf{S}_{r_2} \cap \cdots \mathcal{R}_{r_{n-1}}^{-1}(\mathbf{S}_{r_n}) \cdots)$$

are exponentially small in n, and their intersection is the $\mathcal{R}$-invariant hyperbolic Cantor set $\mathcal{A}$ parametrized by bi-infinite sequences of $r_i \in \mathbb{N} \cup \{\infty\}$. The set $\mathcal{A}$ is a "horseshoe" attractor for the renormalization operator,

$$\operatorname{dist}(\mathcal{R}^n \zeta, \mathcal{A}) \to 0$$

at a geometric rate. The action of $\mathcal{R}$ on $\mathcal{A}$ is conjugate to the two-sided shift.
Let us conclude this exposition by formalizing the Hyperbolicity Conjecture as follows:

Renormalization Hyperbolicity Conjecture (Lanford's Program). *The renormalization operator $\mathcal{R}$ in the space of critical circle maps possesses a "horseshoe" attractor $\mathcal{A}$ on which its action is conjugated to the two-sided shift. Moreover, there exists a renormalization-invariant space of critical circle maps with the structure of an infinite dimensional smooth manifold, with respect to which $\mathcal{A}$ is a hyperbolic set with one-dimensional expanding direction. If t denotes the expanded coordinate in a local chart, then the dependence $t \mapsto \rho(t, \cdot)$ is continuous and (not strictly) monotone.*

3. Outline of the results

3.1. **Known results.** The early efforts to prove the Renormalization Hyperbolicity Conjecture culminated in an argument of Mestel [Mes], who established the existence and local hyperbolicity properties of a fixed point of $\mathcal{R}_1$ in C^∞. The argument was based on rigorous computer estimates. This approach was developed by Lanford in the context of renormalization of unimodal maps. One of its major drawbacks is its local nature; neither the uniqueness of the fixed point, nor global hyperbolicity of $\mathcal{R}_1$ are established.

Construction of the renormalization horseshoe. In his address to ICM-86 in Berkeley Sullivan [Sul1] outlined a program of constructing the invariant set of renormalization of unimodal maps and its stable set by means of Teichmüller theory, which he subsequently carried out (see [Sul2],[MvS]). Sullivan's program has been adapted to the setting of critical circle mappings by de Faria [dF1],[dF2]. De Faria has defined, in particular, extensions of Epstein commuting pairs to holomorphic dynamical systems in the plane, which are analogous to quadratic-like mappings in the unimodal renormalization theory. We discuss these extensions, called *holomorphic commuting pairs*, in §4.1. Using Sullivan's Riemann surface laminations machinery, de Faria constructed a horseshoe attractor $\mathcal{A}_B$ for the renormalization operator acting on the space of Epstein commuting pairs of type bounded by some constant B (this corresponds to restricting the operator $\mathcal{R}$ to the finite collection of vertical strips $\cup \mathbf{S}_r$, $r \leq B$ in Figure 1). In a recent paper [Ya2] we generalized the results of de Faria to Epstein commuting pairs of unbounded topological type. Before formulating the result, let us make some further notation. Before formulating our results let us make some further notation. Let Σ be the space of bi-infinite sequence of natural numbers, and denote by $\sigma : \Sigma \to \Sigma$ the shift on this space:

$$\sigma : (r_i)_{-\infty}^{\infty} \mapsto (r_{i+1})_{-\infty}^{\infty}.$$

Let us also complement the natural numbers with the symbol ∞ with the conventions $\infty + x = \infty$, $1/\infty = 0$; and denote by $\bar{\Sigma}$ the space $(\mathbb{N} \cup \{\infty\})^{\mathbb{Z}}$, and by $\bar{\Sigma}^+$ the space $(\mathbb{N} \cup \{\infty\})^{\mathbb{N}}$. Our theorem completely settles the first part of Lanford's Program:

Theorem 3.1 (Renormalization horseshoe)**.** *There exists an $\mathcal{R}$-invariant set $\mathcal{I} \subset \mathcal{E}$ consisting of pairs with irrational rotation numbers with the following properties. The action of $\mathcal{R}$ on $\mathcal{I}$ is topologically conjugate to the two-sided shift $\sigma : \bar{\Sigma} \to \bar{\Sigma}$:*

$$i \circ \mathcal{R} \circ i^{-1} = \sigma$$

and if $\zeta = i^{-1}(\ldots, r_{-k}, \ldots, r_{-1}, r_0, r_1, \ldots, r_k, \ldots)$ then $\rho(\zeta) = [r_0, r_1, \ldots, r_k, \ldots]$. The set $\mathcal{I}$ is pre-compact, its closure $\mathcal{A} \subset \mathcal{E}$ is the horseshoe attractor for the renormalization. That is, for any $\zeta \in \mathcal{E}$ with irrational rotation number we have

$$\mathcal{R}^n \zeta \to \mathcal{A}$$

in Carathéodory topology. Moreover, for any two pairs ζ, $\zeta' \in \mathcal{E}$ with $\rho(\zeta) = \rho(\zeta')$ we have

$$\operatorname{dist}(\mathcal{R}^n \zeta, \mathcal{R}^n \zeta') \to 0$$

for the uniform distance between analytic extensions of the renormalized pairs on compact sets.

The proof of Theorem 3.1 combines the ideas of McMullen from [McM2] with *a priori* bounds obtained in [Ya1]. We consider the *geometric limits* of the rescaled sequences of renormalizations

$$\zeta, \mathcal{R}\zeta, \ldots, \mathcal{R}^n \zeta.$$

These limits which we call *bi-infinite limiting renormalization towers* (see §4.1 for the definition) should be thought of as bi-infinite sequences of nested holomorphic commuting pairs in the plane. They are naturally equipped with rotation numbers which are bi-infinite sequences in $\bar{\Sigma}$. Each tower corresponds to a single point in the attractor $\mathcal{A}$, and the result follows from a Rigidity Theorem [Ya2]:

Theorem 3.2 (Tower rigidity). *For each bi-infinite sequence $\bar{s} \in \bar{\Sigma}$ there exists a unique (up to a real homothety) bi-infinite tower with $\bar{s}$ as its rotation number.*

An important difference with the situation considered by McMullen and its adaptation to the case of circle maps carried out in [dFdM2] is the presense of *parabolic commuting pairs* in a limiting tower. Parabolic elements of the attractor $\mathcal{A}$ are the pairs ζ in the closure of $\mathcal{I}$ on which renormalization is not defined. Necessarily, $\rho(\zeta) = 0$, and ζ posesses a unique fixed point, which has eigenvalue 1. We call any commuting pair with this property *parabolic*, and construct a family of natural extensions of $\mathcal{R}$ to parabolic commuting pairs, the *parabolic renormalizations* $\mathcal{P}_\theta$, $\theta \in \mathbb{T}$. We may extend Theorem A to parabolic pairs as follows. For every parabolic commuting pair $\zeta \in \mathcal{E}$ and every $\rho \in \mathbb{T}$ there exists a unique $\theta(\zeta, \rho)$ with the property that $\rho(\mathcal{P}_{\theta(\zeta,\rho)}\zeta) = \rho$, and moreover, if ρ is rational then $\mathcal{P}_{\theta(\zeta,\rho)}\zeta$ is also a parabolic pair. Let us define

$$\mathcal{X} = \overline{\{\zeta \in \mathcal{E},\ \rho(\zeta) \in \mathbb{R} \setminus \mathbb{Q}\}} \subset \mathcal{E},$$

and blow-up the boundary of this space by replacing every boundary element ζ by $\zeta \times \bar{\Sigma}^+$. To topologize this space, let us replace every $\zeta \in \mathcal{X}$ with $\rho(\zeta) \in \mathbb{R} \setminus \mathbb{Q}$ with $(\zeta, i^{-1}(\zeta))$, and in the case $\zeta \in \partial\mathcal{X}$, and $\rho(\zeta) = [r_1, \ldots, r_k]$, for $\bar{s} = (s_1, s_2, \ldots)$ let us express the corresponding point in the blown-up space as $(\zeta, \bar{t})$ where $\bar{t} = (r_1, \ldots, r_k, s_1, s_2, \ldots)$. Now we use the product topology on $\mathcal{X} \times \bar{\Sigma}^+$ to topologize the blow-up of $\mathcal{X}$, and call the resulting space $\hat{\mathcal{X}}$. Let $\hat{\mathcal{A}} \subset \hat{\mathcal{X}}$ be the blow-up of $\mathcal{A}$. Define the *generalized renormalization operator* $\mathcal{G} : \hat{\mathcal{X}} \to \hat{\mathcal{X}}$ setting $\mathcal{G}(\zeta) = \mathcal{R}(\zeta)$ in the case $\rho(\zeta) \in \mathbb{R} \setminus \mathbb{Q}$, and $\mathcal{G}(\zeta, (r_1, r_2, \ldots)) = (\mathcal{P}_\theta(\zeta), (r_2, r_3, \ldots))$ where $\theta = \theta(\zeta, [r_1, r_2, \ldots])$. With our definitions we have the following:

Theorem 3.3. *The set $\hat{\mathcal{A}}$ is invariant under $\mathcal{G}$. The map $i : \Sigma \to \mathcal{I}$ extends homeomorphically to $\bar{\Sigma} \to \hat{\mathcal{A}}$, so that $\mathcal{G} \circ i = i \circ \sigma$. The generalized renormalizations $\mathcal{G}^n(x)$ converge to $\hat{\mathcal{A}}$ for any $x \in \hat{\mathcal{X}}$.*

In the case of bounded type rotation number, de Faria and de Melo [dFdM2] strengthened the original result of de Faria, utilizing a technique of McMullen [McM2] to show that the rate of converegence to $\mathcal{A}_B$ is geometric:

Theorem 3.4 ([dFdM2])**.** *For every $B \in \mathbb{N}$ and every $r \in \mathbb{N} \cup 0$ there exists $\alpha_{r,B} < 1$ such that for every $\zeta \in \mathcal{E}$ with $\rho(\zeta) \in \mathbb{R} \setminus \mathbb{Q}$ of type bounded by B and for every $\hat{\zeta} \in \mathcal{A}_B$ with $\rho(\hat{\zeta}) = \rho(\zeta)$ we have*

$$\mathrm{dist}_{C^r}(\mathcal{R}^n \zeta, \mathcal{R}^n \hat{\zeta}) < \mathrm{const} \cdot (\alpha_{r,B})^n.$$

Despite the difficulty in imposing a manifold structure on the space of commuting pairs compatible with the analyticity of $\mathcal{R}$, it has long been known how one may construct an unstable direction in such a manifold. A version of this construction was first shown to us by Folkert Tangerman, who participated in its discussion in Sullivan's seminar at the Graduate Center of CUNY.

Cylinder renormalization operator, and hyperbolicity of periodic orbits. The main result of [Ya3] is the following:

Theorem 3.5 (**Hyperbolicity of Periodic Orbits of $\mathcal{R}_{cyl}$**)**.** *There exists a real-symmetric analytic operator $\mathcal{R}_{cyl}$ from an open subset of $\mathbf{C}_U$ to $\mathbf{C}_U$ (thus $\mathcal{R}_{cyl}$ maps an open subset of $\mathbf{M}$ into $\mathbf{M}$), such that the following holds. For every $r \neq \infty$ there is a canonical homeomorphism onto the image*

$$\mathbf{S}_r \cap \mathcal{E}_s / \underset{conf}{\sim} \xrightarrow{\iota_r} \mathbf{M}$$

such that $\rho(\iota_r(\zeta)) = \rho(\mathcal{R}(\zeta))$, and $\iota_r \circ \mathcal{R}^N \circ \iota_r^{-1} \equiv \mathcal{R}_{cyl}$. For every periodic point $p \in \mathbf{S}_r$ of the operator $\mathcal{R}^N$ there exists $\ell \in \mathbb{N}$ such that the point $\iota_r(p)$ is a hyperbolic fixed point of $\mathcal{R}^\ell_{cyl}|_{\mathbf{M}}$, with one-dimensional unstable direction.

In addition, in the paper [EY] we have considered the cylinder version of the parabolic renormalization $\mathcal{P}_\theta$ (which we will again denote $\mathcal{P}_\theta$ to avoid a cumbersome notation). In this setting, we gave a much simplified argument for a part of Theorem 3.3. Moreover, denoting $\mathcal{G}_{\mathrm{cyl}}$ the cylinder version of the generalized renormalization operator, we have the following:

Theorem 3.6. *The family of projections ι_r can be complemented by*

$$\iota_\infty : \mathbf{S}_\infty \cap \hat{\mathcal{X}} / \underset{conf}{\sim} \to \mathbf{M},$$

so that $\iota_\infty \circ \mathcal{G}^N \circ \iota_\infty^{-1} \equiv \mathcal{G}_{cyl}$. *Let* $\bar{s} = (\dots, s_{-n}, \dots, s_0, \dots, s_n, \dots) \in \bar{\Sigma} \setminus \Sigma$ *be a periodic sequence, and set*

$$\kappa = (\zeta_{\bar{s}}, (s_0, \dots, s_n, \dots)) = i(\bar{s}) \in \hat{\mathcal{X}}.$$

Denote $f = \iota_{s_0}(\kappa) \in \mathbf{M}$ *the corresponding critical cylinder map, which in this case posesses a parabolic periodic orbit. Let* k *be the period of* f *as a periodic point of* $\mathcal{G}_{cyl}$. *Then there exists a codimension one complex-analytic submanifold of* $\mathbf{C}_U$ *containing* f *in which* $\mathcal{G}_{cyl}^k$ *is an analytic contraction.*

3.2. **Statement of the main result.** In the present paper we complete our investigation of Lanfords Program by showing the following:

Theorem 3.7. *The set*

$$\hat{\mathcal{I}} \equiv \bigcup_r i_r(\mathcal{I} \cap \mathbf{S}_r) \subset \mathbf{M}$$

is uniformly hyperbolic for the analytic operator $\mathcal{R}_{cyl}$, *with one-dimensional unstable direction.*

A key ingredient of the proof is the construction of the stable lamination of the renormalization operator:

Theorem 3.8. *There exists a lamination* $\mathcal{L} \subset \mathbf{C}_U$ *of a neihborhood of the closure of* $\hat{\mathcal{I}}$ *whose leaves are codimension one analytic submanifolds with the following properties.*

- *For every point* $f \in \hat{\mathcal{I}}$ *there is a unique leaf* L_f *passing through* f. *All the points in* L_f *are infinitely cylinder renormalizable, and lie in the stable set of* f.
- *Moreover, every* $g \in L_f$ *is quasiconformally conjugate to* f *on some neighborhood of the equator (which varies with* g*).*
- *The collection of leafs* $\{L_f\}$ *is dense in* $\mathcal{L}$, *and leafs in* $\mathcal{L} \setminus \cup L_f$ *are submanifolds consisting of critical cylinder maps with parabolic periodic orbits.*
- *Denoting* $\hat{\mathcal{L}}$ *the blow-up of* $\mathcal{L}$ *constructed as in Theorem 3.3 we have that* $\hat{\mathcal{L}}$ *is* $\mathcal{G}_{\text{cyl}}$*-invariant and its leaves are again stable subsets of* $\mathcal{G}_{\text{cyl}}$.

4. Preliminary considerations

4.1. **Holomorphic commuting pairs.** De Faria [dF1, dF2] introduced holomorphic commuting pairs to apply Sullivan's Riemann surface laminations technique to renormalization of critical circle maps. They are suitably defined holomorphic extensions of critical commuting pairs which replace Douady-Hubbard polynomial-like maps [DH2]. A critical commuting pair $\zeta = (\eta|_{I_\eta}, \xi|_{I_\xi})$ extends to a *holomorphic commuting pair* $\mathcal{H}$ if there exist four simply-connected $\mathbb{R}$-symmetric domains Δ, D, U, V such that

- $\bar{D}$, $\bar{U}$, $\bar{V} \subset \Delta$, $\bar{U} \cap \bar{V} = \{0\}$; $U \setminus D$, $V \setminus D$, $D \setminus U$, and $D \setminus V$ are nonempty connected and simply-connected regions, $U \cap \mathbb{R} = I_\eta$, $V \cap \mathbb{R} = I_\xi$;
- mappings $\eta : U \to (\Delta \setminus \mathbb{R}) \cup \eta(I_\eta)$ and $\xi : V \to (\Delta \setminus \mathbb{R}) \cup \xi(I_\xi)$ are onto and univalent;

- $\nu \equiv \eta \circ \xi : D \to (\Delta \setminus \mathbb{R}) \cup \nu(I_D)$ is a three-fold branched covering with a unique critical point at zero, where $I_D = D \cap \mathbb{R}$.

We shall call ζ the *commuting pair underlying* $\mathcal{H}$, and write $\zeta \equiv \zeta_{\mathcal{H}}$. The domain $D \cup U \cup V$ of a holomorphic commuting pair $\mathcal{H}$ will be denoted Ω or $\Omega_{\mathcal{H}}$, the range will be denoted Δ or $\Delta_{\mathcal{H}}$. The closure of the set of points whose orbit under $\mathcal{H}$ is contained in Ω will be referred to as the *filled Julia set of* $\mathcal{H}$, denoted $K(\mathcal{H})$. The *Julia set of* $\mathcal{H}$ is defined as $J(\mathcal{H}) = \partial K(\mathcal{H})$.

It is easy to see directly from the definition (cf. [dF2]) that:

Proposition 4.1. *Let ζ be a commuting pair with $\chi(\zeta) < \infty$. Suppose ζ is a restriction of a holomorphic commuting pair $\mathcal{H}$, that is $\zeta = \zeta_{\mathcal{H}}$. Then there exists a holomorphic commuting pair $\mathcal{G}$ with range $\Delta_{\mathcal{H}}$, such that $\zeta_{\mathcal{G}} = \mathcal{R}\zeta$.*

In this paper we will slightly generalize the definition of holomorphic commuting pairs. Namely, we will say that $\mathcal{H}$ is a *generalized* holomorphic commuting pair if $\mathcal{H}$ is conjugated to a holomorphic commuting pair $\mathcal{G}$ by a quasiconformal homeomorphism. All the relevant definitions naturally extend to this case.

We say that a real commuting pair (η, ξ) with an irrational rotation number has *complex* a priori *bounds*, if all its renormalizations extend to holomorphic commuting pairs with *bounded moduli*:

$$\mathrm{mod}(\Delta \setminus \Omega) > \mu > 0.$$

For $\mu \in (0, 1)$ let $\mathbf{H}(\mu)$ denote the space of holomorphic commuting pairs $\mathcal{H} : \Omega_{\mathcal{H}} \to \Delta_{\mathcal{H}}$, with $\mathrm{mod}(\Delta_{\mathcal{H}} \setminus \Omega_{\mathcal{H}}) > \mu$, $\min(|I_\eta|, |I_\xi|) > \mu$ and $\mathrm{diam}(\Delta_{\mathcal{H}}) < 1/\mu$.
The existense of complex *a priori* bounds is a key analytic issue of renormalization theory. In the case of critical circle maps it is settled by the following theorem:

Theorem 4.2. *There exist universal constants $\mu > 0$ and $K > 1$ such that the following holds. Let $\zeta \in \mathbf{C}$ be a critical commuting pair with an irrational rotation number. Then there exists $N = N(\zeta)$ such that for all $n \geq N$ the commuting pair $\mathcal{R}^n\zeta$ extends to a holomorphic commuting pair $\mathcal{H}_n : \Omega_n \to \Delta_n$ in $\mathbf{H}(\mu)$. The range Δ_n is a Euclidean disk, and the regions $\Omega_n \cap (\pm\mathbb{H})$ are K-quasidisks.*

We first proved this theorem in [Ya1] for commuting pairs ζ in an Epstein class $\mathcal{E}_s$, in which case $N = N(s)$. Our proof was later adapted by de Faria and de Melo [dFdM2] to the case of a non-Epstein critical commuting pair.
Let ζ be at least n times renormalizable critical commuting pair. For the lack of a better term, let us say that the pair of numbers $\tau_n(\zeta) = (r_{n-1}, r_{n-2})$ forms the *history* of the pair $\mathcal{R}^n\zeta$. Based on the above theorem and a detailed analysis of the shapes of the domains Ω_n we proved the following in [Ya1]:

Theorem 4.3 ([Ya1])**.** *There exists a universal constant $K_1 > 1$ such that the following holds. Let $\zeta_1 = (\eta_1, \xi_1)$ and $\zeta_2 = (\eta_2, \xi_2)$ be two critical commuting pairs with irrational rotation numbers. Let $n > max(N(\zeta_1), N(\zeta_2)) + 1$ as above. Assume that the n-th renormalizations of ζ_1, ζ_2 have the same rotation number and the same history. Then their* *holomorphic commuting pair extensions* $\mathcal{H}^1_n$, $\mathcal{H}^2_n$ *are* K_1-*quasiconformally conjugate. The conjugating map is conformal on the filled Julia set.*

For commuting pairs of type bounded by B this theorem was first proved by de Faria [dF1, dF2], with "K_1" depending on the value of B.

4.2. **Limiting renormalization towers.** Sullivan's original proof of the existence and uniqueness of the Feigenbaum fixed point used the Teichmüller theory for Riemann surface laminations associated to quadratic-like maps, which he himself developed. McMullen [McM2] has replaced this approach with an elegant argument based on a rigidity result for dynamical objects he called renormalization towers. The renormalization towers of critical commuting pairs were studied in [dFdM2] for pairs of bounded type, and later in [Ya2] without the restriction on the type. Their definition is as follows. Let $\{\mathcal{H}_k\}_1^\infty$ be a sequence of generalized holomorphic commuting pairs. Set $\zeta_k = \zeta_{\mathcal{H}_k}$. Suppose $m(k) > n(k) \to \infty$ are such that ζ_k is $m(k)$-times renormalizable. Moreover, the pre-renormalizations $p\mathcal{R}^{n(k)+i}\zeta_k$ have holomorphic pair extensions $\mathcal{H}^i_k$ such that $\mathcal{H}^i_k \to \mathcal{H}_i \in \mathbf{H}$, for $i \in \mathbb{Z}$. Then the bi-infinite sequence $\mathcal{T} = (\mathcal{H}_i)^\infty_{-\infty}$ is referred to as a *bi-infinite limiting renormalization tower*. The *rotation number* $\rho(\mathcal{T})$ is naturally defined as the bi-infinite sequence of heights $(\chi(\zeta_{\mathcal{H}_i}))^\infty_{-\infty}$. The precise formulation of Theorem 3.2 is:

Tower Rigidity Theorem. [Ya2] *For any bi-infinite sequence* $\bar\rho = (\chi_n)^\infty_{-\infty}$, $\chi_n \in \mathbb{N}\cup\{\infty\}$ *there exists a bi-infinite limiting renormalization tower* $\mathcal{T}$ *with* $\rho(\mathcal{T}) = \bar\rho$. *Moreover, for any two limiting renormalization towers* $\mathcal{T}_1 = (\mathcal{H}^1_i)^\infty_{-\infty}$, $\mathcal{T}_2 = (\mathcal{H}^2_i)^\infty_{-\infty}$ *with* $\rho(\mathcal{T}_1) = \rho(\mathcal{T}_2)$ *we have*

$$\zeta_{\mathcal{H}^1_i} \equiv \zeta_{\mathcal{H}^2_i} \text{ for all } i.$$

4.3. **Parabolic maps and their perturbations.** Let us begin with a statement about limits of renormalizable maps in the Epstein class:

Lemma 4.4 (Lemma 2.13, [Ya2])**.** *Let* $\zeta = (\eta, \xi) \in \mathcal{E}$ *be a critical commuting pair with* $\rho(\zeta) = 0$, *which appears as a limit of a sequence* $\{\zeta_n\} \subset \mathcal{E}$ *with* $\rho(\zeta_n) \in \mathbb{R}\setminus\mathbb{Q}$. *Then the map* η *has a unique fixed point in the interval* I_η, *which is necessarily parabolic, with multiplier one.*

We will call a commuting pair $\zeta = (\eta, \xi) \in \mathcal{E}$ *parabolic* if the map η has a unique fixed point in I_η, which has a unit multiplier; this point will usually be denoted p_η. Note, that by virtue of its uniqueness, p_η has to be globally attracting on one side for the interval homeomorphism $\eta|_{I_\eta}$, it is globally attracting on the other side under η^{-1}. A review of the theory of parabolic bifurcations, as applied, in particular, to maps in the Epstein class is given in [Ya2], we summarize some basic facts here. Let us fix a parabolic pair $\eta_0 \in \mathcal{E}$ and set $p = p_{\eta_0}$. We will denote $\Phi^A : U^A \to \mathbb{C}$ and $\Phi^R : U^R \to \mathbb{C}$ the attracting and repelling Fatou coordinates of η_0, C^A and C^R the corresponding Fatou cylinders, and $\pi_A : U^A \to C^A$ and $\pi_R : U^R \to C^R$ the natural projections. Any conformal *transit* *homeomorphism* $\tau : C^A \to C^R$ is a translation in suitable coordinates. Lifiting it produces a map $\bar{\tau} : U^A \to \mathbb{C}$ satisfying

$$\tau \circ \pi_A = \pi_R \circ \bar{\tau}.$$

We will sometimes write $\tau \equiv \tau_\theta$, and $\bar{\tau} = \bar{\tau}_\theta$, where

$$\Phi^R \circ \bar{\tau} \circ (\Phi^A)^{-1}(z) \equiv z + \theta \operatorname{mod}(z).$$

Suppose for an Epstein map η in a sufficiently small neighborhood of η_0 the parabolic point splits into a complex conjugate pair of repelling fixed points $p_\eta \in \mathbb{H}$ and $\bar{p}_\eta$ with multipliers $e^{2\pi i \pm \alpha(\eta)}$. In this situation one may still speak of attracting and repelling petals U^A_η, U^R_η, and define the Douady coordinates

$$\Phi^A_\eta : U^A \to \mathbb{C} \text{ and } \Phi^R_\eta : U^R_f \to \mathbb{C},$$

such that

$$\Phi^A_\eta(\eta(z)) = \Phi^A_\eta(z) + 1 \text{ and } \Phi^R_\eta(\eta(z)) = \Phi^R_\eta(z) + 1.$$

The Fatou and Douady coordinates can be specified uniquely with an arbitrary choice of real basepoints $a \in U^A$ and $r \in U^R$, by requiring that $\Phi^A(a) = \Phi^A_\eta(a) = 0$, and $\Phi^R(r) = \Phi^R_\eta(r) = 0$. With these normalization, the Douady coordinates Φ^A_η, and Φ^R_η depend continuously on η with respect to the compact-open topology on the image (in fact, the dependence is analytic in the appropriate sense), and converge to the respective Fatou coordinates as $\eta \to \eta_0$. Moreover, select the smallest $n(\eta) \in \mathbb{N}$ for which $\eta^{n(\eta)}(a) \geq r$. Then

$$\eta^{n(\eta)}(z) = (\Phi^R_\eta)^{-1} \circ T_{\theta(\eta)+K} \circ \Phi^A_\eta$$

wherever both sides are defined. In this formula $T_a(z)$ denotes the translation $z \mapsto z + a$, $\theta(\eta) \in [0, 1)$ is given by

$$\theta(\eta) = 1/\alpha(\eta) + \underset{\alpha(\eta)\to\infty}{o(1)} \mod 1,$$

and the real constant K is determined by the choice of the basepoints a, r. Thus for a sequence $\{\eta_k\}$ converging to η_0, the iterates $\eta_k^{n(\eta_k)}$ converge locally uniformly if and only if there is a convergence $\theta(\eta) \to \theta$, and the limit in this case is a certain lift of the transit homeomorphism τ_θ for the parabolic map η_0.

4.4. **Parabolic renormalization of maps of the cylinder.** We again refer the reader to [Ya3] for the definition of the spaces $\mathbf{A}_U$, $\mathbf{B}_U$, the Banach manifold $\mathbf{C}_U$ and the cylinder renormalization $\mathcal{R}_{cyl}$.

To define ι_∞ as in Theorem 3.6, let $\zeta = (\eta, \xi) \in \mathcal{E}_s$ have a parabolic fixed point p (as shown in [Ya2], such a point is necessarily unique). Let C^A_ζ, and C^R_ζ be the attracting and repelling Écalle-Voronin cylinders of this point, and denote E_ζ the first return map from a neighborhood of $\mathbb{T} \subset C^R_\zeta$ to C^A_ζ. Let us normalize it so that its critical point on $\mathbb{T}$ is at 0. Then E_ζ is a critical circle map in $\mathbf{C}^{\mathbb{R}}_U$. As before, we see that it is independent of the choice of representative of $[\zeta]_{\underset{\text{conf}}{\sim}}$. For $\theta \in \mathbb{C}/\mathbb{Z}$ denote $\tau_\theta : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ the transit isomorphism $z \mapsto z + \theta \operatorname{mod} \mathbb{Z}$. Monotonicity considerations (see [Ya2]) imply that for every element $\bar{s} \in \bar{\Sigma}^+$ there exists a unique choice of $\theta = \theta(\bar{s}) \in \mathbb{T}$ for which the composition $\tau_\theta \circ E_f$

has rotation number with continued fraction $\bar{s}$, and when $\bar{\sigma} \in \Sigma^+ \setminus \Sigma^+$, it has a parabolic periodic orbit in $\mathbb{T}$. We set

$$\iota_\infty(\zeta, \bar{s}) = f_{\zeta,\bar{s}} = \tau_{\theta(\bar{s})} \circ E_\zeta.$$

Let $\pi_\zeta^A : U_\zeta^A \to C_\zeta^A$, $\pi_\zeta^R : U_\zeta^R \to C_\zeta^R$ denote the natural projections. Let $N \geq 0$ be such that $\eta^N(\xi(0)) \in C^A$. Fix a branch π_R^{-1}, and for every $\theta \in \mathbb{T}$ take the smallest M for which

$$\eta^M \circ \pi_R^{-1} \circ \tau_\theta \circ \pi_A \circ \eta^N \circ \xi(0) \in [0, \eta(0)].$$

The composition

$$\gamma_\theta \equiv \eta^M \circ \pi_R^{-1} \circ \tau_\theta \circ \pi_A \circ \eta^N \circ \xi$$

has a well-defined extension to the whole interval $[0, \eta(0)]$ which is independent of the choice of the branch of π_R^{-1}.

Now let $\bar{s} \in \bar{\Sigma}^+ \setminus \Sigma^+$. The generalized renormalization of an element $(\zeta, \bar{s}) \in \hat{\mathcal{X}}$ is defined in [Ya2] as

$$\mathcal{G}(\zeta, \bar{s}) = (\widetilde{(\gamma_{\theta(\bar{s})}|_{[\eta(0),0]}, \eta}|_{[0,\gamma_{\theta(\bar{s})}(0)]})$$

Finally, let us define the generalized cylinder renormalization of a pair $(f_{\zeta,\bar{s}}, \sigma(\bar{s}))$ as

$$\mathcal{G}_{\text{cyl}}(f_{\zeta,\bar{s}}, \sigma(\bar{s})) = f_{\mathcal{G}^N(\zeta,\bar{s})}.$$

Denote Z the subset of $\mathbf{C}_U$ consisting of maps with a parabolic orbit having the same period as the parabolic orbit of f_ζ. As before, $\mathcal{G}_{\text{cyl}}$ extends to an analytic operator from a neighborhood of f_ζ in Z (which is an analytic codimension one submanifold of $\mathbf{C}_U$ by the Implicit Function Theorem) to $\mathbf{C}_U$.

5. Stable lamination of $\mathcal{R}_{cyl}$

In this section we shall prove Theorem 3.8. For a positive reduced rational fraction p/q let us define $\mathcal{S}_{p/q}$ to be the set of elements $g \in \mathbf{C}_U$ possessing an invariant equatorial quasi-circle $T_g \subset U$ containing the critical point 0, and such that the rotation number $\rho(g|_{T_g}) = p/q$. Further, let $\mathcal{D}_{p/q} \subset \mathcal{S}_{p/q}$ be the collection of maps $g \in \mathcal{S}_{p/q}$ with $g^q(0) = 0$; and let $\mathcal{P}_{p/q}$ be the set of maps $g \in \mathcal{S}_{p/q}$ with a parabolic periodic point in T_g. Let us also define $\mathcal{V}_q \supset \mathcal{D}_{p/q}$ as the subset of $\mathbf{C}_U$ given by the condition $g^q(0) = 0$.
Our first Lemma is the following:

Lemma 5.1. *There exists $N \in \mathbb{N}$ and an open set $\mathcal{Y} \subset \mathbf{C}_U$, which contains $\mathbf{C}_U^{\mathbb{R}}$, such that for all p/q whose continued fraction expansion contains at least $N+2$ terms the following holds:*

(I) *the set $\mathcal{D}_{p/q}$ is open in $\mathcal{V}_q \cap \mathcal{Y}$, and is a codimenion one submanifold of $\mathbf{C}_U$;*
(II) *for a given p/q the quasicircle T_g can be chosen to move holomorphically with $g \in \mathcal{D}_{p/q} \cap \mathcal{Y}$.*

Proof. The argument we give makes use of the complex *a priori* bounds of [Ya1]. As follows from that work, there exists a Euclidean disk $D \subset U$ around the origin, and $N \in \mathbb{N}$ such that for all p/q as above, and every $g \in \mathcal{D}_{p/q} \cap \mathbf{C}_U^{\mathbb{R}}$ the following holds. Denote p_i/q_i the i-th continued fraction approximation of p/q. Then there exist simply-connected domains

Ω_1^g containing $g^{-q_N}(0)\cap T_g=\mathbb{T}$ and $\Omega_2^g\ni g^{-q_{N+1}}(0)\cap T_g=\mathbb{T}$ such that $\Omega_i^g\Subset D$, and moreover, $\operatorname{mod}(D\setminus(\Omega_1^g\cup\Omega_2^g))>\epsilon$ for some universal bound $\epsilon>0$, and

$$g^{q_N}(\overline{\Omega_1^g})=D,\ g^{q_{N+1}}(\overline{\Omega_2^g})=D$$

where both maps are 3-to-1 on the interior.

Now we may select an open neighborhood $\mathcal{Y}$ of $\mathbf{C}_U^{\mathbb{R}}$ small enough so that for all maps $g\in\mathcal{D}_{p/q}\cap\mathcal{Y}$ there are two domains $\Omega_i^g\Subset D$ containing the appropriate preimages of 0 and having the same properties. Let $B_g:U_g(0)\to\mathbb{D}$ denote the Böttcher coordinate of g, which conjugates g^q to $z\mapsto z^3$ on a neighborhood of the origin. We may choose a particular normalization of B_g so that the dependence $g\mapsto B_g$ is analytic. Let l_g denote the B_g-preimage of the straight segment $[x,x^3)$ for some small $x>0$. Inductively define l_g^m to be the g^q-preimage of l_g^{m-1} adjacent to it. Due to the choice of $\mathcal{Y}$ we have made, and since g^q factors through g^{q_N}, $g^{q_{N+1}}$, all l_g^m are contained in D, and thus exist for all m. For obvious reasons these curves do not intersect. Observe that they converge to a repelling periodic point p_g of g. We set

$$L_g=\overline{(\cup_{m=0}^{\infty}l_g^m)\cup(\cup_{m=0}^{\infty}g^{mq}(l_g))},\ \text{and}\ T_g=\cup_{n=0}^{q-1}g^n(L_g).$$

Since B_g analytically depends on g, the dependence $g\mapsto T_g$ is a holomorphic motion, and by construction, for $g\in\mathbf{C}_U^{\mathbb{R}}$, $T_g=\mathbb{T}$. For every $g\in\mathcal{D}_{p/q}$ we may choose a neighborhood $U(g)\subset\mathcal{Y}$ small enough so that the orbit p_g remains repelling throughout this neighborhood. Therefore, the above construction remains true for every $f\in U(g)\cap\mathcal{V}_q$, hence $f\in\mathcal{D}_{p/q}$. By the Implicit Function Theorem, $\mathcal{D}_{p/q}\cap\mathcal{Y}$ is a codimension one submanifold of $\mathbf{C}_U$, and the proof is complete. □

Let us give several further definitions. For a critical cylinder map $g\in\mathbf{C}_U$ we denote $\mathcal{M}_U(g)$ the space of Beltrami differentials μ on $\mathbb{C}/\mathbb{Z}$ with $||\mu||_\infty<\infty$ which satisfy the invariance condition

$$g^*\mu(z)=\mu(g(z))\ \text{for almost every}\ z,g(z)\in U.$$

Equipped with L^∞-norm, $\mathcal{M}_U(g)$ is a Banach space. As usual, $\sigma_0\in\mathcal{M}_U(g)$ will stand for the trivial Beltrami differential in $\mathbb{C}/\mathbb{Z}$, specifying the standard complex structure. As before, we denote $\tilde{U}=\pi^{-1}(U)\subset\mathbb{C}$, and identify the tangent space to $\mathbf{C}_U$ at any point f with $\mathbf{B}_U$. Let $\mathcal{C}_1$ be an open cone in $\mathbf{B}_U$ given by

$$\mathcal{C}_1=\{v\in\mathbf{B}_U,\ \text{such that}\ \inf_{x\in\mathbb{R}}\operatorname{Re}v(x)>\frac{1}{2}||v||\}.$$

Let us take a rotation number $p/q\in\mathbb{Q}$ whose continued fraction has at least $N+2$ terms, and fix $f\in\mathcal{D}_{p/q}\cap\mathcal{Y}$. To every tangent vector $v\in T_f\mathcal{D}_{p/q}$ we will associate we will associate an element of $\mathcal{M}_U(f)$ which produces the infinitesimal quasiconformal deformation of f in the direction of v. To do so, consider $g\in\mathcal{D}_{p/q}\cap\mathcal{Y}$. Utilizing the complex bounds again, we see that $p\mathcal{R}^Ng$ has a generalized holomorphic commuting pair extension $\mathcal{H}_g:\Omega_g\to\Delta$. Here the fundamental region $W_g=\Delta\setminus\Omega_g$ is bounded by the arcs of the circle $\partial\Delta$, and the invariant quasicircle T_g, as well as a finite number of their preimages. In view of the previous lemma, the dependence $g\mapsto W_g$ is a holomorphic motion, equivariant with respect to the dynamics of $\mathcal{H}_g$. We may now select an analytic family of K-quasiconformal maps $h_g : W_f \to W_g$ which conjugate the holomorphic commuting pairs $\mathcal{H}_g$ and $\mathcal{H}_f$ on the boundary. By the standard pull-back argument (see e.g. [Ya1]) we can extend h_g to K-quasiconformal conjugacies of the holomorphic pairs $\mathcal{H}_f$ and $\mathcal{H}_g$. Using the dynamics of f and g we can propagate this conjugacy to a K-quasiconformal conjugacy $\tilde{h}_g$ mapping a fixed equatorial annulus V onto an annulus V_g around T_g. Note that by construction, the dependence $g \mapsto \tilde{h}_g$ is analytic. Finally, we may extend $\tilde{h}_g$ to a quasiconformal homeomorphism of $\mathbb{C}/\mathbb{Z}$, again so that the dependence on g is analytic, and denote $\mu_g \in \mathcal{M}_U(f)$ its Beltrami differential.

Now let $v \in T_f\mathcal{D}_{p/q}$. Select an analytic family of maps $g_t \in \mathcal{D}_{p/q}$ with the property

$$g_t = \pi(\pi \circ f \circ \pi^{-1} + \mathrm{Id} + tv + o(t)),$$

and construct $\tilde{h}_{g_t}$ and μ_{g_t} as above. Differentiate the analytic map $t \mapsto \mu_{g_t}$ to obtain $\mu_{f,v} \in \mathcal{M}_U(f)$. A standard computation shows that v satisfies the following equivariance relation:

$$\pi(\mathrm{Id} + v) = -f'\vartheta + \vartheta \circ f, \text{ where } \bar{\partial}\vartheta = \mu_{f,v}. \tag{5.1}$$

Lemma 5.2. *A vector field $v \in T_f\mathcal{D}_{p/q}$ if and only if it satisfies the equivariance condition (5.1) with $\mu_{f,v} \in \mathcal{M}_U(f)$.*

Proof. We have seen the "only if" part. To show the converse, starting with v satisfying (5.1) consider the family of Beltrami differentials $\mu_t = t\mu_{f,v}$. Denoting ψ_t the normalized solution of the Beltrami equation $\psi_t^*\sigma_0 = \mu_t$, and $g_t = \psi_t \circ f \circ \psi_t^{-1} \in \mathcal{D}_{p/q}$, we have

$$g_t = \pi(\pi \circ f \circ \pi^{-1} + \mathrm{Id} + tv + o(t)).$$

□

Lemma 5.3. *There exists an open subset $\mathcal{Y}_1 \subset \mathcal{Y}$, which again contains $\mathbf{C}_U^{\mathbb{R}}$, such that for all p/q with the continued fraction expansion of length at least $N+2$, and every $f \in \mathcal{D}_{p/q} \cap \mathcal{Y}_1$ the tangent space $T_f\mathcal{D}_{p/q} \cap \mathcal{C}_1 = \emptyset$. The same is true for $f \in \mathcal{P}_{p/q} \cap \mathcal{Y}_1$ for an arbitrary p/q.*

Proof. Let $h \in \mathbf{C}_U^{\mathbb{R}} \cap \mathcal{D}_{p/q}$, and denote ϵ the sup-distance between f and h. Fix a subannulus $V \Subset U$. Let ϵ_0 be such that if $\epsilon < \epsilon_0$, and $v \in T_f\mathcal{D}_{p/q}$, $\hat{v} \in T_h\mathcal{D}_{p/q}$, with unit norms, and

$$\mu_{f,v} \equiv \mu_{h,\hat{v}} \text{ in } \Delta \setminus \Omega_h,$$

then $||v - \hat{v}|| < 1/8$.

Let us now take $\mathcal{Y}_1$ to be the ϵ_0 neighborhood of $\mathbf{C}_U^{\mathbb{R}}$. Let us show that for $f \in \mathcal{Y}_1$, and $v \in \mathcal{C}_1$,

$$v \notin T_f\mathcal{D}_{p/q}.$$

Assume the contrary. Then we may choose $\mu = \mu_{h,\hat{v}} \in \mathcal{M}_U(h)$ such that

$$\pi(\mathrm{Id} + \hat{v}) = -h'\vartheta_1 + \vartheta_1 \circ h, \text{ where } \bar{\partial}\vartheta_1 = \mu,$$

and

$$\inf_{x\in\mathbb{R}} \operatorname{Re} \hat{v}(x) > \frac{1}{4}||\hat{v}|_V||. \tag{5.2}$$

Indeed, we may simply take $\mu \equiv \mu_{f,v}$ on $\Delta_h \setminus \Omega_h$, and then extend it dynamically to a Beltrami differential on $\mathbb{C}/\mathbb{Z}$ as above. The h-invariant Beltrami differential μ gives rise to an analytic curve $\{h_t\} \subset \mathbf{C}_V^{\mathbb{R}}$ beginning at $h|_V$ whose tangent vector is $\hat{v}|_V \in \mathbf{B}_V$. Clearly, no vector field which satisfies (5.2) can be tangent to a map in $\mathbf{C}_U^{\mathbb{R}}$, and the claim follows.

For $f \in \mathcal{P}_{p/q}$ the result follows by continuity. □

The last lemma implies that the analytic submanifolds $\mathcal{D}_{p/q} \cap \mathcal{Y}_1$ uniformly converge to a codimension one lamination $\mathcal{L} \subset \mathbf{C}_U$. It obviously satisfies the properties required in Theorem 3.8.

6. Proof of global hyperbolicity

6.1. **Unstable direction.** Let us select an annulus $U' \Supset U$ such that for a renormalizable $f \in \hat{\mathcal{I}}$

$$\mathcal{R}_{cyl} f \in \mathbf{C}_{U'}.$$

Define a cone field $\mathcal{C}$ on $\hat{\mathcal{I}}$ as follows:

$$\mathcal{C}_f = \{v \in T_f\mathbf{C}_U;\ v \text{ analytically extends to } U' \text{ and } \inf_{x\in\mathbb{R}} v(x) > 0\}.$$

Fix a renormalizable map $f \in \hat{\mathcal{I}}$. We first have

Proposition 6.1. *The operator $D_f\mathcal{R}_{cyl} : \mathbf{B}_U \to \mathbf{B}_U$ is compact.*

Proof. See Proposition 9.1 of [Ya3]. □

Lemma 6.2.

(I) *The cone field $\mathcal{C}$ is renormalization-invariant: $D\mathcal{R}_{cyl} : \mathcal{C} \to \mathcal{C}$,*

(II) *Moreover, there exists $\alpha > 0$ and $k \in \mathbb{N}$ such that for any $f \in \hat{I}$ and vector field $v \in \mathcal{C}_f$*
$$\inf_{x\in\mathbb{R}} D\mathcal{R}_{cyl}^k(v(x)) > (1+\alpha) \inf_{x\in\mathbb{R}} v(x)$$

(III) *Finally, there exists $\ell \in \mathbb{N}$ such that if $v \in \mathbf{B}_U$ belongs to the closure of $\mathcal{C}$, and $v \neq 0$, then $D\mathcal{R}_{cyl}^\ell(v) \in \mathcal{C}$.*

Proof. Let $\tilde{f} = \pi^{-1} \circ \hat{f} \circ \pi$. As follows from an elementary computation, if $\tilde{f}_t(x) = \tilde{f}(x) + tv(x) + o(t)$, then

$$\tilde{f}_t^2(x) = \tilde{f}^2(x) + tv_2(x) + o(t) = \tilde{f}^2(x) + t(\tilde{f}'(\tilde{f}(x))v(x) + v(\tilde{f}(x))) + o(t), \tag{6.1}$$

and more generally, if we write

$$\tilde{f}_t^n(x) = \tilde{f}^n(x) + tv_n(x) + o(t), \text{ then } v_n(x) = f'(f^{n-1}(x))v_{n-1}(x) + v(f^{n-1}(x)) \tag{6.2}$$

Let us denote $C_k \Subset U$ the fundamental crescent of f corresponding to the k-th cylinder renormalization $\mathcal{R}_{cyl}^k f$, and $J_k = C_k \cap \mathbb{T}$. Let $\phi_k(x)$ be the corresponding uniformizing coordinate $C_k \to \mathbb{C}/\mathbb{Z}$, and $\Phi_k : C_n \to \mathbb{C}$ its lift. In the local chart given by ϕ_k^{-1}, the k-th cylinder renormalization $\mathcal{R}_{cyl}^k f$ is represented by the iterate $g_k = f^{q_{mk}}$ for some $m \in \mathbb{N}$.

To prove the first claim, observe that by (6.2),

$$\inf_{x\in\mathbb{R}} v_n(x) \geq \inf_{x\in\mathbb{R}} v(x) > 0. \tag{6.3}$$

The image

$$D\mathcal{R}_{cyl} v = [(\Phi_k)' \circ g_k \cdot (v_{q_{mk}}|_{J_k})] \circ (\Phi_k)^{-1} \tag{6.4}$$

Since $(\Phi_k)' > 0$, (I) follows.

To prove (II), note that by the real *a priori* bounds, the sizes of the intervals J_k decrease geometrically with k. On the other hand, applying the Koebe Distortion Theorem to the conformal extension of Φ_k to C_k and its two neighboring iterates, we see that the distortion of Φ_k on the interval $g_k(J_k)$ is bounded uniformly in k nd f. Hence, there exists $\alpha > 0$ and $l \in \mathbb{N}$ independent of f such that for all $k \geq l$

$$(\Phi_k)'|_{g_k(J_k)} > 1 + \alpha,$$

and (II) follows from (6.4).

Finally, to see (III), observe that if $v(x) \not\equiv 0$, then there exist ℓ, $n \leq q_{m\ell}$ such that $v(x) > 0$ for every $x \in f^n(J_\ell)$. The claim now follows from (6.2). □

We conclude:

Proposition 6.3. *There exists $k \in \mathbb{N}$, $\alpha > 0$ such that for every $f \in \hat{\mathcal{I}}$ the operator $D_f\mathcal{R}_{cyl}^k$ has an eigenvalue $\lambda_f > 1 + \alpha$.*

Proof. Let k, α be as in the previous lemma. Since the spectrum of the compact operator $D_f\mathcal{R}_{cyl}^k$ is discrete, its every non-zero element is an eigenvalue. Hence part (II) of the previous lemma implies that the spectral radius $R_{\mathrm{Sp}}(D_f\mathcal{R}_{cyl}^k) > 1 + \alpha$. □

6.2. Geometric contraction.

Theorem 6.4. *There exists $\rho < 1$, $C > 0$ such that for every $f \in \mathcal{L}$*

$$\mathrm{dist}_U(\mathcal{R}_{cyl}^k(f), \hat{\mathcal{I}}) < C\rho^k.$$

Moreover, there exists $k \in \mathbb{N}$ such that $\forall f \in \hat{\mathcal{I}}$ for the differential

$$||D\mathcal{R}_{cyl}^k|_{L_f}|| < \rho.$$

Proof. The argument we use is due to Lyubich [Lyu4],[Lyu5]. Firstly, note that the convergence to $\hat{\mathcal{I}}$ is uniform in $\mathcal{L}$. Indeed, otherwise there exists $\epsilon > 0$ such that for an arbitrary large $k \in \mathbb{N}$ we may find two maps f and g in the same leaf of $\mathcal{L}$ whose first k renormalizations are ϵ-apart in the uniform metric. Using a normality argument, we may then construct two distinct bi-infinite limiting renormalization towers in the sense of [Ya2] with the same rotation number. This contradicts the Tower Rigidity Theorem of [Ya2]. By the Schwarz Lemma in a Banach manifold, the uniform convergence implies strict contraction in the leaves of $\mathcal{L}$, and the claim follows. □

6.3. **Concluding remarks.** It has been shown by Świątek [Sw] in 1988, with the use of the methods of real dynamics, that in smooth and monotone ($\partial/\partial t > 0$) families $\{f_t\}$ of critical circle maps the parameter values corresponding to irrational rotation numbers occupy a set of measure zero. In a straightforward way, it can be shown that the global hyperbolicity of the renormalization horseshoe proved in this paper implies the same result (compare this with the case of unimodal maps, where the corresponding statement is proved by Lyubich in [Lyu5] with the use of the renormalization hyperbolicity).

A question indirectly involved in the analysis of the above problem is the size of the parameter intervals H_n for which $\rho(f_t) = [n, \ldots]$. We investigate it in [Ya4], and prove in effect that $|H_n| \sim n^2$.

An open question which remains in the theory of critical circle maps is:

$C^{1+\beta}$**-Rigidity Problem.** *When is the conjugacy between two critical circle maps with the same irrational rotation number which maps* 0 *to* 0, $C^{1+\beta}$*-smooth?*

de Faria and de Melo [dFdM2] showed that if the two maps are analytic and of bounded type, then the conjugacy has the desired smoothness. Their argument stems from their result on geometric convergence of renormalizations for maps of bounded type. We expect that given the geometric convergence for all types obtained in this paper, the result can be extended to arbitrary analytic maps. It is worth noting, that for non-analytic maps of unbounded type the result is known to be false (see [dFdM1, dFdM2]).

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