

RICHARDSON TABLEAUX AND SCHUBERT POSITIVITY

HUNTER SPINK AND VASU TEWARI

ABSTRACT. We compute the Schubert cycle expansion of those irreducible components of Springer fibers equal to Richardson varieties. This generalizes work of Güemes in the case of a hook shape and answers a question of Karp–Precup.

Let $B, B^- \subset \mathrm{GL}_n$ denote the Borel and opposite Borel subgroups of upper triangular and lower triangular matrices respectively, and we denote by $\mathfrak{b}, \mathfrak{b}^-, \mathfrak{gl}_n$ for the respectively Lie algebras. Karp and Precup [24] recently studied the interplay between two distinguished classes of subvarieties of the complete type A flag variety $\mathrm{Fl}_n = \mathrm{GL}_n/B$.

- (1) The first class are the *Springer fibers*

$$\mathcal{B}_\lambda = \{gB \in \mathrm{Fl}_n \mid g^{-1}Mg \in \mathfrak{b}\},$$

for $M \in \mathfrak{gl}_n$ a fixed nilpotent operator of Jordan type the partition $\lambda \vdash n$ [44]. The irreducible components are indexed by standard Young tableaux (SYTs) of shape λ [43], and Springer [45] showed the cohomology rings of Springer fibers carry an action of the symmetric group which is irreducible in the top dimension, connecting the geometric study of Springer fibers to representation theory.

- (2) The second class are the *Richardson varieties*

$$X_v^w := \overline{BwB} \cap \overline{B^-vB},$$

indexed by pairs of permutations (v, w) where $v \leq_B w$ in Bruhat order. These varieties are irreducible, and are obtained by intersecting Schubert varieties $X^w = \overline{BwB}$ with opposite Schubert varieties $X_v = \overline{B^-vB}$. Richardson varieties play a crucial role in the connection between algebraic geometry and Schubert calculus because of the identity

$$[X_v^w] = \sum_u c_{uv}^w [X^u] \in H_\bullet(\mathrm{Fl}_n)$$

which decomposes the homology class of the Richardson variety into the homological basis of Schubert varieties, with coefficients c_{uv}^w the structure constants for Schubert polynomial multiplication $\mathfrak{S}_u \mathfrak{S}_v = \sum_w c_{uv}^w \mathfrak{S}_w$. Showing the nonnegativity of c_{uv}^w via combinatorial means for arbitrary triples (u, v, w) is an outstanding open problem in algebraic combinatorics [48, Problem 11]. The answer, outside of the celebrated Littlewood–Richardson rule

HS and VT acknowledge the support of the NSERC, respectively [RGPIN-2024-04181] and [RGPIN-2024-05433].

for Grassmannian permutations [27, 30, 40], is known in special cases. See for instance [18, 21, 22, 28, 38] for manifestly nonnegative rules, [36] for a signed rule, and [37] for a complexity-theoretic perspective.

Motivated by Lusztig’s study [33] of the totally nonnegative Springer fiber, Karp and Precup classified the SYTs which they called *Richardson tableaux* that index irreducible components of Springer fibers which are equal to Richardson varieties [24, Theorem 1.5].

It is unknown what the coefficients of the expansion of arbitrary Springer fibers and their irreducible components into Schubert cycles are in full generality. For tableaux of hook shape the decomposition was determined by Güemes [20], and all such tableaux are Richardson tableaux [24, §10]. In [24, Problem 10.4] Karp and Precup ask in the special case of a Richardson tableau T whether there was a combinatorial way to compute the Schubert cycle expansion of the corresponding Springer fiber component/Richardson variety $X_{v_T}^{w_T}$. We answer this question:

Theorem 0.1. Given a Richardson tableau T , there is a combinatorial rule for computing $c_{u,v_T}^{w_T}$.

Our strategy is to describe a family of pairs of permutations (v, w) we call *well-aligned*, for which we can combinatorially compute the Schubert structure coefficients $c_{u,v}^w$, and show that (v_T, w_T) are well-aligned when T is a Richardson tableau. For pairs of well-aligned permutations we give two distinct combinatorial ways of computing these coefficients.

- (1) One way is to show that well-aligned pairs give Bruhat intervals $[v, w]$ that are translation-equivalent to $[v', w']$ with v' a 132-avoiding permutation, which guarantees that $c_{u,v}^w = c_{u,v'}^{w'}$, and then using the fact that $\mathfrak{S}_{v'}$ is a dominant monomial we can compute the Schubert structure coefficients by iterating the “simplest” case of Sottile’s Pieri rule [42].
- (2) Our second way is to show that the $c_{u,v}^w$ can be obtained by successively applying Schubert positive maps to $\mathfrak{S}_u$, one of which is the divided difference operation $\partial_i f = \frac{f - s_i \cdot f}{x_i - x_{i+1}}$ and the other is the Bergeron–Sottile map [6, 34]

$$R_i f = f(x_1, \dots, x_{i-1}, 0, x_i, x_{i+1}, \dots).$$

Both ways have geometric interpretations. The first method corresponds to showing that X_v^w is a left-translate of $X_{v'}^{w'}$ by the permutation matrix for $v'v^{-1}$. The second method corresponds to geometrically building X_v^w in a manner similar to Schubert varieties via geometric push-pull operations, interspersed with inclusions of flag varieties into larger flag varieties associated to Bergeron–Sottile pattern maps [6], a technique that has been exploited by the authors in collaboration with Bergeron, Gagnon, and Nadeau in a series of papers studying quasisymmetric coinvariants in algebraic geometry [4, 5, 34, 35].

Finally, Karp and Precup show that $X_{v_T}^{w_T}$ are smooth. We generalize this result by showing that the pairs (v_T, w_T) satisfy a finer property we call *very well-aligned*, and we show for very well-aligned pairs that X_v^w is always a smooth variety.

Outline: In Section 1 we collect the combinatorial background. In Section 2 we introduce our central combinatorial object, well-aligned pairs, and then proceed to establish that the Bruhat interval $[v, w]$ coming from a well-aligned (v, w) is equivalent by left translation to an interval $[v', w']$ where v' is 132-avoiding. In Section 3 we give two different nonnegative combinatorial expansions for $c_{u,v}^w$ for well-aligned (v, w) . In Section 4 we show that the pairs of permutations indexing the Richardson varieties of Karp and Precup are well-aligned. In Section 5 we reinterpret our combinatorial results geometrically. We conclude with some enumerative speculation in Section 6.

Acknowledgements. We are very grateful to Martha Precup and Steven Karp for discussions about their work, for sharing notes as well as an early draft, and for directing us to relevant portions of their article. We are also grateful to Allen Knutson for helpful discussions on descent cycling. Finally, VT is particularly thankful to the organizers of the “Combinatorics and Enumerative Geometry” workshop held at the IAS in February 2025, as it provided ample food for thought.

1. COMBINATORIAL BACKGROUND

Throughout $[n] = \{1, \dots, n\}$ for n a positive integer. We refer the reader to standard texts [17, 47] for any undefined terminology.

1.1. Young tableaux. Recall that a *partition* λ is a weakly decreasing sequence of positive integers. We denote the size of λ , i.e. the sum of its entries by $|\lambda|$. If $|\lambda| = n$, we denote this by $\lambda \vdash n$. We represent λ using its *Young diagram* in English notation. Given partitions $\mu \subseteq \lambda$, we define the *skew shape* λ/μ as the set-theoretic difference of the Young diagrams of λ and μ . A *column strip* is a skew shape with no two boxes occupying the same row.

Given $\lambda \vdash n$, a *standard Young tableau* (henceforth SYT) of shape λ is a filling of the Young diagram of λ bijectively with numbers drawn from $[n]$, so that the entries increase strictly from left to right along rows and from top to bottom along columns. We let $\text{SYT}(\lambda)$ denote the set of standard Young tableaux of shape λ .

Given $T \in \text{SYT}(\lambda)$ we define two reading words associated with it. The *reading word* is obtained by reading the entries of the tableau row-wise bottom to top, with each row read left to right. The *top-down reading word* is defined similarly except that the rows are read from top to bottom.

We now describe Schützenberger’s evacuation operator [39]. Given $T \in \text{SYT}(\lambda)$, define the evacuation tableau $\text{evac}(T) \in \text{SYT}(\lambda)$ as follows. Delete the entry in the top left cell of T and decrement the remaining entries by 1. Then perform jeu-de-taquin slides to rectify the resulting Young tableau of skew shape thereby obtaining an SYT T' on $n - 1$ boxes with shape $\lambda \setminus \{c\}$, where c is a corner box in λ . We then place n in c and consider it “frozen” for the remainder of

the procedure. We repeat this with T' and continue until all boxes are frozen. The final tableau is $\text{evac}(T)$.

Example 1.1. Shown below are an SYT T and its evacuation.

1	2	5	7	10
3	8	11		
4	9			
6	12			

1	4	7	9	12
2	5	8		
3	10			
6	11			

The reading word of T and the top-down reading word of $\text{evac}(T)$ are as follows:

$$\begin{array}{cccccccccccc} 6 & 12 & 4 & 9 & 3 & 8 & 11 & 1 & 2 & 5 & 7 & 10 \\ 1 & 4 & 7 & 9 & 12 & 2 & 5 & 8 & 3 & 10 & 6 & 11. \end{array}$$

1.2. Permutations. We denote the symmetric group comprising permutations of $[n]$ by S_n . It is generated by the simple transpositions $s_i = (i, i+1)$ for $1 \leq i \leq n-1$. A *reduced word* for $w \in S_n$ is a minimal length expression $w = s_{i_1} \cdots s_{i_\ell}$ as a product of simple transpositions. An *inversion* in $w \in S_n$ is an ordered pair (i, j) where $i < j$ and $w(i) > w(j)$. The *length* of w , denoted by $\ell(w)$, is the number of inversions in w . A *descent* of w is an index $i \in [n-1]$ such that $w(i) > w(i+1)$. We denote the set of descents of w by $\text{Des}(w)$. We will typically write our permutations in one-line notation, e.g. $w = w(1) \cdots w(n)$.

Definition 1.2. The *Bruhat order* $\leq_B$ on S_n is obtained as the transitive closure of the cover relation $\prec_B$ defined by $u \prec_B v$ if and only if $ut = v$, where t is a transposition and $\ell(v) - \ell(u) = 1$.

Since we need the tableau criterion [10] for Bruhat order, we recall it here.

Theorem 1.3 (Tableau criterion). Given $u, v \in S_n$, we say that $u \leq_B v$ if and only if for all $u_{i,k} \leq v_{i,k}$ for all $1 \leq k \leq n$, where $u_{i,k}$ is the i -th entry in the increasing rearrangement of $u(1), \dots, u(k)$ and similarly for $v_{i,k}$. In fact, it suffices to take $k \in \text{Des}(u)$ to check these inequalities, a statement known as the *improved tableau criterion* [9].

Since we will need to apply the aforementioned criterion to establish incomparability in Bruhat order, the following definition will be useful.

Definition 1.4. If $u \not\leq_B v$, then we call $p \in \text{Des}(u)$ a *witness* if $u_{i,p} > v_{i,p}$ for some $1 \leq i \leq p$.

Given $u \leq_B v$, we let $[u, v]$ be the interval in Bruhat order containing w satisfying $u \leq_B w \leq_B v$. We say that Bruhat intervals $[u, v]$ and $[p, q]$ are *translation-equivalent* if

$$\{u^{-1}w \mid w \in [u, v]\} = \{p^{-1}r \mid r \in [p, q]\}.$$

We record here a simple criterion that allows for producing certain translation-equivalent pairs. For the general statement for Coxeter groups we refer the reader to [12, Proposition 1.9] or [16].

Proposition 1.5. Suppose $v \leq_B w$ where $v, w \in S_n$. Suppose $1 \leq i \leq n-1$ is such that

- (i) $v^{-1}(i) < v^{-1}(i+1)$,
- (ii) $w^{-1}(i) < w^{-1}(i+1)$,
- (iii) $s_i v \not\leq_B w$.

Then left multiplication by the simple transposition s_i induces a poset isomorphism between the Bruhat intervals $[v, w]$ and $[s_i v, s_i w]$.

1.3. Rothe diagrams and dominant permutations. Consider an $n \times n$ collection of boxes with rows labeled from 1 to n top to bottom and columns labeled from 1 to n left to right. The *Rothe diagram* of $w \in S_n$ is the collection of boxes

$$\mathcal{R}(w) = \{(w(j), i) \mid i < j, w(i) > w(j)\}.$$

Alternatively, $\mathcal{R}(w)$ is obtained by marking all boxes $b_i = (w(i), i)$ and crossing out all boxes below and to the right of b_i . The boxes that do not get crossed out form $\mathcal{R}(w)$. The number of boxes in $\mathcal{R}(w)$ is equal to $\ell(w)$. See Figure 1 for two examples; the shaded boxes correspond to the cells of the Rothe diagram.

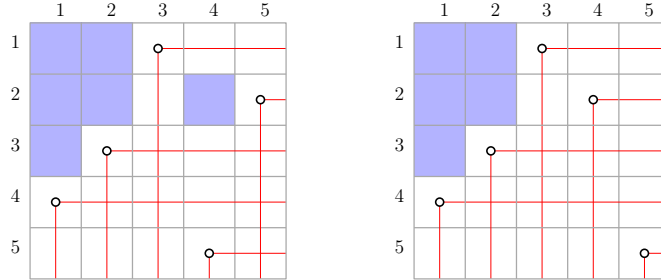


FIGURE 1. The Rothe diagrams for 43152 (left) and dominant 43125 (right).

Definition 1.6. A permutation is *132-avoiding* or *dominant* if there does not exist $i < j < k$ with $w(i) < w(k) < w(j)$.

It is well known [25] that the set of dominant permutations in S_n is enumerated by the Catalan number $\text{Cat}_n := \frac{1}{n+1} \binom{2n}{n}$. Dominant permutations have Rothe diagram a Young diagram, whose row lengths are given by the Lehmer code $(k_1, \dots, k_p, 0, \dots, 0)$ of v^{-1} , i.e. with $k_i = \#\{j > i \mid v^{-1}(j) > v^{-1}(i)\}$. See the right panel in Figure 1 for an example. In this case, we have $v^{-1} = 34215$ and so its Lehmer code is $(2, 2, 1, 0, 0)$.

1.4. Schubert polynomials. We denote by $S_\infty = \langle s_1, s_2, \dots \rangle$ the permutations of $\mathbb{N}$ with finite support, with $s_i = (i, i+1)$ the adjacent transpositions. We view $S_n \subset S_\infty$ as the subgroup generated by $s_1, \dots, s_{n-1}$. We denote by $\mathbb{Z}[\mathbf{x}_n] := \mathbb{Z}[x_1, \dots, x_n]$, and $\mathbb{Z}[\mathbf{x}] := \mathbb{Z}[x_1, x_2, \dots] = \bigcup_{n=1}^\infty \mathbb{Z}[\mathbf{x}_n]$.

The group S_∞ acts on $\mathbb{Z}[\mathbf{x}]$ by having s_i swap x_i and x_{i+1} . The *divided difference operations* are defined by $\partial_i f = \frac{f - s_i f}{x_i - x_{i+1}}$. These satisfy the *nil-Coxeter relations* $\partial_i \partial_j = \partial_j \partial_i$ for $|i - j| \geq 2$, $\partial_i^2 = 0$, and the braid relations $\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$. Because of this, we can define $\partial_w := \partial_{i_1} \cdots \partial_{i_k}$ for any reduced word decomposition $w = s_{i_1} \cdots s_{i_k}$ of w .

The *Schubert polynomials* $\{\mathfrak{S}_u \mid u \in S_\infty\} \subset \mathbb{Z}[\mathbf{x}]$ are the unique homogenous polynomials with the property that $\text{ev}_0 \partial_w \mathfrak{S}_u = \delta_{w,u}$. Alternately, they are characterized by $\mathfrak{S}_{\text{id}} = 1$ and

$$\partial_i \mathfrak{S}_u = \begin{cases} \mathfrak{S}_{us_i} & i \in \text{Des}(u) \\ 0 & \text{otherwise.} \end{cases}$$

If $u \in S_n$ and $w_o = n(n-1) \cdots 1$ is the longest word in S_n , then $\mathfrak{S}_{w_o} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$, and we may explicitly compute $\mathfrak{S}_u = \partial_{u^{-1}w_o} \mathfrak{S}_{w_o}$.

We define the *generalized Littlewood–Richardson coefficients* $c_{u,v}^w$ as the integers arising from the expansion

$$\mathfrak{S}_u \mathfrak{S}_v = \sum_w c_{u,v}^w \mathfrak{S}_w.$$

Alternately, we may write $c_{u,v}^w = \text{ev}_0 \partial_w (\mathfrak{S}_u \mathfrak{S}_v)$. It is known that if $u \not\leq_B w$ or $v \not\leq_B w$ then $c_{u,v}^w = 0$.

The following fact will be essential for us, and follows from the combinatorial model for computing $c_{u,v}^w$ via Monk's rule in the Bruhat interval $[v, w]$.

Fact 1.7. If $[v, w]$ is translation equivalent to $[v', w']$ then $c_{u,v}^w = c_{u,v'}^{w'}$.

2. WELL-ALIGNED PAIRS

The following map that inserts 1 into a permutation shall be relevant to us. Given a permutation $w \in S_{n-1}$ and a positive integer $j \in \{1, \dots, n\}$, let $\varepsilon_j(w) \in S_n$ be the permutation obtained by inserting a 1 in position j in w and then incrementing the previously existing numbers by 1. Similarly, given $w \in S_n$ we define $\delta(w) \in S_{n-1}$ to be the permutation obtained by deleting 1 from w and then decrementing the remaining numbers by 1. Alternatively

$$(\varepsilon_i w)(j) = \begin{cases} w(j) + 1 & j < i \\ 1 & j = i \\ w(j-1) + 1 & j > i \end{cases} \quad \delta(w)(i) = \begin{cases} w(j) - 1 & j < w^{-1}(1) \\ w(j+1) - 1 & j \geq w^{-1}(1). \end{cases}$$

Writing permutations in one line notation, we have for instance that $\varepsilon_1(25143) = 136254$, $\varepsilon_2(25143) = 316254$, and $\delta(25143) = 1432$.

Definition 2.1. A pair of permutations $(v, w) \in S_n \times S_n$ is *aligned* if the following hold:

- (1) $v^{-1}(1) \leq w^{-1}(1)$,
- (2) all indices $v^{-1}(1) \leq i \leq w^{-1}(1) - 1$ are ascents in v , i.e. satisfy $v(i) < v(i+1)$.

We call $(v, w) \in S_n \times S_n$ is *well-aligned* if it is aligned and $(\delta(v), \delta(w))$ is well-aligned. We let $WA_n \subset S_n \times S_n$ denote the set of well-aligned pairs.

Example 2.2. We have $(15726348, 75182364)$ is aligned because $1 < 5 < 7$, and is in fact well-aligned, as successive applications of δ yield

$$\begin{aligned} (15726348, 75182364) &\mapsto (4615237, 6471253) \mapsto (354126, 536142) \mapsto (24315, 42531) \\ &\mapsto (1324, 3142) \mapsto (213, 231) \mapsto (12, 12) \mapsto (1, 1). \end{aligned}$$

Example 2.3. A class of well-aligned pairs arises organically in a geometric context in the authors' previous work [34] joint with Nadeau. Let $c = s_{n-1} \cdots s_2 s_1 \in S_n$ be a standard Coxeter element. Then (v, vc) is well-aligned for all $v \in S_n$ satisfying $v(n) = n$.

The main result of this section will be Proposition 2.12, that for a well-aligned pair (v, w) , there is a distinguished dominant permutation $v^\uparrow$ with $[v, w]$ translation-equivalent to $[v^\uparrow, v^\uparrow v^{-1}w]$.

2.1. Translation-equivalent Bruhat intervals and well-aligned pairs. Our next lemma relates well-aligned pairs to Bruhat order.

Lemma 2.4. For $v, w \in S_n$, if $(v, w) \in WA_n$ then $v \leq_B w$.

Proof. We establish the claim by induction on n . Let $i = v^{-1}(1)$ and $j = w^{-1}(1)$, so that $v = s_i \cdots s_{j-1} \varepsilon_j(\delta(v))$ and $w = \varepsilon_j(\delta(w))$. We have

$$v \leq_B v s_i \leq_B \cdots v s_i \cdots s_{j-1} = \varepsilon_j(\delta(v)) \leq_B \varepsilon_j(\delta(w)),$$

where the first string of inequalities is because $(v s_i \cdots s_{k-1})(k) = v(i) = 1 < (v s_i \cdots s_{k-1})(k+1)$ and the last inequality is by the inductive hypothesis $\delta(v) \leq_B \delta(w)$ and the fact that ε_j preserves the Bruhat order by the tableaux criterion. $\square$

The next result provides a converse when v is dominant.

Lemma 2.5. Let $v \in S_n$ be dominant. Then $v \leq_B w$, if and only if $(v, w) \in WA_n$.

Proof. The previous lemma shows the reverse implication, so it remains to establish the forward implication.

Let $j = v^{-1}(1)$. Since v is dominant we must have that $1 = v(j) < v(j+1) < \cdots < v(n)$. It is well known (see for instance [23, §2]) that for v dominant,

$$(2.1) \quad v \leq_B w \iff \mathcal{R}(v) \subseteq \mathcal{R}(w).$$

In particular $w^{-1}(1) \geq j$, and so (v, w) is aligned.

To finish the proof we need to show that $(\delta(v), \delta(w)) \in WA_{n-1}$. By induction it suffices to show that $\delta(v) \leq_B \delta(w)$. Note that $\delta(v)$ is dominant. Furthermore, since $\mathcal{R}(v) \subseteq \mathcal{R}(w)$, we have that $\mathcal{R}(\delta(v)) \subseteq \mathcal{R}(\delta(w))$. This implies that $\delta(v) \leq_B \delta(w)$ as desired. See Figure 2 for an illustration of the inclusion of the Rothe diagrams. $\square$

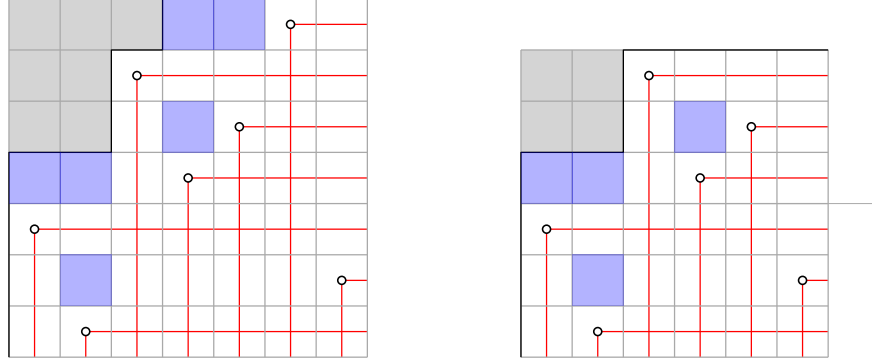


FIGURE 2. The gray shaded cells correspond to the Rothe diagram of the dominant permutations $v = 4521367$ and $\delta(v) = 341256$ respectively, while the union of the gray and blue shaded cells correspond to the Rothe diagrams of $w = 5724316$ and $\delta(w) = 461325$ respectively.

Define

$$\text{WA}_n^{132} := \{(v, w) \in \text{WA}_n \mid v \text{ is dominant}\}.$$

Remark 2.6. Pairs $(v, w) \in S_n \times S_n$ with $v \leq_B w$ where v is dominant are in bijection with matchings on $2n$ points [23, §2]. Thus $|\text{WA}_n^{132}| = (2n - 1)!!$.

We emphasize a special instance of 132-patterns. Given $w \in S_n$ we call $1 \leq i \leq n - 1$ *critical* if w possesses a subsequence of the form $\cdots i \cdots j \cdots (i + 1) \cdots$ where $j > i + 1$. We let the set of critical values of w be denoted by $C(w)$. It is straightforward to check that

$$C(w) \text{ is empty} \iff w \text{ is dominant.}$$

Lemma 2.7. Suppose $(v, w) \in \text{WA}_n$. Then we have $w^{-1}(i) < w^{-1}(i + 1)$ for all $i \in C(v)$.

Proof. We first consider the case $i = 1$. Since (v, w) is aligned, the criticality of 1 implies that we must have $w^{-1}(1) < v^{-1}(2)$. We claim this implies that $w^{-1}(1) < w^{-1}(2)$. Indeed, if $w^{-1}(1) > w^{-1}(2)$ then $(\delta(v), \delta(w))$ cannot be aligned which contradicts the well-aligned-ness of (v, w) .

Now suppose $i > 1$. Note that if $i \in C(v)$, then $i - 1 \in C(\delta(v))$. Since $(\delta(v), \delta(w))$ is well-aligned, our inductive hypothesis implies that $(\delta(w))^{-1}(i - 1) < (\delta(w))^{-1}(i)$. But then we necessarily have $w^{-1}(i) < w^{-1}(i + 1)$. $\square$

Example 2.8. For $(v, w) = (15726348, 75182364) \in \text{WA}_8$, we have $C(v) = \{1, 2, 5\}$. It is easily checked that for each $i \in C(v)$ we have i lying to the left of $i + 1$ in w .

Lemma 2.9. Suppose $(v, w) \in \text{WA}_n$. Then we have that $(s_i v, s_i w) \in \text{WA}_n$ for all $i \in C(v)$.

Proof. It is straightforward to see that $(s_i v, s_i w)$ is aligned, so it remains to show $(\delta(s_i v), \delta(s_i w)) \in \text{WA}_{n-1}$. We show this by induction on i . For $i = 1$, begin by observing that $\delta(s_1 v)$ and $\delta(s_1 w)$

are obtained by deleting the 2 from the one line notation of s_1v and s_1w respectively and then decrementing all entries greater than two by one. This implies that $(\delta(s_1v), \delta(s_1w))$ is aligned. Furthermore we have

$$(\delta\delta(s_1v), \delta\delta(s_1w)) = (\delta\delta(v), \delta\delta(w)),$$

and the latter is well-aligned because (v, w) is. Thus $(\delta\delta(s_1v), \delta\delta(s_1w)) \in \text{WA}_{n-2}$, which in turn implies $(\delta(s_1v), \delta(s_1w)) \in \text{WA}_{n-1}$ and concludes the base step of our induction.

Now suppose $i > 1$. Note that $(\delta(s_iv), \delta(s_iw)) = (s_{i-1}\delta(v), s_{i-1}\delta(w))$. Furthermore, observe that $(\delta(v), \delta(w)) \in \text{WA}_{n-1}$ with $i-1 \in C(\delta(v))$. The inductive hypothesis thus implies that $(s_{i-1}\delta(v), s_{i-1}\delta(w)) \in \text{WA}_{n-1}$, and therefore that $(\delta(s_iv), \delta(s_iw)) \in \text{WA}_{n-1}$. This concludes the induction step. $\square$

Lemma 2.10. Suppose $(v, w) \in \text{WA}_n$. For all $i \in C(v)$, we have that $[s_iv, s_iw]$ is translation-equivalent to $[v, w]$.

Proof. We verify conditions in Proposition 1.5(i)–(iii). Condition (i) holds because $i \in C(v)$. Condition (ii) is the content of Lemma 2.7. It remains to check Condition (iii) that $s_iv \not\leq_B w$.

Suppose $i = 1$. Let $j = v^{-1}(2)$ and $k = w^{-1}(1)$, and note that $1 \in C(v)$ implies that $j > k$. Therefore $\min\{(s_1v)(1), \dots, (s_1v)(k)\} = 1 < 2 = \min\{w(1), \dots, w(k)\}$ and we conclude $s_1v \not\leq_B w$ by the tableau criterion.

Now suppose $i > 1$. Then $i-1 \in C(\delta(v))$ so our inductive hypothesis yields $s_{i-1}\delta(v) \not\leq_B \delta(w)$. Let $p \in \text{Des}(s_{i-1}\delta(v))$ be a witness for this incomparability; see Definition 1.4. Suppose that

$$s_iv = \varepsilon_a(s_{i-1}\delta(v)) \quad \text{and} \quad w = \varepsilon_b(\delta(w)).$$

Note that $a \leq b$ since (v, w) is aligned. If furthermore $a \leq p$ then $b \leq p+1$ by alignedness of (v, w) , so $p+1 \in \text{Des}(s_iv)$ is a witness for $s_iv \not\leq_B w$. On the other hand if $a > p$, then $b > p$ by alignedness of (v, w) so $p \in \text{Des}(s_iv)$ witnesses the incomparability $s_iv \not\leq_B w$. $\square$

We illustrate various aspects of the preceding lemmas with the following example.

Example 2.11. For $(v, w) = (15726348, 75182364) \in \text{WA}_8$, we have $C(v) = \{1, 2, 5\}$. It is easily checked that $s_1v \not\leq_B w$.

Consider $i = 2$. We will show how a witness for $s_1\delta(v) \not\leq_B \delta(w)$ produces one for $s_2v \not\leq_B w$. We have $\delta(v) = 4615237$ and $\delta(w) = 6471253$. Since the first four entries of $s_1\delta(v)$ in increasing order 2, 4, 5, 6 while those of $\delta(w)$ are 1, 4, 6, 7, we have that $p = 4$ witnesses $s_1\delta(v) \not\leq_B \delta(w)$. Now note that

$$s_2v = 15736248 = \varepsilon_1(s_1\delta(v)) \quad \text{and} \quad w = \varepsilon_3(\delta(w)).$$

Therefore, $p+1 = 5$ witnesses $s_2v \not\leq_B w$.

Finally, consider $i = 5$. Since the first two entries of $s_4\delta(v) = 5614237$ in increasing order are 5, 6 while those of $\delta(w)$ are 4, 6, we have that $p = 2$ witnesses $s_4\delta(v) \not\leq_B \delta(w)$. Now note that

$$s_5v = 16725348 = \varepsilon_1(s_4\delta(v)) \quad \text{and} \quad w = \varepsilon_3(\delta(w)).$$

Therefore, $p + 1 = 3$ witnesses $s_5v \not\leq_B w$.

2.2. Labeled plane binary trees and dominant permutations. Recall that a *plane binary tree* T is a rooted tree where every internal (i.e. non-leaf) node has a left and right subtree. We denote the set of such trees with n internal nodes by $\mathcal{T}_n$. A decreasing labeling $\mathcal{L}$ of $T \in \mathcal{T}_n$ is a bijection of the internal nodes with $[n]$ such that whenever w is a descendent of v , the label of w is less than the label of v . We now recall a folklore bijection [46, pp. 23–24]

$$\begin{aligned} S_n &\rightarrow \{(T, \mathcal{L}) \mid T \in \mathcal{T} \text{ and } \mathcal{L} \text{ is a decreasing labeling of } T\} \\ w &\mapsto (\psi(w), \mathcal{L}(w)) \end{aligned}$$

though we follow the conventions in [31, §2.4].

The bijection is obtained by recursively applying the following procedure, starting from the one-line notation $w(1) \cdots w(n)$ of w : for a word of distinct numbers z we write $z = z^{(1)}mz^{(2)}$ with $m = \max(z)$ and associate to it the tree whose root is labeled m and whose left and right subtrees are given recursively by applying this procedure to the words $z^{(1)}$ and $z^{(2)}$ respectively.

Given $T \in \mathcal{T}_n$, consider the fiber

$$Z_T = \{w \in S_n \mid \psi(w) = T\}.$$

It is well known that Z_T has a unique maximal element $w^\uparrow$ under weak order, characterized by the fact that it is the unique dominant permutation in Z_T (see [32, Theorem 2.5] and also [1, §1.2]). In particular for dominant permutations we have $w = w^\uparrow$. We can describe $w^\uparrow$ explicitly as the unique permutation such that $T(w)$ has the property that for any node v , the smallest label in the left subtree of v is greater than the largest label in the right subtree of v . Figure 3 shows an unlabeled plane binary tree T on the left and the set Z_T on the right, with the dominant permutation highlighted.

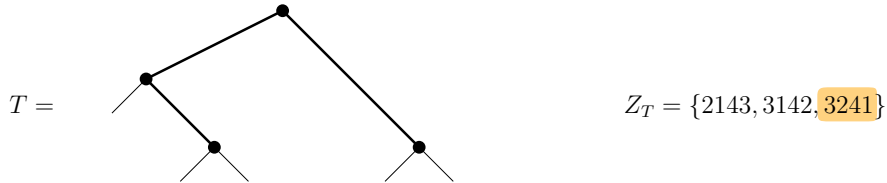


FIGURE 3. A plane binary tree T and the set Z_T .

Note that $i \in C(w)$ is equivalent to stating that the nodes labeled i and $i + 1$ are incomparable in T with the node labeled i appearing before that labeled $i + 1$ in order. Because all linear extensions

of a poset can be obtained by swapping labels of adjacent incomparable nodes, by repeatedly applying simple transpositions corresponding to the entries in the critical set we may transform w to $w^\uparrow$ without altering the underlying tree. Figure 4 illustrates this starting with $w = 2143$ and applying swaps corresponding to the critical values.

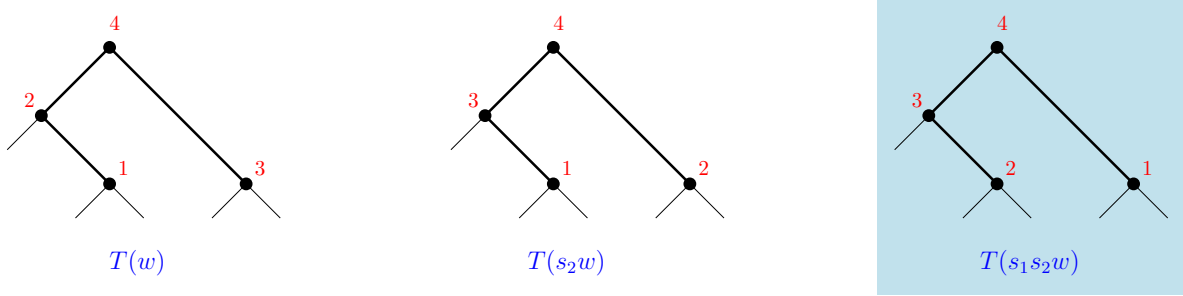


FIGURE 4. Three decreasing trees with the leftmost tree associated to $w = 2143$ and the rightmost corresponding to the dominant permutation $w^\uparrow = s_1s_2w = 3241$.

We are now ready to prove the main result of this section.

Proposition 2.12. Suppose $(v, w) \in \text{WA}_n$. Then the following hold.

- (1) $(v^\uparrow, v^\uparrow v^{-1}w) \in \text{WA}_n^{132}$.
- (2) The intervals $[v, w]$ and $[v^\uparrow, v^\uparrow v^{-1}w]$ are translation-equivalent.

Proof. If v is dominant then there is nothing to show as $v = v^\uparrow$. Thus we may assume that v is not dominant, i.e. $C(v) \neq \emptyset$. Pick any $i \in C(v)$. Then $(s_i v, s_i w) \in \text{WA}_n$ by Lemma 2.9. Furthermore we know that $\ell(s_i v) = \ell(v) + 1$ and $\ell(s_i w) = \ell(w) + 1$. Repeating this procedure eventually produces a $(v', w') \in \text{WA}_n^{132}$ so that $[v', w']$ is translation-equivalent to $[v, w]$. Since v and v' are in the same fiber of ψ , we have $v' = v^\uparrow$. Both claims now follow. $\square$

Example 2.13. For $(v, w) = (15726348, 75182364) \in \text{WA}_8$, we have $v^\uparrow = 56734128$. Proposition 2.12 guarantees that $[v, w]$ is translation-equivalent to $[v^\uparrow, v^\uparrow v^{-1}w] = [56734128, 76583142]$.

Remark 2.14. We revisit the well-aligned pairs from Example 2.3. In this case, Proposition 2.12 guarantees that the Bruhat interval $[v, vc]$ is translation-equivalent to $[v^\uparrow, v^\uparrow c]$ where $v^\uparrow \in S_n$ is dominant and satisfies $v^\uparrow(n) = n$ as v itself does. In particular there are Cat_n many such intervals, and it is a fact they are all pairwise non-equivalent with respect to translation. Translating these special intervals by $(v^\uparrow)^{-1}$ produces noncrossing partitions, and this connection is the first step toward understanding quasisymmetry in the context of Fl_n [5, 34].

3. WELL-ALIGNED SCHUBERT STRUCTURE COEFFICIENTS

Recall that we want to determine the structure coefficients $c_{u,v}^w$ where $(v, w) \in \text{WA}_n$.

3.1. Coefficients via translation-equivalence. We recall Sottile's Pieri rule [42] in a special case as it will be the only combinatorial gadget that we need. We momentarily work in the infinite symmetric group S_∞ . For $1 \leq a < b$, let $t_{ab} \in S_\infty$ be the transposition swapping a and b .

Definition 3.1. Given a nonnegative integer k , we say that a permutation u is covered by ut_{ab} in k -Bruhat order if $a \leq k < b$ and $\ell(ut_{ab}) = \ell(u) + 1$. The k -Bruhat order $u \leq_B^k w$ is obtained as the transitive closure of these covers.

The saturated chains that matter to us are

$$u \leq_B^k u t_{a_1 b_1} \leq_B^k u t_{a_1 b_1} t_{a_2 b_2} \leq_B^k \cdots \leq_B^k u t_{a_1 b_1} \cdots t_{a_k b_k} = w$$

where the $t_{a_i b_i}$ are transpositions with a_1 through a_k distinct and each $a_i \leq k < b_i$. We denote a chain as above by $u \xrightarrow{k} w$.

Theorem 3.2 ([42, Theorem 1]). Fix k a nonnegative integer and let $u \in S_\infty$. Then we have

$$x_1 \cdots x_k \mathfrak{S}_u = \sum_{u \xrightarrow{k} w} \mathfrak{S}_w.$$

Note that for v a dominant permutation with Lehmer code of v^{-1} given by $(k_1, \dots, k_p, 0, 0, \dots)$, we have

$$\mathfrak{S}_v = \prod_{1 \leq i \leq p} x_1 \cdots x_{k_i},$$

a dominant monomial. Iterating this Pieri rule tells us how to compute the product of an arbitrary Schubert polynomial with such a dominant monomial in the Schubert basis. Suppose we have a sequence $\mathbf{k} = (k_1 \geq \cdots \geq k_p)$ of nonnegative integers and a permutation $u \in S_\infty$. We write $u \xrightarrow{\mathbf{k}} w$ if there is a sequence of permutations $u = u_0, u_1, \dots, u_p = w$ such that $u_{i-1} \xrightarrow{k_i} u_i$ for $1 \leq i \leq p$. Then we have the following corollary.

Corollary 3.3. Let $v \in S_\infty$ be dominant with Lehmer code of v^{-1} given by $(k_1, \dots, k_p, 0, 0, \dots)$. Then we have

$$\mathfrak{S}_v \mathfrak{S}_u = \sum_{u \xrightarrow{\mathbf{k}} w} \mathfrak{S}_w.$$

In particular the Schubert structure coefficients $c_{u,v}^w$ can be computed combinatorially.

We are now in a position to state our main result.

Theorem 3.4. Let $(v, w) \in \text{WA}_n$. Then

$$c_{u,v}^w = c_{u, v^\uparrow}^{v^\uparrow v^{-1} w},$$

and these coefficients can be computed combinatorially.

Proof. The first half of the claim is a consequence of Proposition 2.12 which states that $[v, w]$ and $[v^\uparrow, v^\uparrow v^{-1}w]$ are translation equivalent. The second half follows from the fact that the classes of must agree. Finally, the combinatorial computation of the coefficients $c_{u, v^\uparrow}^{v^\uparrow v^{-1}w}$ follows from Corollary 3.3 since $v^\uparrow$ is dominant. $\square$

3.2. Coefficients via Bergeron–Sottile maps. We now proceed to study the well-aligned Schubert structure coefficients $c_{u, v}^w$ in a different way, using the Bergeron–Sottile maps. Say that a map $f : \mathbb{Z}[x_1, \dots, x_n] \rightarrow \mathbb{Z}[x_1, \dots, x_n]$ is *Schubert positive* if

$$f(\mathfrak{S}_u) = \sum a_u^v \mathfrak{S}_v \text{ with } a_u^v \geq 0.$$

We will say that f is *combinatorially Schubert positive* if we can combinatorially determine a_u^v to be nonnegative. The functional $\Phi_v^w : \mathbb{Z}[x_1, \dots, x_n] \rightarrow \mathbb{Z}$ given by

$$\Phi_v^w(f) = \text{ev}_0 \partial_w(f \mathfrak{S}_v)$$

is Schubert positive because $\Phi_v^w(\mathfrak{S}_u) = c_{u, v}^w$. To show the combinatorial nonnegativity of $c_{u, v}^w$ for well-aligned pairs $(v, w) \in \text{WA}_n$ is tantamount to showing that Φ_v^w is in fact combinatorially Schubert positive.

We will show the combinatorial Schubert positivity of Φ_v^w by showing it is a composite of combinatorially Schubert positive operations. The operations ev_0, ∂_i are manifestly combinatorially Schubert positive operations; we will need one further combinatorially Schubert positive operation.

Definition 3.5. For $f \in \mathbb{Z}[x_1, x_2, \dots]$ and a positive integer i , we define the *i th Bergeron–Sottile operator* by

$$R_i f = f(x_1, \dots, x_{i-1}, 0, x_i, x_{i+1}, \dots, x_{n-1})$$

As shown in Bergeron–Sottile [6], the result of applying R_i on Schubert polynomials can be computed combinatorially using Sottile’s Pieri rule – in particular, R_i is a combinatorially nonnegative operation. We verify this here quickly.

Fact 3.6. $R_1 \mathfrak{S}_w = \delta_{w(1)=1} \mathfrak{S}_{\delta(w)}$ and

$$R_i f = R_1 \partial_1 \cdots \partial_{i-1} x_1 \cdots x_{i-1} f.$$

Proof. The statement for R_1 is well-known – one way to see it is that because $\partial_i R_1 = \partial_{i+1}$ for all i , we have

$$\text{ev}_0 \partial_{w'} R_1 \mathfrak{S}_w = \text{ev}_0 \partial_{\varepsilon_1(w')} \mathfrak{S}_w = \delta_{\varepsilon_1(w'), w} = \delta_{w(1)=1} \delta_{\delta(w), w'}.$$

For the more general statement, because ∂_j commutes with polynomials symmetric in x_j, x_{j+1} we may rewrite

$$R_1 \partial_1 \cdots \partial_i x_1 \cdots x_{i-1} f = R_1 \partial_1 x_1 \partial_2 x_2 \cdots \partial_{i-1} x_{i-1} f$$

and then repeatedly apply the (easily verified) fact $R_j \partial_j x_i = R_{j+1}$. $\square$

Lemma 3.7. Consider $(v, w) \in \text{WA}_n$. Suppose $i = v^{-1}(1)$ and $j = w^{-1}(1)$. Then

$$\Phi_v^w = \Phi_{\delta(v)}^{\delta(w)} R_i \partial_i \partial_{i+1} \cdots \partial_{j-1}.$$

Proof. If $i = j$, then $\Phi_v^w = \Phi_{\delta(v)}^{\delta(w)} R_i$ is the statement of [34, Proposition 8.1]. Otherwise because (w, v) is well-aligned we have $j - 1 \in \text{Des}(w)$ and $j - 1 \notin \text{Des}(v)$ so

$$\Phi_v^w f = \text{ev}_0 \partial_w \mathfrak{S}_v f = \text{ev}_0 \partial_{ws_j} \partial_j \mathfrak{S}_v f = \text{ev}_0 \partial_{ws_j} \mathfrak{S}_v (\partial_j f) = \Phi_v^{ws_j} \partial_{j-1} f,$$

where the second last equality follows because $\mathfrak{S}_v$ is symmetric in the variables x_{j-1}, x_j (this idea is a special case of Knutson's descent-cycling [26]). Since $(v, ws_j) \in \text{WA}_n$, we can apply this repeatedly until the positions of 1 align and then apply the case $i = j$. $\square$

Note that since $(\delta(v), \delta(w)) \in \text{WA}_{n-1}$ if $(v, w) \in \text{WA}_n$, iterating the previous lemma, we obtain a formula for $\text{ev}_0 \partial_w (\mathfrak{S}_v \mathfrak{S}_u)$ as a composite of R_i s and ∂_j s applied to $\mathfrak{S}_u$. We make this explicit in the case where $(v, w) \in \text{WA}_n^{132}$. In this case, this composite can be read off from the skew Rothe diagram $\mathcal{R}(w) \setminus \mathcal{R}(v)$.

Proposition 3.8. Let $(v, w) \in \text{WA}_n^{132}$ and let $D := \mathcal{R}(w) \setminus \mathcal{R}(v)$. For $1 \leq i \leq n$, let r_i be the number of boxes in the i th row of D , considered from top to bottom. Suppose further that $(c_1, \dots, c_n) = \text{lcode}(v^{-1})$. Then we have

$$\Phi_v^w = \text{ev}_0 R_{c_n+1} \partial_{c_n+1} \cdots \partial_{c_n+r_n} R_{c_{n-1}+1} \partial_{c_{n-1}+1} \cdots \partial_{c_{n-1}+r_{n-1}} \cdots R_{c_1+1} \partial_{c_1+1} \cdots \partial_{c_1+r_1}.$$

Proof. This is a consequence of the fact that applying δ to each of v and w corresponds to striking out the top row of the bounding $n \times n$ box and then the column containing the 1 in v . $\square$

Example 3.9. Referring back to the well-aligned pair $(v, w) = (4521367, 5724316)$ depicted on the left in Figure 2, we have

$$(r_1, \dots, r_7) = (2, 0, 1, 2, 0, 1, 0) \quad \text{and} \quad (c_1, \dots, c_7) = (3, 2, 2, 0, 0, 0, 0) = \text{lcode}(4351267).$$

Thus we have

$$\Phi_v^w = \text{ev}_0 R_1 R_1 \partial_1 R_1 R_1 \partial_1 \partial_2 R_3 \partial_3 R_3 R_4 \partial_4 \partial_5.$$

In particular, the boundary of the grey region corresponding to v determines the subscripts of the R_i s and the skew Rothe diagram D (given by the blue shaded cells) determines the subscripts of the ∂_j s.

4. RICHARDSON TABLEAUX AND WELL-ALIGNED PAIRS

The main subfamily of well-aligned pairs in this article comes from a subfamily of SYTs comprising Richardson tableaux. Given a partition λ , we let $\text{crop}(\lambda)$ denote the partition obtained by omitting the first row of λ . For $T \in \text{SYT}(\lambda)$, we let $\text{crop}(T)$ denote the SYT of shape $\text{crop}(\lambda)$ obtained from the subtableau of T determined from rows 2 and below, where we naturally standardize said filling so that the entries are precisely $\{1, \dots, |\text{crop}(\lambda)|\}$.

- (1) for every entry j in the second row $j - 1$ is in the first row, and
- (2) $\text{crop}(T)$ is a Richardson tableau.

The preceding definition is not the original definition of Richardson tableau by Karp and Precup [24, Definition 1.3]; that the aforementioned definition is an alternative characterization is shown by the same authors [24, Corollary 4.18].

Diagram illustrating the decomposition of a 4x4 grid into four 2x2 grids. The 4x4 grid contains numbers 1 through 12, with the last two cells empty. The four 2x2 grids contain the following numbers:

- Grid 1: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
- Grid 2: $\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$
- Grid 3: $\begin{bmatrix} 9 & 10 \\ 11 & 12 \end{bmatrix}$
- Grid 4: $\begin{bmatrix} & \\ & \end{bmatrix}$

- (1) Richardson tableaux are characterized by the fact that each evacuation slide is an *L-slide* [24, Theorem 5.3]. That is, as one computes the evacuation tableau for T a Richardson tableau, the path traced by the “hole” during a single jeu-de-taquin slide is *L-shaped*.
- (2) A tableau T is Richardson if and only if $\text{evac}(T)$ is Richardson [24, Corollary 3.23].

$$\text{crop}(\text{evac}(T)) = \text{evac}(\text{crop}(T)).$$

- (1) v_T^{-1} is the top down reading word of $\text{evac}(T)$,
- (2) $w_o w_T^{-1} w_o$ is the reading word of T .

$$\begin{array}{cccccccccccc} v_T = & 1 & 6 & 9 & 2 & 7 & 11 & 3 & 8 & 4 & 10 & 12 & 5 \\ w_T = & 11 & 6 & 1 & 9 & 7 & 2 & 12 & 3 & 10 & 8 & 4 & 5. \end{array}$$

We now proceed to an alternative recursive description of v_T and w_T that will be more convenient for our purposes. We begin with an elementary lemma about Richardson tableaux that relies on their characterization in terms of L -slides.

Lemma 4.6. Let $\lambda \vdash n$ and $T \in \text{RT}(\lambda)$. Suppose m is the largest number such that the entries $\{1, 2, \dots, m\}$ appear in the first column of T . Then the first m evacuation slides terminate in rows $r_1, \dots, r_m$ where $r_1 > r_2 > \dots > r_m = 1$. Consequently, the entries $\{n - m + 1, \dots, n\}$ occupy a column strip in $\text{evac}(T)$ with $n - m + 1$ appearing at the end of the first row.

Proof. We shall repeatedly use the L -slide characterization of Richardson tableaux without explicit mention. The case $m = 1$ is trivial. Suppose $m > 1$. If $\lambda = (1^n)$ then the claim is clear. So we assume that $\lambda_1 > 1$. By our condition on m we know that $m + 1$ appears in the first row next to 1. In particular the m th evacuation slide must terminate at the end of the first row. So $r_m = 1$.

To establish the claim we only need to show that $r_1 > r_2$, as induction does the rest. If the r_1 th row is of length 1, again the claim is clear. So we assume that the r_1 th row has length at least 2. Let j be the entry in the r_1 th row and the second column. Since T is standard we know that j is strictly larger than the entry immediately above it; call it k . After the first evacuation slide, j moves to the first column whilst remaining in row r_1 , whereas k does not move. This configuration guarantees that the next evacuation slide terminates in a row strictly above r_1 . This concludes the proof. $\square$

Example 4.7. Shown below is a Richardson tableau T (left) and the result of applying the first three evacuation slides to it. The shaded entries belong to the evacuation tableau and are frozen. In this case $m = 3$, $r_1 = 5$, $r_2 = 2$, and $r_3 = 1$. The entries $\{10, 11, 12\}$ occupy a column strip in $\text{evac}(T)$ with 10 appearing at the end of the first row.

1	4	7	9	12
2	5	8		
3	10			
6	11			

1	3	6	8	11
2	4	7		
5	9			
10	12			

1	2	5	7	10
3	6	11		
4	8			
9	12			

1	4	6	9	10
3	6	11		
4	8			
9	12			

Informally, the preceding lemma states that a column of entries $\{1, \dots, m\}$ on the top left in a Richardson tableau T creates a distinguished column strip in the Richardson tableau $\text{evac}(T)$. To employ this observation recursively, we first show that there is a natural *column strip decomposition* of any Richardson tableau completely determined by the entries in its first row.

Definition 4.8. For $T \in \text{SYT}(\lambda)$, we let $\text{First}(T) = \{1 = a_1 < \dots < a_{\lambda_1}\}$ denote the entries in the first row of T read from left to right. We further declare that $a_{\lambda_1+1} = |\lambda| + 1$.

Lemma 4.9. Let $T \in \text{RT}(\lambda)$. Let $\text{First}(T) = \{1 = a_1 < \dots < a_{\lambda_1}\}$. For $1 \leq i \leq \lambda_1$, the cells occupied by the entries in $\{a_i, \dots, a_{i+1} - 1\}$ form a column strip in T .

Proof. Let Bad denote the set of Richardson tableaux that do not have the desired property. Our goal is to show that $\text{Bad} = \emptyset$. We proceed by contradiction. Suppose $T \in \text{Bad}$ is minimal, in the sense that T has the fewest boxes among all elements of Bad . Let $X = \{a_i, \dots, a_{i+1} - 1\}$ be the set of entries in T that do not form a column strip, and let S denote the skew shape consisting of these entries.

We claim that S must contain at least two boxes in the second row. Indeed, if this were not the case, then $\text{crop}(T)$ would also be a Richardson tableau lying in Bad , contradicting the minimality of T . Let j be the smallest element of X that lies in the second row. Since T is Richardson, we know that $j - 1$ must appear in the first row.

Now consider the entry j' immediately to the right of j in T . We must have $j' > j + 1$. Indeed, if $j' = j + 1$, then by the definition of a Richardson tableau, j would lie in the first row, a contradiction. Observe further that $j' \in X$. But then $j' - 1$ belongs to both the first row and to X . This is impossible, since X contains exactly one entry from the first row. $\square$

Given $T \in \text{RT}(\lambda)$, we let $\Pi(T)$ denote the set partition of $\{1, \dots, |\lambda|\}$ whose blocks are given by the entries in the column strips from Lemma 4.9. We note that $\Pi(T)$ is completely determined by $\text{First}(T)$. As a corollary of the preceding two lemmas we have the following.

Corollary 4.10. Fix $\lambda \vdash n$. For $T \in \text{RT}(\lambda)$, the partition $\Pi(\text{evac}(T))$ is obtained by replacing the entries i in each block of $\Pi(T)$ by $n + 1 - i$.

Example 4.11. Shown below are the column strip decompositions of T and $\text{evac}(T)$ from Example 1.1.

1	2	5	7	10	
3	8	11			
4	9				
6	12				

1	4	7	9	12	
2	5	8			
3	10				
6	11				

For completeness, we record both $\Pi(T)$ and $\Pi(\text{evac}(T))$ below.

$$\begin{aligned} \Pi(T) &= \{1\} \sqcup \{2, 3, 4\} \sqcup \{5, 6\} \sqcup \{7, 8, 9\} \sqcup \{10, 11, 12\}, \\ \Pi(\text{evac}(T)) &= \{12\} \sqcup \{9, 10, 11\} \sqcup \{7, 8\} \sqcup \{4, 5, 6\} \sqcup \{1, 2, 3\}. \end{aligned}$$

We record some additional consequences separately as they will be useful in proving that the pairs (v_T, w_T) satisfy a stronger condition than being well-aligned. We omit the proof as it amounts to unraveling Definition 4.4 which tells us how to obtain (v_T, w_T) from the appropriate reading words.

Corollary 4.12. Fix $\lambda \vdash n$ and let $\lambda_1 = k$. For $T \in \text{RT}(\lambda)$, consider the column strip decomposition of $\text{evac}(T)$. Suppose that m_1 through m_k (resp. M_1 through M_k) are the smallest (resp. largest) entries within each block in increasing order.

- (1) The positions of $1, 2, \dots, k$ in v_T are given by m_1 through m_k . Additionally we have $\text{Des}(v_T) \subseteq \{M_1, \dots, M_{k-1}\}$.
- (2) The positions of $1, 2, \dots, k$ in w_T are given by M_1 through M_k . Additionally we have $\text{Des}(w_T) \subseteq \{M_1, \dots, M_{k-1}, M_k\}^c$.

The parallel between the two statements in the next result is a reflection of the fact that evacuation is an involution. The interested reader is also referred to [24, Proposition 6.4] as an alternative means to deriving the second part from the first part; the result in loc. cit. tell us how w_T may be computed from $v_{\text{evac}(T)}$.

Definition 4.13. We call $(v, w) \in \text{WA}_n$ *very well-aligned* if (vw_o, ww_o) is well-aligned as well.

Since right multiplication by w_o reverses the one line notation of a permutation, we see that (v, w) is very well-aligned if and only if the following hold:

- (i) $v^{-1}(1) \leq w^{-1}(1)$,
- (ii) all indices $v^{-1}(1) \leq i \leq w^{-1}(1) - 1$ are ascents in v ,
- (iii) all indices $v^{-1}(1) \leq i \leq w^{-1}(1) - 1$ are descents in w , and
- (iv) $(\delta(v), \delta(w))$ is very well-aligned.

Lemma 4.14. Fix $\lambda \vdash n$ and let $T \in \text{RT}(\lambda)$. Then (v_T, w_T) is very well-aligned.

Proof. Let $k := \lambda_1$. By Corollary 4.12, we know that $v_T^{-1}(i) = m_i$ and $w_T^{-1}(i) = M_i$ for $1 \leq i \leq k$, and $m_i \leq M_i$ for $1 \leq i \leq k$. We further know that all indices in the range $[m_i, M_i - 1]$ are ascents in v_T and descents in w_T . In particular, Conditions (i)–(iii) hold for $(\delta^i(v_T), \delta^i(w_T))$ all $0 \leq i \leq k - 1$. To establish the claim observe that

$$(\delta^k(v_T), \delta^k(w_T)) = (v_{\text{crop}(T)}, w_{\text{crop}(T)}).$$

We briefly explain this equality. Note that $v_{\text{crop}(T)}$ can be read off from $\text{evac}(\text{crop}(T))$, but this latter tableau equals $\text{crop}(\text{evac}(T))$. Cropping the first row of $\text{evac}(T)$, whose entries correspond to the positions of 1 through k in v_T , and then standardizing the remaining entries to obtain an SYT then amounts to applying δ^k to v_T . Similarly, the positions of 1 through k in w_T , considered from the right, are determined by the first row of T . Cropping the first row of T and then standardizing amounts to applying δ^k to w_T . $\square$

Example 4.15. For the Richardson tableau T in Example 4.2, we have $\text{crop}(T)$ and $\text{evac}(\text{crop}(T))$ equaling the following tableaux.

1	4	6
2	5	
3	7	

1	3	5
2	6	
4	7	

It follows that

$$\begin{aligned} v_{\text{crop}(T)} &= 1 \quad 4 \quad 2 \quad 6 \quad 3 \quad 5 \quad 7 \\ w_{\text{crop}(T)} &= 6 \quad 1 \quad 4 \quad 2 \quad 7 \quad 5 \quad 3, \end{aligned}$$

which the reader can check agree with $\delta^5(v_T)$ and $\delta^5(w_T)$ respectively from Example 4.5.

Example 4.16. We borrow the the well-aligned pair $(v_T, w_T) = (1523467, 7123654)$ from [24, Example 10.9] for ease of comparison, and use Theorem 3.4 to compute the expansion $[X_v^w] = \sum_{u \in S_n} c_{u,v}^w [X_u]$. We have $v_T^\uparrow = 4512367$ and $v_T^\uparrow v_T^{-1} w_T = 7412653$. The Schubert polynomial indexed by v_T is the Schur polynomial $s_{(4)}(x_1, x_2)$ whereas the Schubert polynomial indexed by $v_T^\uparrow$ is the dominant monomial $x_1^3 x_2^3$. For $w_o = n(n-1) \cdots 1$, by the identity $c_{u,v}^w = c_{v, w_o w}^u$ whenever $u, v, w \in S_n$ we have

$$\mathfrak{S}_v \mathfrak{S}_{w_o w} = \sum_u c_{v, w_o w}^u \mathfrak{S}_u = \sum_{u \in S_n} c_{w_o u, v}^w \mathfrak{S}_u + \sum_{u \in S_\infty \setminus S_n} c_{v, w_o w}^u \mathfrak{S}_{w_o u}.$$

and so applying this for $v = v_T^\uparrow$ and $w = v_T^\uparrow v_T^{-1} w_T$, if we restrict this sum to those $u \in S_n$ we can read off the well-aligned Schubert coefficients (after reindexing the permutations $u \mapsto w_o u$). From Figure 5 it follows that

$$\mathfrak{S}_{4512367} \cdot \mathfrak{S}_{1476235} = \mathfrak{S}_{4765123} + \mathfrak{S}_{5763124} + \mathfrak{S}_{6735124} + \mathfrak{S}_{6752134} + \cdots,$$

where we omitted the Schuberts $\mathfrak{S}_u$ with $u \notin S_7$, so

$$[X_v^w] = [X_{4123765}] + [X_{3125764}] + [X_{2153764}] + [X_{2136754}].$$

We note that prior to our work, the cases for which $[X_{v_T}^{w_T}]$ was computed combinatorially in the Schubert basis was limited to tableaux of hook shape [20]; see also work of Graham–Zierau [19] which applies localization techniques but the expressions produced are not combinatorial. We briefly recall the combinatorial interpretation in [20], recast in the language of *Güemes tableaux* in [24, §10]. Given a Richardson tableau T of hook shape, the full expansion involves counting certain semistandard tableaux of staircase shape whose (column) reading words give reduced words. Our expansion, on the other hand, involves counting certain saturated chains in k -Bruhat order, for varying k .

Problem 4.17. Find a bijection between the chains in k -Bruhat order that arise in this way for well-aligned pairs (v_T, w_T) associated to a Richardson tableaux T of hook shape and Güemes tableaux.

For $u = v_T$ for T a Richardson tableau of hook shape of size n , the permutation $u^\uparrow$ is relatively straightforward to compute– it is given by a shuffle of the letters $k, k+1, \dots, n$ and $1, 2, \dots, k-1$, where the letters in the former appear in increasing order and those in the latter appear in decreasing order, and the first letter of $u^\uparrow$ is k . It is easily checked that this is dominant.

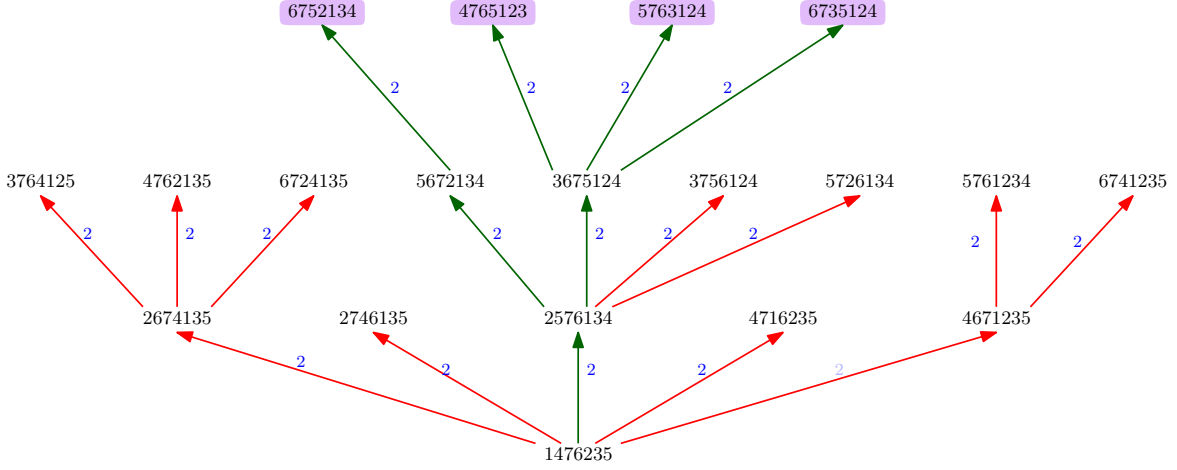


FIGURE 5. The product $x_1^3 x_2^3 \mathfrak{S}_{1476235}$ computed via Corollary 3.3. The green chains in 2-Bruhat order are the ones that contribute. At every level we have only recorded terms indexed by permutations in S_7 .

5. THE GEOMETRY OF WELL AND VERY-WELL ALIGNED RICHARDSON VARIETIES

We refer the reader to [3, 8, 49] for detailed treatment of notions/results in Schubert calculus that we employ but do not define. Throughout we work over $\mathbb{C}$. We denote by GL_n the group of invertible $n \times n$ matrices. Let B and B^- be the subsets of GL_n comprising upper triangular and lower triangular invertible $n \times n$ matrices respectively. A *flag* in $\mathbb{C}^n$ is a sequence of subspaces $V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n$ where $\dim(V_i) = i$. The *complete flag variety* Fl_n is the set of all flags in $\mathbb{C}^n$. The group GL_n acts transitively on Fl_n via its natural action on $\mathbb{C}^n$. The stabilizer of the *standard flag* $E_\bullet = E_1 \subset E_2 \subset \cdots \subset E_n = \mathbb{C}^n$, where $E_i = \mathrm{span}\{e_1, \dots, e_i\}$, is B . Thus we have $\mathrm{Fl}_n \cong \mathrm{GL}_n / B$.

For $v, w \in S_n$, we will denote the *Schubert cycles* in Fl_n by $X^w = \overline{BwB}$, the *opposite Schubert cycles* by $X_v = \overline{B^-vB}$, and for $v \leq w$ the *Richardson varieties* $X_v^w := X^w \cap X_v = \overline{BwB} \cap \overline{B^-vB}$. By work of Borel [11], we have an identification $H^\bullet(\mathrm{Fl}_n) = \mathbb{Z}[x_1, \dots, x_n] / \mathrm{Sym}_n^+$ where Sym_n^+ is the ideal generated by positive degree symmetric polynomials in x_1 through x_n . The classes of Schubert cycles $[X^w] \in H_\bullet(\mathrm{Fl}_n)$ for $w \in S_n$ form a homology basis Kronecker dual to the cohomology basis $\{\mathfrak{S}_v(x_1, \dots, x_n) \mid v \in S_n\} \subset H^\bullet(\mathrm{Fl}_n)$ of Schubert polynomials [29], which themselves represent the Poincaré dual classes to the opposite Schubert cycles X_v in $H^\bullet(\mathrm{Fl}_n)$.

We have an expansion

$$[X_v^w] = \sum c_{u,v}^w [X^u]$$

where $c_{u,v}^w$ are the generalized Littlewood–Richardson coefficients appearing in $\mathfrak{S}_u \mathfrak{S}_v = \sum c_{u,v}^w \mathfrak{S}_w$. More generally, for $f \in H^\bullet(\mathrm{Fl}_n)$ we have

$$\deg_{X_v^w} f = \mathrm{ev}_0 \partial_w (\mathfrak{S}_v f) = \Phi_v^w f.$$

In particular, $\deg_{X_v^w} \mathfrak{S}_u = c_{u,v}^w$.

5.1. Translation equivalence for well-aligned Richardson varieties. Recall that the *generalized Plücker functions* of $M \in \mathrm{GL}_n$ are given by

$$\mathrm{Pl}_w(M) = \prod_{i=1}^n \det M_{w(1), \dots, w(k)}^{1, \dots, k}$$

where M_B^A is the submatrix of M determined by the columns from A and the rows of B . We have $\mathrm{Pl}_w(w'M) = \pm \mathrm{Pl}_{ww'}(M)$, $\mathrm{Pl}_w(Bw'B) = 0$ whenever $w \not\leq_B w'$, and $BwB \subset \{\mathrm{Pl}_w \neq 0\}$. The characterization that follows is well known; we include a proof for completeness.

Lemma 5.1. If $v \leq_B w$ then we can write

$$X_v^w = \bigcap_{u \notin [v, w]} \{\mathrm{Pl}_u = 0\}.$$

Proof. It suffices to show the statement for X^w and X_v . We do this for X^w , the case for X_v is similar. We have to show that

$$X^w = \bigcap_{u \not\leq_B w} \{\mathrm{Pl}_u = 0\}.$$

Recall the Bruhat decomposition $\mathrm{GL}_n/B = \bigsqcup_{u \in S_n} BuB$. Note that X^w and $\bigcap_{u \not\leq_B w} \{\mathrm{Pl}_u = 0\}$ are both closed subvarieties containing BwB , and disjoint from BuB for $u \not\leq_B w$. The claim now follows from the equalities:

$$X^w = \overline{BwB} = \bigcup_{u \leq_B w} BuB. \quad \square$$

Proposition 5.2. If $[v, w]$ is translation-equivalent to $[v', w']$, then $X_{v'}^{w'} = v'v^{-1}X_v^w$.

Proof. This follows from the Plücker vanishing characterization in Lemma 5.1 and the S_n -equivariance of generalized Plücker functions up to sign. $\square$

Corollary 5.3. If $(v, w) \in \mathrm{WA}_n$, then $(v^\uparrow v^{-1})X_v^w = X_{v^\uparrow}^{v^\uparrow v^{-1}w}$. In particular $[X_v^w] = [X_{v^\uparrow}^{v^\uparrow v^{-1}w}] \in H_\bullet(\mathrm{Fl}_n)$.

5.2. Building well-aligned Richardsons from pattern maps and geometric push-pull. Let $\Psi_{1,i} : \mathrm{GL}_{n-1}/B \rightarrow \mathrm{GL}_n/B$ be the which takes MB to $M'B$ where M' is obtained by including M into an $n \times n$ matrix avoiding the first row and i 'th column, and then inserting a 1 into the first row and i 'th column. For example

$$\Psi_{1,2} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ a & 0 & b \\ c & 0 & d \end{bmatrix}.$$

This is an inclusion map, and as shown in [34, Theorem 4.4] we have

$$\Psi_{1,i} X_v^w = X_{\varepsilon_i(w)}^{\varepsilon_i(w)}.$$

Furthermore, we have $\Psi_i^* : H^\bullet(\mathrm{Fl}_n) \rightarrow H^\bullet(\mathrm{Fl}_{n-1})$ is given by R_i , so for $f(x_1, \dots, x_n) \in H^\bullet(\mathrm{Fl}_n)$ we have

$$\deg_{X_{\varepsilon_i(w)}^{\varepsilon_i(w)}} f = \deg_{X_v^w} R_i f.$$

Let $\pi_i : \mathrm{GL}_n/B \rightarrow \mathrm{GL}_n/P_i$ be the projection map where P_i is the minimal parabolic subgroup associated to s_i . Then by the classical computation of Bernstein–Gelfand–Gelfand [7] we have $(\pi_i)^*(\pi_i)_* f = \partial_i f$. We claim that if i is an ascent of both v and w we have

$$\pi_i^{-1} \pi_i X_v^w = X_v^{ws_i},$$

with $\pi_i|_{X_v^w}$ generically injective. Indeed, we have $\pi_i^{-1} \pi_i X_v^w \subset X_v^{ws_i}$ and $\pi_i^{-1} \pi_i X_v \subset X_v$ which shows the left hand side is contained in the right hand side, and the identity $\Phi_v^{ws_i} = \Phi_v^w \partial_i$, verified in the proof of Lemma 3.7 implies that $(\pi_i^*)(\pi_i)_*[X_v^w] = [X_v^{ws_i}]$. In particular, we have

$$\deg_{X_v^{ws_i}} f = \deg_{X_v^w} \partial_i f.$$

Therefore because a well-aligned pair can be obtained by successively applying either $(v, w) \mapsto (\varepsilon_i v, \varepsilon_i w)$ and $(v, w) \mapsto (v, ws_i)$ with i an ascent of both v, w , we see that X_v^w can be obtained as a successive application of $\Psi_{1,i}$ and $\pi_i^{-1} \pi_i$. Furthermore, $\Phi_v^w f = \deg_{X_v^w} f$, so the derivation of Φ_v^w as a successive application of operations R_i and ∂_i follows geometrically from the corresponding degree map facts above.

5.3. Smoothness and very well-aligned-ness. We revisit another result of Karp–Precup in light of ours and offer a generalization. In [24, §9] they establish that the Richardson variety X_{vT}^{wT} is smooth for all Richardson tableaux T . We show that this smoothness property extends to Richardson varieties coming from very well-aligned pairs. Our proof also involves verifying [24, Corollary 9.5] which gives a criterion for smoothness of Richardson varieties by reducing it to testing the smoothness at two special T -fixed points.

Fact 5.4 ([24, Corollary 9.5]). X_v^w is smooth if and only if it is smooth at the points vB and wB .

We will also need Deodhar's criterion [13] for smoothness at T -fixed points of X_v^w . Recall that we denote $t_{ab} \in S_n$ for the transposition swapping a and b , where by convention we always set $a < b$.

Proposition 5.5 ([13]). For $uB \in X_v^w$, we have

$$\#\{t_{ab} \in S_n \mid u t_{ab} \in [v, w]\} \geq \ell(w) - \ell(v)$$

with equality if and only if X_v^w is smooth at uB .

Lemma 5.6. If v is dominant then X_v^w is smooth at wB .

Proof. The boxes in the Rothe diagram $\mathcal{R}(w)$ are in bijection with the inversions t_{ab} of w , i.e. with $wt_{ab} \leq_B w$, by taking such a transposition to the box at position $(w(b), a)$. Furthermore the $\ell(v)$ -many boxes in the Rothe diagram $\mathcal{R}(v)$ are inversions t_{ab} such that wt_{ab} has its permutation matrix occupy a box in $\mathcal{R}(v)$, and hence $v \not\leq_B wt_{ab}$ by (2.1). We conclude by Proposition 5.5 that the remaining $(\ell(w) - \ell(v))$ -many inversions of w have $v \leq_B wt_{ab}$ and X_v^w is smooth at wB . $\square$

Remark 5.7. Geometrically, if v is a dominant permutation then $X_v \subset \text{Fl}_n$ is the subvariety of those gB where the entries of g inside the Rothe diagram are set to zero (this follows from the rank condition characterization of Schubert varieties and is the key observation in the study of Ding partition varieties, see e.g. [15, 14]). In the Bruhat decomposition $\text{GL}_n/B = \bigsqcup_{u \in S_n} BuB$, we have $BuB \cong \mathbb{A}^{\ell(u)}$ as the set of matrices which are 1 in the entries $(u(i), i)$, indeterminate entries in the boxes of the Rothe diagram $\mathcal{R}(u)$, and zero elsewhere. Therefore we have

$$X_v^w = \bigsqcup_{u \leq_B w} X_v \cap BuB = \bigsqcup_{u \in [v, w]} X_v \cap BuB$$

and $X_v \cap BuB \cong \mathbb{A}^{\ell(u) - \ell(v)}$ is the coordinate subspace of $BuB \cong \mathbb{A}^{\ell(u)}$ where we set the entries in $\mathcal{R}(v)$ to 0 (this is well-defined by (2.1)). This gives an affine paving of Richardsons X_v^w with v dominant, and because $X_v \cap BwB$ is a dense open chart around w isomorphic to $\mathbb{A}^{\ell(w) - \ell(v)}$ this gives an alternate geometric way to verify that $wB \in X_v^w$ is smooth.

Theorem 5.8. Let $(v, w) \in \text{WA}_n$ be very well-aligned. Then the Richardson variety X_v^w is smooth.

Proof. Let $[v', w']$ be translation-equivalent to $[v, w]$ with v' dominant. Then $X_v^w = v(v')^{-1}X_{v'}^{w'}$, and so X_v^w is smooth at wB since $X_{v'}^{w'}$ is smooth at $v'B$ by Lemma 5.6.

Since (ww_o, vw_o) is also well-aligned, we know that the interval $[ww_o, vw_o]$ is translation equivalent to $[w''w_o, v''w_o]$ with $w''w_o$ a dominant permutation. Therefore vw_oB is smooth in $X_{vw_o}^{ww_o}$ by Lemma 5.6. Now, right multiplication by w_o is an anti-isomorphism from $[ww_o, vw_o]$ to $[v, w]$. So, by Proposition 5.5, we get that $X_{vw_o}^{ww_o}$ is smooth at vw_oB if and only if X_v^w is smooth at wB . Because X_v^w is smooth at vB and wB , we conclude by Fact 5.4. $\square$

6. ODDS AND ENDS

To conclude this article we collect some numerological observations that may be of interest. One may ask for quantitative information about how many Schubert structure coefficient have we computed. Put differently, one may inquire about the number $|\text{WA}_n|$ of well-aligned pairs for a given nonnegative integer n . We offer the following conjecture.

Conjecture 6.1. Let $\mathcal{W}(x)$ denote the exponential generating function:

$$\mathcal{W}(x) = \sum_{n \geq 0} |\text{WA}_n| \frac{x^n}{n!} = 1 + 1 \frac{x^1}{1!} + 3 \frac{x^2}{2!} + 17 \frac{x^3}{3!} + 147 \frac{x^4}{4!} + 1729 \frac{x^5}{5!} + 25827 \frac{x^6}{6!} + 468593 \frac{x^7}{7!} + \cdots$$

Then $\mathcal{W}(x)$ satisfies the following functional equation:

$$\mathcal{W}'(x) = \frac{\mathcal{W}(x)^2}{2 - \mathcal{W}(x)}.$$

The sequence in question matches [41, A234289] for $0 \leq n \leq 7$. Both sides of the functional equation can be assigned combinatorial meaning easily, which in turn suggests there is a recursive decomposition of “pointed” well-aligned pairs that witnesses the functional equation.

As seen before, the well-aligned pairs in WA_n^{132} are crucial for us. The cardinality of this set equals $(2n - 1)!!$; see Remark 2.6. Nevertheless there is a curious observation to be made. It is the case that there exist $(v, w), (v', w') \in \text{WA}_n^{132}$ such that the Bruhat intervals $[v, w]$ and $[v', w']$ are translation-equivalent. So one may quotient WA_n^{132} further by this equivalence and inquire about the resulting number of equivalence classes. We make the following conjecture.

Conjecture 6.2. The total number of equivalence classes of translation-equivalent intervals is given by [41, A111088], which counts circular planar electrical networks [2].

REFERENCES

- [1] M. Aguiar and F. Sottile. Structure of the Loday-Ronco Hopf algebra of trees. *J. Algebra*, 295(2):473–511, 2006. [10](#)
- [2] J. Alman, C. Lian, and B. Tran. Circular planar electrical networks: posets and positivity. *J. Combin. Theory Ser. A*, 132:58–101, 2015. [24](#)
- [3] D. Anderson and W. Fulton. *Equivariant cohomology in algebraic geometry*, volume 210 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2024. [20](#)
- [4] N. Bergeron, L. Gagnon, P. Nadeau, H. Spink, and V. Tewari. Equivariant quasisymmetry and noncrossing partitions, 2025, 2504.15234. [2](#)
- [5] N. Bergeron, L. Gagnon, P. Nadeau, H. Spink, and V. Tewari. The quasisymmetric flag variety: a toric complex on noncrossing partitions, 2025, 2508.12171. [2](#), [11](#)
- [6] N. Bergeron and F. Sottile. Schubert polynomials, the Bruhat order, and the geometry of flag manifolds. *Duke Math. J.*, 95(2):373–423, 1998. [2](#), [13](#)
- [7] I. N. Bernšteĭn, I. M. Gelfand, and S. I. Gelfand. Schubert cells, and the cohomology of the spaces G/P . *Uspehi Mat. Nauk*, 28(3(171)):3–26, 1973. [22](#)
- [8] S. C. Billey, Y. Gao, and B. Pawlowski. Introduction to the cohomology of the flag variety, 2025, 2506.21064. [20](#)

- [9] A. Björner and F. Brenti. An improved tableau criterion for Bruhat order. *Electron. J. Combin.*, 3(1):Research Paper 22, approx. 5 pp, 1996. [4](#)
- [10] A. Björner and F. Brenti. *Combinatorics of Coxeter groups*, volume 231 of *Graduate Texts in Mathematics*. Springer, New York, 2005. [4](#)
- [11] A. Borel. Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts. *Ann. of Math. (2)*, 57:115–207, 1953. [20](#)
- [12] E. Delanoy. Completely compressible Bruhat intervals and Kazhdan-Lusztig polynomials. *European J. Combin.*, 29(3):746–759, 2008. [4](#)
- [13] V. V. Deodhar. Local Poincaré duality and nonsingularity of Schubert varieties. *Comm. Algebra*, 13(6):1379–1388, 1985. [23](#)
- [14] M. Develin, J.L. Martin, and V. Reiner. Classification of Ding’s Schubert varieties: finer rook equivalence. *Canad. J. Math.*, 59(1):36–62, 2007. [23](#)
- [15] K. Ding. Rook placements and cellular decomposition of partition varieties. *Discrete Math.*, 170(1-3):107–151, 1997. [23](#)
- [16] F. du Cloux. An abstract model for Bruhat intervals. *European J. Combin.*, 21(2):197–222, 2000. [4](#)
- [17] W. Fulton. *Young tableaux*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997. With applications to representation theory and geometry. [3](#)
- [18] Y. Gao and H. Zhu. Boolean Schubert Structure Coefficients, 2025, 2405.05527. [2](#)
- [19] W. Graham and R. Zierau. Smooth components of Springer fibers. *Ann. Inst. Fourier (Grenoble)*, 61(5):2139–2182 (2012), 2011. [19](#)
- [20] J. J. Güemes. On the homology classes for the components of some fibres of Springer’s resolution. *Astérisque*, (173-174):10, 257–269, 1989. [2](#), [19](#)
- [21] D. Huang. Schubert products for permutations with separated descents. *Int. Math. Res. Not. IMRN*, (20):17461–17493, 2023. [2](#)
- [22] D. Huang and P. Pylyavskyy. Bumpless pipe dream RSK, growth diagrams, and Schubert structure constants, 2022, 2206.14351. [2](#)
- [23] M. Josuat-Vergès and J. S. Kim. Generalized Dyck tilings. *European J. Combin.*, 51:458–474, 2016. [7](#), [8](#)
- [24] S. N. Karp and M. E. Precup. Richardson tableaux and components of Springer fibers equal to Richardson varieties, 2025, 2506.20792. [1](#), [2](#), [15](#), [18](#), [19](#), [22](#)
- [25] D. E. Knuth. *The art of computer programming. Vol. 1: Fundamental algorithms*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969. Second printing. [5](#)
- [26] A. Knutson. Descent-cycling in Schubert calculus. *Experiment. Math.*, 10(3):345–353, 2001. [14](#)
- [27] A. Knutson and T. Tao. The honeycomb model of $GL_n(\mathbb{C})$ tensor products. I. Proof of the saturation conjecture. *J. Amer. Math. Soc.*, 12(4):1055–1090, 1999. [2](#)
- [28] A. Knutson and P. Zinn-Justin. Schubert puzzles and integrability iii: separated descents, 2023, 2306.13855. [2](#)
- [29] A. Lascoux and M.-P. Schützenberger. Polynômes de Schubert. *C. R. Acad. Sci. Paris Sér. I Math.*, 294(13):447–450, 1982. [20](#)
- [30] D. E. Littlewood. *The Theory of Group Characters and Matrix Representations of Groups*. Oxford University Press, New York, 1940. [2](#)
- [31] J.-L. Loday and M. O. Ronco. Hopf algebra of the planar binary trees. *Adv. Math.*, 139(2):293–309, 1998. [10](#)
- [32] J.-L. Loday and M. O. Ronco. Order structure on the algebra of permutations and of planar binary trees. *J. Algebraic Combin.*, 15(3):253–270, 2002. [10](#)
- [33] G. Lusztig. Total positivity in Springer fibres. *Q. J. Math.*, 72(1-2):31–49, 2021. [2](#)

- [34] P. Nadeau, H. Spink, and V. Tewari. The geometry of quasisymmetric coinvariants, 2024, 2410.12643. [2](#), [7](#), [11](#), [14](#), [22](#)
- [35] P. Nadeau, H. Spink, and V. Tewari. Quasisymmetric divided differences, 2024, 2406.01510. [2](#)
- [36] I. Pak and C. Robichaux. Signed puzzles for Schubert coefficients, 2025, 2504.17734. [2](#)
- [37] I. Pak and C. Robichaux. Vanishing of Schubert coefficients via the effective Hilbert nullstellensatz. *Forum Math. Sigma*, 13:Paper No. e162, 2025. [2](#)
- [38] O. Pechenik and A. Weigandt. An inverse Grassmannian Littlewood-Richardson rule and extensions. *Forum Math. Sigma*, 12:Paper No. e114, 24 pp, 2024. [2](#)
- [39] M. P. Schützenberger. Quelques remarques sur une construction de Schensted. *Math. Scand.*, 12:117–128, 1963. [3](#)
- [40] M.-P. Schützenberger. La correspondance de Robinson. In *Combinatoire et représentation du groupe symétrique (Actes Table Ronde CNRS, Univ. Louis-Pasteur Strasbourg, Strasbourg, 1976)*, Lecture Notes in Math., Vol. 579, pages 59–113. Springer, Berlin-New York, 1977. [2](#)
- [41] N. J. A. Sloane. The encyclopedia of integer sequences. [24](#)
- [42] F. Sottile. Pieri’s formula for flag manifolds and Schubert polynomials. *Ann. Inst. Fourier (Grenoble)*, 46(1):89–110, 1996. [2](#), [12](#)
- [43] N. Spaltenstein. The fixed point set of a unipotent transformation on the flag manifold. *Indag. Math.*, 38(5):452–456, 1976. *Nederl. Akad. Wetensch. Proc. Ser. A* 79. [1](#)
- [44] T. A. Springer. The unipotent variety of a semi-simple group. In *Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968)*, volume 4 of *Tata Inst. Fundam. Res. Stud. Math.*, pages 373–391. Tata Inst. Fund. Res., Bombay, 1969. [1](#)
- [45] T. A. Springer. Trigonometric sums, Green functions of finite groups and representations of Weyl groups. *Invent. Math.*, 36:173–207, 1976. [1](#)
- [46] R. P. Stanley. *Enumerative combinatorics. Vol. 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original. [10](#)
- [47] R. P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin. [3](#)
- [48] R. P. Stanley. Positivity problems and conjectures in algebraic combinatorics. In *Mathematics: frontiers and perspectives*, pages 295–319. Amer. Math. Soc., Providence, RI, 2000. [1](#)
- [49] A. Woo and A. Yong. Schubert geometry and combinatorics, 2023, 2303.01436. [20](#)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, ON M5S 2E4, CANADA

Email address: hunter.spink@utoronto.ca

DEPARTMENT OF MATHEMATICAL AND COMPUTATIONAL SCIENCES, UNIVERSITY OF TORONTO MISSISSAUGA, MISSISSAUGA, ON L5L 1C6, CANADA

Email address: vasu.tewari@utoronto.ca