THE QUASISYMMETRIC FLAG VARIETY: A TORIC COMPLEX ON NONCROSSING PARTITIONS

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ABSTRACT. We develop the geometric theory of equivariant quasisymmetry via a new "quasisymmetric flag variety". This is a toric complex in the flag variety whose fixed point set is the set of noncrossing partitions, and whose cohomology ring is the ring of quasisymmetric coinvariants.

1. INTRODUCTION

In this paper we study a new geometric object we call the *quasisymmetric flag variety* QFl_n, which is contained in the variety Fl_n of complete flags of subspaces $0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_n = \mathbb{C}^n$. We show that QFl_n plays the same role for the quasisymmetric polynomials of Gessel [24] and Stanley [43] that Fl_n plays for symmetric polynomials. Our main results, summarized below, provide a complete geometric model for the quasisymmetric coinvariants.

Recall that a quasisymmetric polynomial in the variable set $\mathbf{x}_n = \{x_1, \ldots, x_n\}$ is one where the coefficient of $x_1^{a_1} \cdots x_k^{a_k}$ equals that of $x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}$ for all increasing sequences $1 \le i_1 < \cdots < i_k \le n$ and all sequences $(a_1, \ldots, a_k)$ of positive integers. We denote by QSym_n the ring of all quasisymmetric polynomials, and note that since the defining condition for quasisymmetry is a weakening of symmetry, the ring Sym_n of symmetric polynomials in $\mathbf{x}_n$ is a subring of QSym_n.

In [4], Aval–Bergeron–Bergeron initiated the study of the *quasisymmetric coinvariant ring*

$$\mathrm{QSCoinv}_n := \mathbb{Z}[\mathbf{x}_n]/\mathrm{QSym}_n^+ \quad \text{where} \quad \mathrm{QSym}_n^+ := \langle f(\mathbf{x}_n) - f(0, \ldots, 0) \mid f \in \mathrm{QSym}_n \rangle.$$

The graded ring $\mathrm{Coinv}_n := \mathbb{Z}[\mathbf{x}_n]/\mathrm{Sym}_n^+$ has a unimodal symmetric sequence of ranks, which reflects the fact that $H^\bullet(\mathrm{Fl}_n) \simeq \mathrm{Coinv}_n$ and Fl_n is a smooth projective variety. In contrast, the graded space $\mathrm{QSCoinv}_n$ is not rank symmetric for any $n \ge 3$ [4, Theorem 1.1], and thus it cannot arise as the cohomology of any smooth projective variety.

We construct the quasisymmetric flag variety using a collection of $(n-1)$-dimensional smooth toric varieties $X(T)$ parameterized by the set Tree_n of n-leaf planar binary trees,

$$\mathrm{QFl}_n := \bigcup_{T \in \mathrm{Tree}_n} X(T) \subset \mathrm{Fl}_n.$$

NB and LG were supported by the Natural Sciences and Engineering Research Council of Canada (NSERC) and York Research Chair in Applied Algebra. PN was partially supported by French ANR grant ANR-19-CE48-0011 (COMBINÉ). HS and VT acknowledge the support of the NSERC, respectively [RGPIN-2024-04181] and [RGPIN-2024-05433].

Each variety $X(T)$ is a left-translated Richardson variety, and an iterated $\mathbb{P}^1$-bundle following a process determined by the combinatorial structure of the tree T. The moment polytopes of these varieties are both combinatorial cubes and *polypositroids* in the sense of Lam–Postnikov [33].

Theorem A. We have $H^\bullet(\mathrm{QFl}_n) \cong \mathrm{QSCoinv}_n$.

We note that this does not contradict the previous observation on smooth projective varieties as QFl_n is reducible. We give a similar presentation of the torus-equivariant cohomology ring via the coinvariant ring of the *equivariantly quasisymmetric polynomials* defined in [6]; see Section 12.2.

The combinatorics of QFl_n is governed by *noncrossing partitions*. Let S_n denote the symmetric group on n letters. The torus fixed points of QFl_n are exactly the permutations $\mathrm{NC}_n \subseteq S_n$ obtained by treating each block of a noncrossing set partition as a backwards cycle, see Biane [9]. Noncrossing partitions also yield the following intrinsic characterization of QFl_n, which we prove in Section 10.

Theorem B. Denoting by $[\mathrm{Pl}_\sigma]_{\sigma \in S_n}$ the *Plücker functions* on Fl_n, we have that

$$\mathrm{QFl}_n = \bigcap_{\sigma \in S_n \setminus \mathrm{NC}_n} \{\mathrm{Pl}_\sigma = 0\} \subset \mathrm{Fl}_n.$$

Our remaining results establish the following parallels between QFl_n and Fl_n.

(1) The Bruhat decomposition gives a stratification of the flag variety into a union of affine Schubert cells $(X^w)^\circ$ with well-behaved closure relations known as an affine paving. In Section 9, we describe an affine paving

$$\mathrm{QFl}_n = \bigsqcup_{w \in \mathrm{NC}_n} (X^w)^\circ \cap \mathrm{QFl}_n,$$

which shares many combinatorial properties with the Bruhat decomposition. In Section 11 we prove that the closures of our affine cells give a homology basis for $H_\bullet(\mathrm{QFl}_n)$.

(2) Schubert polynomials [34] give a basis of $H^\bullet(\mathrm{Fl}_n)$ in Borel's presentation, and this basis is Kronecker dual to the homology basis of Schubert cycles $[X^w]$. In Section 12 we prove that our homology basis is likewise Kronecker dual to the family of forest polynomials [41, 39] and double forest polynomials [6].

(3) The divided difference formalism for Schubert polynomials admits a geometric interpretation via Bott–Samelson resolutions of Schubert varieties. Forest polynomials satisfy similar formalisms, and we show in Section 12 that these operations can be similarly interpreted using iterated $\mathbb{P}^1$-bundles. We exploit this connection to compute the degree map of each toric variety $X(F)$ in QFl_n using a recursive combinatorial process first defined in [6].

(4) Torus-equivariant versions of (1)–(3) also hold, highlighting parallels between the classical double Schubert polynomials and the double forest polynomials defined in [6].

In fact, many of our "non-equivariant" results are derived by first proving their equivariant analogue. In doing so, we rely crucially on the combinatorial relationship between double forest polynomials and noncrossing partitions developed in [6]. In particular, we show that the Goresky–Kottwitz–MacPherson graph [25] associated to QFl_n under the standard torus action is the Kreweras lattice on NC_n, and specializations of double forest polynomials describe a free basis of the associated graph cohomology ring. This generalizes earlier work of the first and second author [5] on *quasisymmetric orbit harmonics*, in which the main result shows that the ring

$$\mathbb{Q}[\mathbf{x}_n]/\langle f(\mathbf{x}_n) \mid f(w(1),\dots,w(n)) = 0 \text{ for all } w \in \mathrm{NC}_n\rangle$$

has associated graded $\mathrm{QSCoinv}_n$.

We now give a brief outline of the paper. In Section 2 we recall standard facts about Fl_n, noncrossing partitions and binary trees. We then define a family of elementary "building operations" on Fl_n in Section 3 and study the combinatorics of their compositions in Section 4. The building operations are the backbone of our work, and in Section 5 we use them to construct a family of Bott manifolds $X(\widehat{F})$ attached to bicolored nested forests $\widehat{F}$ that includes the $X(T)$ as the top-dimensional case. We provide a precise description of the inclusion order of these varieties in Section 6, highlighting the role of noncrossing partitions, and connect our work to HHMP_n and Richardson varieties in Section 7. In Section 8 we define QFl_n and completely describe the structure of this complex. The final four sections (9, 10, 11, and 12) contain our main results, as described above.

In the remainder of this introduction, we explain the motivation behind the construction of the varieties $X(T)$. In [38] a partial solution to the geometric realization of $\mathrm{QSCoinv}_n$ was obtained via a different construction called the *Ω-flag variety*. Let $S_{n-1} \subset S_n$ comprise permutations w satisfying $w(n) = n$ and denote the backwards long cycle $\boldsymbol{c} = (n\, n-1 \cdots 1) \in S_n$. In [38], the third, fourth, and fifth authors consider the complex of $(n-1)!$ smooth toric Richardson varieties

$$\mathrm{HHMP}_n \coloneqq \bigcup_{w \in S_{n-1}} X_w^{w\boldsymbol{c}}$$

under the torus action of $T_n = (\mathbb{C}^*)^n$ acting on Fl_n via its action on $\mathbb{C}^n$. This complex was first studied in [29, 36] as arising from a toric degeneration of a general T_n-orbit closure in Fl_n. We emphasize that this complex differs from QFl_n, as for $n \geq 3$ it contains more Richardson varieties than there are $X(T)$ varieties. Each toric Richardson in this complex is smooth and has a moment polytope that is a combinatorial cube.

The toric complex HHMP_n is assembled in a combinatorially simple way, equivalent to the decomposition of $[0,1] \times [0,2] \times \cdots \times [0,n-1]$ into unit cubes; see [38, Section 7.1]. One of the main results of [38] was that for the inclusion $\phi\colon\ \mathrm{HHMP}_n \to \mathrm{Fl}_n$, we have

$$\phi^* H^\bullet(\mathrm{Fl}_n) \cong \mathrm{QSCoinv}_n \subset H^\bullet(\mathrm{HHMP}_n).$$

Here the inclusion is strict: multiple cycles $X_w^{w\mathbf{c}}$ are equivalent under the left action of S_n by permutation matrices.

The geometry of QFl_n is constructed to avoid this duplication problem. We show (see also [38, §6]) that the $X(T)$ varieties collapse the left-translation redundancy of the $X_w^{w\mathbf{c}}$ via a surjective map

$$\begin{array}{rcl} \{X_w^{w\mathbf{c}} \mid w \in S_{n-1}\} & \to & \{X(T) \mid T \in \mathrm{Tree}_n\} \\ X_w^{w\mathbf{c}} & \mapsto & w^{-1}X_w^{w\mathbf{c}} \end{array}$$

taking the $(n-1)!$-many Richardson varieties to the $\mathrm{Cat}_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$-many $X(T)$ varieties.

The trade-off in the construction of QFl_n is a more complicated toric complex. In particular, the moment polytopes of our translated Richardson varieties overlap in an irregular manner, and torus orbits associated to partially overlapping faces do not intersect. These phenomena first appears in the case $n = 3$, and Figure 1 contrasts the structure of the moment polytopes for HHMP_n with our "translated" moment polytopes, which reflects the structure of QFl_n. Geometrically HHMP_3 is two Hirzebruch surfaces glued along a common torus-invariant $\mathbb{P}^1$, while QFl_3 is two Hirzebruch surfaces glued along two different and intersecting torus-invariant $\mathbb{P}^1$'s. We revisit this figure in Section 8 where the right panel is redrawn to resemble the Kreweras lattice NC_3.

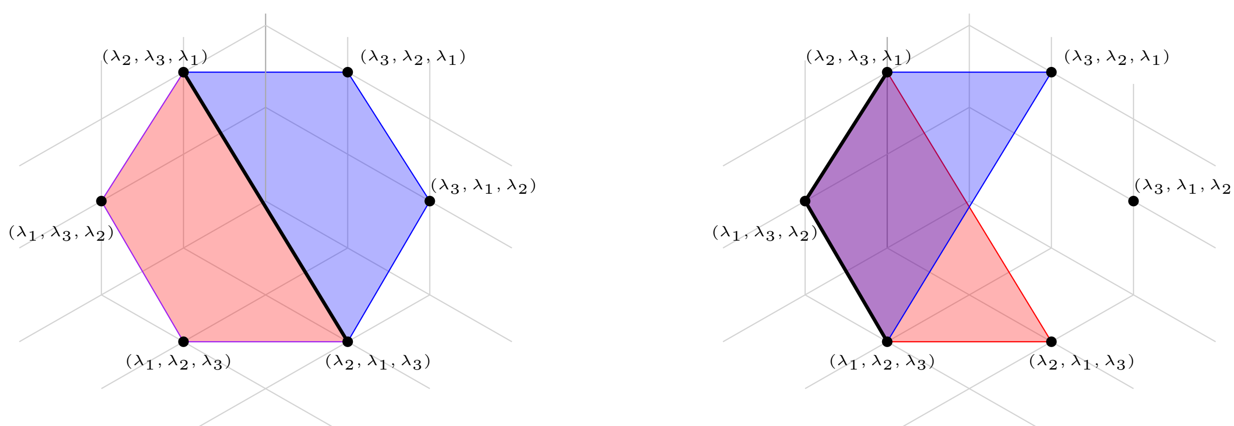


FIGURE 1. The two trapezoids comprising the HHMP subdivision of the $n = 3$ permutahedron (left) and the intersecting $X(T)$ moment polytopes (right).

Acknowledgements. We are very grateful to Allen Knutson and Alex Fink for numerous helpful conversations and their generous sharing of ideas. VT owes a lot to enlightening conversations with Dave Anderson. Finally thanks also go out to Sara Billey, Alexander Woo, and Alejandro Morales for pointers to relevant results in literature.

2. Preliminaries

Let $s_1, \ldots, s_{n-1}$ be the simple transpositions $s_i = (i, i+1)$ generating the symmetric group S_n. We let

$$\boldsymbol{c} \coloneqq (n\, n-1\, \cdots\, 2\, 1) = s_{n-1}s_{n-2}\cdots s_1 \in S_n,$$

denote the backwards long cycle. For all nonnegative integers m we set $[m] \coloneqq \{1, \ldots, m\}$.

2.1. **Recollections on** Fl_n**.** We work over $\mathbb{C}$ and denote by $T_n, B_n, B_n^- \subset \mathrm{GL}_n$ the subsets of diagonal, upper triangular, and lower triangular invertible $n \times n$ matrices. When there is no ambiguity we write T, B, and B^- for T_n, B_n, and B_n^-. We will denote χ_i for the ith standard character of T_n, corresponding to the i'th entry along the diagonal, so that for $a = \mathrm{diag}(a_1, \ldots, a_n) \in T$ we have $\chi_i(a) = a_i$.

We identify the complete flag variety

$$\mathrm{Fl}_n \coloneqq \text{flags of subspaces } \big(\{0\} = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \cdots \subsetneq \mathcal{F}_{n-1} \subsetneq \mathcal{F}_n = \mathbb{C}^n\big).$$

with GL_n/B_n via the transitive action of GL_n with B_n the stabilizer of the *standard coordinate flag* with $\mathcal{F}_i = \langle e_1, \ldots, e_i \rangle$ for $1 \leq i \leq n$. Via this identification ith subspace in the flag associated to hB_n is the column span of the first i columns of h for $1 \leq i \leq n$.

For $w \in S_n$, we will denote the Schubert cycles in Fl_n by $X^v = \overline{BvB}$, the opposite Schubert cycles by $X_u = \overline{B^-uB}$, and for $u \leq v$ in the Bruhat order the Richardson varieties $X_u^v \coloneqq X^v \cap X_u = \overline{BvB} \cap \overline{B^-uB}$. The Bruhat decomposition

$$\mathrm{Fl}_n = \bigsqcup_{w \in S_n} BwB \qquad \text{with} \qquad BwB \cong \mathbb{A}^{\ell(w)}, \tag{2.1}$$

gives an affine paving if the BwB are ordered via any linear extension of the Bruhat order on S_n; see Section 9 for more details.

For $w \in S_n$ and a dominant weight $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n) \in \mathbb{Z}^n$ we have a *Plücker function* defined for $h \in \mathrm{GL}_n$ by

$$\mathrm{Pl}_{\lambda,w}(h) = (\det h_{w(1)})^{\lambda_1 - \lambda_2}(\det h_{w(1),w(2)})^{\lambda_2 - \lambda_3} \cdots (\det h_{w(1),\ldots,w(n)})^{\lambda_n},$$

where $h_{i_1,\ldots,i_k}$ is the submatrix of h with columns $1, \ldots, k$ and rows $i_1, \ldots, i_k$. These functions together define a map $\mathrm{Pl}_\lambda : \mathrm{Fl}_n \to \mathbb{P}^{n!-1}$, which we write simply Pl if λ is the fundamental dominant weight $(n, n-1, \ldots, 1)$. Then Pl_λ is T_n-equivariant with respect to the T_n-action on $\mathbb{P}^{n!-1}$ where $\mathrm{diag}(a_1, \ldots, a_n) \in T_n$ acts in the w-coordinate by $a_{w(1)}^{\lambda_1} a_{w(2)}^{\lambda_2} \cdots a_{w(n)}^{\lambda_n} = a_1^{\lambda_{w^{-1}(1)}} \cdots a_n^{\lambda_{w^{-1}(n)}}$.

The moment polytope of Fl_n under Pl_λ is the generalized permutahedron

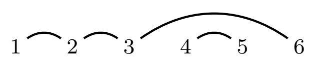
$$\mathrm{Perm}(\lambda) = \mathrm{conv}\{u \cdot \lambda \mid u \in S_n\} \subseteq \mathbb{R}^n,$$

where $u \cdot \lambda = (\lambda_{u^{-1}(1)}, \ldots, \lambda_{u^{-1}(n)})$ and conv denotes taking the convex hull. If $X \subset \mathrm{Fl}_n$ is a T_n-invariant subvariety, then its moment polytope under Pl_λ is

$$P(X;\lambda) = \mathrm{conv}\{u \cdot \lambda \mid u \in X^T\} \subseteq \mathrm{Perm}(\lambda).$$

The moment polytope of a T_n-orbit closure X is always a *flag matroid polytope* (see [22]), meaning its vertices are contained in the vertices of $\mathrm{Perm}(\lambda)$ and all edges are parallel to the type A_{n-1} roots $e_i - e_j$ for various i, j.

2.2. **Noncrossing partitions.** A *combinatorial noncrossing partition* is a partition $A_1 \sqcup \cdots \sqcup A_k = [n]$ such that for $i \neq j$, distinct elements $a, b \in A_i$, and distinct elements $c, d \in A_j$, we never have $a < c < b < d$. We depict combinatorial noncrossing partitions as noncrossing arc diagram; for example we draw $\{1,2,3,6\} \sqcup \{4,5\} = [6]$ as



A *backwards cycle* is a cycle $(b_1\, b_2\, \cdots\, b_r)$ with $b_1 > b_2 > \cdots > b_r$. An *algebraic noncrossing partition* is defined to be a permutation w whose disjoint cycle decomposition $\mathrm{Cyc}(w) \coloneqq C_1 C_2 \cdots C_k$ consists of backwards cycles whose underlying sets define a combinatorial noncrossing partition. For example $w = (6321)(54)$ is the algebraic noncrossing partition associated to the arc diagram above. We denote

$$\mathrm{NC}_n \coloneqq \{\text{algebraic noncrossing partitions of } [n]\} \subset S_n.$$

The *Kreweras order* on NC_n is defined by setting $u \leq_K v$ if the combinatorial noncrossing partition associated to u refines the combinatorial noncrossing partition associated to v. For example $(63)(21)(54) \leq_K (6321)(54)$. The Kreweras order is a lattice, and the lattice structure is induced by the lattice of set partitions under refinement [32].

Going forward, we will identify combinatorial and algebraic noncrossing partitions, using "combinatorial" and "algebraic" only when disambiguation is needed.

We say that permutations $u, w \in S_n$ are *adjacent* if $w = (i\, j)u$ for some transposition $(i\, j)$. Let $\mathrm{Cayley}(S_n)$ be the *Cayley graph* on S_n in which adjacent permutations are connected by an edge. Then $u, w \in \mathrm{NC}_n$ are adjacent in the Kreweras lattice (meaning one element covers the other) if and only if u, w are adjacent in $\mathrm{Cayley}(S_n)$. In this way, we may identify the Hasse diagram of the Kreweras lattice with the induced subgraph of $\mathrm{Cayley}(S_n)$ on NC_n as was first observed by Biane [9]. We denote this induced subgraph by $\mathrm{Cayley}(\mathrm{NC}_n)$.

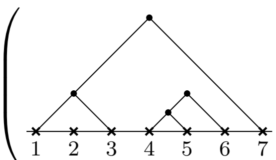
Remark 2.1. The Kreweras order is different from the Bruhat order $\leq$ restricted to NC_n. For example, $\boldsymbol{c}$ is the maximal element of $\leq_K$, whereas w_0 is the maximal element of $\leq$. See [10] for a more detailed study of the interaction between the two orders.

2.3. **Binary trees and nested forests.** A *planar binary tree* is a rooted tree T in which each node v is either an *internal node* with exactly 2 children v_L and v_R (the left and right child), or v is a *leaf* with zero children. Let $\mathrm{IN}(T)$ denote the set of internal nodes in T. We allow for the possibility that $|\mathrm{IN}(T)| = 0$, in which case the unique node is both a root and a leaf.

A *nested forest supported on* $[n]$ is a family $\widehat{F} = (T_C)_{C\in \mathrm{Cyc}(w)}$ of binary trees T_C indexed by the disjoint cycles of a noncrossing partition $w = C_1C_2\cdots C_k \in \mathrm{NC}_n$ such that each T_C has $|C|$ leaves. We identify the leaves of each T_C with the underlying set of C in increasing order and depict $\widehat{F}$ by drawing the T_C in the upper half plane so that the set $[n]$ of all leaves appears in increasing order along the horizontal axis.

The internal nodes of $\widehat{F}$ are $\mathrm{IN}(\widehat{F}) = \bigsqcup_{C\in\mathrm{Cyc}(w)} \mathrm{IN}(T_C)$. The *canonical label* of each $v \in \mathrm{IN}(F)$ is the value of the rightmost leaf descendant of v_L.

We define a map $\mathrm{NCPerm} : \mathrm{NestFor}_n \to \mathrm{NC}_n$ that sends a nested forest $(T_C)_{C\in\mathrm{Cyc}(w)}$ to its underlying noncrossing partition $w = \prod_{C\in\mathrm{Cyc}(w)} C$. For example

$$\mathrm{NCPerm}\left(\vphantom{\Big|} \right.$$


$$\left. \vphantom{\Big|} \right) = (731)(654)(2).$$

Remark 2.2. Nested forests should be seen as an enriched version of noncrossing partitions. In Section 6, we show that each nested forest corresponds, up to certain trivial commutation relations, to a distinguished factorization of its associated noncrossing partition.

3. Building split $\mathbb{P}^1$-bundles

We now introduce the operations used to build $\mathrm{QFl}_n \subset \mathrm{Fl}_n$.

3.1. Ψ_i^- **and** Ψ_i^+**.** We recall certain pattern maps that were studied by Bergeron–Sottile [7] and summarize their essential properties. We refer the reader to [13, 11] for a more general perspective.

Definition 3.1. Let $\Psi_{i,j}\colon \mathrm{Mat}_{m-1\times m-1} \to \mathrm{Mat}_{m\times m}$ be the operation

$$(\Psi_{i,j}M)_{k,\ell} = \begin{cases} M_{k-\delta_{k<i},\ell-\delta_{\ell<j}} & k \neq i \text{ and } \ell \neq j \\ 1 & (k,\ell) = (i,j) \\ 0 & \text{otherwise.} \end{cases}$$

The map $\Psi_{1,i}$ was crucial to the construction in [38, §5]. In contrast, the following two pattern maps will be important to us:

$$\Psi_i^- := \Psi_{i,i} \text{ and } \Psi_i^+ := \Psi_{i,i+1}.$$

For example when $m = 4$ we have

$$\Psi_2^- \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & 0 & b & c \\ 0 & 1 & 0 & 0 \\ d & 0 & e & f \\ g & 0 & h & i \end{bmatrix} \text{ and } \Psi_2^+ \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & 0 & c \\ 0 & 0 & 1 & 0 \\ d & e & 0 & f \\ g & h & 0 & i \end{bmatrix}.$$

By restricting to invertible matrices, the pattern maps descend to closed embeddings

$$\Psi_{i,j} : \mathrm{Fl}_{m-1} \hookrightarrow \mathrm{Fl}_m.$$

Define $\gamma_i : T_m \to T_{m-1}$ to be the map $\mathrm{diag}(a_1, \ldots, a_m) \mapsto \mathrm{diag}(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m)$.

Definition 3.2. If T is a torus and $\gamma : T \to T_m$ is a map of tori then we write Fl_m^γ for Fl_m equipped with the action of T induced by γ.

Fact 3.3. The maps $\Psi_{i,j} : \mathrm{Fl}_{m-1}^{\gamma_i} \to \mathrm{Fl}_m$ are T_m-equivariant closed embeddings, and in particular this is true of $\Psi_i^\pm : \mathrm{Fl}_{m-1}^{\gamma_i} \hookrightarrow \mathrm{Fl}_m$.

Write $\epsilon_i : \mathbb{C}^{m-1} \hookrightarrow \mathbb{C}^m$ for the inclusion $(x_1, \ldots, x_{m-1}) \mapsto (x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_{m-1})$. Then ϵ_i is a T_m-equivariant inclusion if we give $\mathbb{C}^{m-1}$ the action of T_m induced by γ_i, and we have

$$\Psi_i^- \mathcal{F} = \{0\} \subset \epsilon_i(\mathcal{F}_1) \subset \cdots \subset \epsilon_i(\mathcal{F}_i) \subset \epsilon_i(\mathcal{F}_i) \oplus \langle e_i \rangle \subset \epsilon_i(\mathcal{F}_{i+1}) \oplus \langle e_i \rangle \subset \cdots$$
$$\Psi_i^+ \mathcal{F} = \{0\} \subset \epsilon_i(\mathcal{F}_1) \subset \cdots \subset \epsilon_i(\mathcal{F}_i) \subset \epsilon_i(\mathcal{F}_{i+1}) \subset \epsilon_i(\mathcal{F}_{i+1}) \oplus \langle e_i \rangle \subset \cdots.$$

3.2. **Building $\mathbb{P}^1$-bundles with $\mathbb{P}_i$.** We use the subbundle convention for relative projectivization, so that for $\mathcal{V}$ a vector bundle on a variety X we have $\mathsf{Proj}(\mathcal{V})_X := (\mathcal{V} \setminus \{0\})/\mathbb{C}^*$. Consider the sequence of maps

$$\mathrm{Fl}_{m-1}^{\gamma_i} \overset{\Psi_i^\pm}{\rightrightarrows} \mathrm{Fl}_m \xrightarrow{\pi_i} \mathrm{GL}_m/P_i,$$

where $P_i = \langle B, s_i \rangle$ is the ith minimal parabolic subgroup of GL_m, and π_i is the projection map.

The space GL_m/P_i is typically identified with the variety of partial flags $\{0\} \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{i-1} \subset \mathcal{F}_{i+1} \subset \cdots \subset \mathcal{F}_m = \mathbb{C}^m$ with $\dim \mathcal{F}_j = j$. Under this identification π_i becomes the map which forgets the ith subspace of a complete flag.

Fact 3.4. $\mathrm{Fl}_m \to \mathrm{GL}_m/P_i$ is T_m-equivariantly isomorphic to the $\mathbb{P}^1$-bundle $\mathsf{Proj}(\mathcal{F}_{i+1}/\mathcal{F}_{i-1})_{\mathrm{GL}_m/P_i}$.

We now study how the $\mathbb{P}^1$-bundle π_i interacts with the maps $\Psi_i^\pm$. Since $s_i P_i = P_i$, we have the equality $\pi_i \Psi_i^- = \pi_i \Psi_i^+$. Consequently we write

$$\pi_i \Psi_i := \pi_i \Psi_i^+ = \pi_i \Psi_i^-$$

to emphasize that this composite does not depend on $\pm$. The map $\pi_i \Psi_i$ is given by

$$\pi_i \Psi_i(\mathcal{F})_j = \begin{cases} \epsilon_i(\mathcal{F}_j) & j < i \\ \epsilon_i(\mathcal{F}_{j-1}) \oplus \langle e_i \rangle & j > i, \end{cases}$$

and is a closed T_m-equivariant embedding $\mathrm{Fl}_{m-1}^{\gamma_i} \hookrightarrow \mathrm{GL}_m/P_i$ with image

$$\pi_i\Psi_i(\mathrm{Fl}_{m-1}) = \big\{\{\mathcal{F}_j\}_{j\in[m]\setminus i} : \mathcal{F}_{i-1} \subset \{x_i = 0\} \text{ and } e_i \in \mathcal{F}_{i+1}\big\} \subset \mathrm{GL}_m/P_i.$$

For $Z \subset \mathrm{Fl}_{m-1}$, we define

$$\mathbb{P}_i Z \coloneqq \pi_i^{-1}\pi_i\Psi_i Z \subset \mathrm{Fl}_m.$$

Like the pattern maps we can also view $\mathbb{P}_i Z$ in terms of matrices. For an $(m-1)\times(m-1)$ matrix M, let $\mathbb{G}_i M$ be the set obtained from $\Psi_i^+ M$ by replacing the 0 in entry (i,i) with all values $+ \in \mathbb{C}^*$. Then $\mathbb{G}_i$ descends to a map

$$\text{subsets of } \mathrm{Fl}_{m-1} \to \text{subsets of } \mathrm{Fl}_m,$$

and $\mathbb{P}_i Z = (\Psi_i^+ Z) \sqcup (\mathbb{G}_i Z) \sqcup (\Psi_i^- Z)$. For example when $m = 4$ we have

$$\mathbb{P}_2 \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} B_3 = \left\{ \begin{bmatrix} a & b & 0 & c \\ 0 & 0 & 1 & 0 \\ d & e & 0 & f \\ g & h & 0 & i \end{bmatrix} B_4 \right\} \sqcup \left\{ \begin{bmatrix} a & b & 0 & c \\ 0 & + & 1 & 0 \\ d & e & 0 & f \\ g & h & 0 & i \end{bmatrix} B_4 : + \in \mathbb{C}^* \right\} \sqcup \left\{ \begin{bmatrix} a & 0 & b & c \\ 0 & 1 & 0 & 0 \\ d & 0 & e & f \\ g & 0 & h & i \end{bmatrix} B_4 \right\}.$$

Theorem 3.5. Let $Z \subset \mathrm{Fl}_{m-1}$ be a T_{m-1}-invariant subvariety. If we consider Z as a T_m-invariant subvariety of $\mathrm{Fl}_{m-1}^{\gamma_i}$ for some fixed $1 \le i \le m-1$, then the following are true.

(1) The map
$$(\pi_i\Psi_i)^{-1}\pi_i \colon \mathbb{P}_i Z \to Z$$
realizes $\mathbb{P}_i Z$ as a T_m-equivariant $\mathbb{P}^1$-bundle over Z, which is T_m-equivariantly isomorphic to the $\mathbb{P}^1$-bundle $\mathsf{Proj}((\mathcal{F}_i/\mathcal{F}_{i-1}) \oplus \mathbb{C}_{\chi_i})_Z \to Z$.

(2) The closed subsets $\Psi_i^-|_Z$ and $\Psi_i^+|_Z$ correspond to the T_m-equivariant sections $\mathsf{Proj}(\{0\} \oplus \mathbb{C}_{\chi_i})_Z$ and $\mathsf{Proj}((\mathcal{F}_i/\mathcal{F}_{i-1}) \oplus \{0\})_Z$ respectively.

Proof. Both results follow from the case $Z = \mathrm{Fl}_{m-1}$ by restriction, so we consider only $Z = \mathrm{Fl}_{m-1}$.

(1) Since $\mathbb{P}_i\mathrm{Fl}_{m-1}$ can be defined by the pullback diagram

$$\begin{array}{ccc} \mathbb{P}_i\mathrm{Fl}_{m-1} & \longhookrightarrow & \mathrm{Fl}_m = \mathsf{Proj}(\mathcal{F}_{i+1}/\mathcal{F}_{i-1})_{\mathrm{GL}_m/P_i} \\ \downarrow & & \downarrow \pi_i \\ \mathrm{Fl}_{m-1}^{\gamma_i} & \xhookrightarrow{\pi_i\Psi_i} & \mathrm{GL}_m/P_i, \end{array}$$

we have $\mathbb{P}_i\mathrm{Fl}_{m-1}$ is the projectivization of the pullback of $\mathcal{F}_{i+1}/\mathcal{F}_{i-1}$ under the closed embedding $\pi_i\Psi_i$. This pullback is given by

$$(\pi_i\Psi_i)^*(\mathcal{F}_{i+1}/\mathcal{F}_{i-1}) = (\mathcal{F}_i \oplus \mathbb{C}_{\chi_i})/(\mathcal{F}_{i-1} \oplus \{0\}) \cong (\mathcal{F}_i/\mathcal{F}_{i-1}) \oplus \mathbb{C}_{\chi_i} \tag{3.1}$$

so the result follows.

(2) We can realize $\Psi_i^- \mathrm{Fl}_{m-1}$ and $\Psi_i^+ \mathrm{Fl}_{m-1}$ as T_m-equivariant sections of $\pi_i|_{\mathbb{P}_i \mathrm{Fl}_{m-1}}$ taking a partial flag $\{\mathcal{F}_j\}_{j \in [m] \setminus i} \in \pi_i \Psi_i(\mathrm{Fl}_{m-1})$ to respectively

$$\{0\} \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{i-1} \subset \mathcal{F}_{i-1} \oplus \langle e_i \rangle \subset \mathcal{F}_{i+1} \subset \cdots \subset \mathcal{F}_{m-1} \subset \mathbb{C}^m, \text{ and}$$
$$\{0\} \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{i-1} \subset \mathcal{F}_{i+1} \cap \{x_i = 0\} \subset \mathcal{F}_{i+1} \subset \cdots \subset \mathcal{F}_{m-1} \subset \mathbb{C}^m.$$

These two sections correspond to the choice of intermediate sub-bundles $\mathcal{F}_{i-1} \oplus \langle e_i \rangle$ and $\mathcal{F}_{i+1} \cap \{x_i = 0\}$ between $\mathcal{F}_{i-1}|_{\pi_i \Psi_i(\mathrm{Fl}_{m-1})}$ and $\mathcal{F}_{i+1}|_{\pi_i \Psi_i(\mathrm{Fl}_{m-1})}$, which in the pullback bundle (3.1) correspond to $\{0\} \oplus \mathbb{C}_{\chi_i}$ and $(\mathcal{F}_i/\mathcal{F}_{i-1}) \oplus \{0\}$. □

4. THE VARIETIES $X(\widehat{F})$ AND RELATIONS ON THE BUILDING OPERATIONS

In this section we introduce bicolored nested forests as a tool to study compositions of the building operations Ψ_i^-, Ψ_i^+, and $\mathbb{P}_i$. This culminates in Definition 4.9, in which we introduce the varieties $X(\widehat{F})$ used to define QFl_n.

4.1. Combinatorics of bicolored nested forests.

Let RESeq be the set of words from the alphabet

$$\bigcup_{i=1}^{\infty} \{\mathsf{r}_i^-, \mathsf{r}_i^+, \mathsf{e}_i\},$$

and for $\Omega \in \mathrm{RESeq}$ define $|\Omega|$ to be the number of e_i letters in Ω. For example $\mathsf{r}_2^- \mathsf{r}_3^+ \mathsf{e}_2 \in \mathrm{RESeq}$ and $|\mathsf{r}_2^- \mathsf{r}_3^+ \mathsf{e}_2| = 1$. We define a distinguished subset $\mathrm{RESeq}_n \subset \mathrm{RESeq}$ by

$$\mathrm{RESeq}_n \coloneqq \{\mathsf{x}_1 \cdots \mathsf{x}_n \in \mathrm{RESeq} \mid \mathsf{x}_i \in \{\mathsf{r}_1^-, \ldots, \mathsf{r}_i^-, \mathsf{r}_1^+, \ldots, \mathsf{r}_{i-1}^+, \mathsf{e}_1, \ldots, \mathsf{e}_{i-1}\} \text{ for all } i\}.$$

Note that every word of RESeq_n begins with r_1^-.

Definition 4.1. Let $\mathrm{BNestFor}_n$ be the quotient of RESeq_n by the local relations

$$\mathsf{e}_i \mathsf{e}_j = \mathsf{e}_j \mathsf{e}_{i+1} \text{ for } i > j$$
$$\mathsf{r}_i^- \mathsf{r}_j^- = \mathsf{r}_j^- \mathsf{r}_{i+1}^- \text{ for } i \geq j, \qquad \mathsf{r}_i^+ \mathsf{r}_j^+ = \mathsf{r}_j^+ \mathsf{r}_{i+1}^+ \text{ for } i > j,$$
$$\mathsf{r}_i^+ \mathsf{r}_j^- = \mathsf{r}_j^- \mathsf{r}_{i+1}^+ \text{ for } i \geq j, \qquad \mathsf{r}_i^- \mathsf{r}_j^+ = \mathsf{r}_j^+ \mathsf{r}_{i+1}^- \text{ for } i > j,$$
$$\mathsf{e}_i \mathsf{r}_j^- = \mathsf{r}_j^- \mathsf{e}_{i+1} \text{ for } i \geq j, \qquad \mathsf{r}_i^- \mathsf{e}_j = \mathsf{e}_j \mathsf{r}_{i+1}^- \text{ for } i > j$$
$$\mathsf{e}_i \mathsf{r}_j^+ = \mathsf{r}_j^+ \mathsf{e}_{i+1} \text{ for } i > j, \qquad \mathsf{r}_i^+ \mathsf{e}_j = \mathsf{e}_j \mathsf{r}_{i+1}^+ \text{ for } i > j.$$

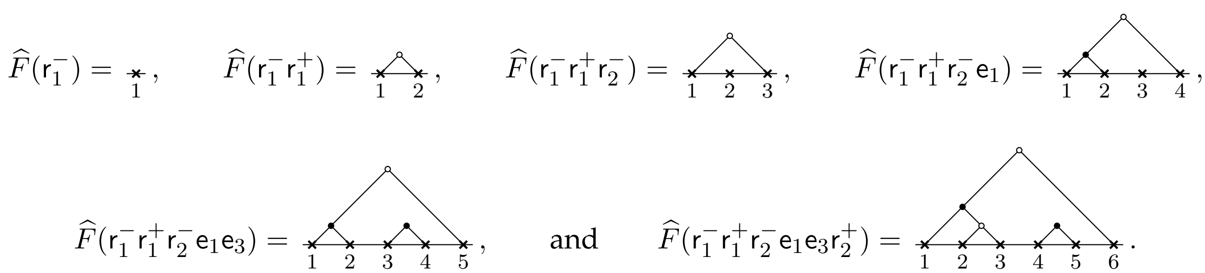
This is well-defined as the relations all preserve RESeq_n.

We recall from [6, Proposition 9.4] how the equivalence classes in $\mathrm{BNestFor}_n$ can be represented diagrammatically. A *bicolored nested forest* is a nested forest in which each internal node has been colored with either black ( ) or white ( ). We now define a map $\widehat{F}$ from RESeq_n to bicolored nested forests with support in $[n]$.

Definition 4.2. For $\Omega \in \mathrm{RESeq}_n$, the associated bicolored nested forest $\widehat{F}(\Omega)$ with support in $[n]$ is defined recursively by $\widehat{F}(\varnothing) = \varnothing$ and

(1) $\widehat{F}(\Omega \cdot \mathsf{r}_i^-)$ is obtained from $\widehat{F}(\Omega)$ by inserting a new tree with no internal nodes as a leaf between $i-1$ and i and relabeling leaves appropriately,
(2) $\widehat{F}(\Omega \cdot \mathsf{r}_i^+)$ is obtained from $\widehat{F}(\Omega)$ by replacing the ith leaf with a white node whose children are leaves and relabeling leaves appropriately, and
(3) $\widehat{F}(\Omega \cdot \mathsf{e}_i)$ is obtained from $\widehat{F}(\Omega)$ by replacing the ith leaf with a black node whose children are leaves and relabeling leaves appropriately.

Example 4.3. We demonstrate the recursive process for $\Omega = \mathsf{r}_1^-\mathsf{r}_1^+\mathsf{r}_2^-\mathsf{e}_1\mathsf{e}_3\mathsf{r}_2^+$:



Theorem 4.4. [6, Proposition 9.4] Two elements $\Omega, \Omega' \in \mathrm{RESeq}_n$ are in the same equivalence class of $\mathrm{BNestFor}_n$ if and only if $\widehat{F}(\Omega) = \widehat{F}(\Omega')$.

Going forward we identify elements of $\mathrm{BNestFor}_n$ with the associated bicolored nested forest, so that we can write $\widehat{K} = \widehat{H} \cdot \mathsf{x}_i$ to mean $\widehat{H} = \widehat{F}(\Omega)$ and $\widehat{K} = \widehat{F}(\Omega \cdot \mathsf{x}_i)$ for some $\Omega \in \mathrm{RESeq}_n$. The defining relations of $\mathrm{BNestFor}_n$ preserve $|\Omega|$, so it makes sense to discuss $|\widehat{F}|$. Diagrammatically, $|\widehat{F}|$ is the number of black nodes in $\widehat{F}$. See Example 4.3 for a bicolored nested forest built from an $\Omega \in \mathrm{RESeq}_6$ with $|\Omega| = 2$.

Definition 4.5. We conclude by identifying some distinguished subsets of $\mathrm{BNestFor}_n$ which will appear in later sections.

(1) Let $\mathsf{Forest}_n = \{\widehat{F} \in \mathrm{BNestFor}_n \mid \text{each } \Omega \in \widehat{F} \text{ has the form } (\mathsf{r}_1^-)^{n-k}\mathsf{e}_{i_1}\cdots\mathsf{e}_{i_k}\}$; elements of this set map to forests without white nodes in which no nesting occurs, so that the support of each tree is a contiguous interval of $[n]$.
(2) Let $\mathrm{Tree}_n = \{\widehat{F} \in \mathrm{BNestFor}_n \mid |\widehat{F}| = n-1\}$; elements of this set have representatives of the form $\mathsf{r}_1^-\mathsf{e}_{i_1}\cdots\mathsf{e}_{i_{n-1}}$ and map to singleton trees $(T_{\mathbf{c}})$ with entirely black nodes.

We note that in the above definition, $\mathrm{Tree}_n \subseteq \mathsf{Forest}_n \subseteq \mathrm{BNestFor}_n$.

4.2. **Combinatorics of building operations.**

Lemma 4.6. We have the relations

$$\mathbb{P}_j\mathbb{P}_i = \mathbb{P}_{i+1}\mathbb{P}_j \text{ for } i > j$$

$$\begin{aligned} \Psi_j^-\Psi_i^- &= \Psi_{i+1}^-\Psi_j^- \text{ for } i \geq j, & \Psi_j^+\Psi_i^+ &= \Psi_{i+1}^+\Psi_j^+ \text{ for } i > j,\\ \Psi_j^-\Psi_i^+ &= \Psi_{i+1}^+\Psi_j^- \text{ for } i \geq j, & \Psi_j^+\Psi_i^- &= \Psi_{i+1}^-\Psi_j^+ \text{ for } i > j,\\ \Psi_j^-\mathbb{P}_i &= \mathbb{P}_{i+1}\Psi_j^- \text{ for } i \geq j, & \mathbb{P}_j\Psi_i^- &= \Psi_{i+1}^-\mathbb{P}_j \text{ for } i > j\\ \Psi_j^+\mathbb{P}_i &= \mathbb{P}_{i+1}\Psi_j^+ \text{ for } i > j, & \mathbb{P}_j\Psi_i^+ &= \Psi_{i+1}^+\mathbb{P}_j \text{ for } i > j. \end{aligned}$$

Moreover, these relations hold when all $\mathbb{P}$'s are replaced by $\mathbb{G}$'s.

Proof. Since $\mathbb{P}_i = \Psi_i^- \sqcup \mathbb{G}_i \sqcup \Psi_i^+$, it suffices to check all of the above relations with $\mathbb{G}_i$ in place of $\mathbb{P}_i$. It suffices to show the stronger statement that these relations hold for the operations Ψ_i^-, Ψ_i^+, and $\mathbb{G}_i$ on subsets of matrices without considering equivalence classes mod B. These are straightforward so we omit their explicit verification. □

Remark 4.7. The relations in Lemma 4.6 are not exhaustive: in Lemma 8.2 we show that two additional relations are needed to describe all interactions between the building operations.

Example 4.8. The following computation witnesses the relation $\mathbb{G}_1\mathbb{G}_2 = \mathbb{G}_3\mathbb{G}_1$:

$$\begin{bmatrix} a & b & c\\ d & e & f\\ g & h & i \end{bmatrix} \xrightarrow{\mathbb{G}_2} \begin{bmatrix} a & b & 0 & c\\ 0 & + & 1 & 0\\ d & e & 0 & f\\ g & h & 0 & i \end{bmatrix} \xrightarrow{\mathbb{G}_1} \begin{bmatrix} + & 1 & 0 & 0 & 0\\ a & 0 & b & 0 & c\\ 0 & 0 & + & 1 & 0\\ d & 0 & e & 0 & f\\ g & 0 & h & 0 & i \end{bmatrix} \xleftarrow{\mathbb{G}_3} \begin{bmatrix} + & 1 & 0 & 0\\ a & 0 & b & c\\ d & 0 & e & f\\ g & 0 & h & i \end{bmatrix} \xleftarrow{\mathbb{G}_1} \begin{bmatrix} a & b & c\\ d & e & f\\ g & h & i \end{bmatrix}.$$

Since the relations of Lemma 4.6 are the opposites to the defining relations of $\mathrm{BNestFor}_n$, the following is well-defined.

Definition 4.9. For $\widehat{F} \in \mathrm{BNestFor}_n$, we define $X(\widehat{F}) \subset \mathrm{Fl}_n$ recursively: $X(\mathsf{r}_1^-) = \mathrm{Fl}_1$ is a single point, and

(1) $X(\widehat{F} \cdot \mathsf{r}_j^{\pm}) = \Psi_j^{\pm} X(\widehat{F})$
(2) $X(\widehat{F} \cdot \mathsf{e}_j) = \mathbb{P}_j X(\widehat{F})$.

Similarly, we define $X^\circ(\widehat{F}) \subset \mathrm{Fl}_n$ recursively: $X^\circ(\mathsf{r}_1^-) = \mathrm{Fl}_1$ is a single point, and

(1) $X^\circ(\widehat{F} \cdot \mathsf{r}_j^{\pm}) = \Psi_j^{\pm} X^\circ(\widehat{F})$
(2) $X^\circ(\widehat{F} \cdot \mathsf{e}_j) = \mathbb{G}_j X^\circ(\widehat{F})$.

Remark 4.10. We conclude by revisiting the distinguished families from Definition 4.5.

(1) In Theorem 7.1, we show that the $X(F)$ for $F \in \mathsf{Forest}_n$ are the *quasisymmetric Schubert cycles* of [38, Definition 6.2].

(2) For $T \in \mathrm{Tree}_n$, the $X(T)$ are the top-dimensional quasisymmetric Schubert cycles, which we use to construct QFl_n.

5. The Bott manifolds $X(\widehat{F})$

We now describe the toric structure of varieties $X(\widehat{F})$ and $X^\circ(\widehat{F})$ in Definition 4.9.

Theorem 5.1. For $\widehat{F} \in \mathrm{BNestFor}_m$, the variety $X^\circ(\widehat{F})$ is a torus orbit in Fl_m of dimension $|\widehat{F}|$. Furthermore, $X(\widehat{F})$ is the closure of the torus orbit $X^\circ(\widehat{F})$ in Fl_m.

We prove the theorem using the following. Recall the meaning of $\mathrm{Fl}_{m-1}^{\gamma_i}$ from Definition 3.2.

Proposition 5.2. Fix $\widehat{F} \in \mathrm{BNestFor}_{m-1}$, and consider $X(\widehat{F}) \subset \mathrm{Fl}_{m-1}^{\gamma_i}$ for some fixed $1 \leq i \leq m$.

(1) The map $\Psi_i^- : X(\widehat{F}) \to X(\widehat{F} \cdot \mathsf{r}_i^-)$ is a T_m-equivariant isomorphism.

(2) If $i < m$, the map $\Psi_i^+ : X(\widehat{F}) \to X(\widehat{F} \cdot \mathsf{r}_i^+)$ is a T_m-equivariant isomorphism.

(3) If $i < m$, there exists a T_m-equivariant isomorphism $X(\widehat{F} \cdot \mathsf{e}_i) \cong \mathsf{Proj}(\mathcal{F}_i/\mathcal{F}_{i-1} \oplus \mathbb{C}_{\chi_i})_{X(\widehat{F})}$. Furthermore, $X(\widehat{F} \cdot \mathsf{r}_i^-)$ and $X(\widehat{F} \cdot \mathsf{r}_i^+)$ are the T_m-equivariant sections $\mathsf{Proj}(\{0\} \oplus \mathbb{C}_{\chi_i})_{X(\widehat{F})}$ and $\mathsf{Proj}((\mathcal{F}_i/\mathcal{F}_{i-1}) \oplus \{0\})_{X(\widehat{F})}$ respectively.

Proof. These are immediate corollaries of Theorem 3.5 and Definition 4.9. □

Proof of Theorem 5.1. The dimension statement follows because each $\mathbb{G}_i$ defining $X^\circ(\widehat{F})$ increases the dimension by 1 and each $\Psi_i^\pm$ preserves the dimension. To show that $X^\circ(\widehat{F})$ is a torus orbit, we induct on m. By Proposition 5.2(1) and (2) if we know $X^\circ(\widehat{F})$ is a torus orbit then so is $X^\circ(\widehat{F} \cdot \mathsf{r}_i^\pm)$. Proposition 5.2 further implies that for any $Y \subset X(\widehat{F})$ we have that $\mathbb{G}_i Y \to Y$ is a $\mathbb{C}^*$-bundle obtained from $\mathbb{P}_i Y$ by removing the two distinguished sections. By applying this for $Y = X^\circ(\widehat{F})$ we conclude that T_m acts transitively on $\mathbb{G}_i X^\circ(\widehat{F}) = X^\circ(\widehat{F} \cdot \mathsf{e}_i)$ from the fact that $T_m \cong \gamma_i(T_m) \times \mathbb{C}^*_{\chi_i}$ with $\gamma_i(T_m)$ acting transitively on the base of the $\mathbb{C}^*$-bundle $X^\circ(\widehat{F} \cdot \mathsf{e}_i) \to X^\circ(\widehat{F})$ and $\mathbb{C}^*_{\chi_i}$ acting transitively on the fibers while fixing the base. Finally, for any $Z \subset \mathrm{Fl}_{m-1}$ we have $\Psi_i^\pm \overline{Z} = \overline{\Psi_i^\pm Z}$ and $\mathbb{P}_i \overline{Z} = \overline{\mathbb{P}_i Z} = \overline{\mathbb{G} Z}$, which shows by induction that $X(\widehat{F})$ is the closure of $X^\circ(\widehat{F})$. □

Recall that a *combinatorial cube* is a polytope whose face lattice is identical to that of a cube of the same dimension. If we have a sequence of varieties $X_1, X_2, \dots, X_m$ where $X_1 = \{\mathrm{pt}\}$ is a single point and $X_i = \mathsf{Proj}(\mathcal{L}_{i-1} \oplus \mathbb{C})_{X_{i-1}}$ for $\mathcal{L}_{i-1}$ a toric line bundle on X_{i-1}, then X_m is a smooth projective toric variety whose moment polytope is a combinatorial cube. The toric structure is defined recursively by saying if T_{i-1} is the torus for X_{i-1}, then X_i is a toric variety for $T_i := T_{i-1} \times \mathbb{C}^*$ where T_{i-1} acts trivially on the factor of $\mathbb{C}$ and $\mathbb{C}^*$ acts by scaling this $\mathbb{C}$ factor. The dense torus orbit can also be obtained by taking $\mathsf{Proj}(\mathcal{L}_{i-1} \oplus \mathbb{C})_{T_i}$ and removing the two distinguished sections. Such an X_m obtained in this way is called a *Bott manifold* (see [37]).

Definition 5.3. For $\Omega_1, \Omega_2 \in \mathrm{RESeq}_n$, we say $\Omega_1 \leq_{re} \Omega_2$ if Ω_1 is obtained from Ω_2 by switching some letters e_i to $\mathsf{r}_i^{\pm}$.

Proposition 5.4. For $\Omega_1 \mathsf{e}_i \Omega_2 \in \mathrm{RESeq}_n$,

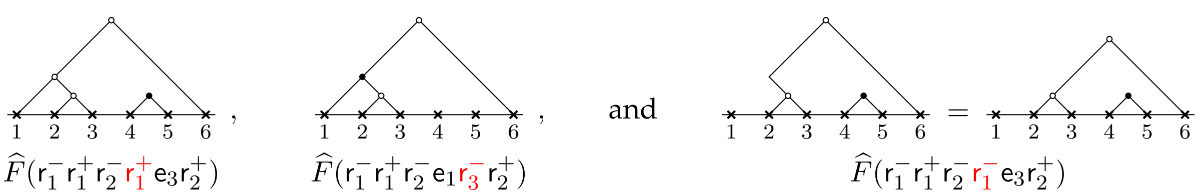
(1) $\widehat{F}(\Omega_1 \mathsf{r}_i^+ \Omega_2)$ is obtained from $\widehat{F}(\Omega_1 \mathsf{e}_i \Omega_2)$ by changing the black node associated to e_i to a white node and
(2) $\widehat{F}(\Omega_1 \mathsf{r}_i^- \Omega_2)$ is obtained from $\widehat{F}(\Omega_1 \mathsf{e}_i \Omega_2)$ by deleting the left edge of the black node associated to e_i and contracting the resulting node.

In particular, the relation $\leq_{re}$ descends to a partial order on $\mathrm{BNestFor}_n$.

Proof. This follows immediately from the way that bicolored nested forests are created recursively from sequences in RESeq_n. □

We define the operation of *left edge deletion* at a node in $\mathrm{IN}(\widehat{F})$ to be the operation described in Proposition 5.4(2). We sometimes emphasize that a forest is obtained by left edge deletion at v by drawing the contracted edge through the former location of v, as shown below.

Example 5.5. We demonstrate Proposition 5.4 using the nested forest $\widehat{F}(\mathsf{r}_1^- \mathsf{r}_1^+ \mathsf{r}_2^- \mathsf{e}_1 \mathsf{e}_3 \mathsf{r}_2^+)$ from Example 4.3. Three examples of $\leq_{re}$-smaller nested forests are:



Definition 5.6. For $\widehat{F} \in \mathrm{BNestFor}_n$, we define

$$\mathrm{Face}(\widehat{F}) = \{\widehat{G} : \widehat{G} \leq_{re} \widehat{F}\} \text{ and}$$
$$\mathrm{Vert}(\widehat{F}) = \{\widehat{G} : \widehat{G} \leq_{re} \widehat{F} \text{ and } |\widehat{G}| = 0\}.$$

Restricting $\leq_{re}$ to $\mathrm{Face}(\widehat{F})$ gives the face poset of a $|\widehat{F}|$-dimensional cube. Indeed, the choice of whether to change each black node white or to perform left edge deletion is equivalent to choosing one from a pair of opposite faces. Figure 4 shows in the left panel two such cubes and the elements of $\mathrm{Face}(\widehat{F})$ associated with each face. We now give this interpretation a geometric meaning.

Theorem 5.7. For $\widehat{F} \in \mathrm{BNestFor}_n$, $X(\widehat{F}) \subset \mathrm{Fl}_n$ is a Bott manifold of dimension $|\widehat{F}|$ with dense torus orbit $X^\circ(\widehat{F})$. The distinct torus orbits of $X(\widehat{F})$ are given by $\{X^\circ(\widehat{G}) : \widehat{G} \in \mathrm{Face}(\widehat{F})\}$, and the distinct torus orbit closures are given by $\{X(\widehat{G}) : \widehat{G} \in \mathrm{Face}(\widehat{F})\}$.

Proof. By Proposition 5.2, any $\Omega \in \mathrm{RESeq}_n$ representing $\widehat{F}$ induces a Bott manifold structure on $X(\widehat{F})$ with dense torus orbit $X^\circ(\widehat{F})$ as described above, and the dimension is $|\widehat{F}|$ by Theorem 5.1.

As Theorem 5.1 shows the closure of the torus orbit $X^\circ(\widehat{G})$ is $X(\widehat{G})$, it remains to describe the distinct torus-orbit closures. We proceeed inductively. Suppose the result holds for all $\widehat{F} \in \mathrm{BNestFor}_{n-1}$; we aim to prove it for each $\widehat{F} \cdot \mathsf{x}_i \in \mathrm{BNestFor}_n$.

First, note that $\Psi_i^\pm : X(\widehat{F}) \to X(\widehat{F} \cdot \mathsf{r}_i^\pm)$ is an isomorphism and moreover induces a bijection between torus orbit closures via

$$X(\widehat{G}) \mapsto \Psi_i^\pm X(\widehat{G}) = X(\widehat{G} \cdot \mathsf{r}_i^\pm),$$

so we conclude the result for $\widehat{F} \cdot \mathsf{r}_i^\pm$ as $\mathrm{Face}(\widehat{F} \cdot \mathsf{r}_i^\pm) = \{\widehat{G} \cdot \mathsf{r}_i^\pm : \widehat{G} \in \mathrm{Face}(\widehat{F})\}$. Consider now the $\mathbb{P}^1$-bundle $X(\widehat{F} \cdot \mathsf{e}_i) \to X(\widehat{F})$. For any projective toric variety X with a toric line bundle $\mathcal{L}$, the torus orbit closures on the toric variety $\mathsf{Proj}(\mathcal{L} \oplus \mathbb{C})$ are given by the $\mathbb{P}^1$-bundles over the torus orbit closures in X, together with the images of the torus orbit closures in X in the two disjoint sections of the split projective bundle. Consequently by Proposition 5.2 the torus orbit closures in $X(\widehat{F})$ are given by

$$\bigsqcup_{\widehat{G} \in \mathrm{Face}(\widehat{F})} \{\Psi_i^- X(\widehat{G}), \Psi_i^+ X(\widehat{G}), \mathbb{P}_i X(\widehat{G})\} = \bigsqcup_{\widehat{G} \in \mathrm{Face}(\widehat{F})} \{X(\widehat{G} \cdot \mathsf{r}_i^-), X(\widehat{G} \cdot \mathsf{r}_i^+), X(\widehat{G} \cdot \mathsf{e}_i)\}$$

and we conclude as $\mathrm{Face}(\widehat{F} \cdot \mathsf{e}_i) = \bigcup_{\widehat{G} \in \mathrm{Face}(\widehat{F})} \{\widehat{G} \cdot \mathsf{r}_i^-, \widehat{G} \cdot \mathsf{r}_i^+, \widehat{G} \cdot \mathsf{e}_i\}$. □

6. Torus fixed points of $X(\widehat{F})$

In this section we describe the combinatorics of the fixed point sets

$$I_{\widehat{F}} := X(\widehat{F})^T \subset S_n.$$

As the following theorem shows, elements of $I_{\widehat{F}}$ always lie in NC_n.

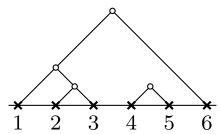
Theorem 6.1. For any $\widehat{F} \in \mathrm{BNestFor}_n$, we have

$$I_{\widehat{F}} = \{\mathrm{NCPerm}(\widehat{G}) \mid \widehat{G} \in \mathrm{Vert}(\widehat{F})\} \subset \mathrm{NC}_n\,.$$

In particular, if $\widehat{G} \in \mathrm{BNestFor}_n$ has $|\widehat{G}| = 0$, then $X_{\widehat{G}} = \{\mathrm{NCPerm}(\widehat{G})\}$.

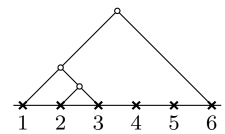
Example 6.2. We apply the theorem to the fixed point set for the nested forest $\widehat{F}(\mathsf{r}_1^-\mathsf{r}_1^+\mathsf{r}_2^-\mathsf{e}_1\mathsf{e}_3\mathsf{r}_2^+)$ from Examples 4.3 and 5.5. We have:

$(6321)(54) \leftrightarrow$



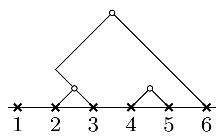
$\widehat{F}(\mathsf{r}_1^-\mathsf{r}_1^+\mathsf{r}_2^-\mathsf{r}_1^+\mathsf{r}_3^+\mathsf{r}_2^+)$

$(6321) \leftrightarrow$



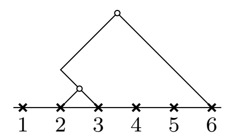
$\widehat{F}(\mathsf{r}_1^-\mathsf{r}_1^+\mathsf{r}_2^-\mathsf{r}_1^+\mathsf{r}_3^-\mathsf{r}_2^+)$

$(632)(54) \leftrightarrow$



$\widehat{F}(\mathsf{r}_1^-\mathsf{r}_1^+\mathsf{r}_2^-\mathsf{r}_1^-\mathsf{r}_3^+\mathsf{r}_2^+)$

$(632) \leftrightarrow$



$\widehat{F}(\mathsf{r}_1^-\mathsf{r}_1^+\mathsf{r}_2^-\mathsf{r}_1^-\mathsf{r}_3^-\mathsf{r}_2^+)$

Going forward, we will abuse notation and treat Ψ_i^- and Ψ_i^+ as maps on permutations rather than just permutation matrices. These are given by the group homomorphism $\Psi_i^- : S_{n-1} \hookrightarrow S_n$, induced by the increasing injection $\{1, \ldots, n-1\} \hookrightarrow \{1, \ldots, i-1, i+1, \ldots, n\}$ onto the subgroup of S_n with $u(i) = i$, and $\Psi_i^+ w = (\Psi_i^- w)s_i$.

Proof of Theorem 6.1. We know from Theorem 5.7 that the fixed point set $I_{\widehat{F}}$ is given by the points $X_{\widehat{G}}$ such that $\widehat{G} \in \mathrm{Vert}(\widehat{F})$. Therefore we need only verify the the statement when $|\widehat{G}| = 0$, meaning that $\widehat{G}$ is represented by a sequence of $\mathsf{r}_i^\pm$.

For $n = 1$ we have $\widehat{G} = \mathsf{r}_1^-$ and $X_{\widehat{G}} = \{\mathrm{id}_{S_1}\} = \{\mathrm{NCPerm}(\widehat{G})\}$. For $n > 1$, by the recursive construction of $X_{\widehat{G}}$ in Definition 4.9 it suffices to show that

$$\mathrm{NCPerm}(\widehat{F} \cdot \mathsf{r}_i^-) = \Psi_i^-\big(\mathrm{NCPerm}(\widehat{F})\big) \quad \text{and} \quad \mathrm{NCPerm}(\widehat{F} \cdot \mathsf{r}_i^+) = \Psi_i^+\big(\mathrm{NCPerm}(\widehat{F})\big). \tag{6.1}$$

We check these by comparing the definition of $\widehat{F} \cdot \mathsf{r}_i^\pm$ with that of $\Psi_i^\pm$: the former is straightforward, and the latter follows from that fact that for a backwards cycle C on a set A containing i but not $i+1$, the product $C(i\; i+1)$ is the backwards cycle on $A \sqcup \{i+1\}$. □

We now characterize $I_{\widehat{F}}$ in a manner that connects nested forests to factorizations of noncrossing partitions, as mentioned in Remark 2.2. For each internal node $v \in \mathrm{IN}(\widehat{F})$, let τ_v denote the transposition $(i\, j)$ of the rightmost leaf descendant i of v_L and the rightmost leaf descendant j of v, as shown in Figure 2.

Say that a total order on $\mathrm{IN}(\widehat{F})$ is a *linear extension* of $\widehat{F}$ if each $v \in \mathrm{IN}(\widehat{F})$ is preceded by all of its ancestors. As we now explain, the τ_v have the property that any two product orders on

$$\prod_{v \in \mathrm{IN}(\widehat{F})} \tau_v \tag{6.2}$$

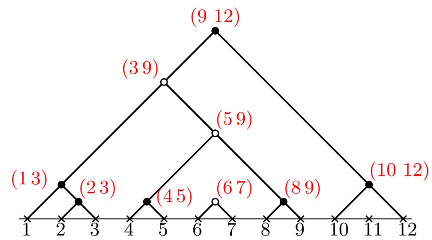


FIGURE 2. A bicolored nested forest with internal nodes labeled by transpositions τ_v

which are linear extensions of $\widehat{F}$ are related entirely by commuting factors past one another. Indeed, any two linear extensions are related by repeatedly swapping adjacent elements v, v' which are not ancestor and descendant, and for such a pair the transpositions $\tau_v = (i,j)$ and $\tau_{v'} = (k,\ell)$ are disjoint and hence commute.

Thus (6.2) is unambiguously defined as the product of the vertices of $\widehat{F}$ taken in any order dictated by a linear extension of $\widehat{F}$. For the same reasons, one can define the product $\prod_{v\in S}\tau_v$ for any $S \subset \mathrm{IN}(\widehat{F})$.

Theorem 6.3. For each $\widehat{F} \in \mathrm{BNestFor}_n$, we have

$$I_{\widehat{F}} = \left\{ \prod_{v\in \mathrm{IN}(\widehat{F})\setminus S} \tau_v \mid S \subseteq \{\text{black vertices of } \widehat{F}\} \right\}.$$

In particular, $\mathrm{NCPerm}(\widehat{F}) = \prod_{v\in\mathrm{IN}(\widehat{F})}\tau_v$.

Proof. We first show the claim holds when $\widehat{F}$ has only white nodes, so that $I_{\widehat{F}} = \{\mathrm{NCPerm}(\widehat{F})\}$ by Theorem 6.1 and the claim amounts to $\mathrm{NCPerm}(\widehat{F}) = \prod_{v\in\mathrm{IN}(\widehat{F})}\tau_v$. We proceed by induction on $|\mathrm{IN}(\widehat{F})|$. If $\widehat{F}$ has no nodes, then the claim holds trivially. Otherwise let v_0 be the root of a tree in $\widehat{F}$ that is not nested under any other tree, so that v_0 is the first element in some linear extension of $\widehat{F}$. Let $\widehat{G}$ be the forest obtained by deleting v_0 and all incident edges. As $|\mathrm{IN}(\widehat{G})| < |\mathrm{IN}(\widehat{F})|$, our inductive hypothesis guarantees that $\mathrm{NCPerm}(\widehat{G}) = \prod_{v\in\mathrm{IN}(\widehat{G})}\tau_v$ is the unique element of $I_{\widehat{G}}$. Moreover, $\mathrm{NCPerm}(\widehat{G}) = \prod_{v\in\mathrm{IN}(\widehat{F})-\{v_0\}}\tau_v$, since for $v \in \mathrm{IN}(\widehat{G})$ the value of τ_v does not depend on whether we consider v as a node of $\widehat{F}$ or $\widehat{G}$. It therefore suffices to show that $\tau_{v_0}\,\mathrm{NCPerm}(\widehat{F}) = \mathrm{NCPerm}(\widehat{G})$, which follows from that fact that if c_A and c_B are backwards cycles with $\max A < \min B$, then $(\max A, \max B)c_Ac_B = c_{A\sqcup B}$.

We now consider the general case of $\widehat{F} \in \mathrm{BNestFor}_n$. By Theorem 6.1, we have that $I_{\widehat{F}} = \{\mathrm{NCPerm}(\widehat{G}) \mid \widehat{G} \in \mathrm{Vert}(\widehat{F})\}$. Further, each $\widehat{G} \in \mathrm{Vert}(\widehat{F})$ is obtained precisely by performing left edge deletion at some subset S of black nodes from $\widehat{F}$ and changing the remaining black nodes to white nodes. Thus applying the special case proved above to each $\widehat{G} \in \mathrm{Vert}(\widehat{F})$, we see that the theorem holds for $\widehat{F}$. $\square$

Recall that the moment polytope for each $X(\widehat{F})$ is a combinatorial cube with vertices corresponding to the fixed point set $I_{\widehat{F}}$.

Corollary 6.4. For $\widehat{F} \in \mathrm{BNestFor}_n$, $I_{\widehat{F}}$ is an induced Boolean sublattice in the Kreweras order, and $w \mapsto w \cdot \lambda$ maps $I_{\widehat{F}}$ onto the vertices of the moment polytope of $X(\widehat{F})$ for the dominant weight λ in such a way that the Hasse diagram of $I_{\widehat{F}}$ is identified with the 1-skeleton of the polytope.

Proof. By Theorem 6.1, $I_{\widehat{F}} \subseteq \mathrm{NC}_n$. By [30, Lemma 2.11], this is an induced Boolean sublattice of the Kreweras order. Moreover, using the description of $I_{\widehat{F}}$ given in Proposition 6.3, the edges of the moment polytope connect exactly those pairs of elements of $I_{\widehat{F}}$ which differ by the inclusion of a single τ_v. □

7. Translated Richardsons and polypositroids

In this section we relate our $X(\widehat{F})$ to certain Richardson varieties previously studied in [39, 40] and the quasisymmetric Schubert cycles of [38]; see Remark 7.3. We then use this connection to describe the moment polytope of each $X(\widehat{F})$ as a polypositroid [33].

Recall that a Richardson variety is the intersection $X_w^v = X^v \cap X_w$, which is nonempty if and only if $w \leq v$. It is straightforward to see that $w \leq w\boldsymbol{c}$ if and only if $w \in S_{n-1}$, and in this case the Richardson variety $X_w^{w\boldsymbol{c}}$ is known to be an $(n-1)$-dimensional toric variety [38]. Consider now the image of each $X_w^{w\boldsymbol{c}}$ under left multiplication by w^{-1}.

Theorem 7.1. We have

$$\{X(T) \mid T \in \mathrm{Tree}_n\} = \{w^{-1}X_w^{w\boldsymbol{c}} \mid w \in S_{n-1}\}.$$

In particular, there are Cat_{n-1} distinct translated Richardson varieties, one for each $T \in \mathrm{Tree}_n$.

For $T \in \mathrm{Tree}_n$ recall that we have

$$X(T) = \mathbb{P}_{i_{n-1}}\mathbb{P}_{i_{n-2}} \cdots \mathbb{P}_{i_1}\{\mathrm{pt}\} \subset \mathrm{Fl}_n$$

for any sequence $\mathsf{r}_1^-\mathsf{e}_{i_1}\mathsf{e}_{i_2}\cdots\mathsf{e}_{i_{n-1}}$ associated to T. For any m, let ε_i be the map from $S_{m-1} \to S_m$ which, in one-line notation, inserts a 1 into the ith position and increases the remaining numbers by 1. For example $\varepsilon_3 15684237 = 261795348$. Note that this coincides with the map $\Psi_{1,i}^-$ restricted to permutation matrices. However, unlike $\Psi_{1,i}^-$, we shall reserve ε_i for use on permutations only.

Lemma 7.2. Let $Y(u,v) := u^{-1}X_u^v$. Then $\mathbb{P}_iY(u,v) = Y(\varepsilon_i u, \varepsilon_{i+1}v)$.

Proof. In [38, §4] it was shown that $\pi_i^{-1}\pi_i\Psi_{1,i}X_u^v = X_{\varepsilon_i u}^{\varepsilon_{i+1}v}$. Since π_i is equivariant with respect to left multiplication, we get the following sequence of equalities

$$\begin{aligned}Y(\varepsilon_i u, \varepsilon_{i+1}v) =& (\varepsilon_i u)^{-1}X_{\varepsilon_i u}^{\varepsilon_{i+1}v} = (\varepsilon_i u)^{-1}\pi_i^{-1}\pi_i\Psi_{1,i}X_u^v \\ =& \pi_i^{-1}\pi_i(\varepsilon_i u)^{-1}\Psi_{1,i}X_u^v = \pi_i^{-1}\pi_i\Psi_i^- u^{-1}X_u^v = \mathbb{P}_iY(u,v),\end{aligned}$$

where we use the fact that $(\varepsilon_i u)^{-1}\Psi_{1,i} = \Psi_i^- u^{-1}$. □

Proof of Theorem 7.1. Suppose first that $T \in \mathrm{Tree}_n$ is associated to the sequence $\mathsf{r}_1^- \mathsf{e}_{i_1} \cdots \mathsf{e}_{i_{n-1}} \in \mathrm{RESeq}_n$. Let $v = \varepsilon_{i_{n-1}+1} \cdots \varepsilon_{i_1+1} \mathrm{id}_{S_1}$ and $u = \varepsilon_{i_{n-1}} \cdots \varepsilon_{i_1} \mathrm{id}_{S_1}$. By [38, Proposition B.4(2)] we have $v = u\boldsymbol{c}$ and $u(n) = n$. By repeated applications of Lemma 7.2 we have $X(T) = u^{-1}X_u^v = u^{-1}X_u^{u\boldsymbol{c}}$.

Conversely given $u \in S_{n-1}$, by induction one can show that $u = \varepsilon_{i_{n-1}} \cdots \varepsilon_{i_1} \mathrm{id}_{S_1}$ for some sequence $i_1, \ldots, i_{n-1}$ with $i_j \le j$, and the tree $T \in \mathrm{Tree}_n$ associated to the sequence $\mathsf{r}_1^- \mathsf{e}_{i_1} \cdots \mathsf{e}_{i_{n-1}} \in \mathrm{RESeq}_n$ has $X(T) = u^{-1}X_u^{u\boldsymbol{c}}$.

Showing that there are Cat_{n-1}-many distinct translated Richardson varieties amounts to showing that the $X(T)$ for $T \in \mathrm{Tree}_n$ are distinct. This follows either from the identification with the quasisymmetric Schubert cycles of [38] as described in Remark 7.3. A second proof can be obtained using the results of Section 8: we characterize when two forests in $\mathrm{BNestFor}_n$ produce the same torus-orbit closure, and in particular show that this does not occur for any two trees. □

Remark 7.3. In [38] certain translates $u^{-1}X_u^v$ of Richardson varieties called *quasisymmetric Schubert cycles* were defined for any forest $F \in \mathsf{Forest}_n$. The description of $X(T)$ for $T = \mathsf{r}_1^- \mathsf{e}_{i_1} \cdots \mathsf{e}_{i_{n-1}} \in \mathrm{Tree}_n$ as a translated Richardson variety is exactly the same as the description of the quasisymmetric Schubert cycle associated to T in [38] (matching the notation, T would have been described as associated to a sequence $\mathsf{r}_1 \mathsf{t}_{i_1} \cdots \mathsf{t}_{i_{n-1}} \in \mathrm{RTSeq}_n$). More generally for $F \in \mathsf{Forest}_n$, applying Lemma 7.2 recursively to $X((\mathsf{r}_1^-)^{n-k}) = X_{\mathrm{id}_{S_{n-k}}}^{\mathrm{id}_{S_{n-k}}} \subset \mathrm{Fl}_{n-k}$ realizes each $X(F)$ as the quasisymmetric Schubert cycle associated to F.

Recall that faces of Bruhat interval polytopes are themselves Bruhat interval polytopes. Since the torus orbit closures in a fixed toric Richardson variety correspond to faces of the associated Bruhat interval polytope, we infer [44, Proposition 7.12] that every torus orbit closure in a toric Richardson variety is also a toric Richardson variety. This yields the following corollary.

Corollary 7.4. Every $X(\widehat{F})$ is the left-translate of a toric Richardson variety by an element of S_n.

Remark 7.5. Theorem 7.1 provides an alternate perspective on the presence of noncrossing partitions arising as torus fixed points of $X(\widehat{F})$. Indeed, consider a fixed point $u \in I_T$ for $T \in \mathrm{Tree}_n$. Using the fact that $X(T)$ is a translated Richardson variety we will show that $u \in \mathrm{NC}_n$. To begin, choose a maximal chain from w to $w\boldsymbol{c}$ containing wu in the Bruhat order. Left translation by w^{-1} gives a factorization of $\boldsymbol{c}$ as a product of transpositions $\tau_1 \cdots \tau_{n-1}$ with $u = \tau_1 \cdots \tau_m$ for $m = \ell(u) - \ell(w)$. Hence $u \in \mathrm{NC}_n$ by the characterization of NC_n due to Biane [9].

The description of each $X(T)$ as a translated Richardson variety also leads to a description of the defining hyperplanes for the moment polytope of $X(T)$. Recall the canonical labelling of T given in Section 2.3 and suppose that i is the label of an internal node. Let $\mathrm{Right}(T, i)$ (resp. $\mathrm{Left}(T, i)$) denote the set containing i and the labels of each internal node in the right (resp. left) subtree of i. These sets are necessarily intervals of $\mathbb{N}$ containing i.

Recall that a polytope is a generalized permutahedron if its edges are parallel to vectors of the form $e_i - e_j$ for distinct i and j, and that a polytope is an alcoved polytope if its facet normals are parallel to vectors of the form $e_i + e_{i+1} + \cdots + e_j$ for $i \leq j$. Following [33], a *polypositroid* is a polytope that is both a generalized permutahedron and an alcoved polytope. The following is essentially the content of [40, Remark 6.11].

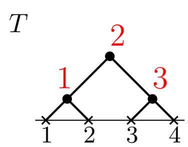
Theorem 7.6. The moment polytope of $X(T)$ in the hyperplane $z_1 + \cdots + z_n = \lambda_1 + \cdots + \lambda_n$ is the polypositroid defined by the following inequalities: for each $i \in \{1, \ldots, n-1\}$ we have

$$\sum_{j \in \mathrm{Right}(T,i)} z_j \geq \sum_{j \in \mathrm{Right}(T,i)} \lambda_{j+1}, \quad \text{and} \quad \sum_{j \in \mathrm{Left}(T,i)} z_j \leq \sum_{j \in \mathrm{Left}(T,i)} \lambda_j.$$

As faces of polypositroids are polypositroids, we have in fact showed that the moment polytope of every $X(\widehat{F})$ is a polypositroid. By Theorem B (proved in Section 10.1), the $X(\widehat{F})$ are the only irreducible subvarieties of Fl_n whose torus fixed points are contained in NC_n. Thus the moment polytopes of our $X(\widehat{F})$ account for all flag matroid polytopes which have vertices in NC_n and moreover have "geometric origin."

Conjecture 7.7. Every polypositroid—and more generally every flag matroid polytope—whose vertices are contained in NC_n is the moment polytope of some $X(\widehat{F})$.

Example 7.8. Figure 3 (left) depicts $T \in \mathrm{Tree}_4$ with the canonical labeling of $\mathrm{IN}(T)$. On the right is the facet description inside the hyperplane $z_1 + z_2 + z_3 + z_4 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$.



i	1	2	3
$\mathrm{Left}(T,i)$	$\{1\}$	$\{1,2\}$	$\{3\}$
$\mathrm{Right}(T,i)$	$\{1\}$	$\{2,3\}$	$\{3\}$
Inequality (L)	$z_1 \leq \lambda_1$	$z_1 + z_2 \leq \lambda_1 + \lambda_2$	$z_3 \leq \lambda_3$
Inequality (R)	$z_1 \geq \lambda_2$	$z_2 + z_3 \geq \lambda_3 + \lambda_4$	$z_3 \geq \lambda_4$

FIGURE 3. The facet inequalities for a particular moment polytope

By Theorem 6.3 and Corollary 6.4, the set of vertices of this polytope is given by

$$\{u \cdot \lambda \mid u \text{ a subword of the product } (2,4)(3,4)(1,2)\}.$$

8. THE QUASISYMMETRIC FLAG VARIETY

We define the *quasisymmetric flag variety* as the toric complex

$$\mathrm{QFl}_n \coloneqq \bigcup_{T \in \mathrm{Tree}_n} X(T) \subset \mathrm{Fl}_n.$$

The union defining QFl_n is not disjoint as there is some overlap between distinct $X(T)$, $X(T')$. In this section we characterize this overlap in terms of the torus fixed point sets $I_{\widehat{F}}$ described in Section 6.

Theorem 8.1. For $\widehat{F}, \widehat{G} \in \mathrm{BNestFor}_n$ we have $X(\widehat{G}) \subset X(\widehat{F})$ if and only if $I_{\widehat{G}} \subset I_{\widehat{F}}$.

The criterion therein provides a combinatorial model for the toric structure of QFl_n, see Figure 4. First take the disjoint union of the moment polytopes for each $X(T)$, which we showed in Section 5 are $(n-1)$-dimensional combinatorial cubes. Then create a polyhedral complex $\mathrm{Complex}(\mathrm{QFl}_n)$ by identifying faces from distinct polytopes that are equal in the sense that they share the same set of vertices. After identification, the faces of $\mathrm{Complex}(\mathrm{QFl}_n)$ are then in bijection with the distinct torus orbit closures in QFl_n.

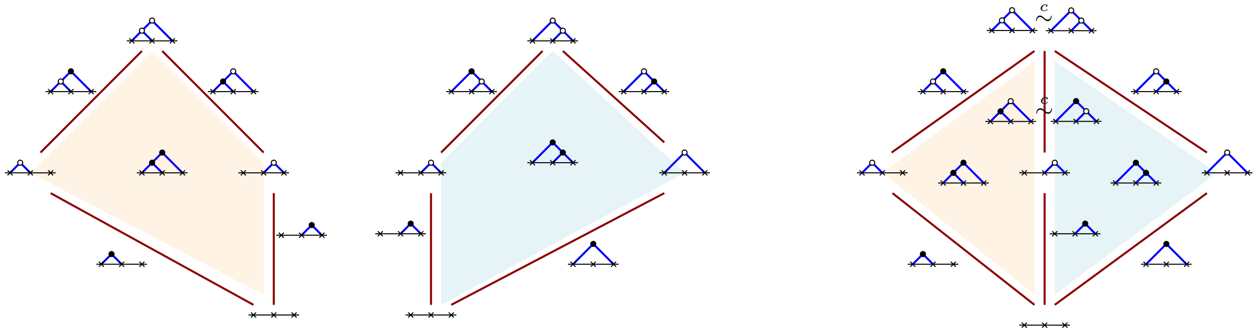


FIGURE 4. The combinatorial cubes corresponding to the toric orbit closures in each of the two components of QFl_3 (left) and the global complex $\mathrm{Complex}(\mathrm{QFl}_3)$ encoding the inclusion order on all toric closures $X(\widehat{F})$ in QFl_3 (right).

We prove Theorem 8.1 at the end of Section 8.2 after introducing an important equivalence relation on $\mathrm{BNestFor}_n$ in Section 8.1. Section 8.3 contains further enumerative and structural results about $\mathrm{Complex}(\mathrm{QFl}_n)$.

8.1. **Colored Tamari equivalence and normal forms.** Every torus orbit closure in QFl_n is by definition contained in $X(T)$ for some $T \in \mathrm{Tree}_n$. In Section 5, we showed every torus orbit closure in $X(T)$ is of the form $X(\widehat{F})$ for a bicolored nested forest $\widehat{F} \leq_{re} T$. Thus the torus orbit closures in QFl_n can be parametrized by $\mathrm{BNestFor}_n$. However, there is some redundancy in this parametrization as is apparent from Figure 4. This is explained by the following two additional relations that the building operations satisfy.

Lemma 8.2. For all $1 \leq i < n$ we have the relations

$$\Psi^+_{i+1}\mathbb{P}_i = \mathbb{P}_i\Psi^+_i \quad \text{and} \quad \Psi^+_{i+1}\Psi^+_i = \Psi^+_i\Psi^+_i.$$

Proof. Both relations can be verified with elementary matrix computations. □

In the correspondence between words in RTSeq_n and compositions of building operations, these relations correspond to $\mathsf{e}_i\mathsf{r}^+_{i+1} = \mathsf{r}^+_i\mathsf{e}_i$ and $\mathsf{r}^+_i\mathsf{r}^+_{i+1} = \mathsf{r}^+_i\mathsf{r}^+_i$. Considering the relations at the level of binary forests leads to the following definition.

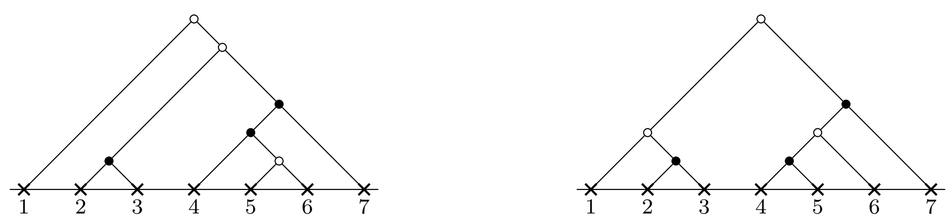
Definition 8.3. We say that $\widehat{F}, \widehat{G} \in \mathrm{BNestFor}_n$ are *colored Tamari equivalent*, denoted by $\widehat{F} \overset{c}{\sim} \widehat{G}$, if one can be transformed into the other by a sequence of *colored Tamari rotations* shown below.

By the preceding discussion, we get the following result.

Proposition 8.4. If $\widehat{F}, \widehat{G} \in \mathrm{BNestFor}$ satisfy $\widehat{F} \overset{c}{\sim} \widehat{G}$, then $X(\widehat{F}) = X(\widehat{G})$, and in particular $I_{\widehat{F}} = I_{\widehat{G}}$.

Definition 8.5. We say that $\widehat{F} \in \mathrm{BNestFor}_n$ is in *normal form* if every right child in $\mathrm{IN}(\widehat{F})$ is a black node. Let $\mathrm{BNestFor}_n^{\mathrm{nf}}$ be the set of bicolored nested forests that are in normal form.

Every element of $\mathrm{BNestFor}_n$ can be transformed to some element of $\mathrm{BNestFor}_n^{\mathrm{nf}}$ by applying colored Tamari rotations repeatedly. We will prove that this normal form is unique in the next section. Figure 5 depicts a bicolored tree as well as its colored Tamari equivalent normal form.



$(1\,7)\,(3\,7)\,\mathbf{(6\,7)}\,\mathbf{(4\,6)}\,(5\,6)\,\mathbf{(2\,3)}$ $\qquad$ $(3\,7)\,(1\,3)\,\mathbf{(6\,7)}\,(5\,6)\,\mathbf{(4\,5)}\,\mathbf{(2\,3)}$

FIGURE 5. A bicolored tree which is not in normal form (left), its Tamari-equivalent normal form (right), and the associated factorizations of $\boldsymbol{c}$ for each tree (below).

Remark 8.6. Proposition 6.3 relates bicolored nested forests to factorizations of noncrossing partitions, where the colored Tamari equivalence allows certain relations $(i\,k)(j\,k) = (j\,k)(i\,j)$.

8.2. **The Bruhat maximal element of $X(\widehat{F})$ and uniqueness of normal form.** We will need to understand the Bruhat order on $I_{\widehat{F}}$. Since $X(\widehat{F})$ is a torus orbit closure in Fl_n, we have that $I_{\widehat{F}}$ is a flag matroid [22]. Hence we have the following fact.

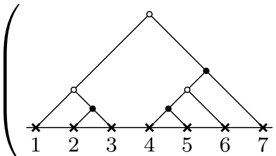
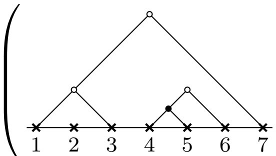
Fact 8.7 ([16, § 1.9])**.** For any $\widehat{F} \in \mathrm{BNestFor}_n$, $I_{\widehat{F}}$ has a unique Bruhat-maximum element. This element is characterized by the property that all adjacent vertices are lower in the Bruhat order.

In order to describe this distinguished element, we define a new map. For $\widehat{F}$ in $\mathrm{BNestFor}_n^{\mathrm{nf}}$, let $\mathrm{RC}(\widehat{F})$ denote the set of internal nodes that are right children. By definition of normal form, we note that all nodes in $\mathrm{RC}(\widehat{F})$ are black. We define a map

$$\begin{array}{rccc} \mathrm{ForToNC}: & \mathrm{BNestFor}_n^{\mathrm{nf}} & \to & \mathrm{NC}_n \\ & \widehat{F} & \mapsto & \mathrm{NCPerm}\left(\widehat{F} \setminus \mathrm{RC}(\widehat{F})\right) \end{array}$$

where $\widehat{F} \setminus \mathrm{RC}(\widehat{F})$ denotes the bicolored nested forest obtained by left edge deletion for each node $v \in \mathrm{RC}(\widehat{F})$ and contracting it in the sense of Proposition 5.4. This is a natural extension of the map on Forest_n which we defined in [6, Section 7.2] using the same notation.

Example 8.8. We have

ForToNC () = NCPerm () $= (731)(654)(2)$.

Lemma 8.9. For $\widehat{F} = \widehat{G} \cdot \mathsf{x}_i \in \mathrm{BNestFor}_n^{\mathrm{nf}}$ with $\widehat{G} \in \mathrm{BNestFor}_{n-1}^{\mathrm{nf}}$, we have

$$\mathrm{ForToNC}(\widehat{F}) = \begin{cases} \Psi_i^- \,\mathrm{ForToNC}(\widehat{G}) & \text{if } \mathsf{x}_i = \mathsf{r}_i^- \text{ or leaf } i \text{ is a right child in } \widehat{G} \\ \Psi_i^+ \,\mathrm{ForToNC}(\widehat{G}) & \text{otherwise.} \end{cases}$$

Proof. From the definitions we immediately verify

$$\widehat{F} \setminus \mathrm{RC}(\widehat{F}) = \begin{cases} (\widehat{G} \setminus \mathrm{RC}(\widehat{G})) \cdot \mathsf{r}_i^- & \text{if } \mathsf{x}_i = \mathsf{r}_i^- \text{ or leaf } i \text{ is a right child in } \widehat{G} \\ (\widehat{G} \setminus \mathrm{RC}(\widehat{G})) \cdot \mathsf{x}_i & \text{otherwise} \end{cases}$$

after which the formulas (6.1) complete the proof. □

Proposition 8.10. If $\widehat{F} \in \mathrm{BNestFor}_n^{\mathrm{nf}}$, then $\mathrm{ForToNC}(\widehat{F})$ is the Bruhat-maximum element of $I_{\widehat{F}}$.

Proof. We proceed by induction on n. For $n = 1$, $\mathrm{ForToNC}(\widehat{F})$ is the only element of $I_{\widehat{F}}$. For $n > 1$, we have $\widehat{F} = \widehat{G} \cdot \mathsf{x}_i$ for $\mathsf{x}_i \in \{\mathsf{r}_i^{\pm}, \mathsf{e}_i\}$ and $\widehat{G} \in \mathrm{BNestFor}_{n-1}^{\mathrm{nf}}$. Setting $w = \mathrm{ForToNC}(\widehat{F})$ and $u = \mathrm{ForToNC}(\widehat{G})$, Lemma 8.9 states that $w = \Psi_i^{\epsilon}(u)$ for $\epsilon \in \{+, -\}$.

For $\mathsf{x}_i = \mathsf{r}_i^{\epsilon}$, we note that both Ψ_i^- and Ψ_i^+ preserve the Bruhat order. This follows, for instance, from the tableau criterion [14, Theorem 2.6.3]. Thus resorting to our inductive hypothesis on u, we have that $w = \Psi_i^{\epsilon}(u)$ is the Bruhat maximum of $I_{\widehat{F}} = \Psi_i^{\epsilon}(I_{\widehat{G}})$.

Now suppose that $\mathsf{x}_i = \mathsf{e}_i$. By Fact 8.7, it suffices to show that w is greater than all adjacent fixed points in $I_{\widehat{F}}$. As $I_{\widehat{F}}$ is a combinatorial cube and Ψ_i^{ϵ} is a face inclusion, all but one of these adjacent elements are contained in $\Psi_i^{\epsilon}(I_{\widehat{G}})$ and therefore covered by the previous argument. The remaining adjacent fixed point is ws_i, so what remains is to show that $w(i) > w(i+1)$. Let v be the (black) node associated to x_i. If v is a right child in $\widehat{F}$, then $w(i) = i$ and $w(i+1) < i+1$. If v is not a right child, then i is the smallest element of its cycle, which must also contain $i+1$, so again $w(i) > w(i+1)$. □

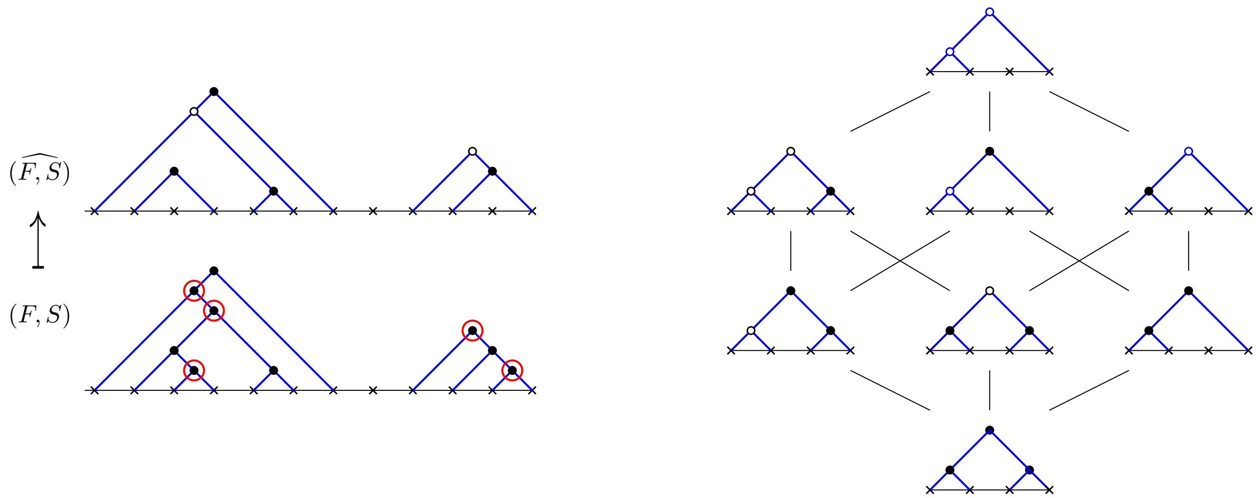


FIGURE 6. An example of the construction in Proposition 8.11 for $n = 12$ (left) and all elements $\widehat{(F,S)} \in \mathrm{BNestFor}_4^{\mathrm{nf}}$ for $F = \widehat{F}(\mathsf{r}_1^- \mathsf{e}_1 \mathsf{e}_1 \mathsf{e}_3) \in \mathsf{Forest}_4$ (right).

We need an alternative construction related to the map $\mathrm{ForToNC}$. Let (F,S) be a pair consisting of an indexed forest $F \in \mathsf{Forest}_n$ and a subset $S \subseteq \mathrm{IN}(F)$. We define

$$\widehat{(F,S)} \in \mathrm{Face}(F) \subset \mathrm{BNestFor}_n^{\mathrm{nf}}$$

by performing left-edge deletion on all vertices in $S \cap \mathrm{RC}(F)$, and coloring the remaining vertices of S white. An example is shown in Figure 6 (left).

Proposition 8.11. The map $(F,S) \mapsto \widehat{(F,S)}$ is a bijection from $\{(F,S) \mid F \in \mathsf{Forest}_n, S \subseteq \mathrm{IN}(F)\}$ onto $\mathrm{BNestFor}_n^{\mathrm{nf}}$. Furthermore $\mathrm{ForToNC}\,\widehat{(F,S)} = \mathrm{ForToNC}(F)$ for all (F,S).

Proof. The fact that $\mathrm{ForToNC}\,\widehat{(F,S)} = \mathrm{ForToNC}(F)$ follows from the definition of $\mathrm{ForToNC}$. For fixed F the forests $\widehat{(F,S)}$ are distinct as they correspond to distinct elements of $\mathrm{Face}(F)$: given a fixed sequence $(\mathsf{r}_1^-)^{n-k} e_{i_1} \cdots e_{i_k}$ for F the choice of S determines the subset of $e_{i_1}, \ldots, e_{i_k}$ to transform to $\mathsf{r}_i^{\pm}$. This shows that the map $(F,S) \mapsto \widehat{(F,S)}$ is an injection, so it remains to construct an inverse map. Given $\widehat{G}$ in $\mathrm{BNestFor}_n^{\mathrm{nf}}$ we create $G \in \mathsf{Forest}_n$ by coloring all of its vertices black and then connecting the remaining nested trees using the procedure described below.

For each $v \in \mathrm{IN}(\widehat{G})$, let $T_1, \ldots, T_k$ be the outermost trees which are nested in the subtree below v, listed from left to right. Denoting their roots $w_1, \ldots, w_k$ we create new black nodes $v'_1, \ldots, v'_k$ in the interior of the edge from v to v_R in this order, and then for each i we connect w_i to v'_i.

The resulting forest G does not depend on the order in which we apply the connection procedure. Setting $S \subseteq \mathrm{IN}(G)$ to be set consisting of the newly created nodes together with the black nodes that came from white nodes of $\widehat{G}$, $\widehat{G} \mapsto (G,S)$ is the inverse map. ☐

Proposition 8.12. Let $\widehat{F}, \widehat{G} \in \mathrm{BNestFor}_n^{\mathrm{nf}}$. If $I_{\widehat{F}} = I_{\widehat{G}}$ then $\widehat{F} = \widehat{G}$.

Proof. We show that we can reconstruct $\widehat{F}$ from $I = I_{\widehat{F}}$. First, we recover $w = \mathrm{ForToNC}(\widehat{F})$ as the maximal element in I with respect to the Bruhat order by Proposition 8.10. By Proposition 8.11, this means $\widehat{F} = \widehat{(F,S)}$ for the unique $F \in \mathsf{Forest}_n$ with $\mathrm{ForToNC}(F) = w$ and some unique $S \subset \mathrm{IN}(F)$ so it remains to show that the $I_{\widehat{(F,S)}}$ are distinct for fixed F. Indeed, by Proposition 8.11 again, as we vary S we obtain distinct $\widehat{(F,S)} \in \mathrm{Face}(F)$, so we conclude the vertex sets $I_{\widehat{(F,S)}}$ are distinct as a face is determined by its vertex set. □

Remark 8.13. One can show that $I_{\widehat{(F,S)}}$ is the smallest sublattice of I_F which contains $\mathrm{ForToNC}(F)$ and $\mathrm{NCPerm}(\widehat{(F,S)} \setminus A)$ for $A \subset \mathrm{IN}(\widehat{(F,S)})$ the subset of black nodes that are not right children.

Proposition 8.14. Every colored Tamari equivalence class in $\mathrm{BNestFor}_n$ has a unique normal form representative in $\mathrm{BNestFor}_n^{\mathrm{nf}}$. Moreover, for $\widehat{F}, \widehat{G} \in \mathrm{BNestFor}_n$, the following are equivalent:

1. $X(\widehat{F}) = X(\widehat{G})$,
2. $\widehat{F} \overset{c}{\sim} \widehat{G}$, and
3. $I_{\widehat{F}} = I_{\widehat{G}}$.

Proof. If $\widehat{F} \overset{c}{\sim} \widehat{G}$ are both in normal form, then by Proposition 8.4 we have $X(\widehat{F}) = X(\widehat{G})$, and in particular $I_{\widehat{F}} = I_{\widehat{G}}$ and so by Proposition 8.12 we conclude that $\widehat{F} = \widehat{G}$.

We know already that (2) implies (1) by Proposition 8.4, and obviously (1) implies (3). It remains only to show that (3) implies (2). By Proposition 8.4 colored Tamari rotations preserve $I_{\widehat{F}}$. Let $\widehat{F}'$ and $\widehat{G}'$ be the normal form colored Tamari equivalents to $\widehat{F}$ and $\widehat{G}$. Then $I_{\widehat{F}'} = I_{\widehat{G}'}$ and so by Proposition 8.12 we deduce that $\widehat{F}' = \widehat{G}'$, and so conclude that $\widehat{F} \overset{c}{\sim} \widehat{G}$. □

We are now ready to prove Theorem 8.1.

Proof of Theorem 8.1. Clearly $X(\widehat{G}) \subset X(\widehat{F})$ implies $I_{\widehat{G}} \subset I_{\widehat{F}}$. Suppose now $I_{\widehat{G}} \subset I_{\widehat{F}}$. By Corollary 6.4, both $I_{\widehat{G}}$ and $I_{\widehat{F}}$ are boolean lattices inside the Kreweras lattice. Recall the absolute length of a noncrossing partition $w \in \mathrm{NC}_n$ is n minus the number of cycles of w, which is the minimal number of transpositions needed to multiply to w. Let $u, v \in I_{\widehat{G}}$ be the top and bottom elements in the Kreweras order, of absolute lengths a and b respectively, so that $|I_{\widehat{G}}| = 2^{a-b}$. On the other hand the set of permutations in $I_{\widehat{F}}$ between u and v in the Kreweras order is a subinterval of $I_{\widehat{F}}$ with 2^{a-b} elements. It follows that $I_{\widehat{G}}$ is this subinterval of $I_{\widehat{F}}$, so there exists $\widehat{G}' \in \mathrm{Face}(\widehat{F})$ with $I_{\widehat{G}} = I_{\widehat{G}'}$. By Proposition 8.14 this implies $X(\widehat{G}') = X(\widehat{G}) \subset X(\widehat{F})$. □

8.3. **Combinatorics of torus orbit closures.** In this section we describe several enumerative aspects of the complex $\mathrm{Complex}(\mathrm{QFl}_n)$. We moreover show in Proposition 8.15 that the number of such orbits for $n = 1, 2, 3, \ldots$ is given by generalized Catalan numbers [42, A064062], while counting them according to dimension gives a refinement known as *Borel's triangle* [42, A234950].

We now describe the cell structure of $\mathrm{Complex}(\mathrm{QFl}_n)$. In general the intersection of two torus orbit closures under inclusion is a union of one or more torus orbit closures. As shown on the right in Figure 4, the intersection of the top-dimensional orbit closures in $\mathrm{Complex}(\mathrm{QFl}_3)$ is the (non-disjoint) union of two intervals. Theorem 8.1 shows that faces of $\mathrm{Complex}(\mathrm{QFl}_n)$ are in bijection with the fixed point sets $I_{\widehat{F}} \subseteq \mathrm{NC}_n$ for $\widehat{F} \in \mathrm{BNestFor}_n$, with inclusion of faces corresponding to inclusion of sets. The faces are also in bijection with normal form forests in $\mathrm{BNestFor}_n^{\mathrm{nf}}$, but inclusion is harder to compute with these objects; in particular it is strictly stronger than the restriction of the order $\leq_{re}$.

Proposition 8.15. Let $G(z,u) = \sum_{n,k\geq 0} f_{n,k} z^n u^k$ where $f_{n,k}$ is the number of torus orbits in QFl_n of dimension k. Then

$$G(z,u) = \frac{1+2u-\sqrt{1-4(u+1)z}}{2(z+u)}. \tag{8.1}$$

Proof. We have that $f_{n,k}$ is the number of $\widehat{F} \in \mathrm{BNestFor}_n^{\mathrm{nf}}$ with k black nodes. First let $G_{\boldsymbol{c}}(z,u)$ be the generating function for trees $T \in \mathrm{BNestFor}_n^{\mathrm{nf}}$, so that $\mathrm{NCPerm}(T) = \boldsymbol{c}$. Decomposition at the root gives a quadratic functional equation for $G_{\boldsymbol{c}}(z,u)$ that has the solution

$$G_{\boldsymbol{c}}(z,u) = \frac{1+2u-z-\sqrt{1+z^2-2(2u+1)z}}{2u}. \tag{8.2}$$

For the general case, note that each $\widehat{F} \in \mathrm{BNestFor}_n^{\mathrm{nf}}$ is given by the choice of an element $w \in \mathrm{NC}_n$ and for each cycle $C = (c_1 \ \cdots \ c_k)$ of w, a tree in $\mathrm{BNestFor}_k^{\mathrm{nf}}$. As in [18, §2.2], we can therefore apply the R-transform from free probability to obtain the equation $G(z,u) = G_{\boldsymbol{c}}(zG(z,u),u)$. This can be solved using the quadratic equation for $G_{\boldsymbol{c}}(z,u)$, from which we obtain the desired result. □

The expression (8.1) is the generating function for Borel's triangle [42, A234950] whose entries have the explicit closed form

$$f_{n,k} = \frac{1}{n}\binom{2n}{n-k-1}\binom{n+k-1}{k}.$$

From the expression (8.2), it follows that the enumeration for trees is given by the classical *large Schröder numbers* [42, A006318] and the refined version according to $|\widehat{F}|$ is [42, A088617].

Remark 8.16. For comparison, every face of the complex attached to the complex HHMP_n discussed in the introduction is a cube, and intersections of faces are faces. These are indexed bijectively by the words in RESeq_n without any r_i^+, and a face F_1 contains a face F_2 if the word for F_1 can be transformed to that for F_2 by changing some letters e_i to either r_i^- or r_{i+1}^-. The total number of faces is $1\cdot 3\cdot 5\cdots(2n-1)$ [42, A001147], refined according to dimension by the generating polynomial $(1)(2+t)(3+2t)\cdots(n+(n-1)t)$.

9. The affine paving of QFl_n

We now describe a family of affine charts for QFl_n around each of its torus fixed points. We begin in Section 9.1 by defining the charts in terms of the Bott manifold structure of the $X(\widehat{F})$ and showing that they partition QFl_n. Then in Section 9.2 we explicitly construct each chart and show that our partition can be equivalently obtained by intersecting QFl_n with the Bruhat decomposition of Fl_n.

9.1. **The paving.** For each $F \in \mathsf{Forest}_n$, the Bott manifold structure of $X(F)$ from Section 5 gives an affine chart $C(F)$ around the T-fixed point $\mathrm{ForToNC}(F) \in X(F)$. Explicitly, $C(F)$ is isomorphic to an affine space $\mathbb{A}^{|F|}$ of dimension $|F|$ and decomposes into sub-torus-orbits of $X(F)$ as

$$C(F) = \bigsqcup_{\substack{\widehat{H}\in\mathrm{Face}(F)\\ \mathrm{ForToNC}(F)\in I_{\widehat{H}}}} X^{\circ}(\widehat{H}). \tag{9.1}$$

Theorem 9.1. The affine charts form a partition of QFl_n:

$$\mathrm{QFl}_n = \bigsqcup_{F\in\mathsf{Forest}_n} C(F). \tag{9.2}$$

Moreover, for any total ordering $F_1, F_2, \ldots, F_{|\mathrm{NC}_n|}$ of Forest_n that extends the pullback of the Bruhat order via ForToNC, we have $\bigsqcup_{i=1}^{k} C(F_i) = \bigcup_{i=1}^{k} X(F_i)$.

We prove the theorem after the following remark and lemma.

Remark 9.2. The decomposition in Theorem 9.1 can be interpreted as a partition of $\mathrm{Complex}(\mathrm{QFl}_n)$ by associating each $C(F)$ with the half-open subspace of the moment polytope for $X(F)$ around the vertex $\mathrm{ForToNC}(F)$; see Figure 7.

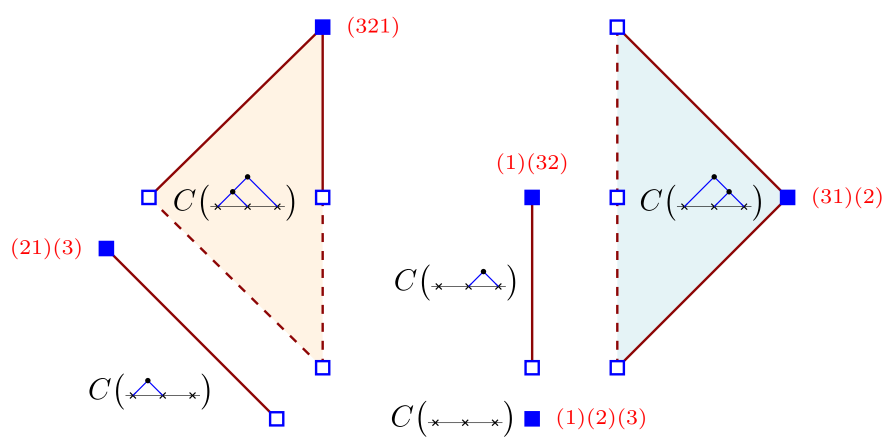


FIGURE 7. The decomposition of $\mathrm{Complex}(\mathrm{QFl}_3)$ induced by our affine paving of QFl_3 as described in Remark 9.2; compare to Figure 4.

Lemma 9.3. Let $F \in \mathsf{Forest}_n$ and $\widehat{H} \in \mathrm{BNestFor}_n^{\mathrm{nf}}$. Then $\mathrm{ForToNC}(\widehat{H}) = \mathrm{ForToNC}(F)$ if and only if $\widehat{H} \in \mathrm{Face}(F)$ and $\mathrm{ForToNC}(F) \in I_{\widehat{H}}$.

Proof. If $\mathrm{ForToNC}(\widehat{H}) = \mathrm{ForToNC}(F)$, then H corresponds to a pair of the form (F, S) by the construction of Proposition 8.11. It follows from the definition of $\leq_{re}$ that $\widehat{H} \in \mathrm{Face}(F)$ and likewise $\mathrm{ForToNC}(F) \in I_{\widehat{H}}$. Conversely, if $\widehat{H} \in \mathrm{BNestFor}_n^{\mathrm{nf}}$ satisfies these two conditions, we know that $I_{\widehat{H}} \subset I_F$ by Theorem 8.1. Since $\mathrm{ForToNC}(F)$ is the Bruhat-maximum element of I_F by Proposition 8.10, it must be the Bruhat-maximum element of $I_{\widehat{H}}$ as well, which implies $\mathrm{ForToNC}(\widehat{H}) = \mathrm{ForToNC}(F)$ by Proposition 8.10 again. □

Proof of Theorem 9.1. By Proposition 8.14, we have

$$\mathrm{QFl}_n = \bigsqcup_{\widehat{F} \in \mathrm{BNestFor}_n^{\mathrm{nf}}} X^\circ(\widehat{F}). \tag{9.3}$$

By Lemma 9.3 and Equation (9.1), we have

$$C(F) = \bigsqcup_{\substack{\widehat{H} \in \mathrm{BNestFor}^{\mathrm{nf}} \\ \mathrm{ForToNC}(\widehat{H}) = \mathrm{ForToNC}(F)}} X^\circ(\widehat{H}). \tag{9.4}$$

As ForToNC is surjective when restricted to Forest_n, it follows immediately that Equation (9.4) coarsens the partition in Equation (9.3) into the one in Equation (9.2).

We now show that $\bigcup_{i=1}^k C(F_i) = \bigcup_{j=1}^k X(F_j)$ for any k. Since $C(F) \subset X(F)$ for any $F \in \mathsf{Forest}$, we only have to show that any $\mathcal{F} \in X(F_j)$ for some $j \leq k$ is included in $C(F_i)$ for some $i \leq k$. We have that $\mathcal{F} \in X^\circ(\widehat{G})$ for some $\widehat{G} \in \mathrm{BNestFor}_n^{\mathrm{nf}}$ such that $X^\circ(\widehat{G}) \subset X(F_j)$. By Theorem 8.1, this implies that $I_{\widehat{G}} \subset I_{F_j}$. In particular, using the characterization of Proposition 8.10, we have $\mathrm{ForToNC}(\widehat{G}) \leq \mathrm{ForToNC}(F_j)$ in Bruhat order. By our choice of total order we then have $\mathrm{ForToNC}(\widehat{G}) = \mathrm{ForToNC}(F_i)$ for some $i \leq j$. Because of (9.4) we then have $X^\circ(\widehat{G}) \subset C(F_i)$, and thus $\mathcal{F} \in C(F_i)$, which concludes the proof. □

Remark 9.4 (Points over $\mathbb{F}_q$)**.** The definition of QFl_n and all arguments used so far make sense over any field, not just $\mathbb{C}$, and so using (9.2) and (9.3) we may count the number of points of QFl_n over a finite field $\mathbb{F}_q$. In this case $C(F) \simeq \mathbb{F}_q^{\mathrm{IN}(F)}$ has cardinality $q^{|F|}$. Summing over all $F \in \mathsf{Forest}_n$ we get

$$\#\mathrm{QFl}_n(\mathbb{F}_q) = \sum_{F \in \mathsf{Forest}_n} q^{|F|} = \sum_{k=0}^{n-1} c_{n,k} q^k$$

where $c_{n,k} = \frac{n-k}{n+k}\binom{n+k}{k}$ [41]. Using (9.3) we get the alternative expression

$$\#\mathrm{QFl}_n(\mathbb{F}_q) = \sum_{F \in \mathrm{BNestFor}_n} (q-1)^{|F|} = \sum_{k=0}^{n-1} f_{n,k}(q-1)^k$$

where the numbers $f_{n,k}$ were introduced in Section 8.3. These two expressions are polynomial in q, and thus one can extract $f_{n,k} = \sum_{m=k}^{n} \binom{m}{k} c_{n,m}$. This gives another proof that the numbers $f_{n,k}$ are given by Borel's triangle [42, A234950] as seen in Section 8.3.

$$M(w) = \begin{bmatrix} * & 1 & 0 & 0 & 0 & 0 \\ * & 0 & 1 & 0 & 0 & 0 \\ * & 0 & 0 & * & * & 1 \\ * & 0 & 0 & * & 1 & 0 \\ * & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad M_{\mathrm{NC}}(w) = \begin{bmatrix} * & 1 & 0 & 0 & 0 & 0 \\ * & 0 & 1 & 0 & 0 & 0 \\ * & 0 & 0 & * & 0 & 1 \\ 0 & 0 & 0 & * & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

FIGURE 8. The Bruhat cell and noncrossing Bruhat cell for $w = 612543$.

9.2. **Paving with noncrossing Bruhat cells.** We now give a combinatorial description of our affine paving for QFl_n. Throughout, we use the convention that a matrix M with entries in $\mathbb{C} \cup \{*, +\}$ represents the set of all matrices whose entries are, depending on the corresponding entry of M, either a particular complex number (if in $\mathbb{C}$) or taken freely from either $\mathbb{C}$ (if $*$) or $\mathbb{C}^\times$ (if $+$).

We first recall the combinatorial construction of Bruhat cells in Fl_n. The *inversion set* of $w \in S_n$ is $\mathrm{Inv}(w) = \{(i,j) \mid i < j \text{ and } w(i) > w(j)\}$. For $w \in S_n$, let $M(w)$ be the matrix with 1's in positions $(w(i), i)$, $*$'s in positions $(w(j), i)$ for $(i,j) \in \mathrm{Inv}(w)$, and 0's elsewhere; see for example Figure 8. Then $M(w)$ is isomorphic to an affine space where each $*$ represents a coordinate, and this gives a complete set of representatives for the Bruhat cell $BwB \subseteq \mathrm{Fl}_n$. In order to reproduce the standard action on Fl_n, we have T act on elements of $M(w)$ by scaling the k, ℓ entry by the character $\chi_k \chi_{w(\ell)}^{-1}$. Thus as a T-representation

$$M(w) \cong \bigoplus_{(i,j) \in \mathrm{Inv}(w)} \mathbb{C}_{\chi_{w(j)} \chi_{w(i)}^{-1}}. \tag{9.5}$$

In order to state a similar result for QFl_n, we introduce an important subset of the inversion set of a noncrossing partition.

Definition 9.5. The *noncrossing inversion set* of $w \in \mathrm{NC}_n$ is

$$\mathrm{Inv}_{\mathrm{NC}}(w) \coloneqq \{(i,j) \in \mathrm{Inv}(w) : w(i,j) \in \mathrm{NC}_n\}.$$

Definition 9.6. The *noncrossing Bruhat cell* for $w \in \mathrm{NC}_n$ is the set represented by the matrix $M_{\mathrm{NC}}(w)$ with entries 1 in position $(w(i), i)$ for each $i \in [n]$, $*$ in position $(w(j), i)$ for each $(i,j) \in \mathrm{Inv}_{\mathrm{NC}}(w)$, and 0 elsewhere. For $F \in \mathsf{Forest}_n$, we define

$$M_{\mathrm{NC}}(F) \coloneqq M_{\mathrm{NC}}(\mathrm{ForToNC}(F)).$$

For $w = \mathrm{ForToNC}(F)$, we have a canonical identification

$$M_{\mathrm{NC}}(w) \cong \bigoplus_{(i,j) \in \mathrm{Inv}_{\mathrm{NC}}(w)} \mathbb{C}_{\chi_{w(j)} \chi_{w(i)}^{-1}} \subset \bigoplus_{(i,j) \in \mathrm{Inv}(w)} \mathbb{C}_{\chi_{w(j)} \chi_{w(i)}^{-1}} \cong M(w).$$

See for example Figure 8. We now state the main result of the section.

Theorem 9.7. For $w \in \mathrm{NC}_n$ we have $\mathrm{QFl}_n \cap BwB = M_{\mathrm{NC}}(w)B$ and

$$\mathrm{QFl}_n = \bigsqcup_{w \in \mathrm{NC}_n} M_{\mathrm{NC}}(w)B. \tag{9.6}$$

Moreover, if for any total ordering $w_1, w_2, \ldots, w_{|\,\mathrm{NC}_n\,|}$ of NC_n that extends the Bruhat order we set

$$X_k = \bigcup_{i=1}^{k} M_{\mathrm{NC}}(w_i)B,$$

then each X_k is closed, $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_{|\,\mathrm{NC}_n\,|} = \mathrm{QFl}_n$, and $X_{k+1} \setminus X_k = M_{\mathrm{NC}}(w_{k+1})B$.

We will prove this result at the end of the section after some preparation.

Remark 9.8. The second part of Theorem 9.7 shows that the X_i form an affine paving of QFl_n; see Section 11.1 for precise definitions.

Our first step is to extend a combinatorial characterization of the noncrossing inversion set stated in [6, Remark 8.18]. For $\widehat{F} \in \mathrm{BNestFor}_n$, the *spread* of $v \in \mathrm{IN}(\widehat{F})$ is the pair (i,j) consisting of the leftmost leaf descendant i of v and the rightmost leaf descendant j of v. Figure 9 depicts a bicolored nested forest in which each internal node is labeled by its spread. Note that the spread of an internal node v is not necessarily the transposition τ_v assigned to each $v \in \mathrm{IN}(\widehat{F})$ earlier in Section 6.

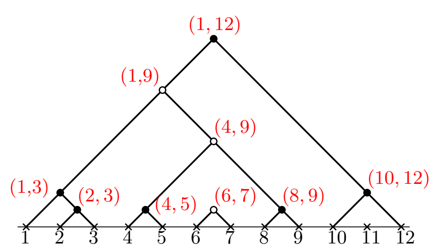


FIGURE 9. A bicolored nested forest with spreads recorded for each internal node

Proposition 9.9. If $\widehat{F} \in \mathrm{BNestFor}_n^{\mathrm{nf}}$ and $w = \mathrm{ForToNC}(\widehat{F})$, then

$$\{(i,j) \in \mathrm{Inv}_{\mathrm{NC}}(w) \mid w(i,j) \in I_{\widehat{F}}\} = \{\text{spreads } (i,j) \text{ of black nodes } v \in \mathrm{IN}(\widehat{F})\}.$$

In particular, if $F \in \mathsf{Forest}_n$ then $\mathrm{Inv}_{\mathrm{NC}}(w)$ is the set of all spreads for $v \in \mathrm{IN}(F)$.

Proof. By Proposition 8.10, it is sufficient to show that the set of elements of $I_{\widehat{F}}$ that are adjacent to w give the spreads of all black nodes in $\widehat{F}$. We do so using induction on n. As $n = 1$ is vacuous, assume that $n > 1$ and write $\widehat{F} = \widehat{G} \cdot \mathsf{x}_k$ and $u = \mathrm{ForToNC}(\widehat{G})$ with $\widehat{G} \in \mathrm{BNestFor}_{n-1}^{\mathrm{nf}}$. We split into cases according to Lemma 8.9. First, suppose that either $\mathsf{x}_k = \mathsf{r}_k^-$ or that leaf k is a right child of $\widehat{G}$ (so in particular $\mathsf{x}_k \neq \mathsf{e}_k$). Then for any $v \in \mathrm{IN}(\widehat{G})$ with spread (i,j) we have

$\Psi_k^-\big(u(i\,j)\big) = w\Psi_k^-\big((i\,j)\big) = w(i+\delta_{i\geq k}, j+\delta_{j\geq k})$, and the node in $\mathrm{IN}(\widehat{F})$ corresponding to v has spread $(i+\delta_{i\geq k}, j+\delta_{j\geq k})$ so the claim holds.

Now suppose $\mathsf{x}_k \neq \mathsf{r}_k^-$ and k is not a right child in $\widehat{G}$. If $\mathsf{x}_k = \mathsf{r}_k^+$, $\Psi_k^+(u(ij)) = ws_k(i+\delta_{i\geq k}, j+\delta_{j\geq k})s_k = w(i+\delta_{i\geq k+1}, j+\delta_{j\geq k+1})$, and we conclude as before. If $\mathsf{x}_k = \mathsf{e}_k$ then the same reasoning applies except that $\widehat{F}$ has one more black node than $\widehat{G}$, with spread $(k, k+1)$, and w has the additional adjacent element ws_k. □

We next give a recursive characterization of the charts in Fl_n determined by the $M_{\mathrm{NC}}(F)$ using the building operations from Section 3.

Proposition 9.10. The representatives of the noncrossing Bruhat cells in Fl_n are characterized recursively by $M_{\mathrm{NC}}(\mathsf{r}_1^-)B = \mathrm{Fl}_1$ and for $F = G\cdot\mathsf{e}_i \in \mathsf{Forest}_n$,

$$M_{\mathrm{NC}}(F)B = \begin{cases} \mathbb{G}_i M_{\mathrm{NC}}(G)B \sqcup \Psi_i^- M_{\mathrm{NC}}(G)B & \text{if leaf } i \text{ is a right child in } G \\ \mathbb{G}_i M_{\mathrm{NC}}(G)B \sqcup \Psi_i^+ M_{\mathrm{NC}}(G)B & \text{otherwise.} \end{cases}$$

We prove the proposition using a technical argument involving a modified version of the $\mathbb{G}_i$ operation. Given a matrix M with column i equal to the jth basis vector for $j < i$, let $\mathbb{G}'_i M$ be obtained from $\Psi_i^- M$ by setting the j,i-entry to $+$.

Example 9.11. We have

$$\mathbb{G}_2 \begin{bmatrix} a & 1 & b \\ c & 0 & d \\ e & 0 & f \end{bmatrix} = \begin{bmatrix} a & 1 & 0 & b \\ 0 & + & 1 & 0 \\ c & 0 & 0 & d \\ e & 0 & 0 & f \end{bmatrix} \qquad \text{and} \qquad \mathbb{G}'_2 \begin{bmatrix} a & 1 & b \\ c & 0 & d \\ e & 0 & f \end{bmatrix} = \begin{bmatrix} a & + & 1 & b \\ 0 & 1 & 0 & 0 \\ c & 0 & 0 & d \\ e & 0 & 0 & f \end{bmatrix}$$

Lemma 9.12. If M is an $(m-1)\times(m-1)$ matrix whose ith column is e_j for some $j < i$, then we have an equality of sets $(\mathbb{G}_i M)B_m = (\mathbb{G}'_i M)B_m$.

Proof. For $x \in \mathbb{C}^\times$, let $(\mathbb{G}_i M)(x)$ and $(\mathbb{G}'_i M)(x^{-1})$ denote the matrices obtained by setting the newly introduced $+$'s to x and x^{-1}, respectively. Let c_i and c_{i+1} be the ith and $i+1$st columns of $(\mathbb{G}_i M)(x)$. Then $(\mathbb{G}'_i M)(x^{-1})$ is obtained from $(\mathbb{G}_i M)(x)$ by performing the column operations $c_i \mapsto x^{-1}c_i$ followed by $c_{i+1} \mapsto xc_i - xc_{i+1}$. Both forward column operations correspond to right multiplication by elements of B_m, so we have $(\mathbb{G}_i M)(x)B_m = (\mathbb{G}'_i M)(x^{-1})B_m$. We conclude as $x \mapsto x^{-1}$ is a bijection on $\mathbb{C}^*$. □

Proof of Proposition 9.10. Let $w = \mathrm{ForToNC}(F)$ and $u = \mathrm{ForToNC}(G)$. By Lemma 8.9, we have $w = \Psi_i^\pm(u)$. Further, by Proposition 9.9, we have

$$\mathrm{Inv}_{\mathrm{NC}}(w) = \{(j+\delta_{j\geq i}, k+\delta_{k\geq i}) \mid (j,k) \in \mathrm{Inv}_{\mathrm{NC}}(u)\} \cup \{(i, i+1)\}.$$

Thus, if we write $M_{\mathrm{NC}}(w;x)$ for the matrix obtained from $M_{\mathrm{NC}}(w)$ by setting the $(w(i+1),i)$ entry equal to x, then we have

$$M_{\mathrm{NC}}(w;0) = \begin{cases} \Psi_i^- M_{\mathrm{NC}}(u) & \text{if leaf } i \text{ is a right child in } G \\ \Psi_i^+ M_{\mathrm{NC}}(u) & \text{otherwise.} \end{cases}$$

Moreover, $M_{\mathrm{NC}}(w) = M_{\mathrm{NC}}(w;0) \sqcup M_{\mathrm{NC}}(w;+)$, so we complete the proof by showing that

$$\mathbb{G}_i M_{\mathrm{NC}}(u)B = M_{\mathrm{NC}}(w;+)B.$$

If leaf i is not a right child in G, then $w(i) > w(i+1)$ and we have a direct equality $M_{\mathrm{NC}}(w;+) = \mathbb{G}_i M_{\mathrm{NC}}(u)$. If leaf i is a right child in G, then $w(i) < w(i+1)$ and $M_{\mathrm{NC}}(w;+) = \mathbb{G}_i' M_{\mathrm{NC}}(u)$, so we must use Lemma 9.12 to relate $\mathbb{G}_i'$ and $\mathbb{G}_i$. Indeed, as a right child i can never be the left endpoint of an internal node, Proposition 9.9 implies that u has no noncrossing inversion ending in i. Thus $M_{\mathrm{NC}}(G)$ has no $*$'s in column i and the hypotheses of Lemma 9.12 are satisfied, giving $M_{\mathrm{NC}}(w;+)B = \mathbb{G}_i M_{\mathrm{NC}}(u)B$. □

Proposition 9.13. Let $w \in \mathrm{NC}_n$, and let F be the unique forest in Forest_n such that $w = \mathrm{ForToNC}(F)$. We have $M_{\mathrm{NC}}(F)B = C(F)$, and as a consequence there is a T-equivariant isomorphism

$$C(F) \cong M_{\mathrm{NC}}(w) \cong \bigoplus_{(i,j)\in \mathrm{Inv}_{\mathrm{NC}}(w)} \mathbb{C}_{\chi_{w(j)}\chi_{w(i)}^{-1}}.$$

Proof. We show that $C(F)$ agrees with the recursive characterization of $M_{\mathrm{NC}}(F)B$ given in Proposition 9.10. As $C((\mathsf{r}_1^-)^{n-k}) = \mathrm{id}_{S_{n-k}} \in \mathrm{Fl}_{n-k}$, this amounts to showing that for $F = G\cdot \mathsf{e}_i \in \mathsf{Forest}_n$,

$$C(F) = \begin{cases} \mathbb{G}_i C(G) \sqcup \Psi_i^- C(G) & \text{if leaf } i \text{ is a right child in } G \\ \mathbb{G}_i C(G) \sqcup \Psi_i^+ C(G) & \text{otherwise.} \end{cases}$$

By the definition of $\mathrm{Face}(\widehat{F})$ and the $\leq_{re}$ order, we have

$$\mathrm{Face}(\widehat{F}) = \{\widehat{H}'\cdot \mathsf{x}_i \mid \mathsf{x}_i \in \{\mathsf{r}_i^-, \mathsf{r}_i^+, \mathsf{e}_i\} \text{ and } \widehat{H}' \in \mathrm{Face}(G)\}.$$

If we furthermore write $w = \mathrm{ForToNC}(F)$ and $u = \mathrm{ForToNC}(G)$, then by Lemma 8.9 we have $w = \Psi_i^\epsilon(u)$ for $\epsilon \in \{+,-\}$. Thus

$$\{\widehat{H} \in \mathrm{BNestFor}_n \mid w \in I_{\widehat{H}}\} = \{\widehat{H}'\cdot \mathsf{x}_i \mid \mathsf{x}_i \in \{\mathsf{r}_i^\epsilon, \mathsf{e}_i\}, \widehat{H}' \in \mathrm{BNestFor}_{n-1} \text{ and } u \in I_{\widehat{H}'}\}.$$

By intersecting the sets above, we arrive at the following description of $C(F)$:

$$C(F) = \bigsqcup_{\substack{\widehat{H}'\leq_{re} G\\ u\in I_{\widehat{H}}}} \left(X^\circ(\widehat{H}\cdot \mathsf{e}_i) \sqcup X^\circ(\widehat{H}\cdot \mathsf{r}_i^\epsilon)\right) = \bigsqcup_{\substack{\widehat{H}'\leq_{re} G\\ u\in I_{\widehat{H}}}} \left(\mathbb{G}_i X^\circ(\widehat{H}) \sqcup \Psi_i^\epsilon X^\circ(\widehat{H})\right) = \mathbb{G}_i C(G) \sqcup \Psi_i^\epsilon C(G)$$

This completes the proof as by Lemma 8.9 $\epsilon = -$ if leaf i is a left child and $+$ otherwise. □

We can now complete the proof of Theorem 9.7.

Proof of Theorem 9.7. By Theorem 9.1, QFl_n is the disjoint union of the charts $C(F)$ for $F \in \mathsf{Forest}_n$. By Proposition 9.13, these charts are exactly the $M_{\mathrm{NC}}(w)B$ for $w \in \mathrm{NC}_n$, giving Equation (9.6). As $M_{\mathrm{NC}}(w)B \subseteq M(w)B = BwB$, and the BwB form a partition of Fl_n by (2.1), it follows that $M_{\mathrm{NC}}(w)B = \mathrm{QFl}_n \cap M(w)B$.

Now for any $w \in S_n$, the Schubert variety $X^w = \overline{BwB}$ consists of all BuB with $u \leq w$ in Bruhat order. It follows that $X_k = \mathrm{QFl}_n \cap \left(\bigcup_{i=1}^k X^{w_i}\right)$ is closed, which concludes the proof. □

10. Intrinsic characterizations of QFl_n

We now give two intrinsic characterizations of QFl_n. These characterizations are independent of one another and are presented in Sections 10.1 and 10.2.

10.1. Characterization with Plücker functions.

This section proves Theorem B, which states that

$$\mathrm{QFl}_n = \bigcap_{w \in S_n \setminus \mathrm{NC}_n} \{\mathrm{Pl}_w = 0\}.$$

The proof is given at the end of the section. We begin with a classical observation, which can be found for instance in [35, Proposition 2.6].

Proposition 10.1. For $\mathcal{F} \in \mathrm{Fl}_n$, the set of torus fixed points in $\overline{T \cdot \mathcal{F}}$ is $\{wB \mid \mathrm{Pl}_w(\mathcal{F}) \neq 0\}$.

Every element of BwB has a unique representative h in the set $M(w)$ defined in Section 9. Thus when restricted to BwB, we can view the Plücker functions as polynomials in the matrix entries which are not uniformly 0 or 1 across all of $M(w)$, namely the $h_{w(j),i}$ for $(i,j) \in \mathrm{Inv}(w)$. Going forward, we define a $\mathbb{Z}^n$-grading on such polynomials by setting

$$\mathrm{degree}(h_{w(j),i}) = e_{w(j)} - e_{w(i)},$$

where e_k denotes the kth standard basis vector. This grading is the weight of the character of the T-action on each entry of $M(w)$ under the action of T as described in Section 9.

Observation 10.2. Let $w \in S_n$. Expressed in the entries of $h \in M(w)$, $\mathrm{Pl}_u(h)/\mathrm{Pl}_w(h)$ is a homogeneous polynomial of degree $\sum (u^{-1}(i) - w^{-1}(i))e_{n+1-i}$ for each $u \in S_n$.

Proof. We compute directly that $\mathrm{Pl}_w(h) = 1$, so $\mathrm{Pl}_u(h)/\mathrm{Pl}_w(h) = \mathrm{Pl}_u(h)$ has denominator 1. The claim now follows from the fact that the Plücker functions are T-equivariant and the weight of the character by which T scales $\mathrm{Pl}_u(h)/\mathrm{Pl}_w(h)$ is $\sum u^{-1}(i)e_{n+1-i} - \sum w^{-1}(i)e_{n+1-i}$. □

In order to perform a more granular analysis on the degree of each Plücker function, we establish some notation using the root system of type A_{n-1}. The positive roots in this system are the vectors $e_i - e_j$ for $1 \leq i < j \leq n$ and the negative roots are $e_j - e_i$ for $1 \leq i < j \leq n$. Given $F \in \mathsf{Forest}_n$, we define the polyhedral cone Cone_F by

$$\mathrm{Cone}_F = \mathbb{R}_{\geq 0}\{e_j - e_i \mid (i,j) \text{ a spread in } F\},$$

and for $w \in \mathrm{NC}_n$ we define $\mathrm{Cone}_w = w \cdot \mathrm{Cone}_F$ where $F \in \mathsf{Forest}_n$ is the unique forest such that $w = \mathrm{ForToNC}(F)$. In view of Proposition 9.9

$$\mathrm{Cone}_w = \mathbb{R}_{\geq 0}\{e_{w(j)} - e_{w(i)} \mid (i,j) \in \mathrm{Inv}_{\mathrm{NC}}(w)\}.$$

Spreads are characterized by that fact that if (i,j) is a spread in $F \in \mathsf{Forest}_n$, then no spread has the form (j,k). Such sets (and their associated cones) were first studied in [23] and are commonly known as *noncrossing alternating forests*; see for example [1]. The following result can be found in [23, §6].

Proposition 10.3. For each $w \in \mathrm{NC}_n$, Cone_w is simplicial and the only roots it contains are the generators $e_{w(j)} - e_{w(i)}$ for $(i,j) \in \mathrm{Inv}_{\mathrm{NC}}(w)$.

The sets $(w(b), w(a))$ for (a,b) a spread in F also appear in the literature as a generalization of noncrossing alternating forests. Specifically, [31, §6] shows that these are canonically in bijection with the set of $\boldsymbol{c}$-clusters.

Proof of Theorem B. First, we take $\mathcal{F} \in \mathrm{QFl}_n$. By Theorem 6.1, the torus fixed points in $\overline{T \cdot \mathcal{F}}$ are all noncrossing partitions. By Proposition 10.1, this means that the nonvanishing Plücker functions of $\mathcal{F}$ are also indexed by elements of NC_n.

Conversely, suppose that $\mathrm{Pl}_u \mathcal{F} = 0$ for all $u \in S_n \setminus \mathrm{NC}_n$. Let $w \in S_n$ be such that $\mathcal{F} \in BwB$ and let $h \in \mathrm{GL}_n$ be the representative of $\mathcal{F}$ in $M(w)$. As $\mathrm{Pl}_w \mathcal{F} \neq 0$ on BwB we conclude that $w \in \mathrm{NC}_n$. We now claim that for each $(a,b) \in \mathrm{Inv}(w) \setminus \mathrm{Inv}_{\mathrm{NC}}(w)$ we have $h_{w(b),a} = 0$.

We proceed by induction on $w(a) - w(b)$. Let $u = w(a\,b)$ and let $\alpha = (b-a)(e_{w(b)} - e_{w(a)})$. We consider the set S consisting of all multisubsets M of $\mathrm{Inv}(w)$ with the property that $\sum_{(i,j) \in M} e_{w(j)} - e_{w(i)} = \alpha$ so that by Observation 10.2 we have

$$\mathrm{Pl}_u(h)/\mathrm{Pl}_w(h) = \sum_{M \in S} c_M \prod_{(i,j) \in M} h_{w(j),i} \qquad \text{for some } c_M \in \mathbb{Z}.$$

We now describe some properties of the elements $M \in S$. First, by considering the first and last nonzero coordinate in the sum $\sum_{(i,j) \in M} e_{w(j)} - e_{w(i)} = \alpha$, every $e_{w(j)} - e_{w(i)} \in M$ has either $w(j) - w(i) < w(b) - w(a)$ or $(i,j) = (a,b)$. Second, by Proposition 10.3, M must contain at least one element of the form $e_{w(j)} - e_{w(i)}$ for $(i,j) \in \mathrm{Inv}(w) \setminus \mathrm{Inv}_{\mathrm{NC}}(w)$. Finally, a direct computation of $\mathrm{Pl}_u(h)/\mathrm{Pl}_w(h)$ shows that the coefficient of $(h_{w(b),a})^{b-a}$ is nonzero: for $i < a$ or $i \geq b$, $\det(h_{u(1),\ldots,u(i)}) = \det(h_{w(1),\ldots,w(i)}) = \pm 1$, while for $a \leq i < b$, $\det(h_{u(1),\ldots,u(i)})$ contains $h_{w(b),a}$ with a coefficient of ± 1. Thus by our assumption on Pl_u and our inductive hypothesis, we have

$0 = (h_{w(b),a})^{b-a}$. This proves the claim. This shows $h \in M_{\mathrm{NC}}(w)$ and finally $\mathcal{F} \in \mathrm{QFl}_n$ by Proposition 9.13. □

10.2. **Characterization via equivalence of flags.** We now give our second characterization of QFl_n as flags that can be obtained from $\mathrm{id}_{\mathrm{Fl}_n}$, the standard coordinate flag $\{0\} \subset \{e_1\} \subset \{e_1, e_2\} \subset \cdots \subset \{e_1, \ldots, e_n\}$, via certain elementary operations.

Definition 10.4. Define $\sim$ to be the equivalence relation on complete flags generated by the relations $\sim_i$ for $1 \leq i \leq n-1$ given by $\mathcal{F} \sim_i \mathcal{G}$ if

(1) $\mathcal{F}_j = \mathcal{G}_j$ for all $j \neq i$, and
(2) $e_i \in \mathcal{F}_{i+1}$ and $\mathcal{F}_{i-1} \subset \{x_i = 0\}$.

We have that $\mathcal{F} \sim_i \mathcal{G}$ for $1 \leq i \leq n-1$ if and only if there exists $\mathcal{H} \in \mathrm{Fl}_{n-1}$ such that $\mathcal{F}, \mathcal{G} \in \mathbb{P}_i\mathcal{H}$, where $\mathbb{P}_i$ is defined in Section 3.2. In particular note that $\Psi_i^- \mathcal{H} \sim_i \Psi_i^+ \mathcal{H}$.

Theorem 10.5. The quasisymmetric flag variety $\mathrm{QFl}_n \subset \mathrm{Fl}_n$ is the equivalence class of $\sim$ containing the standard coordinate flag.

We prove this at the end of the subsection after a preparatory lemma.

Lemma 10.6. Let $\widehat{F} \in \mathrm{BNestFor}_n$. Suppose there exists i such that every element $w \in I_{\widehat{F}}$ satisfies $i \in \{w(i), w(i+1)\}$. Then we have $X(\widehat{F}) \subset \mathbb{P}_i X(\widehat{G})$ for some $\widehat{G} \in \mathrm{BNestFor}_{n-1}$.

Proof. We will prove the following statements, from which the conclusion follows immediately.

(i) If $w(i) = i$ for all $w \in I_{\widehat{F}}$, then $\widehat{F} = \widehat{G} \cdot \mathsf{r}_i^-$ for some $\widehat{G} \in \mathrm{BNestFor}_{n-1}$.
(ii) If $w(i+1) = i$ for all $w \in I_{\widehat{F}}$, then $\widehat{F} \overset{c}{\sim} \widehat{G} \cdot \mathsf{r}_i^+$ for some $\widehat{G} \in \mathrm{BNestFor}_{n-1}$.
(iii) If $i \in \{w(i), w(i+1)\}$ for all $w \in I_{\widehat{F}}$, but we are not in a scenario covered by Cases (i) and (ii), then $\widehat{F} \overset{c}{\sim} \widehat{G} \cdot \mathsf{e}_i$ for some $\widehat{G} \in \mathrm{BNestFor}_{n-1}$.

If we are in case (i), then $\mathrm{NCPerm}(\widehat{F}) \in I_{\widehat{F}}$ has i as a fixed point, implying that the leaf labeled i is a singleton tree in $\widehat{F}$. This immediately yields $\widehat{F} = \widehat{G} \cdot \mathsf{r}_i^-$.

Now suppose we are in a situation described in (ii). Let $v \in \mathrm{IN}(\widehat{F})$ have canonical label i. Let P_1 (respectively P_2) be the path beginning from the leaf labeled i (respectively $i+1$) and terminating in v. Observe that all but the final edge in P_1 connect a right child to its parent node and similarly all edges but the final edge in P_2 connect a left child to its parent. We claim that v must be white. Indeed, if v were black, then left edge deletion at v would result in an element in $I_{\widehat{F}}$ that has i and $i+1$ in different cycles. For similar reasons we infer that all nodes in P_2 are necessarily white. Thus we are in a situation depicted on the left in Figure 10 where the "half-filled" nodes could be black or white. By performing colored Tamari rotation as in Definition 8.3, first along P_1 and then along P_2 as in Figure 10, one can obtain a bicolored nested forest wherein v has left and right children being leaves with labels i and $i+1$. Then as described in Definition 4.2, there exists a $\widehat{G} \in \mathrm{BNestFor}_{n-1}$ satisfying $\widehat{F} = \widehat{G} \cdot \mathsf{r}_i^+$.

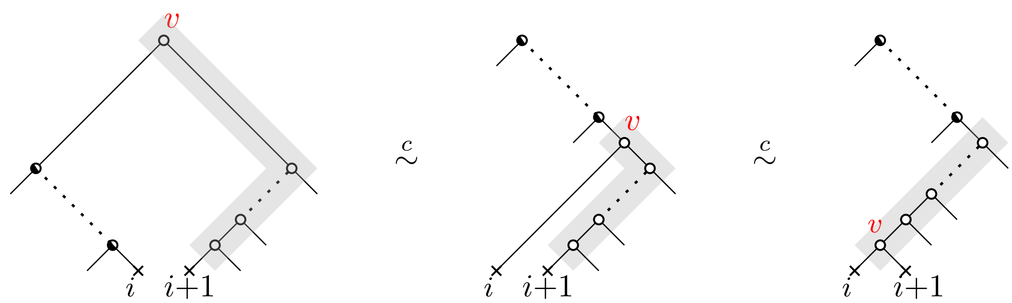


FIGURE 10. Case (ii) (left) and a bicolored nested forest that is colored Tamari equivalent. The half-filled nodes could be either black or white.

Finally we consider (iii). Let $v \in \mathrm{IN}(\widehat{F})$ have canonical label i, like before. For the condition in (iii) to hold, v must be black. Indeed if v were white then no element of $I_{\widehat{F}}$ has i as a fixed point. For this same reason the left child of v is necessarily the leaf labeled i. As above, the path from the leaf labeled $i+1$ to v can only contain white nodes, as this ensures that i and $i+1$ are in the same cycle if i is not a fixed point. We use colored Tamari rotations exactly as in Case (ii) to obtain a bicolored nested forest where v has left and right children given by i and $i+1$. Left edge deletion at v now gives $\widehat{G} \in \mathrm{BNestFor}_{n-1}$ satisfying $\widehat{F} = \widehat{G} \cdot \mathsf{e}_i$. Figure 11 outlines this case. ☐

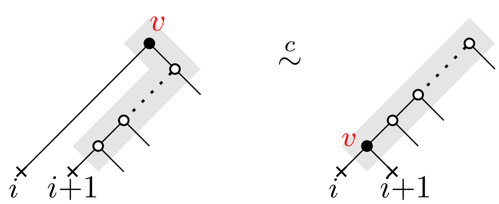


FIGURE 11. Case (iii) (left) and a bicolored nested forest that is colored Tamari equivalent

Proof of Theorem 10.5. First we show that QFl_n is closed under these relations. Suppose first that $\mathcal{F} \in \mathrm{QFl}_n$ satisfies condition (2) in Definition 10.4 for a fixed i, which means that $\mathcal{F} \in \mathbb{P}_i\mathrm{Fl}_{n-1}$.

We claim that in that case $\mathcal{F} \in \mathbb{P}_i\mathrm{QFl}_{n-1}$. Condition (2) is closed and invariant under the action of T, so it is satisfied by all elements of the torus-orbit closure $\overline{T \cdot \mathcal{F}}$. By Theorem 5.1, $\overline{T \cdot \mathcal{F}} = X(\widehat{F})$ for some $\widehat{F} \in \mathrm{BNestFor}_n$, so the conditions (2) also apply to the set of torus fixed points $I_{\widehat{F}} \subset \mathrm{NC}_n$. This means that i is a fixed point or $i, i+1$ are in the same cycle for each element of $I_{\widehat{F}}$, so Lemma 10.6 guarantees the existence of $\widehat{G} \in \mathrm{BNestFor}_{n-1}$ such that $X(\widehat{F}) \subset \mathbb{P}_i X(\widehat{G})$. This proves the claim since $\mathcal{F} \in X(\widehat{F})$ and $X(\widehat{G}) \subset \mathrm{QFl}_{n-1}$.

Now if $\mathcal{F} \sim_i \mathcal{H}$, then by condition (1) we get $\mathcal{H} \in \mathbb{P}_i X(\widehat{G}) \subset \mathbb{P}_i\mathrm{QFl}_{n-1}$ as well, and thus $\mathcal{H} \in \mathrm{QFl}_n$.

Conversely, we now show that we can reduce every element of QFl_n to $\mathrm{id}_{\mathrm{Fl}_n}$ using these relations. We do this by induction on n, the case $n = 2$ being trivial.

Let $\mathcal{F} \in \mathrm{QFl}_n$. Since $\mathrm{QFl}_n = \mathbb{P}_1\mathrm{QFl}_{n-1} \cup \cdots \cup \mathbb{P}_{n-1}\mathrm{QFl}_{n-1}$ and every element of $\mathbb{P}_i\mathcal{H}$ is $\sim_i$-equivalent to $\Psi_i^-\mathcal{H}$, there exist $j \in \{1, \ldots, n-1\}$ and $\mathcal{G} \in \mathrm{QFl}_{n-1}$ such that $\mathcal{F} \sim_j \Psi_j^-\mathcal{G}$. By the inductive hypothesis we know that $\mathcal{G} \sim \mathrm{id}_{\mathrm{Fl}_{n-1}}$.

We claim that for any i, $\mathcal{H} \sim_i \mathcal{H}'$ in QFl_{n-1} implies $\Psi_j^-\mathcal{H} \sim \Psi_j^-\mathcal{H}'$ in QFl_n. To prove this, we let $\mathcal{K} \in \mathrm{QFl}_{n-2}$ be such that $\mathcal{H}, \mathcal{H}' \in \mathbb{P}_i\mathcal{K}$ and consider different cases.

(i) If $i \geq j$ then by Lemma 4.6 we have $\Psi_j^-\mathbb{P}_i\mathcal{K} = \mathbb{P}_{i+1}\Psi_j^-\mathcal{K}$ and so $\Psi_j^-\mathcal{H} \sim_{i+1} \Psi_j^-\mathcal{H}'$.

(ii) If $j \geq i+2$ then by Lemma 4.6 we have $\Psi_j^-\mathbb{P}_i\mathcal{K} = \mathbb{P}_i\Psi_{j-1}^-\mathcal{K}$ and so $\Psi_j^-\mathcal{H} \sim_i \Psi_j^-\mathcal{H}'$.

(iii) Finally suppose $j = i+1$. By Lemma 8.2 we have $\Psi_{i+1}^+\mathbb{P}_i\mathcal{K} = \mathbb{P}_i\Psi_i^+\mathcal{K}$. This in turn implies

$$\Psi_{i+1}^-\mathcal{H} \sim_{i+1} \Psi_{i+1}^+\mathcal{H} \sim_i \Psi_{i+1}^+\mathcal{H}' \sim_{i+1} \Psi_{i+1}^-\mathcal{H}'.$$

Thus $\Psi_j^-\mathcal{H} \sim \Psi_j^-\mathcal{H}'$ in all cases and the claim is proved. By induction we get that if $\mathcal{H} \sim \mathcal{H}'$ in QFl_{n-1} then $\Psi_j^-\mathcal{H} \sim \Psi_j^-\mathcal{H}'$. We apply this to $\mathcal{G}$ and $\mathrm{id}_{\mathrm{Fl}_{n-1}}$ and get $\Psi_j^-\mathcal{G} \sim \Psi_j^-\,\mathrm{id}_{\mathrm{Fl}_{n-1}} = \mathrm{id}_{\mathrm{Fl}_n}$, which concludes the proof since $\mathcal{F} \sim_j \Psi_j^-\mathcal{G}$. □

11. The GKM presentation of $H^\bullet_{T_n}(\mathrm{QFl}_n)$

In this section we give a presentation of $H^\bullet_{T_n}(\mathrm{QFl}_n)$ and $H^\bullet(\mathrm{QFl}_n)$ in terms of a certain combinatorially defined "graph cohomology ring," and describe a free $\mathbb{Z}[\mathbf{t}_n]$-basis for this ring. These results will be used in the next section to give a Borel-type presentation.

We appeal to GKM theory, which is a technique for computing the equivariant cohomology ring of a variety X under the action of an algebraic torus under suitable hypothesis. While originally developed for rational cohomology by Goresky, Kottwitz, and MacPherson [25] with inspiration from Chang and Skjelbred [17], we present a variant with stricter hypotheses that computes integral cohomology.

Throughout we use the case $X = \mathrm{Fl}_n$ as a motivating example.

11.1. **The GKM ring.** Without loss of generality we take $T = T_n$. As is standard in algebraic combinatorics, we denote by t_i the *negative* first Chern class $-c_1^T(\mathbb{C}_{\chi_i}) \in H^2_{T_n}(\mathrm{pt})$. We then have a homomorphism of abelian groups

$$\begin{array}{rcl} -c_1^T(\mathbb{C}_{(-)}) : \{\text{Characters of } T\} & \to & H^2_{T_n}(\mathrm{pt}) \\ \chi_1^{a_1}\chi_2^{a_2}\cdots\chi_n^{a_n} & \mapsto & -(a_1t_1 + a_2t_2 + \cdots + a_nt_n). \end{array}$$

The equivariant cohomology ring of a point is freely generated by the t_i and we identify $H^\bullet_{T_n}(\mathrm{pt}) = \mathbb{Z}[\mathbf{t}_n]$, so that all T_n-equivariant cohomology rings are $\mathbb{Z}[\mathbf{t}_n]$-algebras.

For an edge labeled graph G with vertices V, edges E, and edge labels given by a function

$$\chi\colon E \to \text{linear nonzero polynomials in } \mathbb{Z}[\mathbf{t}_n]/\pm,$$

we define the *graph cohomology ring* for G to be the $H^\bullet_{T_n}(\mathrm{pt})$-algebra

$$H^\bullet_{T_n}(G) \coloneqq \{(f_v)_{v\in V} \mid \chi(uv) \text{ divides } f_v - f_u \text{ for all } uv \in E\} \subset \mathbb{Z}[\mathbf{t}_n]^{\oplus V}$$

with multiplication defined pointwise.

We now describe sufficient conditions for $H^\bullet_{T_n}(G)$ to be the cohomology ring for a variety X. Say that X has a *good affine paving* if there is a filtration $\varnothing = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_\ell = X$ by closed subvarieties X_i such that for each $i \geq 1$ the following hold.

(1) The set $X_i \setminus X_{i-1}$ contains a unique T-fixed point p_i, and there is a T-equivariant isomorphism of algebraic varieties $X_i \setminus X_{i-1} \cong V_i$ for some linear T-representation V_i.

(2) The representation V_i decomposes into a direct sum of one-dimensional T-representations

$$V_i = \bigoplus_{j\in A_i} V_{i,j} \qquad \text{where} \qquad A_i \subset \{1,\dots,i-1\}$$

such that $\overline{V_{i,j}} = V_{i,j} \cup \{p_j\}$ and topologically $\overline{V_{i,j}} \cong \mathbb{P}^1$.

(3) For each $j \in A_i$, $f_{i,j} = -c_1^T(V_{i,j}) \in H^\bullet_{T_n}(G)$ satisfies:
  (a) $f_{i,j} \neq \pm f_{i,k}$ for $j \neq k$, and
  (b) $f_{i,j}$ is reduced, meaning that if $f_{i,j} = a_1t_1 + a_2t_2 + \cdots + a_nt_n$, then $\gcd(a_1, a_2, \dots, a_n) = 1$.

A good affine paving on X defines a *GKM graph*, which is an undirected, edge-labeled graph G_X with vertex set given by the fixed points $X^T = \{p_1, \dots, p_\ell\}$. For each one-dimensional summand $V_{i,j}$ in (2), G_X has an edge p_ip_j, and this edge is labeled by $-c_1^T(V_{i,j})$.

Example 11.1. Let $X = \mathrm{Fl}_n$. For any total order $w_1, \dots, w_{n!}$ of S_n that extends the Bruhat order, $X_k = \bigcup_{i=1}^k Bw_iB$ defines a good affine paving. Then $V_i \cong M(w)$ with $p_k = w_kB$, and following Equation (9.5) we have $j \in A_i$ if and only if $w_j = w_i(a\,b)$ for $(a, b) \in \mathrm{Inv}(w)$ and $f_{i,j} = t_{w(b)} - t_{w(a)}$. It follows that the GKM graph is obtained from $\mathrm{Cayley}(S_n)$ by labeling edges of the form w to $(i, j)w$ by $t_j - t_i$.

We now consider QFl_n. Let $\mathrm{Cayley}(\mathrm{NC}_n)$ denote the Hasse diagram of the Kreweras order on NC_n as defined in Section 2.2, which is an induced subgraph of $\mathrm{Cayley}(S_n)$.

Theorem 11.2. Theorem 9.7 gives a good affine paving of QFl_n, and its GKM graph is obtained from $\mathrm{Cayley}(\mathrm{NC}_n)$ by labeling edges of the form w to $(i, j)w$ by $t_j - t_i$.

Proof. Theorem 9.7 verifies condition (1) directly and shows that $\mathrm{QFl}_n^{T_n} = \{wB \mid w \in \mathrm{NC}_n\}$. Conditions (2) and (3) then follow from the fact that our filtration is obtained by intersecting QFl_n with a good affine paving for Fl_n. The same reasoning computes the edges and edge labels for the GKM graph. ☐

Theorem 11.3. If X has a good affine paving, then:

(1) X has a T-invariant homology basis given by the classes $[\overline{X_i \setminus X_{i-1}}] \in H_\bullet(X)$,

(2) $H_T^\bullet(X) \cong H_T^\bullet(G_X)$, the graph cohomology ring, and
(3) if $H_T^\bullet(X)$ is a free $\mathbb{Z}[\mathbf{t}_n]$-module then $H_T^\bullet(X)/\langle t_1, \ldots, t_n\rangle \cong H^\bullet(X)$.

While variants of Theorem 11.3 appear as [28, Theorem 2.3] and [25, Theorem 1.2.2], we did not find the exact statement required for QFl_n. Therefore we include a proof for completeness.

Proof. The first part follows from [21, see Example 1.9.1 and 19.1.11] (in fact the existence of the filtration where each $X_i \setminus X_{i-1}$ is isomorphic to an affine space suffices). The second part follows from [28, Theorem 3.1]. The last part follows from the implication $(iii) \implies (i)$ of [20, Theorem 1.1] after noting that $(S^1)^n$-equivariant cohomology is identical to T_n-equivariant cohomology because $\mathbb{C}^* \cong \mathbb{R} \times S^1$ and $\mathbb{R}$ is contractible. ☐

Example 11.4. By Theorem 11.3, $H_{T_n}^\bullet(\mathrm{Fl}_n)$ is isomorphic to the graph cohomology ring

$$H_{T_n}^\bullet\big(\mathrm{Cayley}(S_n)\big) = \{(f_w)_{w\in S_n} \mid t_i - t_j \text{ divides } f_w - f_{(i\,j)w} \text{ for all } (i \neq j)\} \subseteq \mathbb{Z}[\mathbf{t}_n]^{\oplus S_n}.$$

Moreover, the $[X^w] = [\overline{BwB}]$ give a homology basis for $H_\bullet(\mathrm{Fl}_n)$.

From Theorem 11.2, we obtain the following corollary about QFl_n.

Corollary 11.5. We have

$$H_{T_n}^\bullet(\mathrm{QFl}_n) \cong \{\big(f_w\big)_{w\in \mathrm{NC}_n} \mid t_b - t_a \text{ divides } f_w - f_{(a\,b)w} \text{ whenever } w, (a\,b)w \in \mathrm{NC}_n\} \subseteq \mathbb{Z}[\mathbf{t}_n]^{\oplus\,\mathrm{NC}_n}.$$

Further recall that the affine charts from Proposition 9.13 have closures $X(F)$ for $F \in \mathsf{Forest}_n$.

Corollary 11.6. The homology group $H_\bullet(\mathrm{QFl}_n)$ has a homology basis $[X(F)]$ for $F \in \mathsf{Forest}_n$.

Remark 11.7. Simple generalizations of Theorem 11.3 exist to compute generalized cohomology theories such as equivariant K-theory. However, determining a good basis for the resulting rings is a combinatorially specific task which does not transfer easily between theories.

11.2. **Flowup bases and double forest polynomials.** The following definition characterizes a distinguished subset of $H_{T_n}^\bullet(G)$; the reader should compare this to the definition of *generating family* by Guillemin–Zara [27, Definition 2.3] or that of *canonical classes* by Tymoczko [45, §2.2].

Definition 11.8. Let $G = (V, E, \chi)$ be a GKM graph. Given a partial ordering $\leq$ on V, a *flowup basis* for $H_{T_n}^\bullet(G)$ is a collection of elements $\{f_v \mid v \in V\} \subset H_{T_n}^\bullet(G)$ such that

(1) $(f_v)_w = 0$ if $v \not\leq w$, and
(2) $(f_v)_v = \pm \prod_{uv\in E \text{ and } u\leq v} \chi(uv)$.

The following fact is classical and a key tool for producing $\mathbb{Z}[\mathbf{t}_n]$-bases for $H_{T_n}^\bullet(G)$.

Proposition 11.9. Any flowup basis is a free $\mathbb{Z}[\mathbf{t}_n]$-basis for $H_{T_n}^\bullet(G)$.

Proof. An outline of the classical proof can be found in [45, §2.2]. ☐

In this section we describe a flowup basis for $H^\bullet_{T_n}(\mathrm{QFl}_n)$ using the double forest polynomials defined in [6]. For $w \in S_n$, let

$$\begin{array}{rcl} \mathrm{ev}_w : \mathbb{Z}[\mathbf{t}_n][\mathbf{x}_n] & \to & \mathbb{Z}[\mathbf{t}_n] \\ f(\mathbf{x}_n;\mathbf{t}_n) & \mapsto & f(t_{w(1)}, t_{w(2)}, \dots, t_{w(n)}; \mathbf{t}_n). \end{array} \tag{11.1}$$

Example 11.10. Consider the graph cohomology ring for Fl_n from Example 11.4. Taking $\leq$ to be the Bruhat order on S_n, we have a flowup basis with

$$(f_v)_w = \mathrm{ev}_w\left(\mathfrak{S}_v(\mathbf{x}_n;\mathbf{t}_n)\right), \qquad \text{for } v, w \in S_n$$

where $\mathfrak{S}_v(\mathbf{x}_n;\mathbf{t}_n)$ is the double Schubert polynomial; see Section 12.1. The fact that conditions (1) and (2) in Definition 11.8 are met is nontrivial but follows from the AJS–Billey theorem [2, 12].

The analogous statement for QFl_n makes use of the double forest polynomials defined in [6, §4], which we denote by $\mathfrak{P}_F(\mathbf{x}_n,\mathbf{t}_n) \in \mathbb{Z}[\mathbf{x}_n][\mathbf{t}_n]$ for each $F \in \mathsf{Forest}_n$. As with Schubert polynomials, we postpone a precise definition of double forest polynomials to Section 12.2.

Theorem 11.11. *Taking $\leq$ to be the Bruhat order restricted to NC_n, double forest polynomials define a flowup basis for the graph cohomology ring $H^\bullet_{T_n}(\mathrm{Cayley}(\mathrm{NC}_n))$. Specifically, for $v, w \in \mathrm{NC}_n$ we have*

$$(f_v)_w = \mathrm{ev}_w\left(\mathfrak{P}_F\right)$$

where $F \in \mathsf{Forest}_n$ is the unique forest such that $v = \mathrm{ForToNC}(F)$.

Proof. The claim follows from the analogue of the AJS–Billey theorem for double forest polynomials proved in [6, §8]. In particular, [6, Theorem 8.14] shows that $(f_v)_w = 0$ whenever $v \not\leq w$, and [6, Theorem 8.17] shows that $(f_v)_v = \prod_{(i,j)\in \mathrm{Inv}_{\mathrm{NC}}(v)} (t_{v(j)} - t_{v(i)})$. □

Figure 12 shows one element of the flowup basis for $H^\bullet_{T_n}(\mathrm{Cayley}(\mathrm{NC}_n))$.

12. The Borel presentation of $H^\bullet_{T_n}(\mathrm{QFl}_n)$

12.1. **Recollections on the equivariant cohomology of Fl_m.** We begin by briefly reviewing key aspects of the equivariant cohomology of the complete flag variety that are relevant to us. The reader is referred to [3] for a more complete exposition.

The following is due to Borel [15]. The equivariant cohomology ring $H^\bullet_{T_m}(\mathrm{Fl}_m)$ is generated by the character lattice of T_m and the Chern classes $c_1^{T_m}(\mathcal{F}_i/\mathcal{F}_{i-1}) \in H^2_{T_m}(\mathrm{Fl}_m)$. We therefore have a map

$$\begin{array}{rcl} \mathbb{Z}[\mathbf{t}_m][\mathbf{x}_m] & \to & H^\bullet_{T_m}(\mathrm{Fl}_m) \\ x_i & \mapsto & -c_1^{T_m}(\mathcal{F}_i/\mathcal{F}_{i-1}) \\ t_i & \mapsto & -c_1^{T_m}(\mathbb{C}_{\chi_i}). \end{array}$$

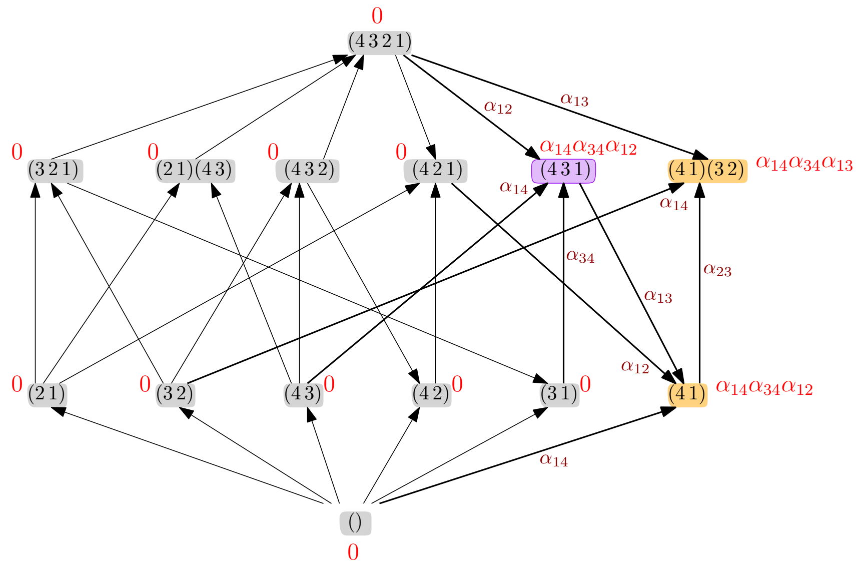


FIGURE 12. The flowup basis element of $H^\bullet_{T_4}(\mathrm{Cayley}(\mathrm{NC}_4))$ indexed by $(4\,3\,1)$. This is an evaluation of the double forest polynomial for $F = \widehat{F}(\mathsf{r}_1^- \mathsf{e}_1\mathsf{e}_1\mathsf{e}_2)$. For clarity we have set $\alpha_{ij} := t_j - t_i$.

The kernel of this map is the ideal $\mathrm{ESym}_m^+ = \langle f(x_1,\dots,x_m) - f(t_1,\dots,t_m) : f \in \mathrm{Sym}_m\rangle$, where Sym_m denotes the ring of symmetric polynomials in $\mathbf{x}_m$. Thus going forward we identify

$$H^\bullet_{T_m}(\mathrm{Fl}_m) \cong \mathbb{Z}[\mathbf{t}_m][\mathbf{x}_m]/\,\mathrm{ESym}_m^+. \tag{12.1}$$

In light of the GKM presentation of $H^\bullet_{T_m}(\mathrm{Fl}_m)$ given in Example 11.4, Borel's presentation has the following meaning. The inclusion of each torus fixed point $wB \in \mathrm{Fl}_m$ gives a pullback map

$$\mathrm{ev}_w : H^\bullet_{T_m}(\mathrm{Fl}_m) \to H^\bullet_{T_m}(wB) \cong \mathbb{Z}[\mathbf{t}_m].$$

In Borel's presentation, we have $\mathrm{ev}_w(x_i) = t_{w(i)}$ and $\mathrm{ev}_w(t_i) = t_i$, so we can identify ev_w with the map of the same name defined on polynomials in (11.1). In other words, the map

$$\prod_{w\in S_m} \mathrm{ev}_w : \mathbb{Z}[\mathbf{t}_m][\mathbf{x}_m] \to \mathbb{Z}[\mathbf{t}_m]^{\oplus S_m}$$

surjects onto the graph cohomology ring $H^\bullet_{T_m}(\mathrm{Cayley}(S_m)) \subseteq \mathbb{Z}[\mathbf{t}_m]^{\oplus S_m}$.

Now let X be an algebraic variety with an action of T_m and $Z \subseteq X$ a T_m-invariant subvariety. For a cohomology class $f \in H^\bullet_{T_m}(X)$, we denote the T_m-equivariant degree of f on Z by

$$\int_Z f := \kappa^Z_*(\iota^* f) = \kappa^X_*(\mathbb{1}_Z f) \in \mathbb{Z}[\mathbf{t}_m].$$

where κ^Z denotes the unique map $Z \to \mathrm{pt}$, $\mathbb{1}_Z \in H^\bullet_{T_m}(X)$ is the pushforward of $1 \in H^\bullet_{T_m}(Z)$ along the inclusion $\iota : Z \to X$, and the equality $\kappa^Z_*(\iota^* f) = \kappa^X_*(\mathbb{1}_Z f)$ is the push–pull formula.

Borel's presentation provides a simple way to compute the degree on a Schubert variety using the *divided difference operations* $\partial_i : \mathbb{Z}[\mathbf{t}_m][\mathbf{x}_m] \to \mathbb{Z}[\mathbf{t}_m][\mathbf{x}_m]$ defined by

$$\partial_i f(\mathbf{x}_m; \mathbf{t}_m) = \frac{f(\mathbf{x}_m, \mathbf{t}_m) - f(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_m; \mathbf{t}_m)}{x_i - x_{i+1}}.$$

Recall the forgetful map $\pi_i \colon \mathrm{Fl}_n \to \mathrm{GL}_m/P_i$ from Section 3.2. The following is due to Bernstein–Gelfand–Gelfand [8] and Demazure [19]; see [3, Chapter 10, Lemma 6.5] for textbook treatment.

Proposition 12.1. The map $(\pi_i)^*(\pi_i)_* : H^\bullet_{T_m}(\mathrm{Fl}_m) \to H^\bullet_{T_m}(\mathrm{Fl}_m)$ is given by $f \mapsto \partial_i f$. Moreover, for $w \in S_m$ with $w(i) < w(i+1)$, we have

$$\int_{X^{ws_i}} f = \int_{X^w} \partial_i f.$$

The *double Schubert polynomials* are the unique family of polynomials $\{\mathfrak{S}_w(\mathbf{x}_m; \mathbf{t}_m) \mid w \in S_m\}$ such that each $\mathfrak{S}_w(\mathbf{x}_m; \mathbf{t}_m)$ does not depend on x_m and moreover satisfies

$$\mathfrak{S}_w(\mathbf{t}_m; \mathbf{t}_m) = \delta_{w, \mathrm{id}_{S_m}} \qquad \text{and} \qquad \partial_i \mathfrak{S}_w(\mathbf{x}_m; \mathbf{t}_m) = \begin{cases} \mathfrak{S}_{ws_i}(\mathbf{x}_m; \mathbf{t}_m) & \text{if } w(i) > w(i+1), \\ 0 & \text{otherwise.} \end{cases}$$

We therefore have a T_m-equivariant Kronecker duality between double Schubert polynomials and the homology basis of Schubert cycles $[X^w]$ described in Example 11.4, as we have

$$\int_{X^w} \mathfrak{S}_v(\mathbf{x}_m; \mathbf{t}_m) = \delta_{w,w'}.$$

In the following subsections, we tell a parallel story for QFl_m using equivariantly quasisymmetric polynomials and double forest polynomials.

Remark 12.2. Borel's presentation for ordinary cohomology is $H^\bullet(\mathrm{Fl}_m) \cong \mathbb{Z}[\mathbf{x}_m]/\,\mathrm{Sym}_m^+$, which can be recovered from the equivariant version by setting $t_i \mapsto 0$, as is formalized in Theorem 11.3.

12.2. **Borel's theorem for equivariant quasisymmetry.** In order to state our analogue of Borel's theorem, we review several aspects of equivariant quasisymmetry from [6]. In Section 12.3 we will give geometric interpretations which motivate these definitions. We define the *equivariant Bergeron–Sottile maps*

$$\mathsf{R}_i^- f = f(x_1, \ldots, x_{i-1}, t_i, x_i, x_{i+1}, \ldots; \mathbf{t})$$
$$\mathsf{R}_i^+ f = f(x_1, \ldots, x_{i-1}, x_i, t_i, x_{i+1}, \ldots; \mathbf{t}),$$

and the *equivariant quasisymmetric divided difference*

$$\mathsf{E}_i f = \frac{\mathsf{R}_i^+ f - \mathsf{R}_i^- f}{x_i - t_i} = \mathsf{R}_i^- \partial_i f = \mathsf{R}_i^+ \partial_i f.$$

The *double forest polynomials* are then the unique family of polynomials $\{\mathfrak{P}_F(\mathbf{x}_n;\mathbf{t}_n) \mid F \in \mathsf{Forest}_n\}$ such that each $\mathfrak{P}_F(\mathbf{x}_n;\mathbf{t}_n)$ does not depend on x_n and moreover satisfies

$$\mathfrak{P}_F(\mathbf{t}_n;\mathbf{t}_n) = \delta_{F,\varnothing} \qquad \text{and} \qquad \mathsf{E}_i\mathfrak{P}_F = \begin{cases} \mathfrak{P}_G(\mathbf{x}_n;\mathbf{t}_{[n]/\{i\}}) & \text{if } F = G\cdot \mathsf{e}_i, \\ 0 & \text{otherwise} \end{cases}$$

where $\mathbf{t}_{[n]/\{i\}} = (t_1,\dots,t_{i-1},t_{i+1},\dots,t_n)$.

Remark 12.3. While it is not immediately obvious from the definition, [6, Corollary 4.8] shows that double forest polynomials have the following stability property: for each $F \in \mathsf{Forest}_{n-1}$, we have $\mathfrak{P}_F(\mathbf{x}_{n-1};\mathbf{t}_{n-1}) = \mathfrak{P}_{F\cdot \mathsf{r}_n^-}(\mathbf{x}_n;\mathbf{t}_n)$ as polynomials.

The ring of *equivariantly quasisymmetry polynomials* EQSym_n is

$$\mathrm{EQSym}_n = \{f(\mathbf{x}_n;\mathbf{t}_n) \in \mathbb{Z}[\mathbf{t}_n][\mathbf{x}_n] \mid \mathsf{R}_i^- f = \mathsf{R}_i^+ f \text{ for } 1 \le i \le n-1\} \subseteq \mathbb{Z}[\mathbf{t}_n][\mathbf{x}_n],$$

which is a subring of $\mathbb{Z}[\mathbf{t}_n][\mathbf{x}_n]$. We also denote $\mathrm{EQSym}_n^+ = \langle f(\mathbf{x}_n;\mathbf{t}_n) - f(\mathbf{t}_n;\mathbf{t}_n) \mid f \in \mathrm{EQSym}_n\rangle$.

Theorem 12.4. We have

$$H^\bullet_{T_n}(\mathrm{QFl}_n) \cong \mathbb{Z}[\mathbf{t}_n][\mathbf{x}_n]/\mathrm{EQSym}_n^+,$$

and moreover the double forest polynomials $\mathfrak{P}_F(\mathbf{x}_n;\mathbf{t}_n)$ where $F \in \mathsf{Forest}_n$ give a free $\mathbb{Z}[\mathbf{t}_n]$-basis for this ring.

Our proof primarily relies on results from Section 11, but we make use of one result which is deferred to Appendix A for ease of exposition.

Proof. By Theorem 11.3, we know that $H^\bullet_{T_n}(\mathrm{QFl}_n)$ is isomorphic to the graph cohomology ring $H^\bullet_{T_n}(\mathrm{Cayley}(\mathrm{NC}_n)) \subseteq \mathbb{Z}[\mathbf{t}_n]^{\mathrm{NC}_n}$. We prove the theorem by showing that the map

$$\begin{array}{rcl} \mathbf{ev}_{\mathrm{NC}} : \mathbb{Z}[\mathbf{t}_n][\mathbf{x}_n] & \to & \mathbb{Z}[\mathbf{t}_n]^{\mathrm{NC}_n} \\ f(\mathbf{x}_n;\mathbf{t}_n) & \mapsto & \left(\mathrm{ev}_w(f)\right)_{w\in \mathrm{NC}_n} \end{array}$$

induces the desired isomorphism onto the graph cohomology ring.

The image of $\mathbf{ev}_{\mathrm{NC}}$ is contained in the graph cohomology ring, as for any permutation w, $t_b - t_a$ divides $\mathrm{ev}_w(f) - \mathrm{ev}_{(a\,b)w}(f)$. Moreover, by Theorem 11.11, the double forest polynomials map to a free (flowup) basis of $H^\bullet_{T_n}(\mathrm{Cayley}(\mathrm{NC}_n))$, so $\mathbf{ev}_{\mathrm{NC}}$ is surjective.

What remains is to show that $\ker(\mathbf{ev}_{\mathrm{NC}}) = \mathrm{EQSym}_n^+$, which is Theorem A.1 in Appendix A. □

We finally consider ordinary cohomology, proving Theorem A. As in the introduction write QSym_n^+ for the ideal generated by quasisymmetric polynomials with no constant term.

Corollary 12.5. We have

$$H^\bullet(\mathrm{QFl}_n) \cong \mathbb{Z}[\mathbf{x}_n]/\mathrm{QSym}_n^+.$$

Moreover, the forest polynomials $\mathfrak{P}_F(\mathbf{x}_n;0)$ where $F \in \mathsf{Forest}_n$ give a free $\mathbb{Z}$-basis for this ring.

Proof. By Theorem 11.3, we can obtain $H^\bullet(\mathrm{QFl}_n)$ from $H^\bullet_{T_n}(\mathrm{QFl}_n)$ by performing a change of scalars along the homomorphism $\mathbb{Z}[\mathbf{t}_n] \to \mathbb{Z}$ given by $t_i \mapsto 0$. Applying the base change to our Borel presentation, we have a canonical identification between the images of $\mathbb{Z}[\mathbf{t}_n][\mathbf{x}_n]$, EQSym_n, EQSym_n^+, and $\mathfrak{P}_F(\mathbf{x}_n;\mathbf{t}_n)$ with $\mathbb{Z}[\mathbf{x}_n]$, QSym_n, QSym_n^+, and $\mathfrak{P}_F(\mathbf{x}_n;0)$. □

12.3. **Geometric realizations of $\mathsf{R}_i^\pm$ and E_i.** We now show that the equivariant Bergeron–Sottile maps correspond to equivariant cohomology operations that are adjoint to the building maps $\Psi_{i,j}$ and $\mathbb{P}_i$ defined in Section 3.

Fact 12.6. Let $\gamma : T \to T'$ be a coordinate projection between two algebraic tori T and T'. For X a variety with a T-action, we have

$$H^\bullet_T(X) = H^\bullet_T(\mathrm{pt}) \otimes_{H^\bullet_{T'}(\mathrm{pt})} H^\bullet_{T'}(X).$$

Furthermore, if $\Phi : X \to Y$ is a T'-equivariant map of varieties with a T'-action, then $\Phi^* : H^\bullet_T(Y) \to H^\bullet_T(X)$ and $\Phi_* : H^\bullet_T(X) \to H^\bullet_T(Y)$ are given by extending the corresponding maps on T'-equivariant cohomology by the identity map on $H^\bullet_T(\mathrm{pt})$.

As in Definition 3.2 we write $\gamma_i : T_m \to T_{m-1}$ for the ith coordinate projection and $\mathrm{Fl}_{m-1}^{\gamma_i}$ for Fl_{m-1} with the action of T_m induced by γ_i. With Fact 12.6, the GKM presentation of the T_{m-1}-equivariant cohomology of Fl_{m-1} implies that

$$H^\bullet_{T_m}(\mathrm{Fl}_{m-1}^{\gamma_i}) \cong \{(g_w)_{w\in S_{m-1}} | t_{j+\delta_{j\geq i}} - t_{k+\delta_{k\geq i}} \text{ divides } g_w - g_{(j,k)w}, \forall (j\neq k)\} \subset \mathbb{Z}[\mathbf{t}_m]^{\oplus S_{m-1}}$$

and the Borel presentation similarly implies

$$H^\bullet_{T_m}(\mathrm{Fl}_{m-1}^{\gamma_i}) \cong \frac{\mathbb{Z}[\mathbf{t}_m][\mathbf{x}_{m-1}]}{\langle f(\mathbf{x}_{m-1}) - f(t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_m) \mid f \in \mathrm{Sym}_{m-1}\rangle}.$$

The isomorphism between these two presentations is given by

$$f \mapsto \left(f(t_{w(1)+\delta_{w(1)\geq i}},\ldots,t_{w(m-1)+\delta_{w(m-1)\geq i}};\mathbf{t}_m)\right)_{w\in S_{m-1}}.$$

Proposition 12.7. The pullback map $\Psi_{i,j}^* \colon H^\bullet_{T_m}(\mathrm{Fl}_m) \to H^\bullet_{T_m}(\mathrm{Fl}_{m-1}^{\gamma_i})$ sends $t_k \mapsto t_k$ for all k, and

$$x_k \mapsto \begin{cases} x_{k-\delta_{k>j}} & k \neq j \\ t_i & k = j. \end{cases}$$

Proof. We have that $\Psi_{i,j}^* t_k = t_k$ holds since $\Psi_{i,j}$ is a T_m-equivariant map, so it suffices to show the result for x_k. To avoid notational overlap we will let $x_1^{\gamma_i},\ldots,x_{m-1}^{\gamma_i}$ denote the x_i generators in $\mathrm{Fl}_{m-1}^{\gamma_i}$. It suffices to show that for all $w \in S_{m-1}$

$$\mathrm{ev}_w(\Psi_{i,j}^* x_k) = \begin{cases} \mathrm{ev}_w(x^{\gamma_i}_{k-\delta_{k>j}}) & k \neq j \\ t_i & k = j. \end{cases}$$

Note that

$$\Psi_{i,j}(w)(k) = \begin{cases} w(k) + \delta_{w(k)\geq i} & k < j \\ t_i & k = j \\ w(k-1) + \delta_{w(k-1)\geq i} & k > j. \end{cases}$$

For $w \in S_{m-1}$ we have $\mathrm{ev}_w(x_\ell^{\gamma_i}) = \gamma_i^* t_{w(\ell)} = t_{w(\ell)+\delta_{w(\ell)\geq i}}$, and therefore for $k \neq j$ we have

$$\mathrm{ev}_w(x_{k-\delta_{k>j}}^{\gamma_i}) = t_{\Psi_{i,j}(w)(k)}.$$

On the other hand, because ev_w and $\mathrm{ev}_{\Psi_{i,j}(w)}$ are pullbacks under the inclusions $\{w\} \hookrightarrow \mathrm{Fl}_{m-1}^{\gamma_i}$ and $\{\Psi_{i,j}(w)\} \hookrightarrow \mathrm{Fl}_m$, we have

$$\mathrm{ev}_w(\Psi_{i,j}^* x_k) = \mathrm{ev}_{\Psi_{i,j}(w)}(x_k) = t_{\Psi_{i,j}(w)(k)} = \begin{cases} \mathrm{ev}_w(x_{k-\delta_{k>j}}^{\gamma_i}) & k \neq j \\ t_i & k = j. \end{cases} \qquad \square$$

Proposition 12.8. The maps $(\Psi_i^{\pm})^* : H_{T_m}^{\bullet}(\mathrm{Fl}_m) \to H_{T_m}^{\bullet}(\mathrm{Fl}_{m-1}^{\gamma_i})$ are given by

$$(\Psi_i^{\pm})^* f = \mathsf{R}_i^{\pm} f. \tag{12.2}$$

The map $(\Psi_i^{\pm})^*(\pi_i)^*(\pi_i)_* : H_{T_m}^{\bullet}(\mathrm{Fl}_m) \to H_{T_m}^{\bullet}(\mathrm{Fl}_{m-1}^{\gamma_i})$ is given by

$$(\Psi_i^{\pm})^*(\pi_i)^*(\pi_i)_* f = \mathsf{E}_i f. \tag{12.3}$$

Proof. Specializing the computation of $\Psi_{i,j}^*$ in Proposition 12.7 to $j = i, i+1$ gives (12.2). Since $(\pi_i)^*(\pi_i)_*$ computes ∂_i by Proposition 12.1, we obtain (12.3) from the identity $\mathsf{E}_i f = \mathsf{R}_i^{\pm}\partial_i f$. $\square$

Theorem 12.9. Let $Z \subset \mathrm{Fl}_{m-1}^{\gamma_i}$ be a T_m-invariant subvariety. Then for $f \in H_{T_m}^{\bullet}(\mathrm{Fl}_m)$, we have equalities of T_m-equivariant degrees

$$\int_{\Psi_i^{\pm} Z} f = \int_Z \mathsf{R}_i^{\pm} f \qquad \text{and} \qquad \int_{\mathbb{P}_i Z} f = \int_Z \mathsf{E}_i f. \tag{12.4}$$

Proof. As the $\Psi_i^{\pm}$ are closed embeddings we have $\mathbb{1}_{\Psi_i^{\pm} Z} = (\Psi_i^{\pm})_* \mathbb{1}_Z \in H_{T_m}^{\bullet}(\mathrm{Fl}_m)$. The first degree equation now comes from

$$\int_{\Psi_i^{\pm} Z} f = \int_{\mathrm{Fl}_m} \left((\Psi_i^{\pm})_* \mathbb{1}_Z\right) f = \int_{\mathrm{Fl}_{m-1}^{\gamma_i}} \mathbb{1}_Z (\Psi_i^{\pm})^* f = \int_Z \mathsf{R}_i^{\pm} f,$$

where we used the push-pull formula in all three equalities. For the second degree equation, because the $\pi_i \Psi_i$ are closed embeddings, we have

$$\mathbb{1}_{\mathbb{P}_i Z} = \mathbb{1}_{\pi_i^{-1}\pi_i \Psi_i Z} = \pi_i^*(\pi_i \Psi_i)_* \mathbb{1}_Z = \pi_i^*(\pi_i)_*(\Psi_i^-)_* \mathbb{1}_Z \in H_{T_m}^{\bullet}(\mathrm{Fl}_m).$$

Therefore using a similar argument as above,

$$\int_{\mathbb{P}_i Z} f = \int_{\mathrm{Fl}_m} \left(\pi_i^*(\pi_i)_*(\Psi_i^-)_* \mathbb{1}_Z\right) f = \int_{\mathrm{Fl}_{m-1}^{\gamma_i}} \mathbb{1}_Z (\Psi_i^-)^*(\pi_i)^*(\pi_i)_* f = \int_Z \mathsf{E}_i f,$$

where the last equality uses (12.3). This establishes the second part of (12.4). $\square$

12.4. **The degree on $X(\widehat{F})$ using the $\star$-monoid.** We now describe a combinatorial procedure for computing the degree on the $X(\widehat{F})$ varieties using the *$\star$-monoid $\mathcal{S}$* from [6, §9.2]. For each $A \subseteq [n]$, writing $A = (i_1 < \cdots < i_k)$ we define a map

$$\gamma_A \coloneqq \gamma_{i_1} \cdots \gamma_{i_k} : \quad T_n \quad \to \quad T_{n-|A|},$$

the coordinate projection away from $i_1, \ldots, i_k$. To compose these maps, we define an operation on subsets $A, B \subset [n]$ with $|A| + |B| \leq n$:

$$A \star B = \{([n] \setminus B)_i \mid i \in A\} \cup B,$$

where $([n] \setminus B)_i$ denotes the ith element of $[n] \setminus B$ in increasing order. The following proposition is straightforward and we omit the proof.

Proposition 12.10. For $A, B \subseteq [n]$ with $|A| + |B| \leq n$:, $\gamma_A \circ \gamma_B = \gamma_{A \star B}$. In particular, $\gamma_i \circ \gamma_A = \gamma_{i \star A}$.

We let $\mathrm{Fl}_{[n]/A}$ denote $\mathrm{Fl}^{\gamma_A}_{n-|A|}$, and let $X(\widehat{G})_{[n]/A}$ denote $X(\widehat{G}) \subset \mathrm{Fl}^{\gamma_A}_{n-|A|}$ equipped with the torus action of T_n induced by γ_A. Taken in conjunction with Fact 12.6, this allows us to transfer our results about the T_m-varieties Fl_m and $\mathrm{Fl}^{\gamma_i}_{m-1}$ to results about the T_n-varieties $\mathrm{Fl}_{[n]/A}$ and $\mathrm{Fl}_{[n]/(i \star A)}$.

For $A \subseteq [n]$, we denote $t_{i,A} = t_{([n] \setminus A)_i}$ and $\mathbf{t}_{[n]/A} = (t_{1,A}, t_{2,A}, \ldots, t_{n-|A|,A})$. We have isomorphisms

$$H^{\bullet}_{T_n}(\mathrm{Fl}_{[n]/A}) \cong \frac{\mathbb{Z}[\mathbf{t}_n][\mathbf{x}_{n-|A|}]}{\langle f(\mathbf{x}_{n-|A|}) - f(\mathbf{t}_{[n]/A}) : f \in \mathrm{Sym}_{n-|A|} \rangle}$$

and a GKM presentation

$$H^{\bullet}_{T_n}(\mathrm{Fl}_{[n]/A}) \cong \left\{ (g_w)_{w \in S_{n-|A|}} \mid t_{j,A} - t_{k,A} \text{ divides } g_w - g_{(j,k)w} \text{ for all } (j \neq k) \right\} \subset \mathbb{Z}[\mathbf{t}_n]^{\oplus S_{n-|A|}}$$

with the isomorphism between the Borel presentation and the GKM presentation given by

$$f \mapsto (f(t_{w(1),A}, \ldots, t_{w(n-|A|),A}; t_1, \ldots, t_n))_{w \in S_{n-|A|}}.$$

For $f \in \mathbb{Z}[\mathbf{t}_n][\mathbf{x}_n]$ define

$$\begin{aligned}
\mathsf{R}^-_{i,A} f(\mathbf{x}_n; \mathbf{t}_n) &= f(x_1, \ldots, x_{i-1}, t_{i,A}, x_i, x_{i+1}, \ldots, x_{n-1}; \mathbf{t}_n) \\
\mathsf{R}^+_{i,A} f(\mathbf{x}_n; \mathbf{t}_n) &= f(x_1, \ldots, x_{i-1}, x_i, t_{i,A}, x_{i+1}, \ldots, x_{n-1}; \mathbf{t}_n) \\
\mathsf{E}_{i,A} f(\mathbf{x}_n; \mathbf{t}_n) &= \frac{\mathsf{R}^+_{i,A} f(\mathbf{x}_n; \mathbf{t}_n) - \mathsf{R}^-_{i,A} f(\mathbf{x}_n; \mathbf{t}_n)}{x_i - t_{i,A}}.
\end{aligned}$$

Definition 12.11. For $A \subseteq [n]$ and $\Omega \in \mathrm{RESeq}_{n-|A|}$, let

$$[\Phi_\Omega]_A = \begin{cases} \mathrm{id} & \text{if } \Omega = \varnothing \\ [\Phi_{\Omega'}]_{i \star A} \circ \mathsf{R}^{\pm}_{i,A} & \text{if } \Omega = \Omega' \cdot \mathsf{r}^{\pm}_i \\ [\Phi_{\Omega'}]_{i \star A} \circ \mathsf{E}_{i,A} & \text{if } \Omega = \Omega' \cdot \mathsf{e}_i. \end{cases}$$

For $\Omega \in \mathrm{RESeq}_n$ we write $[\Phi_\Omega] = [\Phi_\Omega]_\varnothing$.

As was shown in [6, Theorem 10.5], the operation $[\Phi_\Omega]$ only depends on the colored Tamari equivalence class of the bicolored nested forest $\widehat{F}(\Omega)$ associated to Ω, we we can write $[\Phi_{\widehat{F}}]$ for $\widehat{F} \in \mathrm{BNestFor}_n$ without ambiguity.

Theorem 12.12. For $\widehat{F} \in \mathrm{BNestFor}_n$ and $f \in H^\bullet_{T_n}(\mathrm{Fl}_n)$, we have

$$\int_{X(\widehat{F})} f = [\Phi_{\widehat{F}}]f.$$

In particular, the double forest polynomials $\mathfrak{P}_F(\mathbf{x}_n; \mathbf{t}_n)$ are Kronecker dual to the homology basis $[X(F)]$ of $H^\bullet_{T_n}(\mathrm{QFl}_n)$ given in Corollary 11.6.

Proof. By Theorem 12.9, we have that for $A \subset [n]$ and $\widehat{F} = \widehat{G} \cdot \mathsf{x}_i \in \mathrm{BNestFor}_{n-|A|}$, we have

$$\int_{X(\widehat{F})_{[n]/A}} f = \begin{cases} \int_{X(\widehat{G})_{[n]/(i\star A)}} \mathsf{R}^{\pm}_{i,A} f & \text{if } \mathsf{x}_i = \mathsf{r}^{\pm}_i, \\ \int_{X(\widehat{G})_{[n]/(i\star A)}} \mathsf{E}_{i,A} f & \text{if } \mathsf{x}_i = \mathsf{e}_i \end{cases}$$

after which the theorem follows recursively from the definition of $[\Phi_{\widehat{F}}]$; see [6, §10] for more details on applying these operators to double forest polynomials. ☐

Example 12.13. Let $\Omega = \mathsf{r}^-_1 \mathsf{r}^+_1 \mathsf{e}_2 \mathsf{e}_1 \mathsf{r}^+_2 \mathsf{e}_3$. Then

$$[\Phi_\Omega] = \mathsf{R}^-_{1,\{1,2,3,4,5\}}\, \mathsf{R}^+_{1,\{1,2,3,5\}}\, \mathsf{E}_{2,\{1,2,3\}}\, \mathsf{E}_{1,\{2,3\}}\, \mathsf{R}^+_{2,\{3\}}\, \mathsf{E}_3.$$

A polynomial $a(\mathbf{t}_n) \in \mathbb{Z}[\mathbf{t}_n]$ is called *Graham-positive* if it lies in $\mathbb{Z}_{\geq 0}[t_2 - t_1, \dots, t_n - t_{n-1}]$. As shown by Graham [26], for any T-invariant subvariety $X \subset \mathrm{Fl}_n$, the decomposition

$$[X] = \sum a_w(\mathbf{t}_n)\, [X^w]$$

into Schubert cycles has Graham-positive coefficients $a_w(\mathbf{t}_n) = \int_X \mathfrak{S}_w(\mathbf{x}_n; \mathbf{t}_n)$. The Graham-positivity of $[\Phi_{\widehat{F}}]\mathfrak{S}_w(\mathbf{x}_n; \mathbf{t}_n)$ was shown through purely combinatorial means in [6, Theorem 11.4], which we can now interpret geometrically.

Corollary 12.14 ([6, Theorem 11.4]). For $F \in \mathsf{Forest}_n$, the coefficient

$$a_w(\mathbf{t}_n) = \int_{X(\widehat{F})} \mathfrak{S}_w(\mathbf{x}_n; \mathbf{t}_n) = [\Phi_{\widehat{F}}]\mathfrak{S}_w(\mathbf{x}_n; \mathbf{t}_n)$$

is Graham positive.

Remark 12.15. In [6, Theorem 11.6] we also show that the product of two double forest polynomials has a Graham positive expansion as a sum of double forest polynomials for combinatorial reasons. While the analogous result for double Schubert polynomials has a geometric explanation due to Graham [26], we do not have a geometric explanation for this positivity. We also show in [6, Theorem 11.4] that $[\Phi_{\widehat{F}}]\mathfrak{P}_G(\mathbf{x}_n; \mathbf{t}_n)$ is Graham positive, and we also do not have a geometric explanation for this positivity.

APPENDIX A. DOUBLE FOREST POLYNOMIALS

The purpose of this appendix is to give an alternative description for the ideal EQSym_n^+ as the kernel of the map

$$\mathbf{ev}_{\mathrm{NC}} = \prod_{w \in \mathrm{NC}_n} \mathrm{ev}_w : \mathbb{Z}[\mathbf{t}_n][\mathbf{x}_n] \to \mathbb{Z}[\mathbf{t}_n]^{\oplus \mathrm{NC}_n}.$$

Theorem A.1. We have $\mathrm{EQSym}_n^+ = \ker(\mathbf{ev}_{\mathrm{NC}})$ and as a consequence

$$\mathbb{Z}[\mathbf{t}_n][\mathbf{x}_n] = \mathrm{EQSym}_n^+ \oplus \bigoplus_{F \in \mathsf{Forest}_n} \mathbb{Z}[\mathbf{t}_n]\mathfrak{P}_F(\mathbf{x}_n; \mathbf{t}_n). \tag{A.1}$$

The proof appears at the end of the appendix.

Remark A.2. Theorem A.1 belongs to a family of results known as "Orbit Harmonics." In [5], the first two authors show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{QSCoinv}_n$ is the associated graded of the coordinate ring for the set of noncrossing partitions, considered as points using one-line notation. Specializing $t_i \mapsto i$ in Theorem A.1, we recover this result and find a new cohomological interpretation for it.

We begin by extending the definition of forest polynomials to a basis of the full polynomial ring $\mathbb{Z}[\mathbf{t}_n][\mathbf{x}_n]$. First define

$$\mathsf{Forest} = \bigsqcup_{m \geq 0} \mathsf{Forest}_m \Big/ \{F \sim G \text{ if } F \in \mathrm{BNestFor}_m \text{ and } G = F \cdot (\mathsf{r}_{m+1}^-)^k\},$$

so that we identify two forests if they differ only by some number of trailing isolated leaves. There is an obvious bijection between the internal nodes of any two forests identified in this manner, so we can speak of the internal nodes of $F \in \mathsf{Forest}$ without ambiguity. Say that an internal node v of $F \in \mathsf{Forest}$ is *terminal* if both of its children are leaves, and let

$$\mathrm{LTer}(F) = \{i \mid F \text{ has a terminal node with children } i \text{ and } i+1\},$$

so that $i \in \mathrm{LTer}(F)$ if and only if $F = G \cdot \mathsf{e}_i$ for some $G \in \mathsf{Forest}$. We then define

$$\mathrm{LTForest}_n = \{F \in \mathsf{Forest} \mid \mathrm{LTer}(F) \subseteq [n]\}.$$

We now consider double forest polynomials indexed by $\mathrm{LTForest}_n$. Recall that as described in Remark 12.3, we have $\mathfrak{P}_F(\mathbf{x}_m; \mathbf{t}_m) = \mathfrak{P}_{F \cdot (\mathsf{r}_{m+1}^-)^k}(\mathbf{x}_{m+k}; \mathbf{t}_{m+k})$ for all $m, k \geq 0$ and $F \in \mathsf{Forest}_m$. Thus each class $F \in \mathrm{LTForest}_n$ defines a unique forest polynomial in any set of x and t variables $(\mathbf{x}_m; \mathbf{t}_m)$ such that $m \geq \max \operatorname{supp} F$.

Definition A.3. For $F \in \mathrm{LTForest}_n$, the *n-truncated double forest polynomial* is defined to be

$$\mathfrak{P}_F(\mathbf{x}_n; \mathbf{t}_n) := \mathfrak{P}_F(\mathbf{x}_n, 0, \ldots, 0; \mathbf{t}_n, 0, \ldots, 0),$$

where the right-hand side is the specialization of the unique forest polynomial defined by F.

The truncated forest polynomials have the property that

$$\mathfrak{P}_F(\mathbf{t}_n;\mathbf{t}_n)=\delta_{F,\varnothing}\qquad\text{and}\qquad \mathsf{E}_i\mathfrak{P}_F=\begin{cases}\mathfrak{P}_G(\mathbf{x}_n;\mathbf{t}_{[n]/\{i\}},0) & \text{if } F=G\cdot \mathsf{e}_i,\\ 0 & \text{otherwise,}\end{cases}$$

for $1\le i\le n$, where $\mathsf{E}_1,\dots,\mathsf{E}_{n-1}$ are as defined in Section 12.2 and

$$\mathsf{E}_n f(\mathbf{x}_n;\mathbf{t}_n)=\frac{f(\mathbf{x}_n;\mathbf{t}_n)-f(x_1,\dots,x_{n-1},t_n;\mathbf{t}_n)}{x_n-t_n}.$$

In [6, Corollary 4.7 (2)] we show that

$$\mathbb{Z}[\mathbf{t}_n][\mathbf{x}_n]=\bigoplus_{F\in\mathrm{LTForest}_n}\mathbb{Z}[\mathbf{t}_n]\mathfrak{P}_F(\mathbf{x}_n;\mathbf{t}_n). \tag{A.2}$$

Thus as a consequence, we obtain a $\mathbb{Z}$-basis of $\mathbb{Z}[\mathbf{x}_n]$ consisting of (single) forest polynomials

$$\mathfrak{P}_F(\mathbf{x}_n):-\mathfrak{P}_F(\mathbf{x}_n;0).$$

We prove in [6, Corollary 4.6] that these are the same forest polynomials studied in [39, 41].

Theorem A.4 ([39, Theorem 9.7], [41, Theorem 3.7])**.** The forest polynomials $\mathfrak{P}_F(\mathbf{x}_n;0)$ with $F\in\mathrm{LTForest}_n$ and $n\in\mathrm{LTer}(F)$ are a $\mathbb{Z}$-basis for QSym_n^+.

We also define the set of *zigzag forests* to be

$$\mathsf{ZigZag}_n=\{F\in\mathrm{LTForest}_n\mid \mathrm{LTer}(F)=\{n\}\}.$$

The $\mathfrak{P}_F(\mathbf{x}_n;\mathbf{t}_n)$ for $F\in\mathsf{ZigZag}_n$ are called *double fundamental quasisymmetric polynomials*, and in [6, §4 and §7] we show that they form a basis for EQSym_n. Via [39], [6] also show that the $\mathfrak{P}_F(\mathbf{x}_n;0)$ are the classical fundamental quasisymmetric basis for QSym_n.

Proof of Theorem A.1. Theorem 11.11 shows that

$$\mathbb{Z}[\mathbf{t}_n][\mathbf{x}_n]=\ker(\mathbf{ev}_{\mathrm{NC}})\oplus\bigoplus_{F\in\mathsf{Forest}_n}\mathbb{Z}[\mathbf{t}_n]\mathfrak{P}_F(\mathbf{x}_n;\mathbf{t}_n),$$

so we only need to show that $\mathrm{EQSym}_n^+=\ker(\mathbf{ev}_{\mathrm{NC}})$. By (A.2), it suffices to show the inclusions

$$\bigoplus_{\varnothing\neq F\in\mathrm{LTForest}_n\setminus\mathsf{Forest}_n}\mathbb{Z}[\mathbf{t}_n]\mathfrak{P}_F(\mathbf{x}_n;\mathbf{t}_n)\subseteq\mathrm{EQSym}_n^+\subseteq\ker(\mathbf{ev}_{\mathrm{NC}}). \tag{A.3}$$

For the second inclusion in Equation (A.3) we use the fact, proved in [6, Theorem 7.1], that if $f(\mathbf{x}_n;\mathbf{t}_n)\in\mathrm{EQSym}_n$ then $\mathrm{ev}_w f=\mathrm{ev}_{\mathrm{id}} f$ for all $w\in\mathrm{NC}_n$, so clearly

$$\mathrm{EQSym}_n^+=\{f(\mathbf{x}_n;\mathbf{t}_n)-f(\mathbf{t}_n;\mathbf{t}_n)\mid f\in\mathrm{EQSym}_n\}\subseteq\ker(\mathbf{ev}_{\mathrm{NC}}).$$

We now establish the first inclusion by showing that for all $F\in\mathrm{LTForest}_n\setminus\mathsf{Forest}_n$ we have $\mathfrak{P}_F(\mathbf{x}_n;\mathbf{t}_n)\in\mathrm{EQSym}_n^+$. We proceed by induction on $|F|$. Our base case is $|F|=1$, wherein the assumption $n \in \mathrm{LTer}(T)$ implies that $F = (\mathsf{r}_1^-)^{n-1} \cdot \mathsf{e}_n$, so that $\mathfrak{P}_F(\mathbf{x}_n;\mathbf{t}_n) \in \mathrm{EQSym}_n$ as $F \in \mathsf{ZigZag}_n$ (alternatively as $\mathfrak{P}_F(\mathbf{x}_n;\mathbf{t}_n) = x_1 + \cdots + x_n - t_1 - \cdots - t_n$).

Now assume that $|F| > 1$. By [39, Theorem 9.7], the (single) forest polynomial $\mathfrak{P}_F(\mathbf{x}_n)$ lies in QSym_n^+, which is generated by the fundamental quasisymmetric polynomials $\mathfrak{P}_G(\mathbf{x}_n)$ for $\varnothing \neq G \in \mathsf{ZigZag}_n$. One may then write

$$\mathfrak{P}_F(\mathbf{x}_n) = \sum_{\varnothing \neq G \in \mathsf{ZigZag}_n} f_G(\mathbf{x}_n)\, \mathfrak{P}_G(\mathbf{x}_n).$$

As the double fundamental quasisymmetric polynomial $\mathfrak{P}_G(\mathbf{x}_n;\mathbf{t}_n)$ lies in EQSym_n^+ for $\varnothing \neq G \in \mathsf{ZigZag}_n$, the difference

$$\mathfrak{P}_F(\mathbf{x}_n;\mathbf{t}_n) - \sum_{\varnothing \neq G \in \mathsf{ZigZag}_n} f_G(\mathbf{x}_n)\mathfrak{P}_G(\mathbf{x}_n;\mathbf{t}_n) \tag{A.4}$$

can be written as a $\mathbb{Z}[\mathbf{t}_n]$-linear combination of double forest polynomials $\mathfrak{P}_H(\mathbf{x}_n;\mathbf{t}_n)$ with $H \in \mathrm{LTForest}_n \setminus \mathsf{Forest}_n$. Furthermore, the difference (A.4) contains no monomials consisting entirely of x-variables, so each H must have $|H| < |F|$. By induction, we have now expressed $\mathfrak{P}_F(\mathbf{x}_n;\mathbf{t}_n)$ as an element of EQSym_n^+, completing the proof. ☐

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