QUASISYMMETRIC DIVIDED DIFFERENCES

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ABSTRACT. We develop a quasisymmetric analogue of the combinatorial theory of Schubert polynomials and the associated divided difference operators. Our counterparts are "forest polynomials", and a new family of linear operators, whose theory of compositions is governed by forests and the "Thompson monoid". Our approach extends naturally to m-colored quasisymmetric functions.

We then give several applications of our theory to fundamental quasisymmetric functions, the study of quasisymmetric coinvariant rings and their associated harmonics, and positivity results for various expansions. In particular we resolve a conjecture of Aval–Bergeron–Li regarding quasisymmetric harmonics.

CONTENTS

Key words and phrases. Divided difference operators, harmonics, quasisymmetric polynomials, quasisymmetric coinvariants, volume polynomials.

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1. Introduction

The ring of quasisymmetric functions QSym, first introduced in Stanley's thesis [47] and further developed by Gessel [24], is ubiquitous throughout combinatorics; see [1] for a high-level explanation and [25] for thorough exposition. Truncating to finitely many variables $\{x_1, \ldots, x_n\}$ gives the ring of quasisymmetric polynomials QSym_n. The quasisymmetric polynomials are characterized by a weaker form of variable symmetry, and so contain the ring of symmetric polynomials Sym_n.

Letting Sym_n^+ denote the ideal in $\mathrm{Pol}_n := \mathbb{Z}[x_1, \ldots, x_n]$ generated by positive degree homogenous symmetric polynomials, the coinvariant algebra $\mathrm{Coinv}_n := \mathrm{Pol}_n \,/\, \mathrm{Sym}_n^+$ has been a central object of study for the past several decades. An important reason for this is its distinguished basis of Schubert polynomials [31] and the divided difference operators [12] that interact nicely with this family– see [10, 13, 15, 22, 23, 27, 30, 33] for a sampling of the combinatorics underlying this story. In fact Schubert polynomials lift to a basis of Pol_n. The close relationship between the combinatorics of symmetric and quasisymmetric polynomials leads to the natural question, first posed in [4], of what can be said about the analogous quotient $\mathrm{QSCoinv}_n := \mathrm{Pol}_n \,/ \mathrm{QSym}_n^+$, where QSym_n^+ is the ideal generated by positive degree homogenous quasisymmetric polynomials?

In this paper we develop a quasisymmetric analogue of the combinatorial theory of Schubert polynomials $\mathfrak{S}_w$ and the divided differences ∂_i which recursively generate them. The reader well-versed with the classical story should refer to Table 1 for a comparison. The role of Schubert polynomials $\mathfrak{S}_w$ is played by the *forest polynomials* $\mathfrak{P}_F$ of [38], and the role of the ∂_i operators are played by certain new *trimming operators* T_i. Just as Schubert polynomials generalize Schur polynomials, the forest polynomials generalize fundamental quasisymmetric polynomials, a distinguished basis of QSym_n. The duality between compositions of trimming operators and forest polynomials allows us to expand any polynomial in the basis of forest polynomials. In fact, a special case of our framework gives a remarkably simple method for directly extracting the coefficients of the expansion of a quasisymmetric polynomial in the basis of fundamental quasisymmetric polynomials.

The interaction between forest polynomials and trimming operators descends nicely to quotients by QSym_n^+, and we thus obtain a basis comprising certain forest polynomials for $\mathrm{QSCoinv}_n$ as well. Our techniques are robust enough to gain a complete understanding even in the case one quotients by homogenous quasisymmetric polynomials of degree at least k for any $k \geq 1$. By considering the adjoint operator to trimming under a natural pairing on the polynomial ring, we are able to easily construct QSym_n^+-harmonics, which turn out to have a basis given by the volume polynomials of certain polytopes, answering a question of Aval–Bergeron–Li [5].

In upcoming work [36, 37] we will tie this new combinatorics back to Schubert polynomials by casting new light on known expansions, as well as investigating the underlying geometry. We now proceed to a more detailed description of background as well as results.

Background and results. Let us briefly recall the classical theory of symmetric and Schubert polynomials. Let S_∞ be the permutations of $\mathbb{N} = \{1, 2, \ldots\}$ fixing all but finitely many elements, generated by the adjacent transpositions $s_i := (i, i+1)$, and identify S_n, the permutations of $[n] := \{1, \ldots, n\}$, with the subgroup $\langle s_1, \ldots, s_{n-1} \rangle$ fixing all $i \geq n+1$. Let $\mathrm{Pol}_n := \mathbb{Z}[x_1, \ldots, x_n]$, and denote by $\mathrm{Pol} := \bigcup_n \mathrm{Pol}_n = \mathbb{Z}[x_1, x_2, \ldots]$ for the ring of polynomials in infinitely many variables. S_n acts on Pol_n by permuting variable subscripts, and we denote by $\mathrm{Sym}_n \subset \mathrm{Pol}_n$ for the invariant subring of symmetric polynomials. Two of the most important tools for understanding Pol_n as a Sym_n-module are the $\mathbb{Z}$-basis of Pol given by the *Schubert polynomials* $\mathfrak{S}_w$ of Lascoux–Schützenberger [31], and the *divided difference* operators $\partial_i : \mathrm{Pol} \to \mathrm{Pol}$ given by

$$\partial_i(f) = \frac{f - s_i f}{x_i - x_{i+1}} \tag{1.1}$$

where s_i swaps x_i, x_{i+1}. They are related by the fact that Schubert polynomials are the unique family of homogenous polynomials indexed by $w \in S_\infty$ such that $\mathfrak{S}_{\mathrm{id}} = 1$, and denoting $\mathrm{Des}(w) = \{i : w(i) > w(i+1)\}$ for the descent set of w we have

$$\partial_i \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{ws_i} & \text{if } i \in \mathrm{Des}(w), \\ 0 & \text{otherwise.} \end{cases}$$

The divided differences satisfy the relations $\partial_i^2 = 0$, $\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$ and $\partial_i \partial_j = \partial_j \partial_i$ for $|i - j| \geq 2$. The monoid defined by this presentation is the *nilCoxeter monoid*. These relations imply that $\partial_{i_1} \cdots \partial_{i_k} = 0$ if $s_{i_1} \cdots s_{i_k}$ is not a reduced word, and we may define $\partial_w := \partial_{i_1} \cdots \partial_{i_k}$ where $s_{i_1} \cdots s_{i_k}$ is any reduced word for w. The operators $\{\partial_w \mid w \in S_\infty\}$ are the nonzero composites of the ∂_i, and if we let $\mathrm{ev}_0 f = f(0, 0, \ldots)$ denote the constant term map, then ∂_w and $\mathfrak{S}_w$ satisfy the duality

$$\mathrm{ev}_0 \, \partial_w \mathfrak{S}_{w'} = \delta_{w,w'}.$$

The following are a representative sampling of classical results concerning the relationship between Sym_n and Pol_n which are solved by Schubert polynomials and divided differences.

(Fact 1) (cf. [12, 31]) The Schubert polynomials
- $\{\mathfrak{S}_w \mid \mathrm{Des}(w) \subset [n]\}$ are a $\mathbb{Z}$-basis of Pol_n,
- $\{\mathfrak{S}_w \mid w \notin S_n \text{ and } \mathrm{Des}(w) \subset [n]\}$ are a $\mathbb{Z}$-basis for $\mathrm{Sym}_n^+ \subset \mathrm{Pol}_n$, the ideal generated by positive degree homogenous symmetric polynomials, and
- $\{\mathfrak{S}_w : w \in S_n\}$ are a $\mathbb{Z}$-basis for the coinvariant algebra $\mathrm{Coinv}_n := \mathrm{Pol}_n \,/\, \mathrm{Sym}_n^+$.

(Fact 2) (cf. [35])) The nil-Hecke algebra $\mathrm{End}_{\mathrm{Sym}_n}(\mathrm{Pol}_n)$ of endomorphisms $\phi : \mathrm{Pol}_n \to \mathrm{Pol}_n$ such that $\phi(fg) = f\phi(g)$ whenever $f \in \mathrm{Sym}_n$, is generated as a noncommutative algebra by the divided differences $\partial_1, \ldots, \partial_{n-1}$ and (multiplication by) $x_1, \ldots, x_n$.

(Fact 3) (cf. [12, 49]) The S_n-harmonics HSym_n, defined as the set of polynomials $f \in \mathbb{Q}[\lambda_1, \dots, \lambda_n]$ such that $g(\frac{d}{d\lambda_1}, \dots, \frac{d}{d\lambda_n})f = 0$ whenever $g \in \mathrm{Sym}_n$ is homogenous of positive degree, has a basis given by the "degree polynomials" $\mathfrak{S}_w(\frac{d}{d\lambda_1}, \dots, \frac{d}{d\lambda_n}) \prod_{i<j}(\lambda_i - \lambda_j)$ for $w \in S_n$.

A research program [4, 9, 40, 41] that has garnered attention in recent years revolves around answering the following question, which is the focus of this article.

Question 1.1. How do such results generalize to the *quasisymmetric polynomials* $\mathrm{QSym}_n \subset \mathrm{Pol}_n$?

Recall that $f \in \mathrm{QSym}_n$ if for any sequence $a_1, \dots, a_k \geq 1$, the coefficients of $x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}$ and $x_{j_1}^{a_1} \cdots x_{j_k}^{a_k}$ in f are equal whenever $1 \leq i_1 < \cdots < i_k \leq n$ and $1 \leq j_1 < \cdots < j_k \leq n$. Concretely, just as the ring of symmetric polynomials $\mathrm{Sym}_n \subset \mathrm{Pol}_n$ are invariant under the natural action of the symmetric group S_n permuting variable indices, the quasisymmetric polynomials QSym_n are the ring of invariants under the *quasisymmetrizing* action of S_n on Pol_n due to Hivert [28] where the transposition $(i, i+1)$ acts on monomials $\mathsf{x}^{\mathsf{c}} := x_1^{c_1} \cdots x_n^{c_n}$ by

$$\sigma_i \, \mathsf{x}^{\mathsf{c}} = \begin{cases} s_i \cdot \mathsf{x}^{\mathsf{c}} & \text{if } c_i = 0 \text{ or } c_{i+1} = 0, \\ \mathsf{x}^{\mathsf{c}} & \text{otherwise.} \end{cases} \tag{1.2}$$

Under this action, the orbit of x^{c} is the set of $\mathsf{x}^{\mathsf{c}'}$ where the ordered sequence of nonzero entries of c' is the same as for c, so e.g. $x_1^3x_2 + x_1^3x_3 + x_2^3x_3 \in \mathrm{QSym}_3$.

Pursuing this parallel further, Aval–Bergeron–Bergeron [4] studied the *quasisymmetric coinvariants* $\mathrm{QSCoinv}_n := \mathbb{Z}[x_1, \dots, x_n]/\mathrm{QSym}_n^+$ and produced a basis of monomials indexed by Dyck paths which in particular implies that the dimension of this space is given by the nth Catalan number Cat_n. Subsequent work of Aval [3] and Aval–Chapoton [6] generalized these results to a variant of quasisymmetric polynomials ${}^m\mathrm{QSym}_n$ in several sets of equisized variables called m-quasisymmetric polynomials. On the other hand, in [28] an isobaric quasisymmetric divided difference $\frac{x_{i+1}f - x_i\sigma_i f}{x_{i+1}-x_i}$ was studied, which was obtained by replacing s_i with σ_i in the usual isobaric divided difference $\frac{x_{i+1}f - x_is_i \cdot f}{x_{i+1}-x_i}$ used to define Grothendieck polynomials. Unfortunately, the operators obtained by replacing s_i with σ_i in the definition of ∂_i do not appear to behave well under composition, nor do they descend to $\mathrm{QSCoinv}_n$ (unlike $\partial_1, \dots, \partial_{n-1} \in \mathrm{End}_{\mathrm{Sym}_n}(\mathrm{Pol}_n)$ which descend to endomorphisms of Coinv_n).

We introduce a "quasisymmetric divided difference formalism"[1] built around linear operators $\mathsf{T}_i : \mathrm{Pol} \to \mathrm{Pol}$ satisfying the relations

$$\mathsf{T}_i\mathsf{T}_j = \mathsf{T}_j\mathsf{T}_{i+1} \text{ for } i > j$$

of the (positive) *Thompson monoid* [50], implying that composite operators T_F are indexed by *binary indexed forests* F [38, §3.1] (see also [8]). Just as $\ker(\partial_1|_{\mathrm{Pol}_n}) \cap \cdots \cap \ker(\partial_{n-1}|_{\mathrm{Pol}_n}) = \mathrm{Sym}_n$, we have

[1] Unrelated to the similarly named "quasisymmetric Schubert calculus" of [41].

$\ker(\mathsf{T}_1|_{\mathrm{Pol}_n}) \cap \cdots \cap \ker(\mathsf{T}_{n-1}|_{\mathrm{Pol}_n}) = \mathrm{QSym}_n$, justifying the name, and they descend to operators $\mathsf{T}_1, \ldots, \mathsf{T}_{n-1} : \mathrm{QSCoinv}_n \to \mathrm{QSCoinv}_{n-1}$. We will see that they interact with the family of forest polynomials $\mathfrak{P}_F$ [38] analogously to how ∂_i interacts with $\mathfrak{S}_w$ with the role of ws_i being played by a certain "trimmed forest" F/i, allowing us to tightly follow the classical theory to obtain analogues of all of the above results. In particular, we resolve the following long-standing question.

Question 1.2 (Aval–Bergeron–Li [5])**.** For HQSym_n the analogously defined "quasisymmetric harmonics", find a combinatorially defined basis and show that every element of HQSym_n is in the span of the partial derivatives of the degree $n-1$ quasisymmetric harmonics.

In fact, we will show ${}^m\mathrm{QSym}_n$-analogues of all of the above results. For each m we will define m-quasisymmetric divided differences $\mathsf{T}_i^{\underline{m}}$ satisfying the relations

$$\mathsf{T}_i^{\underline{m}}\,\mathsf{T}_j^{\underline{m}} = \mathsf{T}_j^{\underline{m}}\,\mathsf{T}_{i+m}^{\underline{m}} \text{ for } i > j$$

of the m-Thompson monoid $\mathsf{ThMon}^{\underline{m}}$, whose compositions $\mathsf{T}_F^{\underline{m}}$ are indexed by $(m+1)$*-ary indexed forests* $F \in \mathsf{For}^m$. They interact analogously with a new family of "m-forest polynomials" $\{\mathfrak{P}_F : F \in \mathsf{For}^m\}$ which when $m = 1$ specialize to the aforementioned forest polynomials of [38], and when $m \to \infty$ become the monomial basis.

Outline of article. We refer the reader to Table 1 for a quick overview of where the analogous constructions and results in the theory of symmetric functions appear in this paper. In Section 2 we describe a single-alphabet approach to m-quasisymmetric polynomials and introduce operators R_i and $\mathsf{T}_i^{\underline{m}}$ which can be used to characterize m-quasisymmetry. In Section 3 we describe in depth the combinatorics of certain $(m+1)$-ary forests For^m.

In Section 4 we show that the compositional structure on For^m is given by the "m-Thompson monoid" which has a simple presentation by generators and relations. In Section 5 we define the m-forest polynomials $\mathfrak{P}_F^{\underline{m}}$ for $F \in \mathsf{For}^m$ and show that the $\mathsf{T}_i^{\underline{m}}$ operators give a representation of the m-Thompson monoid, implying their composites $\mathsf{T}_F^{\underline{m}}$ are also indexed by $F \in \mathsf{For}^m$. In Section 6 we show that $\mathsf{T}_i^{\underline{m}}$ interacts with the m-forest polynomials $\mathfrak{P}_F^{\underline{m}}$ in an analogous way to how ∂_i interacts with the Schubert polynomials $\mathfrak{S}_w$. This key fact implies a number of spanning and independence properties for the m-forest polynomials. In particular, we show how to extract individual coefficients in m-forest polynomial expansions. In Section 7 we show a number of positivity results concerning these expansions. In Section 8 we show that the fundamental m-quasisymmetric polynomials are a subset of the m-forest polynomials, and use this to derive a simple formula for the fundamental m-quasisymmetric expansion of an arbitrary $f \in {}^m\mathrm{QSym}_n$. In Section 9 we show how the quasisymmetric divided difference formalism implies analogues of (Fact 1) and (Fact 2) above. In Section 10 we show the m-quasisymmetric analogue of (Fact 3), and resolve Question 1.2. Finally, in Appendix A we prove the interaction between $\mathsf{T}_i^{\underline{m}}$ and $\mathfrak{P}_F^{\underline{m}}$ by using the explicit combinatorial description of $\mathfrak{P}_F^{\underline{m}}$.

Table 1: Comparing the symmetric and m-quasisymmetric stories

§		${}^m\mathrm{QSym}_n$	Sym_n
2	**Divided differences**	T^m_i	∂_i
3	**Indexing combinatorics**	$F \in \mathsf{For}^m$	$w \in S_\infty$
		Fully supported forests For^m_n	S_n
		Forest code $\mathsf{c}(F)$	Lehmer code $\mathrm{lcode}(w)$
		Left terminal set $\mathrm{LTer}(F)$	Descent set $\mathrm{Des}(w)$
		F/i for $i \in \mathrm{LTer}(F)$	ws_i for $i \in \mathrm{Des}(w)$
		Trimming sequences $\mathrm{Trim}(F)$	Reduced words $\mathrm{Red}(w)$
		Zigzag forests $Z \in \mathsf{ZigZag}^m_n$	Grassmannian permutations λ
4	**Monoid**	m-Thompson monoid	nilCoxeter monoid
5	Pol-**basis**	Forest polynomials $\mathfrak{P}_F$	Schuberts $\mathfrak{S}_w$
	Composites	$\mathsf{T}^m_F = \mathsf{T}^m_{i_1} \cdots \mathsf{T}^m_{i_k}$ for $\mathbf{i} \in \mathrm{Trim}(F)$	$\partial_w = \partial_{i_1} \cdots \partial_{i_k}$ for $\mathbf{i} \in \mathrm{Red}(w)$
6	Pol_n-**basis**	$\{\mathfrak{P}_F \mid \mathrm{LTer}(F) \subset [n]\}$	$\{\mathfrak{S}_w \mid \mathrm{Des}(w) \subset [n]\}$
	Duality	$\mathrm{ev}_0\, \mathsf{T}^m_F \mathfrak{P}_G = \delta_{F,G}$	$\mathrm{ev}_0\, \partial_w \mathfrak{S}_{w'} = \delta_{w,w'}$
7	**Positive expansions**	$\mathfrak{P}_F\mathfrak{P}_H = \sum c^G_{F,H}\mathfrak{P}_G,\, c^G_{F,H} \geq 0$	$\mathfrak{S}_u\mathfrak{S}_w = \sum c^v_{u,w}\mathfrak{S}_v,\, c^v_{u,w} \geq 0$
8	**Invariant basis**	Fundamental m-qsyms $\mathfrak{P}_Z$	Schur polynomials s_λ
9	**Coinvariant basis**	$\{\mathfrak{P}_F \mid F \in \mathsf{For}^m_n\}$	$\{\mathfrak{S}_w \mid w \in S_n\}$
	Coinvariant action	$\mathsf{T}^m_i : {}^m\mathrm{QSCoinv}_n \to {}^m\mathrm{QSCoinv}_{n-m}$	$\partial_i : \mathrm{Coinv}_n \to \mathrm{Coinv}_n$
10	**Harmonic basis**	Forest volume polynomials	Degree polynomials

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2. m-QUASISYMMETRIC POLYNOMIALS

The ring QSym_n of quasisymmetric polynomials was recalled in the introduction. Given an integer $m \geq 1$, we will work in the more general context of *m-quasisymmetric polynomials*. Classically, these are defined as certain polynomials in the polynomial ring $\mathbb{Z}[\{z^{(j)}_1, z^{(j)}_2, \ldots\}_{1\leq j\leq m}]$ where $z^{(j)}_1, z^{(j)}_2, \ldots$ are considered the j'th colored variables [3, 6, 7, 42]. Most of these works are in the setting of formal power series instead of polynomials, but we can pass to the finite variable setting by truncating the variable sets. The m-quasisymmetric polynomials are usually defined as the linear span of a basis of "fundamental" m-quasisymmetric polynomials [29, §3.2]; these will ultimately play an important role as the analogue of Schur polynomials in our theory later on (Section 8).

We adopt a slightly different perspective on ${}^m\mathrm{QSym}_n$ which we have not seen in the existing literature despite its naturality. By arranging the variables in order

$$z_1^{(1)}, z_1^{(2)}, \dots, z_1^{(m)}, z_2^{(1)}, z_2^{(2)}, \dots$$

and relabeling them $x_1, x_2, \dots, x_m, x_{m+1}, x_{m+2}, \dots$ we obtain the following description.

Definition 2.1. The *m-quasisymmetric polynomials* ${}^m\mathrm{QSym}_n \subset \mathrm{Pol}_n$ are those polynomials such that for any sequence $a_1, \dots, a_k \geq 1$, the coefficients of $x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}$ and $x_{j_1}^{a_1} \cdots x_{j_k}^{a_k}$ are equal whenever $1 \leq i_1 < \cdots < i_k \leq n$ and $1 \leq j_1 < \cdots < j_k \leq n$ and $i_\ell \equiv j_\ell \bmod m$ for all $1 \leq \ell \leq k$.

When $m = 1$ we recover the quasisymmetric polynomials QSym_n recalled in the introduction. For arbitrary m the equivalence with the description given in [29, §3.2] is straightforward, and we will not comment on this further. Beyond the convenience of having only a single alphabet, the definition also highlights a behavior with respect to translation which is difficult to see in terms of colored alphabets.

Example 2.2. The following polynomials in Pol_4 are 1-quasisymmetric, 2-quasisymmetric, and 3-quasisymmetric respectively:

$$\begin{aligned} &x_3^2x_4 + x_2^2x_4 + x_1^2x_4 + x_2^2x_3 + x_1^2x_3 + x_1^2x_2, \\ &x_3^2x_4 + x_1^2x_4 + x_1^2x_2, \\ &x_3^2x_4. \end{aligned}$$

Let Codes denote the set of all sequences $(c_i)_{i\in\mathbb{N}}$ of nonnegative integers with *finite support*, i.e. there are only finitely many nonzero c_i. Given $\mathsf{c} \in$ Codes we let

$$\mathsf{x}^{\mathsf{c}} := \prod_{i\geq 1} x_i^{c_i}. \tag{2.1}$$

Now note that in Definition 2.1 the condition on the monomials whose coefficients must be equal can be rephrased as saying that the coefficients of x^{c} and $\mathsf{x}^{\mathsf{c}'}$ are equal if c' can be obtained from c by adding or removing consecutive strings of m zeros into c. We now set up an analogous theory of Hivert's quasisymmetrizing action (1.2) for m-quasisymmetry.

Definition 2.3. Let $s_{(i,i+m)}$ swap x_i with x_{i+m}. Then we define

$$\sigma_i^m \mathsf{x}^{\mathsf{c}} = \begin{cases} s_{(i,i+m)} \cdot \mathsf{x}^{\mathsf{c}} & \text{if } c_i = \cdots = c_{i+m-1} = 0 \text{ or } c_{i+1} = \cdots c_{i+m} = 0 \\ \mathsf{x}^{\mathsf{c}} & \text{otherwise.} \end{cases} \tag{2.2}$$

Note that while the $s_{(i,i+m)}$ generate a product of symmetric groups on the residue classes mod m in $\{1, \dots, n\}$, we do not claim anything about the group generated by the σ_i^m operators.

Lemma 2.4 ([28, Proposition 3.15] when $m = 1$)**.** $f(x_1, \ldots, x_n) \in \mathrm{Pol}_n$ is m-quasisymmetric if and only if $f = \sigma_1^m f = \cdots = \sigma_{n-m}^m f$.

Proof. If f is m-quasisymmetric then the coefficients of $x_1^{i_1} \cdots x_k^{i_k}$ and $\sigma_i^m x_1^{i_1} \cdots x_k^{i_k}$ are equal, which guarantees the string of equalities.

Conversely, suppose we have the string of equalities. It suffices to show that we can get from x^{c} to $\mathsf{x}^{\mathsf{c}'}$ by applying the σ_i^m operators whenever c and c' differ by inserting or removing consecutive strings of m zeros. To see this, first note that iteratively applying σ_i^m whenever the exponents of $x_i, \ldots, x_{i+m-1}$ are zero but the exponent of x_{i+m} is nonzero, we eventually terminate at a monomial x^{d} with d uniquely determined as follows. The sequence d has the same nonzero entries as c in the same order with the same residue classes mod m of the indices of the nonzero entries (since each σ_i^m preserves these properties), and furthermore there is no consecutive strings of m or more zeros before the last nonzero d_i. But c' has the same nonzero entries as c in the same order with the same residue classes mod m of the indices of the nonzero entries, so we can also reach x^{d} from $\mathsf{x}^{\mathsf{c}'}$ in the same way. □

2.1. ***m*-quasisymmetry via the Bergeron-Sottile map R_i.** It should be noted that σ_i^m does not respect multiplication even for $m = 1$, so for example this fixed point property does not immediately imply that ${}^m\mathrm{QSym}_n$ is a ring. The following result is at the heart of our understanding of quasisymmetric and m-quasisymmetric functions, which fixes this deficit of σ_i^m by using the equality of certain ring homomorphisms R_i^m to characterize m-quasisymmetry. Even for $m = 1$ this characterization does not seem to be widely known, although it was implicitly used in the study of the connection between quasisymmetric functions and James spaces by Pechenik–Satriano [41]. We call this the *Bergeron–Sottile map* as they were the first to introduce this [10], somewhat surprisingly, in the context of Schubert calculus (see also [11, 32]).

Definition 2.5. For $f \in \mathrm{Pol}$ we define

$$\mathsf{R}_i(f) = f(x_1, \ldots, x_{i-1}, 0, x_i, \ldots). \tag{2.3}$$

In other words, $\mathsf{R}_i(f)$ sets $x_i = 0$ and shifts $x_j \mapsto x_{j-1}$ for all $j \geq i+1$. In particular, for $f \in \mathrm{Pol}_n$ and $i \leq n$ we have $\mathsf{R}_i(f) \in \mathrm{Pol}_{n-1}$ is given by

$$\mathsf{R}_i(f) = f(x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1}). \tag{2.4}$$

Theorem 2.6. $f \in \mathrm{Pol}_n$ has $f \in {}^m\mathrm{QSym}_n$ if and only if $\mathsf{R}_1^m f = \cdots = \mathsf{R}_{n-m+1}^m f$.

Proof. We write 0^m for a list of m zeros, so that $\mathsf{R}_i^m(f) = f(x_1, \ldots, x_{i-1}, 0^m, x_i, \ldots, x_{n-m})$. For $1 \leq i \leq n-m$, and $\mathsf{c} = (c_1, \ldots, c_{n-m})$, the x^{c}-coefficient in $(\mathsf{R}_{i+1}^m - \mathsf{R}_i^m)f$ is the difference of the coefficients of $\mathsf{x}^{\mathsf{c}'}$ and $\mathsf{x}^{\mathsf{c}''}$ where $\mathsf{c}' = (c_1, \ldots, c_{i-1}, c_i, 0^m, c_{i+1}, \ldots, c_{n-m})$ and $\mathsf{c}'' = (c_1, \ldots, c_{i-1}, 0^m, c_i, c_{i+1}, \ldots, c_{n-m})$. This difference is 0 if $f \in {}^m\mathrm{QSym}_n$ and therefore $\mathsf{R}_{i+1}^m f - \mathsf{R}_i^m f = 0$ in that case.

Conversely, the vanishing of $(\mathsf{R}^m_{i+1} - \mathsf{R}^m_i)f$ implies by the above computation that for all c as above we have the difference of the $\mathsf{x}^{\mathsf{c}'}$ and $\mathsf{x}^{\mathsf{c}''}$ coefficients in f is 0. Noting that for each $\mathsf{d} = (d_1, \ldots, d_n)$ we have either $\mathsf{x}^{\mathsf{d}} = \sigma^m_i \mathsf{x}^{\mathsf{d}}$ or $\{\mathsf{x}^{\mathsf{d}}, \sigma^m_i \mathsf{x}^{\mathsf{d}}\} = \{\mathsf{x}^{\mathsf{c}'}, \mathsf{x}^{\mathsf{c}''}\}$ for some $\mathsf{c} = (c_1, \ldots, c_{n-m})$, we deduce $(\mathrm{id} - \sigma^m_i) \cdot f = 0$. Since this is true for $1 \leq i \leq n-m$, by Lemma 2.4 $f \in {}^m\mathrm{QSym}_n$. ☐

Corollary 2.7. ${}^m\mathrm{QSym}_n$ is a ring.

Proof. If $f, g \in {}^m\mathrm{QSym}_n$ then $\mathsf{R}^m_i(fg) = \mathsf{R}^m_i(f)\mathsf{R}^m_i(g) = \mathsf{R}^m_{i+1}(f)\mathsf{R}^m_{i+1}(g) = \mathsf{R}^m_{i+1}(fg)$ for $1 \leq i \leq n-m$, so $fg \in {}^m\mathrm{QSym}_n$. ☐

The case $m = 1$ of the preceding result is classical; see [25, Proposition 5.1.3] for a proof in the setting of quasisymmetric functions. The typical proof that QSym_n is closed under multiplication involves identifying an explicit basis whose multiplication can be explicitly computed. In contrast our algebraic proof only uses that R^m_i respects multiplication.

2.2. ***m*-quasisymmetric divided differences T^m_i.** For $f \in \mathrm{Pol}$ consider the long range divided difference

$$\partial^m_i(f) = \frac{f - f(x_1, \ldots, x_{i-1}, x_{i+m}, x_{i+1}, \ldots, x_{i+m-1}, x_i, x_{i+m+1}, \ldots)}{x_i - x_{i+m}}, \tag{2.5}$$

specializing to ∂_i for $m = 1$. For i in a fixed residue class mod m these operators are just divided difference operators on the x_i variables in this residue class, and in particular the ∂^m_i with i in different residue classes commute with each other.

The quasisymmetric analogues of ∂^m we now define do not have this degenerate feature.

Definition 2.8. We define the *ith m-quasisymmetric divided difference* $\mathsf{T}^m_i : \mathrm{Pol} \to \mathrm{Pol}$ by any of the equivalent expressions

$$\mathsf{T}^m_i f := \mathsf{R}^m_i \partial^m_i f = \mathsf{R}^m_{i+1} \partial^m_i f = \frac{\mathsf{R}^m_{i+1} f - \mathsf{R}^m_i f}{x_i}, \tag{2.6}$$

and when $m = 1$ we write $\mathsf{T}_i := \mathsf{T}^1_i$.

For $f \in \mathrm{Pol}_n$ we have explicitly for $1 \leq i \leq n-m$ that $\mathsf{T}^m_i f \in \mathrm{Pol}_{n-m}$ is given by

$$\mathsf{T}^m_i(f) = \frac{f(x_1, \ldots, x_{i-1}, x_i, 0^m, x_{i+1}, \ldots, x_{n-m}) - f(x_1, \ldots, x_{i-1}, 0^m, x_i, x_{i+1}, \ldots, x_{n-m})}{x_i}. \tag{2.7}$$

Remark 2.9. We can express T^m_i in terms of T_i and R_i via the identity

$$\mathsf{T}^m_i = \mathsf{T}_i \mathsf{R}^{m-1}_{i+1}.$$

Theorem 2.10. $f \in \mathrm{Pol}_n$ is m-quasisymmetric if and only if $\mathsf{T}^m_1 f = \cdots = \mathsf{T}^m_{n-m} f = 0$.

Proof. This is a rephrasing of Theorem 2.6 since we have $\mathsf{T}^m_i(f) = 0 \iff \mathsf{R}^m_{i+1}(f) = \mathsf{R}^m_i(f)$. ☐

Example 2.11. Let $f = x_3^2x_4 + x_1^2x_4 + x_1^2x_2$. Then we can verify by inspection that $f \in {}^2\mathrm{QSym}_4$. Alternatively, we can compute

$$\mathsf{T}_1^2(f) = \frac{1}{x_1}(f(x_1,0,0,x_2) - f(0,0,x_1,x_2)) = -x_1x_2 + x_1x_2 + 0 = 0$$
$$\mathsf{T}_2^2(f) = \frac{1}{x_2}(f(x_1,x_2,0,0) - f(x_1,0,0,x_2)) = 0 - x_1^2 + x_1^2 = 0,$$

which by Theorem 2.10 implies $f \in {}^2\mathrm{QSym}_4$.

The twisted Leibniz rule for ∂_i^m is

$$\partial_i^m(fg) = \partial_i^m(f)g + (s_{(i,i+m)} \cdot f)\partial_i^m(g). \tag{2.8}$$

We describe an analogous rule for T_i^m.

Lemma 2.12 (Twisted Leibniz rule)**.** For $f, g \in \mathrm{Pol}$ we have

$$\mathsf{T}_i^m(fg) = \mathsf{T}_i^m(f)\mathsf{R}_{i+1}^m(g) + \mathsf{R}_i^m(f)\mathsf{T}_i^m(g)$$

Indeed The equality follows by dividing both sides of the following equality by x_i.

$$\mathsf{R}_{i+1}^m(fg) - \mathsf{R}_i^m(fg) = \mathsf{R}_{i+1}^m(f)\mathsf{R}_{i+1}^m(g) - \mathsf{R}_i^m(f)\mathsf{R}_{i+1}^m(g) + \mathsf{R}_i^m(f)\mathsf{R}_{i+1}^m(g) - \mathsf{R}_i^m(f)\mathsf{R}_i^m(g). \tag{2.9}$$

3. Indexed forests

We now discuss our primary data structure: m-indexed forests. For $m = 1$ these forests, along with several combinatorial properties, already appear in [38]. We shall throughout compare our notions with their classical S_∞–counterparts, to which we refer the reader to [34, 35, 48].

The collection of m-indexed forests For^m plays for T_i^m the analogous role of S_∞ for ∂_i. In Section 4 we will describe a natural monoid product $F \cdot G$ on For^m and in Section 5 it will be shown that composites of the T_i^m are indexed by $F \in \mathsf{For}^m$ in such a way that $\mathsf{T}_F^m\mathsf{T}_G^m = \mathsf{T}_{F\cdot G}^m$.

3.1. $(m+1)$-ary trees and m-indexed forests.

A *rooted plane tree* is a rooted tree where the children of a node v are ordered $v_0, \dots, v_k$ with v_0 the *leftmost* child and v_k called the *rightmost* child. An $(m+1)$*-ary rooted plane tree* T is a rooted plane tree where each node has either $m+1$ children $v_0, \dots, v_m$ or 0 children. In the former case the node is called *internal* and we write $\mathrm{IN}(T)$ for the set of internal nodes. Otherwise we say that the node is a leaf. We write $|T| = |\mathrm{IN}(T)|$, and refer to this as the *size* of T.

We write $*$ for the *trivial* singleton $(m+1)$-ary rooted tree with $|*| = 0$, and all other trees we call *nontrivial*. Note $\mathrm{IN}(*) = \varnothing$, and the unique node of $*$ is both a root node and a leaf.

We are now ready to introduce our main combinatorial object.

Definition 3.1. An *m-indexed forest* is an infinite sequence $T_1, T_2, \dots$ of $(m+1)$-ary trees where all but finitely many of the trees are $*$. We write For^m for the set of all m-indexed forests. When $m = 1$ (i.e. the forests are binary) we write $\mathsf{For} := \mathsf{For}^1$.

As an example, Figure 1 depicts an $F \in \mathsf{For}^2$. Note that by labeling leaves of each tree successively, we identify the leaves of F with $\mathbb{N}$, associating the i'th leaf to $i \in \mathbb{N}$.

Notions that apply to trees are now inherited by indexed forests. We write $\mathrm{IN}(F) = \bigcup_{i=1}^{\infty} \mathrm{IN}(T_i)$, and $|F| = |\mathrm{IN}(F)|$. In this way, the totality of nodes in F is identified with $\mathrm{IN}(F) \sqcup \mathbb{N}$. For $v \in \mathrm{IN}(F)$ we always write

$$v_0, \ldots, v_m \in \mathrm{IN}(F) \sqcup \mathbb{N}$$

for the children of v from leftmost to rightmost.

We say that a node $v \in \mathrm{IN}(F)$ is *terminal* if all of its children are leaves. The forest all of whose trees are trivial is called the *empty* forest and denoted by $\varnothing$. Finally, for $F \in \mathsf{For}^m$ we define its *support* $\mathrm{supp}(F)$ to be the set of leaves in $\mathbb{N}$ associated to the nontrivial trees in F, and for fixed $n \geq 1$ we define the class of forests

$$\mathsf{For}_n^m = \{F \in \mathsf{For}^m \mid \mathrm{supp}(F) \subset [n]\}.$$

This class of forests plays the role in our theory of $S_n \subset S_\infty$ for fixed n.

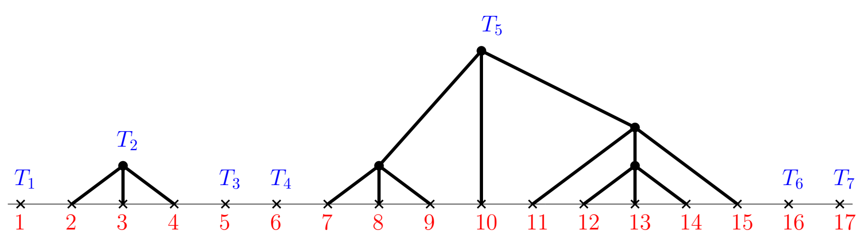

FIGURE 1. A 2-indexed forest in For_{15}^2

In Figure 1, T_2 and T_5 are the only nontrivial trees. The bottom labels in red are the leaves, represented by crosses, identified with $\mathbb{N}$. The size $|F|$ of F equals 5 as there are five internal nodes. Furthermore there are three terminal nodes. Finally the support $\mathrm{supp}(F)$ equals $\{2,3,4\} \sqcup \{7,8,\ldots,14,15\}$, and it follows that F belongs to For_n^2 for any $n \geq 15$.

Remark 3.2. In the case $m = 1$, indexed forests were introduced in [38] with a slightly different notion of support, defined as follows. Given a finite set S of positive integers, an indexed forest with support S is the data of a plane binary tree with leaves $\{a, \ldots, b\}$ for each maximal interval $I = \{a, a+1, \ldots, b-1\}$ in S. By ordering these binary trees from left to right, and interspersing trivial trees given by those leaf labels that are not part of any nontrivial tree, we obtain objects clearly equivalent to the 1-indexed forests of Definition 3.1. This notion of support from [38] was adapted to a "parking function" interpretation, and our notion of support is computed by replacing S with $S \cup \{x : x - 1 \in S\}$. Although this is a slightly coarser notion, it is more suited to the perspective of the current work and extends more naturally to $m > 1$.

The cardinality of For_n^m is given by *Raney numbers* [44], a well known generalization of the Catalan numbers. This follows from an application of the *Cycle Lemma* (cf. [20, §2.1]) when we note the following result.

Fact 3.3. $F \in \mathsf{For}_n^m$ if and only if the leaves of the first $n - m|F|$ trees of F are $\{1, \ldots, n\}$ and these contain all nontrivial trees.

Lemma 3.4. Write $n = mq + r$ where $0 \leq r \leq m-1$. Then we have

$$|\mathsf{For}_n^m| = \frac{r+1}{n+1}\binom{n+q}{q} = \frac{r+1}{(m+1)q+r+1}\binom{(m+1)q+r+1}{q}. \tag{3.1}$$

Observe that when $r = 0$, i.e. n is divisible by m, the right-hand side above becomes the *Fuss–Catalan* number $\frac{1}{n+1}\binom{(m+1)q}{q}$. This at $m = 1$ recovers the usual Catalan number Cat_n, which we know to be the dimension of $\mathrm{QSCoinv}_n$ by [4], a fact that will be reproved in Section 9.

3.2. **The code** $\mathsf{c}(F)$**.** We now discuss an encoding of m-indexed forests by sequences of nonnegative numbers, playing the role of the *Lehmer code* on S_∞. Recall that the Lehmer code of $w \in S_\infty$ is the sequence $\mathrm{lcode}(w) := (c_i)_{i\in\mathbb{N}} \in \mathsf{Codes}$ where $c_i = \#\{j > i \mid w(i) > w(j)\}$.

Let $F \in \mathsf{For}^m$. We define the *flag* $\rho_F : \mathrm{IN}(F) \to \mathbb{N}$ by setting $\rho_F(v)$ to be the label of the leaf obtained by going down left edges starting from v.

Definition 3.5. The *code* $\mathsf{c}(F)$ is defined as

$$\mathsf{c}(F) = (c_i)_{i\in\mathbb{N}} \text{ where } c_i = |\{v \in \mathrm{IN}(F) \mid \rho_F(v) = i\}|.$$

Theorem 3.6. The map $\mathsf{c} : \mathsf{For}^m \to \mathsf{Codes}$ is a bijection.

Proof. There is a well-known encoding (see for example [21]) of $(m+1)$-ary trees with n internal nodes (equivalently having $mn+1$ leaves) and "m-Dyck paths", paths taking steps $U = (1, m)$ and $D = (1, -1)$ which start at $(0, 0)$ and end at $(2mn, 0)$ while always staying weakly above the x-axis. Writing the path as $U^{c_1}DU^{c_2}D\cdots U^{c_n}$, under this encoding the analogously defined code of the tree is $(c_1, \ldots, c_{mn}, c_{mn+1} = 0)$, where c_i counts the number of internal nodes whose leftmost leaf descendant is i. Given an m-indexed forest $T_1, T_2, \ldots$, concatenating the tree codes gives the code of the associated forest in For^m.

Now given a code $(c_1, c_2, \ldots)$, define the path $P = U^{c_1}DU^{c_2}D\cdots$. Then for any $k \geq 1$, the section of P from the first time it touches $y = -k$ to the last time it stays weakly above $y = -k$ is an m-Dyck path, corresponding to a tree T_k via the encoding above. Since P ends with an infinite sequence of D's, T_k is trivial for k large enough. Thus $F = T_1, T_2, \ldots$ is an m-indexed forest, and $(c_1, c_2, \ldots) \mapsto F$ is the inverse of $F \mapsto \mathsf{c}(F)$. $\square$

In particular any mathematical object indexed by For^m can be indexed by Codes.

3.3. **The left terminal set** $\mathrm{LTer}(F)$**.** We now proceed to define a set of indices attached to an m-indexed forest that shall play a role analogous to the descent set $\mathrm{Des}(w)$ for a permutation $w \in S_\infty$. For $F \in \mathsf{For}^m$, let

$$\mathrm{LTer}(F) := \{\rho_F(v) \mid v \text{ a terminal node in } F\}. \tag{3.2}$$

These are precisely the leaves arising as the leftmost children of terminal nodes. For F in Figure 1 we have $\mathrm{LTer}(F) = \{2, 7, 12\}$. In terms of $\mathsf{c}(F) = (c_i)_{i\in\mathbb{N}}$, the following criterion is an immediate consequence of the prefix traversal aspect of our bijection:

$$i \in \mathrm{LTer}(F) \iff c_i > 0 \text{ and } c_{i+1} = \cdots = c_{i+m} = 0. \tag{3.3}$$

In particular,

$$i, j \in \mathrm{LTer}(F) \implies |i - j| \geq m + 1. \tag{3.4}$$

3.4. **Left and right terminally supported forests.** The following class of left-terminally supported forests plays the role of the set of permutations $w \in S_\infty$ with $\mathrm{Des}(w) \subset [n]$ or equivalently $\mathrm{lcode}(w) = (c_1, \ldots, c_n, 0, \ldots)$.

$$\begin{aligned}
\mathsf{LTFor}_n^m &:= \{F \in \mathsf{For}^m \mid \mathrm{LTer}(F) \subset [n]\} \\
&= \{F \in \mathsf{For}^m \mid \mathsf{c}(F) = (c_1, \ldots, c_n, 0, \ldots)\} \\
&= \{F \in \mathsf{For}^m \mid \rho_F(v) \leq n \text{ for all } v \in \mathrm{IN}(F)\}
\end{aligned}$$

where the second equality follows from (3.3). LTFor_n^m thus consists of those $F \in \mathsf{For}^m$ whose leaves arising as leftmost children of internal nodes are supported on $[n]$, or equivalently such that the leftmost leaf descendant of any internal node lies in $[n]$. This latter identification implies

$$\mathsf{For}_n^m \subset \mathsf{LTFor}_n^m.$$

For the forest F in Figure 1, we have $F \in \mathsf{LTFor}_n^2$ for all $n \geq 12$.

More generally for any subset $A \subset \mathbb{N}$, an analogue of the set of permutations $w \in S_\infty$ with $\mathrm{Des}(w) \subset A$ is

$$\mathsf{LTFor}_A^m = \{F \in \mathsf{For}^m \mid \mathrm{LTer}(F) \subset A\},$$

and for $A = [n]$ we recover $\mathsf{LTFor}_n^m = \mathsf{LTFor}_A^m$.

The following class of right-terminally supported forests play the role of the set of permutations $w \in S_\infty$ with $\mathrm{Des}(w) \cap [n-1] = \varnothing$.

For a given $n \geq 1$ and $F \in \mathsf{For}^m$, say that an internal node $v \in \mathrm{IN}(F)$ is supported on $[n]$ if all leaves that are descendants of v lie in $[n]$. In particular $F \in \mathsf{For}_n^m$ if and only if all its internal nodes are supported on $[n]$. In contrast, let

$$\begin{aligned}\mathsf{RTFor}^m_{>n} :=&\{F \in \mathsf{For}^m \mid \text{ no } v \in \mathrm{IN}(F) \text{ is supported on } [n]\}.\\ =&\{F \in \mathsf{For}^m \mid v_m > n \text{ for all terminal } v \in \mathrm{IN}(F)\}\\ =&\mathsf{LTFor}^m_{\{n-m+1,n-m+2,\dots\}}.\end{aligned}$$

To reorient the reader, in terms of leaves we note the following characterizations: $F \in \mathsf{RTFor}^m_{>n}$ (*resp.* LTFor^m_n, *resp.* For^m_n) if and only if all rightmost leaves of F are $> n$ (*resp.* all leftmost leaves are $\leq n$, *resp.* all rightmost leaves are $\leq n$).

3.5. **Zigzag forests.** The final class of forests we consider are the "zigzag forests", which will play an analogous role to the n-Grassmannian permutations $\mathsf{Grass}_n := \{w \in S_\infty \mid \mathrm{Des}(w) \subset \{n\}\}$.

$$\begin{aligned}\mathsf{ZigZag}^m_n &:= \mathsf{LTFor}^m_n \cap \mathsf{RTFor}^m_{>n} = \mathsf{LTFor}^m_{\{n-m+1,n-m+2,\dots,n\}}\\ &= \{F \in \mathsf{For}^m \mid \#\,\mathrm{LTer}(F) \leq 1 \text{ and } \mathrm{LTer}(F) \subset \{n-m+1,\dots,n\}\}.\end{aligned}$$

An element of ZigZag^2_6 is shown in Figure 2. Note that it also belongs to ZigZag^2_7.

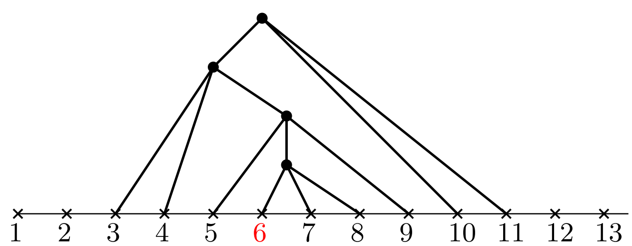

FIGURE 2. A forest $F \in \mathsf{For}^2$ in ZigZag^2_6 and ZigZag^2_7 with $\mathrm{LTer}(F) = \{6\}$

By (3.4) the condition that $\#\,\mathrm{LTer}(F) \leq 1$ is redundant in light of the containment $\mathrm{LTer}(F) \subset \{n-m+1,\dots,n\}$. We refer to these as *zigzag forests*, since they consist of at most one nontrivial tree whose internal nodes form a chain. From the definition it is clear that we have

$$\mathsf{ZigZag}^m_n \subset \mathsf{LTFor}^m_n.$$

When $m = 1$ we write ZigZag_n, which are those forests $F \in \mathsf{For}$ that are empty or have $\mathrm{LTer}(F) = \{n\}$. These were previously considered under the name linear tree in [38].

3.6. **Trimming and blossoming.** We introduce two elementary operations of "blossoming" and "trimming" on forests, which play the role of the transformations $w \mapsto ws_i$ for $w \in S_\infty$ when $i \notin \mathrm{Des}(w)$ and $i \in \mathrm{Des}(w)$ respectively.

Definition 3.7. For $F \in \mathsf{For}^m$ and any i, the *blossomed forest* $F \cdot i$ is obtained by making the ith leaf of F into a terminal node by giving it $m+1$ leaf children. If $i \in \mathrm{LTer}(F)$, we define the *trimmed forest* $F/i \in \mathsf{For}^m$ by removing the terminal node v with $\rho_F(v) = i$.

Clearly we always have $(F \cdot i)/i = F$, and if $i \in \mathrm{LTer}(F)$ we have $(F/i) \cdot i = F$. The reader curious about our choice of notation will find a satisfactory explanation in Section 4.

These operations are easily reflected in terms of codes. If $\mathsf{c}(F) = (c_i)_{i \in \mathbb{N}}$ then for $i \in \mathbb{N}$ we have

$$\mathsf{c}(F \cdot i) := (c_1, \ldots, c_{i-1}, c_i + 1, 0^m, c_{i+1}, c_{i+2}, \ldots). \tag{3.5}$$

In other words we increment the ith part of $\mathsf{c}(F)$ and insert a string of m zeros immediately after. If $i \in \mathrm{LTer}(F)$ then $\mathsf{c}(F) = (c_1, \ldots, c_i, 0^m, c_{i+m+1}, c_{i+m+2}, \ldots)$ with $c_i > 0$ and

$$\mathsf{c}(F/i) := (c_1, \ldots, c_{i-1}, c_i - 1, c_{i+m+1}, c_{i+m+2}, \ldots). \tag{3.6}$$

In words we decrement the ith part of $\mathsf{c}(F)$ and delete the string of m zeros that follows immediately. See Figure 3 depicting the twin operations for $m = 1$. Make note of the shift in the indices comprising the support stemming from the addition/deletion of strings of 0s.

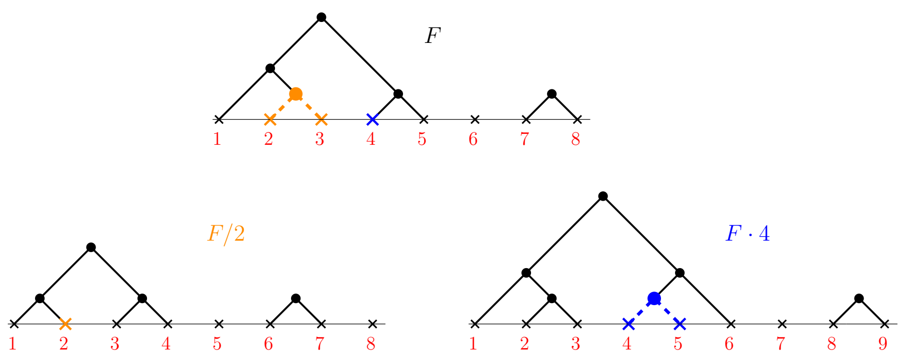

FIGURE 3. An $F \in \mathsf{For}$ with $\mathsf{c}(F) = (2, 1, 0, 1, 0, 0, 1, 0, \ldots)$, and the corresponding $F/2$ and $F \cdot 4$. We have $\mathsf{c}(F/2) = (2, 0, 1, 0, 0, 1, 0, \ldots)$, and $\mathsf{c}(F \cdot 4) = (2, 1, 0, 2, 0, 0, 0, 1, 0, \ldots)$.

Iterating the notion of trimming, we obtain the notion of trimming sequences $\mathrm{Trim}(F)$:

Definition 3.8. For $F \in \mathsf{For}^m$ with $|F| = k$, we define $\mathrm{Trim}(F)$ recursively by setting $\mathrm{Trim}(\varnothing) = \{\varnothing\}$, and for $F \neq \varnothing$ we define

$$\mathrm{Trim}(F) = \{(i_1, \ldots, i_k) \mid (i_1, \ldots, i_{k-1}) \in \mathrm{Trim}(F/i_k) \text{ and } i_k \in \mathrm{LTer}(F)\}.$$

This plays the role of the set of reduced words $\mathrm{Red}(w)$ for $w \in S_\infty$. Note that the elements of $\mathrm{Trim}(F)$ are in obvious bijection with standard decreasing labelings of F, i.e. bijective labelings of $\mathrm{IN}(F)$ with numbers drawn from $\{1, \ldots, |F|\}$ so that the labels decrease going down from root to terminal nodes.

4. Forests and the Thompson monoids

We now develop the combinatorics of the *m-Thompson monoid* $\mathsf{ThMon}^{\underline{m}}$, which we will show in Section 5 governs the composites of the $\mathsf{T}_i^{\underline{m}}$ operators. By identifying this monoid with a monoid structure on For^m, we will be able to index compositions of $\mathsf{T}_i^{\underline{m}}$ operators as $\mathsf{T}_{i_1}^{\underline{m}} \cdots \mathsf{T}_{i_k}^{\underline{m}} = \mathsf{T}_F^{\underline{m}}$ where $F \in \mathsf{For}^m$ and $(i_1, \dots, i_k) \in \mathrm{Trim}(F)$. This is analogous to how we can index compositions of usual divided differences $\partial_{i_1} \cdots \partial_{i_k} = \partial_w$ with $w \in S_\infty$ for $(i_1, \dots, i_k)$ a reduced word.

4.1. A monoid structure on Forests.

Definition 4.1. We define a monoid structure on For^m by taking for $F, G \in \mathsf{For}^m$ the composition $F \cdot G \in \mathsf{For}^m$ to be obtained by identifying the ith leaf of F with the ith root node of G. The empty forest $\varnothing \in \mathsf{For}^m$ is the identity element.

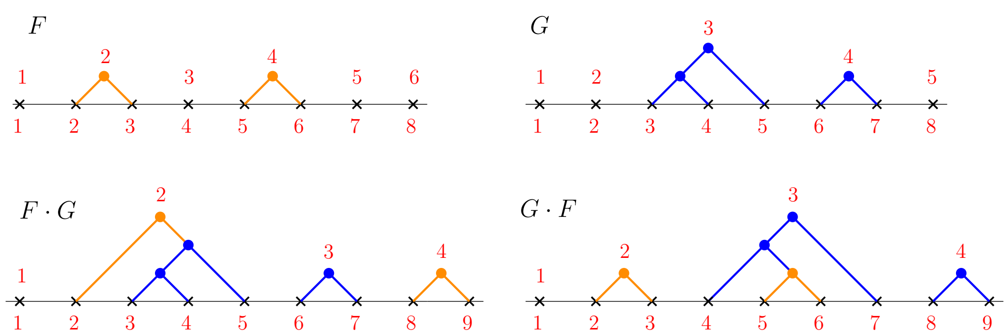

FIGURE 4. The products $F \cdot G$ and $G \cdot F$ for $F, G \in \mathsf{For}$, with both roots and leaves labeled

If $H \in \mathsf{For}^m$, then factorizations $H = F \cdot G$ are in one-to-one correspondence with partitions $\mathrm{IN}(H) = A \sqcup B$ where A is closed under taking parents and B is closed under taking children, and then we may identify $A = \mathrm{IN}(F)$ and $B = \mathrm{IN}(G)$. An example of this is depicted in Figure 4.

Let $\pitchfork$ be the unique $(m+1)$-ary plane rooted tree with $|\pitchfork| = 1$, and define $\underline{i} \in \mathsf{For}^m$ by

$$\underline{i} = \underbrace{* * \cdots *}_{i-1} \pitchfork * * \cdots \tag{4.1}$$

We note that $F \cdot \underline{i}$ agrees with the blossoming $F \cdot i$ defined previously. With this notation, it is clear that for $F \in \mathsf{For}^m$ with $|F| = k$ that

$$\mathrm{Trim}(F) = \{(i_1, \dots, i_k) : F = \underline{i_1} \cdots \underline{i_k}\}.$$

The following shows that the $\underline{i}$ forests play an important role in the monoid For^m.

Proposition 4.2. Every $F \in \mathsf{For}^m$ has a unique expression $F = \underline{1}^{c_1} \cdot \underline{2}^{c_2} \cdots$. The exponents are given by $\mathsf{c}(F) = (c_1, c_2, \dots)$.

Proof. The code map is a bijection by Theorem 3.6, thus it suffices to show that $\mathsf{c}(\underline{1}^{c_1} \cdot \underline{2}^{c_2} \cdots) = (c_i)_{i \in \mathbb{N}}$. We induct on $\sum c_i$. The result is trivial if $\sum c_i = 0$, so suppose that $\sum c_i > 0$. We have

$$\mathsf{c}(\underline{1}^{c_1} \cdot \underline{2}^{c_2} \cdots \underline{n}^{c_n}) = \mathsf{c}(\underline{1}^{c_1} \cdot \underline{2}^{c_2} \cdots \underline{n}^{c_n - 1} \cdot \underline{n}) = \mathsf{c}(F' \cdot \underline{n}) \tag{4.2}$$

where by the inductive hypothesis $\mathsf{c}(F') = (c_1, \ldots, c_{n-1}, c_n - 1, 0, \ldots)$. Hence by (3.5) we have $\mathsf{c}(F' \cdot \underline{n}) = (c_1, \ldots, c_n, 0, \ldots)$ as desired. □

The following says that the monoid For^m is *right-cancellable.*

Proposition 4.3. For fixed $G \in \mathsf{For}^m$, the map $H \mapsto H \cdot G$ is an injection on For^m.

Indeed, by writing $G = \underline{i_1} \cdots \underline{i_k}$, we can recover H from $H \cdot G$ by $H = (((H \cdot G/\underline{i_k})/\underline{i_{k-1}}) \cdots)/\underline{i_1}$. We can thus define the following.

Definition 4.4. For $F, G \in \mathsf{For}^m$, say $F \geq G$ if $F = H \cdot G$ for some $H \in \mathsf{For}^m$. If $F \geq G$ then we write $F/G \in \mathsf{For}^m$ to be the unique indexed forest with $F = (F/G) \cdot G$.

The following is true in any right-cancellable monoid:

Corollary 4.5. If $F \geq H$, then $G \geq F$ if and only if both $G \geq H$ and $G/H \geq F/H$. Under either supposition we have $G/F = (G/H)/(F/H)$.

4.2. **The Thompson monoid.** We consider the following monoid given by generators and relations presentation (see Remark 4.8 for an explanation of the name).

Definition 4.6. The m-Thompson monoid $\mathsf{ThMon}^{\underline{m}}$ is the quotient of the free monoid $\{1, 2, \ldots\}^*$ by the relations $i \cdot j = j \cdot (i + m)$ for $i > j$.

It turns out to describe exactly our monoid structure on For^m.

Theorem 4.7. The map $\mathsf{ThMon}^{\underline{m}} \to \mathsf{For}^m$ given by $i \mapsto \underline{i}$ is a monoid isomorphism.

Proof. The monoid structure on For^m satisfies

$$\underline{i} \cdot \underline{j} = \underbrace{* \cdots *}_{j-1} \pitchfork \underbrace{* \cdots *}_{i-j+m-1} \pitchfork * * \cdots = \underline{j} \cdot \underline{i+m} \text{ whenever } i > j, \tag{4.3}$$

It follows that the map is a well-defined monoid morphism. It is surjective since the indexed forests $\underline{i}$ generate For^m by Proposition 4.2. Using the rules $i \cdot j = j \cdot (i + m)$ for $i > j$, every element $i_1 \cdots i_k \in \mathsf{ThMon}^{\underline{m}}$ can be written as $1^{c_1} \cdot 2^{c_2} \cdots$ for some $c_1, c_2, \ldots$ by moving the smallest i_j to the front and recursing on the remainder of the word. But each $1^{c_1} 2^{c_2} \cdots$ maps to a unique indexed forest $\underline{1}^{c_1} \cdot \underline{2}^{c_2} \cdots$ by Proposition 4.2, which establishes injectivity of the map. □

From now on we will tacitly identify elements $i_1 \cdots i_k \in \mathsf{ThMon}^{\underline{m}}$ of the Thompson monoid and the associated forest $\underline{i_1} \cdots \underline{i_k}$ in For^m, and so omit the underlines from now on.

Remark 4.8. By formally adding inverses to the elements of $\mathsf{ThMon}^{\underline{m}}$ we obtain the Higman–Thompson group

$$G_{m+1,1} := \langle \{r_i\}_{i\in\mathbb{N}} \mid r_ir_j = r_jr_{i+m} \text{ for } i > j\rangle,$$

the group of piecewise-linear homeomorphisms $f : [0,1] \to [0,1]$, all of whose nonsmooth points lie in $\mathbb{Z}[\frac{1}{m+1}]$ and whose slopes are powers of $m+1$ [14, §4]. The elements of $\mathsf{ThMon}^{\underline{m}}$ correspond to those maps whose nonsmooth points have x-coordinates of the form $1 - \frac{1}{(m+1)^k}$, and every element of $G_{m+1,1}$ is uniquely of the form FG^{-1} where $\mathrm{Des}(F) \cap \mathrm{Des}(G) = \varnothing$, see [8, 17] for the case $m = 1$, and for a discussion of the higher $m > 1$ case see [16, 18]. For further combinatorial considerations of Thompson monoids see [19, 50].

4.3. **A monoid factorization.** We will need an analogue for LTFor_n^m of the following canonical decomposition for permutations $w \in S_\infty$ with $\mathrm{Des}(w) \subset [n]$, which index the n-variable Schubert polynomials $\mathfrak{S}_w(x_1,\dots,x_n)$.

Observation 4.9. Fix $n \geq 1$. Every $w \in S_\infty$ can be uniquely written as $w = uv$ where $\mathrm{Des}(u) \cap [n-1] = \varnothing$ and $v \in S_n$. Here $v \in S_n$ is the unique permutation so that $w(v^{-1}(1)) < w(v^{-1}(2)) < \cdots < w(v^{-1}(n))$ and $u = wv^{-1}$.
Moreover $\mathrm{Des}(w) \subset [n]$ if and only if $\mathrm{Des}(u) \subset \{n\}$, i.e. u is an n-Grassmannian permutation.

Let us give an analogue of this factorization for forests, which will be of particular importance when studying quasisymmetric coinvariants in Section 9. To state it, we need the map $\tau : \mathsf{For}^m \to \mathsf{For}^m$ defined by $\tau(F) = *, F$, which shifts the forest one unit to the right. For $G \in \mathsf{For}^m$ of the form $G = *, F$ we also write $\tau^{-1}(G) = F$, which shifts the forest one unit to the left.

Theorem 4.10. Let $n \geq 1$, and $F \in \mathsf{For}^m$. Let $H \leq F$ be the forest induced by all internal nodes of F that are supported on $[n]$. Then $F \mapsto (\tau^{m|H|}(F/H), H)$ is a bijection:

$$\Theta_n : \mathsf{For}^m \to \{(R, H) \in \mathsf{RTFor}^m_{>n} \times \mathsf{For}^m_n \mid R = \varnothing \text{ or } \min\operatorname{supp} R > m|H|\}.$$

It restricts to a bijection

$$\Theta'_n : \mathsf{LTFor}^m_n \to \{(G, H) \in \mathsf{ZigZag}^m_n \times \mathsf{For}^m_n \mid G = \varnothing \text{ or } \min\operatorname{supp} G > m|H|\}.$$

We give an example of Θ'_n in Figure 5.

Proof. Let us first show that Θ_n is well-defined. By construction H is clearly a subforest of F that belongs to For^m_n. By Fact 3.3 its first $n - m|H|$ trees $T_1, \dots, T_{n-m|H|}$ have $[n]$ as the union of their leaves, and the other trees are trivial. As $F = (F/H) \cdot H$, we see that F is obtained by grafting T_1 through $T_{n-m|H|}$ to the first $n - m|H|$ leaves of F/H. None of these first $n - m|H|$ leaves can be the rightmost leaf of a node of F/H, as then the corresponding node in F would be supported on $[n]$. It follows that $F/H \in \mathsf{RTFor}^m_{>n-m|H|}$ and thus $\tau^{m|H|}(F/H) \in \mathsf{RTFor}^m_{>n}$. So Θ_n is well-defined.

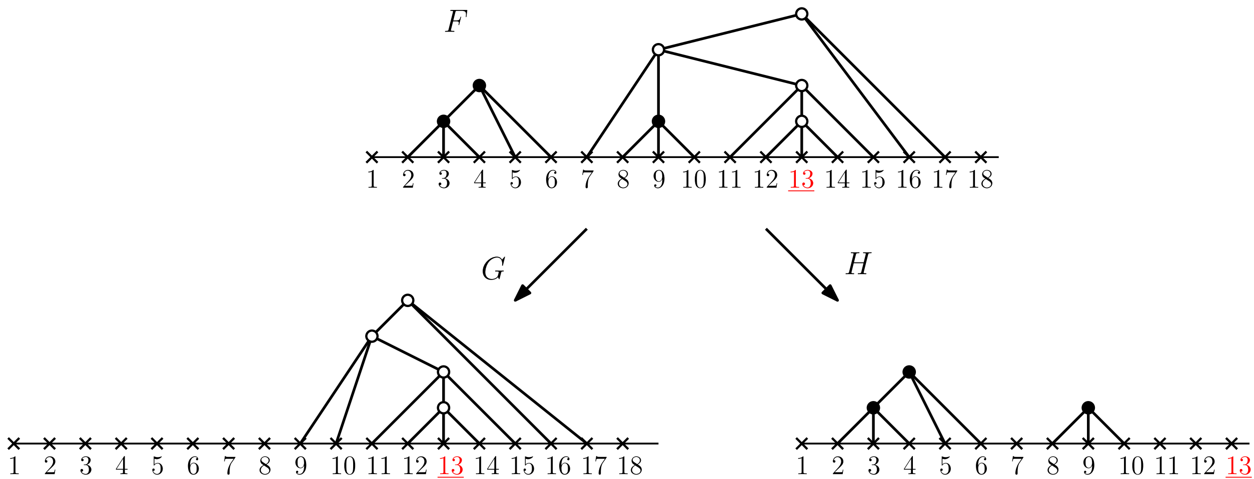

FIGURE 5. Example of the map Θ'_n for $n = 13$. White and black vertices contribute to G and H respectively.

Clearly Θ_n is injective, as if $\Theta_n(F) = (R, H)$ then $F = (\tau^{-m|H|}R) \cdot H$. Let us show surjectivity. Fix $(R, H) \in \mathsf{RTFor}^m_{>n} \times \mathsf{For}^m_n$ with $\min \operatorname{supp} R > m|H|$. By definition of the monoid product, all nodes in $(\tau^{-m|H|}R) \cdot H$ coming from H are supported on $[n]$ since $H \in \mathsf{For}^m_n$. Now fix a node v in $\tau^{-m|H|}R$. Since $\tau^{-m|H|}R \in \mathsf{RTFor}^m_{>n-m|H|}$, the tree rooted at v has a rightmost leaf descendant $>$ $n - m|H|$. Now the first $n - m|H|$ trees in H have leaf set $[n]$, so in $(\tau^{-m|H|}R) \cdot H$ the tree rooted at the node coming from v will have a rightmost leaf descendant $> n$. Thus no node in $(\tau^{-m|H|}R) \cdot H$ coming from $\tau^{-m|H|}R$ is supported on $[n]$. It follows that $\Theta_n((\tau^{-m|H|}R) \cdot H) = (R, H)$.

Assume now $F \in \mathsf{LTFor}^m_n$, so that all leftmost leaves are $\leq n$, and let $\Theta'_n(F) = (G, H)$. If v is a terminal node of $\tau^{-m|H|}G$, then it has a leaf $> n - m|H|$ since $\tau^{-m|H|}G \in \mathsf{RTFor}^m_{>n-m|H|}$. The corresponding node v_F in $F = (\tau^{-m|H|}G) \cdot H$ has a leaf descendant $\leq n$ which implies that v has also a leaf $\leq n - m|H|$. This implies $\rho_{\tau^{-m|H|}G}(v) \in \{n - m|H| - m + 1, \ldots, n - m|H|\}$. Since this holds for all terminal nodes of $\tau^{-m|H|}G$ we have $\tau^{-m|H|}G \in \mathsf{ZigZag}^m_{n-m|H|}$, i.e. $G \in \mathsf{ZigZag}^m_n$. By the same reasoning in reverse we have that $\tau^{-m|H|}G \in \mathsf{ZigZag}^m_{n-m|H|}$ implies that $F \in \mathsf{LTFor}^m_n$, and thus Θ'_n is a bijection. □

5. FOREST POLYNOMIALS $\mathfrak{P}_F$ AND TRIMMING OPERATORS T^m_F

We now introduce a new family of polynomials $\mathfrak{P}_F$ indexed by $F \in \mathsf{For}^m$ which we call m-forest polynomials that specialize to the forest polynomials of the first and third authors [38] when $m = 1$, as well as composites T^m_F of the operators T^m_i indexed by the same set. These will play the roles of $\{\mathfrak{S}_w : w \in S_\infty\}$ and $\{\partial_w : w \in S_\infty\}$ respectively.

5.1. **Forest polynomials $\mathfrak{P}_F$.**

Definition 5.1. For $F \in \mathsf{For}^m$, define $\mathcal{C}(F)$ to be the set of all $\kappa : \mathrm{IN}(F) \to \mathbb{N}$ such that for all $v \in \mathrm{IN}(F)$ with children $v_0, \dots, v_m \in \mathrm{IN}(F) \sqcup \mathbb{N}$ we have

- $\kappa(v) \leq \rho_F(v)$
- If $v_i \in \mathrm{IN}(F)$ then $\kappa(v) \leq \kappa(v_i) - i$
- $\kappa(v) \equiv \rho(v) \bmod m$.

The m-forest polynomial $\mathfrak{P}_F$ is the generating function for $\mathcal{C}(F)$:

$$\mathfrak{P}_F = \sum_{\kappa \in \mathcal{C}(F)} \prod_{v \in \mathrm{IN}(F)} x_{\kappa(v)}.$$

When we need to disambiguate m, we will write the m-forest polynomial as $\mathfrak{P}_F^m$.

For $m = 1$ we recover the definition of forest polynomials as defined in [38, Definition 3.1].

The following fact plays the analogous role that for a Schubert polynomial with Lehmer code c we have

$$\mathfrak{S}_w = \mathsf{x}^{\mathsf{c}} + \sum_{\mathsf{d} < \mathsf{c}} b_{\mathsf{d}} \mathsf{x}^{\mathsf{d}}$$

where the ordering in the sum is the *revlex* (reverse lexicographic) ordering.

Proposition 5.2. For $F \in \mathsf{For}^m$ with code $\mathsf{c}(F) = (c_1, c_2, \dots)$, we have

$$\mathfrak{P}_F = \mathsf{x}^{\mathsf{c}(F)} + \sum_{\mathsf{d} < \mathsf{c}(F)} a_{\mathsf{d}} \mathsf{x}^{\mathsf{d}}$$

where the revlex ordering is used. Furthermore, if $c_i = 0$ for all $i > m$ then $\mathfrak{P}_F = \mathsf{x}^{\mathsf{c}(F)}$.

Proof. The first part follows because the filling $\kappa(v) = \rho_F(v)$ is always valid, and every other filling gives a monomial which is smaller in the revlex ordering. If $c_i = 0$ for all $i > m$, this is the only valid filling since $1 \leq \kappa(v) \leq \rho_F(v) \leq m$ and $\kappa(v) \equiv \rho_F(v) \bmod m$, which implies $\mathfrak{P}_F = \mathsf{x}^{\mathsf{c}(F)}$. $\square$

An immediate corollary is that $\{\mathfrak{P}_F^m : F \in \mathsf{For}^m\}$ is a basis of Pol; even more, $\{\mathfrak{P}_F^m : F \in \mathsf{LTFor}_n^m\}$ is a basis of Pol_n for any $n \geq 1$. We will show this again in Proposition 6.11 using the new divided difference formalism we will introduce shortly.

For $F \in \mathsf{For}^2$ in Figure 6 we have

$$\mathfrak{P}_F = x_2 x_3 x_6^2 + x_2 x_3 x_4 x_6 + x_2 x_3 x_4^2.$$

Note that $\mathsf{c}(F) = (0, 1, 1, 0, 0, 2, 0, \dots)$ and $\mathsf{x}^{\mathsf{c}(F)} = x_2 x_3 x_6^2$ is indeed the revlex leading term.

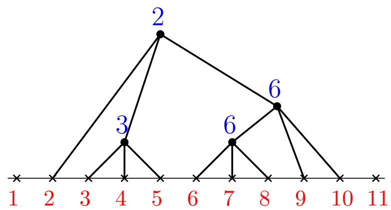

FIGURE 6. An $F \in \mathsf{For}^2$ with the three fillings in $\mathcal{C}(F)$

Nested forests. The content will only be used in Section 10, in the case $m > 1$. An indexed forest $F \in \mathsf{For}^m$ naturally gives rise to an m-colored nested binary forest $\widehat{F}$, as follows. We define the coloring of $F \in \mathsf{For}^m$ as the map $\overline{\rho}_F : \mathrm{IN}(F) \sqcup \mathbb{N} \to \mathbb{Z}/m\mathbb{Z}$ by

$$\overline{\rho}_F(v) = (\rho_F(v) \mod m). \tag{5.1}$$

For $v \in \mathrm{IN}(F)$ with children $v_0, \ldots, v_m \in \mathrm{IN}(F) \sqcup \mathbb{N}$ we have

$$\overline{\rho}_F(v_i) \equiv \overline{\rho}_F(v) + i \bmod m, \tag{5.2}$$

since the number of leaves in the subtree supported by v_j is 1 mod m for $j = 0, \ldots, i-1$.

We define $\widehat{F}$ to be the nested binary plane forest obtained by deleting all edges connecting $v \in \mathrm{IN}(F)$ to one of its internal children $v_1, \ldots, v_{m-1}$, and when referring to $v \in \mathrm{IN}(F)$ as an internal node of $\widehat{F}$, we write $v_L := v_0$ and $v_R := v_m$ for the left and right children of v. From (5.2) it is clear that the connected components of $\widehat{F}$ are monochromatic binary trees, which we can then color with the common color of their vertices, as in Figure 7.

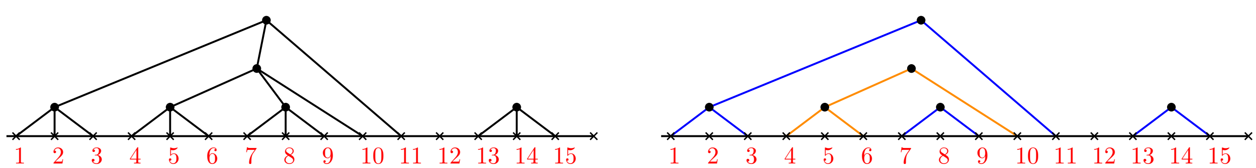

FIGURE 7. A forest $F \in \mathsf{For}^2$ and its associated colored $\widehat{F}$

Given $\kappa : \mathrm{IN}(F) \to \mathbb{N}$, let $\widetilde{\kappa} : \mathrm{IN}(F) \sqcup \mathbb{N} \to \mathbb{N}$ be obtained by extending κ to $\mathbb{N}$ by setting $\widetilde{\kappa}(i) = i$. One checks that in terms of the colored forest $\widehat{F}$, we have $\kappa \in \mathcal{C}(F)$ if and only if:

- For $v \in \mathrm{IN}(F)$ we have $\widetilde{\kappa}(v_L) \geq \widetilde{\kappa}(v) < \widetilde{\kappa}(v_R)$.
- If $v, w \in \mathrm{IN}(F)$ are roots of connected components of $\widehat{F}$ with the tree supported by w nested in the tree supported by v, then $\kappa(v) \geq \kappa(w)$.
- $\widetilde{\kappa}(v) \equiv \overline{\rho}_F(v) \bmod m$ for all $v \in \mathrm{IN}(F)$

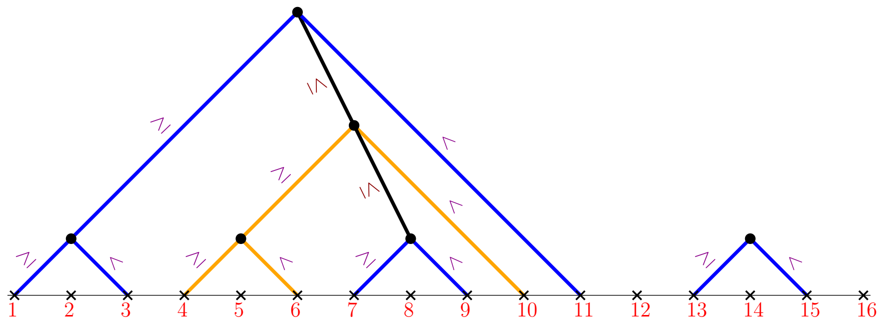

FIGURE 8. Inequalities defining $\mathfrak{P}^{\underline{2}}_F$ from the $\widehat{F}$ perspective for the F in Figure 7

5.2. **Trimming operators $\mathsf{T}^{\underline{m}}_F$.** Let $\mathsf{T}^{\underline{m}} : \mathrm{Pol}_{m+1} \to \mathrm{Pol}_1$ be the operator

$$\mathsf{T}^{\underline{m}}(f) = \frac{f(x, 0^m) - f(0^m, x)}{x}. \tag{5.3}$$

Then viewing $\mathrm{Pol} = \mathrm{Pol}_1^{\otimes\infty}$ we have $\mathsf{T}^{\underline{m}}_i = \mathrm{id}^{\otimes i-1} \otimes \mathsf{T}^{\underline{m}} \otimes \mathrm{id}^{\otimes\infty}$. Because of this, it turns out that composites $\mathsf{T}^{\underline{m}}_{i_1} \cdots \mathsf{T}^{\underline{m}}_{i_k}$ are naturally encoded by the structure of an m-indexed forest. For example, we can write $\mathsf{T}^{\underline{2}}_1\mathsf{T}^{\underline{2}}_1\mathsf{T}^{\underline{2}}_4\mathsf{T}^{\underline{2}}_4\mathsf{T}^{\underline{2}}_7\mathsf{T}^{\underline{2}}_{13}$ as

$$\mathsf{T}^{\underline{2}}(\mathsf{T}^{\underline{2}}(\mathrm{id}^{\otimes 3}) \otimes \mathsf{T}^{\underline{2}}(\mathsf{T}^{\underline{2}}(\mathrm{id}^{\otimes 3}) \otimes \mathsf{T}^{\underline{2}}(\mathrm{id}^{\otimes 3}) \otimes \mathrm{id}) \otimes \mathrm{id}) \otimes \mathrm{id} \otimes \mathsf{T}^{\underline{2}}(\mathrm{id}^{\otimes 3}) \otimes \mathrm{id}^{\otimes\infty}$$

and this latter expression is nested via the parenthesization in a way that is encoded by $F = 1 \cdot 1 \cdot 4 \cdot 4 \cdot 7 \cdot 13 \in \mathsf{For}^2$, the forest in Figure 7 (left).

In this way F can be thought of as encoding a composite $\mathsf{T}^{\underline{m}}$ operator taking inputs in the leaves and producing an output in the roots, which explains why the compositional structure of the $\mathsf{T}^{\underline{m}}_i$ is reflected in the monoid composition on For^m.

Using the m-Thompson monoid gives us a quick way to prove this identification.

Proposition 5.3. $\mathsf{T}^{\underline{m}}_i\mathsf{T}^{\underline{m}}_j = \mathsf{T}^{\underline{m}}_j\mathsf{T}^{\underline{m}}_{i+m}$ for $i > j$. In particular $i \mapsto \mathsf{T}^{\underline{m}}_i$ induces a representation of $\mathsf{ThMon}^{\underline{m}}$ via compositions of the $\mathsf{T}^{\underline{m}}_i$ operators.

Proof. We verify

$$\mathsf{T}^{\underline{m}}_i\mathsf{T}^{\underline{m}}_j = \mathrm{id}^{\otimes j-1} \otimes \mathsf{T}^{\underline{m}} \otimes \mathrm{id}^{i-j-1} \otimes \mathsf{T}^{\underline{m}} \otimes \mathrm{id}^{\otimes\infty} = \mathsf{T}^{\underline{m}}_j\mathsf{T}^{\underline{m}}_{i+m}. \tag{5.4}$$

□

Definition 5.4. For $F \in \mathsf{ThMon}^{\underline{m}}$, define $\mathsf{T}^{\underline{m}}_F := \mathsf{T}^{\underline{m}}_{i_1} \cdots \mathsf{T}^{\underline{m}}_{i_k}$ for any expression $F = i_1 \cdots i_k$.

In the next section we develop the divided difference formalism relating m-forest polynomials $\{\mathfrak{P}_F : F \in \mathsf{For}^m\}$ to the trimming operators $\mathsf{T}^{\underline{m}}_F$.

6. CHARACTERIZING m-FOREST POLYNOMIALS VIA TRIMMING OPERATORS

This section forms the core of this work, the main result being Theorem 6.5. Every result is exactly analogous to a corresponding result for divided differences ∂_w and Schubert polynomials.

The following theorem forms the bedrock for everything that comes after, and is directly analogous to the interaction in (1.1) between ∂_i and Schubert polynomials.

Theorem 6.1. For $F \in \mathsf{For}^m$ and $i \geq 1$ we have

$$\mathsf{T}_i^m \mathfrak{P}_F = \begin{cases} \mathfrak{P}_{F/i} & \text{if } i \in \mathrm{LTer}(F) \\ 0 & \text{otherwise.} \end{cases} \tag{6.1}$$

We defer the proof by explicit computation to Appendix A.

Example 6.2. In Figure 9 we depict successive applications of the trimming operators T_i to a forest polynomial $\mathfrak{P}_F$ with $F \in \mathsf{For}$, which by Theorem 6.1 produces further forest polynomials associated to trimmed forests. If T_i does not appear then its application gives 0.

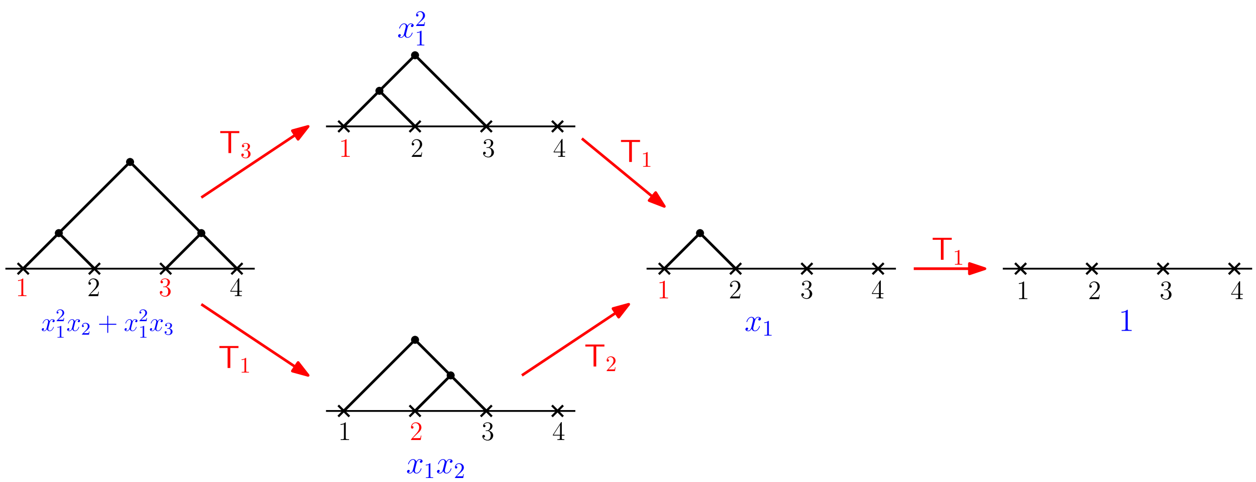

FIGURE 9. Sequences of T_i applied to $\mathfrak{P}_F$ with $F = 1 \cdot 1 \cdot 3 \in \mathsf{For}$

Remark 6.3. The actual definition of m-forest polynomials will play no role in all subsequent proofs. As we will shortly see in Theorem 6.5, the polynomials $\mathfrak{P}_F$ are in fact determined by the condition in Theorem 6.1, homogeneity, and the normalization condition $\mathfrak{P}_\varnothing = 1$. We will use this characterization in proofs, signaling however when a simple alternative proof using the combinatorial definition can be given.

The classical proof that Schubert polynomials exist (i.e. that a homogenous family of polynomials exists satisfying the divided difference relations for ∂_i) is by taking the ansatz $\mathfrak{S}_{w_{0,n}} =$

$x_1^{n-1}\cdots x_{n-1}$ for $w_{0,n}$ the longest permutation in S_n, showing that $\partial_{w_{0,n-1}^{-1}w_{0,n}}\mathfrak{S}_{w_{0,n}} = \mathfrak{S}_{w_{0,n-1}}$ by direct computation, and then defining $\mathfrak{S}_u = \partial_{u^{-1}w_{0,n}}\mathfrak{S}_{w_{0,n}}$ for n sufficiently large so that $u \in S_n$. The m-forest polynomials do not seem to have sufficiently elementary descriptions for some particularly well-chosen sequence of forests F_n such that every other $G \in \mathsf{For}^m$ has $F_n \geq G$, so it does not seem possible to proceed in a similar manner.

Lemma 6.4. $\bigcap_{i\geq n+1}\ker(\mathsf{T}_i^m) = \mathrm{Pol}_n$. In particular, $\bigcap_{i\geq 1}\ker(\mathsf{T}_i^m) = \mathbb{Z}$.

Proof. Clearly $\mathrm{Pol}_n \subset \bigcap_{i\geq n+1}\ker(\mathsf{T}_i^m)$. Conversely, if $k \geq n+1$ and $f(x_1,\ldots,x_k)$ is a polynomial depending nontrivially on k, then $\mathsf{T}_k^m f = \frac{1}{x_k}(f - f|_{x_k=0}) \neq 0$ so $f \notin \ker(\mathsf{T}_k^m)$. ☐

Theorem 6.5. The family of m-forest polynomials $\{\mathfrak{P}_F : F \in \mathsf{For}^m\}$ is uniquely characterized by the properties $\mathfrak{P}_\varnothing = 1$, $\mathfrak{P}_F$ is homogenous, and $\mathsf{T}_i^m\mathfrak{P}_F = \delta_{i\in\mathrm{LTer}(F)}\mathfrak{P}_{F/i}$.

Proof. It follows from the definition of m-forest polynomials and Theorem 6.1 that they satisfy these properties. Suppose there were another such family of polynomials $\{H_F : F \in \mathsf{For}^m\}$. From $\mathsf{T}_i^m H_F = \delta_{i\in\mathrm{LTer}(F)}H_{F/i}$ and $H_\varnothing = 1$ we see by induction that H_F has degree $|F|$. By induction, assume that we know that $H_F = \mathfrak{P}_F$ for $|F| < k$. Then given some $F \in \mathsf{For}^m$ with $|F| = k$ we have $\mathsf{T}_i^m(\mathfrak{P}_F - H_F) = \delta_{i\in\mathrm{LTer}(F)}(\mathfrak{P}_{F/i} - H_{F/i}) = 0$ for all i, and therefore by Lemma 6.4 we have $\mathfrak{P}_F - H_F \in \mathbb{Z}$. But $\mathfrak{P}_F$ and H_F are homogenous of degree $|F| > 1$ so therefore they must be equal. ☐

Corollary 6.6. For $F, G \in \mathsf{For}^m$ we have

$$\mathsf{T}_F^m\mathfrak{P}_G = \begin{cases}\mathfrak{P}_{G/F} & \text{if } G \geq F\\ 0 & \text{otherwise.}\end{cases} \tag{6.2}$$

In particular, $\mathrm{ev}_0\,\mathsf{T}_F^m\mathfrak{P}_G = \delta_{F,G}$.

Proof. We induct on $|F|$. Let $i \in \mathrm{LTer}(F)$, and write $\mathsf{T}_F^m\mathfrak{P}_G = \mathsf{T}_{F/i}^m\mathsf{T}_i^m\mathfrak{P}_G$, which is equal to $\mathfrak{P}_{(G/i)/(F/i)}$ if both $G \geq i$ and $(G/i) \geq (F/i)$, and 0 otherwise. But by Corollary 4.5, both $G \geq i$ and $G/i \geq F/i$ if and only if $G \geq F$, in which case $G/F = (G/i)/(F/i)$. The first part of the result follows.

Now, note that when $G \geq F$, $\mathfrak{P}_{G/F}$ is homogenous of degree $|G/F|$, so the only way that $\mathrm{ev}_0\,\mathfrak{P}_{G/F}$ does not vanish is if $|G/F| = 0$, implying $G = F$. Conversely if $G = F$ then $G/F = \varnothing$ so $\mathrm{ev}_0\,\mathsf{T}_F^m\mathfrak{P}_G = \mathrm{ev}_0\,\mathfrak{P}_\varnothing = \mathrm{ev}_0\,1 = 1$. ☐

Corollary 6.7. The T_i^m operators give a faithful representation of the monoid algebra $\mathbb{Z}[\mathsf{ThMon}^{\underline{m}}]$.

Proof. We know by Proposition 5.3 that they give a representation, so it suffices to show that if $\sum a_F\mathsf{T}_F^m = 0$ then all $a_F = 0$. By applying the linear combination to $\mathfrak{P}_G$ for any G, and then applying ev_0, we obtain indeed

$$0 = \sum a_F\,\mathrm{ev}_0\,\mathsf{T}_F^m\mathfrak{P}_G = \sum a_F\,\delta_{F,G} = a_G. \tag{6.3}$$ ☐

Proposition 6.8. The m-forest polynomials $\{\mathfrak{P}_F : F \in \mathsf{For}^m\}$ form a $\mathbb{Z}$-basis for Pol, and we can write any $f \in \mathrm{Pol}$ in this basis as

$$f = \sum (\mathrm{ev}_0 \, \mathsf{T}^m_F f) \, \mathfrak{P}_F. \tag{6.4}$$

Proof. If we can write $f \in \mathrm{Pol}$ as $f = \sum a_F \mathfrak{P}_F$, then by Corollary 6.6 we have $a_F = \mathrm{ev}_0 \, \mathsf{T}^m_F \mathfrak{P}_F$. Therefore to conclude it suffices to establish the identity $f = \sum(\mathrm{ev}_0 \, \mathsf{T}^m_F f)\mathfrak{P}_F$. We do so by induction on $d := \deg(f)$.

For $d = 0$ the result follows by writing f as a multiple of $\mathfrak{P}_\varnothing = 1$. Assume now $d > 0$ and that the result holds for all smaller degree polynomials. As $\deg(\mathsf{T}^m_i f) < d$ for all i, we have

$$\mathsf{T}^m_i \sum (\mathrm{ev}_0 \, \mathsf{T}^m_F f) \, \mathfrak{P}_F = \sum_{F \geq i} (\mathrm{ev}_0 \, \mathsf{T}^m_F f) \, \mathfrak{P}_{F/i} = \sum_G (\mathrm{ev}_0 \, \mathsf{T}^m_G \mathsf{T}^m_i f) \, \mathfrak{P}_G = \mathsf{T}^m_i f. \tag{6.5}$$

Hence by Lemma 6.4 we have

$$f - \sum (\mathrm{ev}_0 \, \mathsf{T}^m_F f) \, \mathfrak{P}_F \in \bigcap_{i \geq 1} \ker(\mathsf{T}^m_i) = \mathbb{Z}. \tag{6.6}$$

So f and $\sum(\mathrm{ev}_0 \, \mathsf{T}^m_F f) \, \mathfrak{P}_F$ can only differ in their constant term. But in fact both have the same constant term $\mathrm{ev}_0 \, f$, so they are equal. $\square$

Proposition 6.9. A $\mathbb{Z}$-basis for $\ker(\mathsf{T}^m_F)$ is given by $\{\mathfrak{P}_G : G \not\geq F\}$. In particular if $\mathcal{S} \subset \mathsf{For}^m$ is a family of m-forests, then

$$\bigcap_{F \in \mathcal{S}} \ker(\mathsf{T}^m_F) = \mathbb{Z}\{\mathfrak{P}_G \mid G \not\geq F \text{ for all } F \in \mathcal{S}\}. \tag{6.7}$$

Proof. By Proposition 6.8 we know that $\{\mathfrak{P}_G : G \not\geq F\} \subset \ker(\mathsf{T}^m_F)$ so it suffices to show that they span. Given $f \in \ker(\mathsf{T}^m_F)$, we can write it as $f = \sum a_G \mathfrak{P}_G$, and we want to show that $a_G = 0$ for all G such that $G \geq F$. Applying T^m_F we see that

$$0 = \mathsf{T}^m_F f = \sum_{G \geq F} a_G \, \mathfrak{P}_{G/F}. \tag{6.8}$$

The forests G/F are all distinct by Proposition 4.3. Since m-forest polynomials are linearly independent we deduce that (6.8) holds if and only if $a_G = 0$ for all G such that $G \geq F$. $\square$

Corollary 6.10. For $A \subset \mathbb{N}$, a $\mathbb{Z}$-basis for the subring

$$\bigcap_{i \notin A} \ker(\mathsf{T}^m_i) \subset \mathrm{Pol}$$

is given by $\{\mathfrak{P}_G : G \in \mathsf{LTFor}^m_A\}$.

Proof. This is a subring since for each $i \notin A$ we have $\ker(\mathsf{T}^m_i) = \ker(\frac{1}{x_i}(\mathsf{R}^m_{i+1} - \mathsf{R}^m_i))$ is the subalgebra of polynomials on which the two ring maps $\mathsf{R}^m_{i+1}, \mathsf{R}^m_i : \mathrm{Pol} \to \mathrm{Pol}$ agree. The basis fact follows from Proposition 6.9 and the definition of LTFor^m_A. $\square$

Proposition 6.11. $\{\mathfrak{P}_G \mid F \in \mathsf{LTFor}_n^m\}$ is a $\mathbb{Z}$-basis for Pol_n.

By Lemma 6.4, this is the case $A = \{1, \ldots, n\}$ of Corollary 6.10. As noted before it also follows from Proposition 5.2.

We conclude with a proposition concerning the interaction between m-forest polynomials and R_1 which will be useful in our study of quasisymmetric coinvariants.

Proposition 6.12. We have $\mathsf{T}_G^m \mathsf{R}_1^k = \mathsf{R}_1^k \mathsf{T}_{\tau^k G}^m$ and

$$\mathsf{R}_1^k \mathfrak{P}_F = \begin{cases} \mathfrak{P}_{\tau^{-k}F} & \text{if } \tau^{-k}F \text{ exists (i.e. } k < \min \mathrm{supp}(F)) \\ 0 & \text{otherwise.} \end{cases} \tag{6.9}$$

Proof. First, it is direct to check that $\mathsf{T}_i^m \mathsf{R}_1 = \mathsf{R}_1 \mathsf{T}_{i+1}^m$. Therefore for any $G \in \mathsf{For}^m$ with code $(c_1, c_2, \ldots)$ we have

$$\mathsf{T}_G^m \mathsf{R}_1^k = (\mathsf{T}_1^m)^{c_1} (\mathsf{T}_2^m)^{c_2} \cdots \mathsf{R}_1^k = \mathsf{R}_1^k (\mathsf{T}_{1+k}^m)^{c_1} (\mathsf{T}_{2+k}^m)^{c_2} \cdots = \mathsf{R}_1^k \mathsf{T}_{\tau^k G}^m \tag{6.10}$$

since $\mathsf{c}(\tau^k G) = (0^k, c_1, c_2, \ldots)$. Therefore $\mathrm{ev}_0\, \mathsf{T}_G^m \mathsf{R}_1^k \mathfrak{P}_F = \mathrm{ev}_0\, \mathsf{T}_{\tau^k G}^m \mathfrak{P}_F = \delta_{\tau^k G, F}$, which by Proposition 6.8 means that $\mathsf{R}_1^k \mathfrak{P}_F^m = \delta_{\tau^{-k}F \text{ exists}} \mathfrak{P}_{\tau^{-k}F}$. □

Compare the preceding result with its well-known classical analogue: $\mathsf{R}_1^k \mathfrak{S}_w$ for $w \in S_\infty$ equals 0 unless $w(i) = i$ for $1 \leq i \leq k$, i.e. $\mathrm{lcode}(w) = (0^k, c_{k+1}, \ldots)$, and if this holds then $\mathsf{R}_1^k \mathfrak{S}_w = \mathfrak{S}_{w'}$ with $w'(i) = w(i+k) - k$, i.e. $\mathrm{lcode}(w') = (c_{k+1}, \ldots)$.

7. Positive expansions

We say that $f \in \mathrm{Pol}$ is *m-forest positive* if the coefficients a_F in the expansion

$$f = \sum_{F \in \mathsf{For}^m} a_F\, \mathfrak{P}_F \tag{7.1}$$

are nonnegative integers. If, in addition, $a_F \in \{0, 1\}$ then we say that f is *multiplicity-free m*-forest positive.

Lemma 7.1. f is m-forest positive if and only if $\mathsf{T}_i^m f$ is m-forest positive for all i. f is multiplicity free m-forest positive if and only if $\mathsf{T}_i^m f$ is multiplicity free m-forest positive for all i.

Proof. If $f = \sum_F a_F \mathfrak{P}_F$, then $\mathsf{T}_i^m f = \sum_{i \in \mathrm{LTer}(F)} a_F \mathfrak{P}_{F/i}$ which immediately shows both forward directions. Conversely, for any F we have a_F is the coefficient of $\mathfrak{P}_{F/i}$ in $\mathsf{T}_i^m f$ for any $i \in \mathrm{LTer}(F)$ which shows the reverse direction. □

In the remainder of this section our computations will be almost entirely formal consequences of the twisted Leibniz rule $\mathsf{T}_i^m(fg) = \mathsf{T}_i^m(f)\mathsf{R}_{i+1}^m(g) + \mathsf{R}_i^m(f)\mathsf{T}_i^m(g)$ from Lemma 2.12, together with the following identities which may be verified by direct computation:

$$\mathsf{T}_j^m\mathsf{R}_i^m = \begin{cases} \mathsf{R}_{i-m}^m\mathsf{T}_j^m & \text{if } j \le i-m-1 \\ \mathsf{R}_{j+1}^m\mathsf{T}_j^m + \mathsf{R}_j^m\mathsf{T}_{j+m}^m & \text{if } i-m \le j \le i-1 \\ \mathsf{R}_i^m\mathsf{T}_{j+m}^m & \text{if } j \ge i. \end{cases} \tag{7.2}$$

Proposition 7.2. For $F \in \mathsf{For}^m$ we have $\mathsf{R}_i^m\,\mathfrak{P}_F$ is multiplicity-free m-forest positive.

Proof. Induct on $|F|$. By Lemma 7.1 it suffices to show that $\mathsf{T}_j^m\,\mathsf{R}_i^m\,\mathfrak{P}_F$ is multiplicity-free m-forest positive for all j. If $j \le i-m-1$ then by (7.2) we have

$$\mathsf{T}_j^m\,\mathsf{R}_i^m\,\mathfrak{P}_F = \mathsf{R}_{i-m}^m\mathsf{T}_j^m\,\mathfrak{P}_F = \delta_{j\in\mathrm{LTer}(F)}\mathsf{R}_{i-m}^m\,\mathfrak{P}_{F/j}. \tag{7.3}$$

which is multiplicity-free forest positive by induction.

If $j \ge i$ then we have by (7.2)

$$\mathsf{T}_j^m\,\mathsf{R}_i^m\,\mathfrak{P}_F = \mathsf{R}_i^m\,\mathsf{T}_{j+m}^m\,\mathfrak{P}_F = \delta_{j+m\in\mathrm{LTer}(F)}\mathsf{R}_i^m\,\mathfrak{P}_{F/(j+m)}. \tag{7.4}$$

which is multiplicity-free m-forest positive by induction.

Finally if $i-m \le j \le i-1$ then we have by (7.2) that

$$\mathsf{T}_j^m\,\mathsf{R}_i^m\,\mathfrak{P}_F = \mathsf{R}_{j+1}^m\mathsf{T}_j^m\mathfrak{P}_F + \mathsf{R}_j^m\mathsf{T}_{j+m}^m\,\mathfrak{P}_F = \delta_{j\in\mathrm{LTer}(F)}\mathsf{R}_{j+1}^m\mathfrak{P}_{F/j} + \delta_{j+m\in\mathrm{LTer}(F)}\mathsf{R}_j^m\,\mathfrak{P}_{F/(j+m)}. \tag{7.5}$$

Noting that we cannot have both $j, j+m \in \mathrm{LTer}(F)$ by (3.4), this is multiplicity-free m-forest positive by induction. $\square$

The next theorem states that the basis $(\mathfrak{P}_F)_{F\in\mathsf{For}^m}$ of Pol has positive structure constants. The case $m = 1$ was first proved in [38] with a combinatorial interpretation.

Theorem 7.3. For $F, G \in \mathsf{For}^m$ we have $\mathfrak{P}_F\mathfrak{P}_G$ is m-forest positive.

Proof. Induct on $\deg(\mathfrak{P}_F\mathfrak{P}_G) = |F| + |G|$. By Lemma 7.1 it suffices to show that $\mathsf{T}_i^m(\mathfrak{P}_F\mathfrak{P}_G)$ is m-forest positive for all i. By Lemma 2.12 we have

$$\mathsf{T}_i^m(\mathfrak{P}_F\mathfrak{P}_G) = (\mathsf{T}_i^m\,\mathfrak{P}_F)\mathsf{R}_{i+1}^m\mathfrak{P}_G + (\mathsf{R}_i^m\,\mathfrak{P}_F)\mathsf{T}_i^m\,\mathfrak{P}_G. \tag{7.6}$$

So it suffices to show that each term on the right-hand side is forest positive. We do the first, the second is similar.

Note that $\mathsf{T}_i^m\,\mathfrak{P}_F$ is either 0 (in which case we are done) or equals $\mathfrak{P}_{F/i}$ which is homogenous of degree $|F|-1$, and $\mathsf{R}_{i+1}^m\mathfrak{P}_G$ is m-forest positive and homogenous of degree $|G|$, so the result now follows by applying the inductive hypothesis. $\square$

Note that the proof above is combinatorial, while the corresponding statement for Schubert polynomials is only known to hold for geometric reasons, as was recalled in the introduction.

Schubert polynomials are known to satisfy *Monk's rule*, which shows that the Schubert expansion of $\mathfrak{S}_w\mathfrak{S}_{s_i} = \mathfrak{S}_w(x_1 + \cdots + x_i)$ is multiplicity-free. The same holds for m-forest polynomials.

Theorem 7.4 (m-forest polynomial "Monk's Rule")**.** For $F \in \mathsf{For}^m$ we have $\mathfrak{P}^m_{\underline{i}}\mathfrak{P}_F = (x_i + x_{i-m} + x_{i-2m} + \cdots + x_{i \bmod m})\mathfrak{P}_F$ is multiplicity-free m-forest positive. (Here $i \bmod m$ is the representative of i modulo m in $\{1, \ldots, m\}$.)

Proof. We induct on $|F|$. For $|F| = 0$ the result is trivial, so assume that $|F| \geq 1$. Given $G \in \mathsf{For}^m$ with $|G| = |F| + 1 \geq 2$, we want to show that $\mathsf{T}^m_G(\mathfrak{P}^m_{\underline{i}}\mathfrak{P}_F) \in \{0, 1\}$. If there exists $j \in \mathrm{LTer}(G)$ with $j \neq i$ then by Lemma 2.12 we can write

$$\mathsf{T}^m_G(\mathfrak{P}^m_{\underline{i}}\mathfrak{P}_F) = \mathsf{T}^m_{G/j}\mathsf{T}^m_j(\mathfrak{P}^m_{\underline{i}}\mathfrak{P}_F) = \mathsf{T}^m_{G/j}(\mathsf{R}^m_j(\mathfrak{P}^m_{\underline{i}})\,\mathsf{T}^m_j(\mathfrak{P}_F)). \tag{7.7}$$

Now note from direct computation that

$$\mathsf{R}^m_j(\mathfrak{P}^m_{\underline{i}}) = \mathsf{R}^m_j(x_i + x_{i-m} + x_{i-2m} + \cdots + x_{i \bmod m}) = \begin{cases} \mathfrak{P}^m_{\underline{i-m}} & \text{if } j \leq i \\ \mathfrak{P}^m_{\underline{i}} & \text{if } j \geq i+1 \end{cases} \tag{7.8}$$

and $\mathsf{T}^m_j(\mathfrak{P}_F) = \delta_{j \in \mathrm{LTer}(F)}\mathfrak{P}_{F/j}$. So we are done by induction.

Otherwise, we have $\mathrm{LTer}(G) = \{i\}$. As $\mathsf{T}^m_G(\mathfrak{P}^m_{\underline{i}}\mathfrak{P}_F) = \mathsf{T}^m_{G/i}(\mathsf{T}^m_i(\mathfrak{P}^m_{\underline{i}}\mathfrak{P}_F))$, it remains to show that $\mathsf{T}^m_{G/i}(\mathsf{T}^m_i(\mathfrak{P}^m_{\underline{i}}\mathfrak{P}_F))$ is multiplicity-free.

We claim that $\mathrm{LTer}(G/i) = \{j\}$ for some $i - m \leq j \leq i$. Indeed, any $k \in \mathrm{LTer}(G/i)$ must have $k \geq i - m$ since otherwise $k \in \mathrm{LTer}(G)$ as well, and now since $\mathrm{LTer}(G/i) \subset \{i - m, \ldots, i\}$ we conclude $|\mathrm{LTer}(G/i)| = 1$ by (3.4).

If $\mathrm{LTer}(G/i) = \{i\}$ then by Lemma 2.12 and (7.2) we can write $\mathsf{T}^m_{G/i}(\mathsf{T}^m_i(\mathfrak{P}_{\underline{i}}\mathfrak{P}_F))$ as

$$\mathsf{T}^m_{G/i}(\mathsf{R}^m_i(\mathfrak{P}_F) + \mathfrak{P}^m_{\underline{i}}\mathsf{T}^m_i(\mathfrak{P}_F)) = \mathsf{T}^m_{(G/i)/i}(\mathsf{R}^m_i\mathsf{T}^m_{i+m}(\mathfrak{P}_F)) + \mathsf{T}^m_{G/i}(\mathfrak{P}^m_{\underline{i}}\,\mathsf{T}^m_i(\mathfrak{P}_F)). \tag{7.9}$$

At most one of the terms is nonzero since we cannot have both $i, i+m \in \mathrm{LTer}(F)$ by (3.4). If the first term is nonzero then we conclude since $\mathsf{R}^m_i\mathfrak{P}_{F/(i+m)}$ is multiplicity-free, and if the second term is nonzero then we conclude by induction that $\mathfrak{P}^m_{\underline{i}}\mathfrak{P}_{F/i}$ is multiplicity-free.

If $\mathrm{LTer}(G/i) = \{j\}$ with $i - m + 1 \leq j \leq i - 1$ then by Lemma 2.12 and (7.2) we can write $\mathsf{T}^m_{G/i}(\mathsf{T}^m_i(\mathfrak{P}^m_{\underline{i}}\mathfrak{P}_F))$ as

$$\mathsf{T}^m_{G/i}(\mathsf{R}^m_i\mathfrak{P}_F + \mathfrak{P}^m_{\underline{i}}\,\mathsf{T}^m_i(\mathfrak{P}_F)) = \mathsf{T}^m_{(G/i)/j}(\mathsf{R}^m_{j+1}\mathsf{T}^m_j\mathfrak{P}_F + \mathsf{R}^m_j\,\mathsf{T}^m_{j+m}\mathfrak{P}_F) + \mathsf{T}^m_{G/i}(\mathfrak{P}^m_{\underline{i}}\,\mathsf{T}^m_i(\mathfrak{P}_F)). \tag{7.10}$$

At most one of $j, j+m, i \in \mathrm{LTer}(F)$ and so we conclude similarly.

Finally, if $\mathrm{LTer}(G/i) = \{i - m\}$ then by Lemma 2.12 and (7.2), we can write $\mathsf{T}^m_{G/i}(\mathsf{T}^m_i(\mathfrak{P}_{\underline{i}}\mathfrak{P}_F))$ as

$$\mathsf{T}_{G/i}(\mathsf{R}^m_{i+1}\mathfrak{P}_F + \mathfrak{P}^m_{\underline{i-1}}\mathsf{T}^m_i(\mathfrak{P}_F)) = \mathsf{T}_{(G/i)/(i-m)}(\mathsf{R}^m_{i-m+1}\mathsf{T}^m_{i-m}\mathfrak{P}_F) + \mathsf{T}_{G/i}(\mathfrak{P}^m_{\underline{i-1}}\mathsf{T}^m_i(\mathfrak{P}_F)). \tag{7.11}$$

At most one of $i - m, i \in \mathrm{LTer}(F)$ and so we conclude similarly. $\square$

Theorem 7.5. For any $k \geq 1$, m-forest polynomials are km-forest positive. In particular, forest polynomials are m-forest positive.

Proof. We induct on $|F|$ to show that $\mathfrak{P}_F$ is km-forest positive. Note that we can write $\mathsf{T}_i^{mk}\mathfrak{P}_F$ as

$$\begin{aligned}
&\frac{\mathfrak{P}_F(\ldots,x_{i-1},x_i,0^{mk},x_{i+1},\ldots)-\mathfrak{P}_F(\ldots,x_{i-1},0^{mk},x_i,x_{i+1},\ldots)}{x_i}\\
&=\sum_{j=0}^{k-1}\frac{\mathfrak{P}_F(\ldots,x_{i-1},0^{mj},x_i,0^{m(k-j)},x_{i+1},\ldots)-\mathfrak{P}_F(\ldots,x_{i-1},0^{m(j+1)},x_i,0^{m(k-j-1)},x_{i+1},\ldots)}{x_i}\\
&=\sum_{j=0}^{k-1}\mathsf{R}_i^{jm}\mathsf{R}_{i+jm+1}^{(k-j-1)m}\mathsf{T}_{i+jm}^{m}\mathfrak{P}_F, \qquad (7.12)
\end{aligned}$$

which is m-forest positive since T^m_{i+jm}, R^m_i, and R^m_{i+jm+1} all preserve m-forest positivity (the latter two by Proposition 7.2), and by the inductive hypothesis they are therefore km-forest positive as well. The result follows from Lemma 7.1. $\square$

Recall that Schubert polynomials enjoy multiplicity-free *Pieri rules* [46] more generally. These correspond to multiplication by elementary symmetric polynomials $e_k(x_1,\ldots,x_p)$ or homogenous symmetric polynomials $h_k(x_1,\ldots,x_p)$. When $m=1$ these polynomials are also forest polynomials for the forests with codes $(0^{p-k},1^k)$ and $(0^{p-1},k)$ respectively. In view of this it is natural to inquire if $\mathfrak{P}_F\cdot e_k(x_1,\ldots,x_p)$ or $\mathfrak{P}_F\cdot h_k(x_1,\ldots,x_p)$ admits a multiplicity-free expansion in terms of forest polynomials. This is not the case in general, as it is easy to find multiplicities in low degree already.

Remark 7.6. Note that while all of the above positivity proofs unwind to give combinatorially nonnegative algorithms, it would be interesting to obtain the final coefficients directly as the answer to enumerative questions. We leave this to the interested reader.

8. Fundamental m-quasisymmetrics and ZigZag^m_n

The n-Grassmannian permutations $\mathsf{Grass_n}=\{w\in S_\infty \mid \mathrm{Des}(w)\subset\{n\}\}$ parametrize the special subclass of Schubert polynomials $\mathfrak{S}_w$ known as the n-variable *Schur polynomials*, which form a basis of Sym_n.

In our story ZigZag^m_n will play an analogous role to $\mathsf{Grass_n}$. We will show that the associated m-forest polynomials $\{\mathfrak{P}_F : F\in\mathsf{ZigZag}^m_n\}$ lie in ${}^m\mathrm{QSym}_n$ and turn out to form the known basis of ${}^m\mathrm{QSym}_n$ comprising *fundamental m-quasisymmetric polynomials*, giving the well-studied basis of *fundamental quasisymmetric polynomials* [24, 47] at $m=1$. One consequence of this is that we can write down a formula (Corollary 8.7) directly computing the coefficients of an m-quasisymmetric polynomial in its m-fundamental expansion, and even for $m=1$ this is new. The only other direct formula for these coefficients in the literature is in the special case that $f\in\mathrm{Sym}_n$: Gessel

[24, Theorem 3] showed that these coefficients when $m = 1$ can be computed via the Hall inner product of f with a ribbon skew-Schur polynomial.

We first translate the definition of m-fundamental quasisymmetric polynomials [6, 7] to our single alphabet setting. For an integer sequence $a = (a_1, \dots, a_k)$ with $a_i \geq 1$ we define the set of *m-compatible sequences*

$$\mathcal{C}^m(a) = \{(i_1, \dots, i_k) : i_j \equiv a_j \bmod m,\ a_j \geq i_j \geq i_{j+1}, \text{ and if } a_j > a_{j+1} \text{ then } i_j > i_{j+1}\}. \tag{8.1}$$

Given a sequence $\mathbf{i} = (i_1, \dots, i_k)$ we denote $\mathsf{x}_{\mathbf{i}} := x_{i_1} \cdots x_{i_k}$. Then we define the *$m$-slide polynomial* to be the generating function

$$\mathfrak{F}^m_a = \sum_{\mathbf{i} \in \mathcal{C}^m(a)} \mathsf{x}_{\mathbf{i}}. \tag{8.2}$$

The notion of an m-compatible sequence is a straightforward generalization of compatible sequences appearing in the Billey–Jockusch–Stanley formula for Schubert polynomials [13] which correspond to $m = 1$. The definition of an m-slide polynomial is then a straightforward generalization of the notion of (ordinary) slide polynomials [2]. Our indexing conventions agree with [39] and differ from [2] as we use sequences instead of weak compositions. We drop m from our notation of m-slides when $m = 1$, choosing to simply write $\mathfrak{F}_a$.

Example 8.1. Consider $a = 422$ wherein we have omitted commas and parentheses in writing the sequence for readability. For $m = 1$ and $m = 2$ respectively we have

$$\begin{aligned}
\mathfrak{F}_{422} &= \mathsf{x}^{(0,2,0,1)} + \mathsf{x}^{(2,0,0,1)} + \mathsf{x}^{(0,2,1,0)} + \mathsf{x}^{(2,0,1,0)} + \mathsf{x}^{(2,1,0,0)} + \mathsf{x}^{(1,1,0,1)} + \mathsf{x}^{(1,1,1,0)},\\
\mathfrak{F}^2_{422} &= \mathsf{x}^{(0,2,0,1)}.
\end{aligned}$$

The corresponding $\mathcal{C}^m(a)$ are $\{422, 411, 322, 311, 211, 421, 321\}$ and $\{422\}$ respectively.

Like with forest polynomials, it is easy to check that the revlex leading monomial of $\mathfrak{F}^m_a$ is x^{c} where $\mathsf{c} = (c_i)_{i \in \mathbb{N}} \in \mathsf{Codes}$ is determined by $c_i = \#\{a_j = i \mid 1 \leq j \leq k\}$. Furthermore for large m we have the equality $\mathfrak{F}^m_a = \mathsf{x}^{\mathsf{c}}$.

Just as the ordinary fundamental quasisymmetric polynomials constitute a subfamily of slide polynomials [2, Lemma 3.8], so too do the m-fundamental quasisymmetric polynomials constitute a subfamily of the m-slides.

Definition 8.2. We write ${}^m\mathrm{QSeq}_n$ for the set of sequences $(a_1, \dots, a_k)$ satisfying

(i) $a_1 \geq \cdots \geq a_k \geq 1$
(ii) $n \geq a_1 \geq n - m + 1$
(iii) $a_i - a_{i+1} \leq m$ for $1 \leq i \leq k - 1$.

If $(a_1, \dots, a_k) \in {}^m\mathrm{QSeq}_n$ then $\mathfrak{F}^m_a \in \mathrm{Pol}_n$ is called an *m-fundamental quasisymmetric polynomial*.

Up to the change of m alphabets to a single one, as explained in Section 2, this notion corresponds indeed to the one from the literature: see for instance [29, §3.2] for a straightforward comparison. At $m = 1$ the set ${}^m\mathrm{QSeq}_n$ contains sequences $(a_1, \dots, a_k)$ satisfying $a_1 = n$ and $a_i - a_{i+1} \in \{0, 1\}$ for $1 \le i \le k-1$.

Theorem 8.3. The mapping $(a_1, \dots, a_k) \mapsto F = a_k \cdots a_1$ is a bijection ${}^m\mathrm{QSeq}_n \to \mathsf{ZigZag}^m_n$. Under this bijection we have $\mathfrak{F}^m_a = \mathfrak{P}_F$.

Proof. We dispense with the case that $() \mapsto \varnothing$ and assume that all sequences and forests in what follows are nonempty.

First, we show that the map is well-defined. Condition (i) guarantees by Proposition 4.2 that $\mathsf{c}(F) = (c_1, c_2, \dots)$ with $c_i = \#\{j : a_j = i\}$. Condition (iii) implies that the only $c_i \neq 0$ which has m zeros in front of it in $\mathsf{c}(F)$ is c_{a_1}. But this means by (3.3) that $\mathrm{LTer}(F) = \{a_1\}$, and Condition (ii) then forces $F \in \mathsf{ZigZag}^m_n$.

This map is injective because $\mathsf{c}(F)$ determines the sequence of a_i. To show the map is surjective, we show that when we write $F \in \mathsf{ZigZag}^m_n$ as $F = a_k \cdots a_1$ with $a_1 \ge \cdots \ge a_k \ge 1$ that $(a_1, \dots, a_k) \in {}^m\mathrm{QSeq}_n$. To see this, note that $\mathsf{c}(F) = (c_i)_{i \in \mathbb{N}}$ has the property that i such that $c_i \neq 0$ are precisely $i = a_j$ for some j. Because $|\mathrm{LTer}(F)| = 1$ we conclude by (3.3) that $\mathrm{LTer}(F) = \{a_1\}$, and as $\mathrm{LTer}_m(F) \subset \{n - m + 1, \dots, n\}$ we have $n - m + 1 \le a_1 \le n$ verifying Condition (ii). Next, by (3.3) since $\mathrm{LTer}(F) = \{a_1\}$ we must have that when $i \ge 1$ and $a_{i+1} \neq a_1$ that there are at most $m - 1$ consecutive zeros in front of $c_{a_{i+1}}$ in $\mathsf{c}(F)$. This implies $a_i - a_{i+1} \le m$ verifying Condition (iii). We conclude that $(a_1, \dots, a_k) \in {}^m\mathrm{QSeq}_n$.

Finally, to show that $\mathfrak{F}^m_a = \mathfrak{P}_F$, we claim that it suffices to show that

$$\mathsf{T}^m_j \mathfrak{F}^m_a = \delta_{j,a_1} \mathfrak{F}^m_{a'} \tag{8.3}$$

where $a' = (a_2, \dots, a_k)$. Indeed, this implies that $\mathsf{T}^m_F \mathfrak{F}^m_a = \mathsf{T}^m_{a_k \cdots a_1} \mathfrak{F}^m_a = 1$, and for $G \neq F \in \mathsf{For}^m$ with $G = b_k \cdots b_1$ we have $\mathsf{T}^m_G \mathfrak{F}^m_a = \mathsf{T}^m_{b_k} \cdots \mathsf{T}^m_{b_1} \mathfrak{F}^m_a = \delta_{a,b} = 0$ so we conclude by Proposition 6.8.

Clearly $\mathsf{T}^m_j \mathfrak{F}^m_a = 0$ for $j \ge a_1 + 1$ as $\mathfrak{F}^m_a$ only uses variables $x_1, \dots, x_{a_1}$. Next, for $j = a_1$ we note that every element $\mathbf{i} \in \mathcal{C}^m(a)$ has i_1 maximal and $i_1 \le a_1$, so $\mathsf{T}^m_{a_i} \mathsf{x}_{\mathbf{i}} = \frac{1}{\mathsf{x}_{a_1}} \delta_{i_1, a_1} \mathsf{x}_{\mathbf{i}}$. Therefore

$$\mathsf{T}^m_{a_1} \mathfrak{F}^m_a = \frac{1}{x_{a_1}} \sum_{\substack{\mathbf{i} \in \mathcal{C}^m(a) \\ i_1 = a_1}} \mathsf{x}_{\mathbf{i}} = \sum_{\mathbf{i}' \in \mathcal{C}^m(a')} \mathsf{x}_{\mathbf{i}'} = \mathfrak{F}^m_{a'}. \tag{8.4}$$

For $n - m + 1 \le j < a_1$ we note that every element of $\mathcal{C}^m(a)$ either contains an a_1 or every entry is at most $n - m$, so $\mathsf{T}^m_j \mathsf{x}_{\mathbf{i}} = 0$ for all $\mathbf{i} \in \mathcal{C}^m(a)$, implying $\mathsf{T}^m_j \mathfrak{F}^m_a = 0$. Finally, because $\mathfrak{F}^m_a$ is m-quasisymmetric we have by Theorem 2.10 that $\mathsf{T}^m_j \mathfrak{F}^m_a = 0$ for $1 \le j \le n - m$. $\square$

Remark 8.4. The identity $\mathfrak{F}^m_a = \mathfrak{P}_F$ when $F \in \mathsf{ZigZag}^m_n$ also follows directly from the combinatorial definition of the m-forest polynomial: indeed the nodes in $\mathrm{IN}(F)$ form a path with $c_1 = \#\{j : a_j =$

$1\}$ nodes with $\rho_F(v) = 1$, followed by $c_2 = \#\{j : a_j = 2\}$ nodes with $\rho_F(v) = 2$, etc. Then the conditions for a sequence to be in $\mathcal{C}^m(a)$ are easily seen to correspond bijectively to colorings κ of zigzag forest polynomials. We leave the easy verification to the reader.

Example 8.5. Consider the element of ZigZag_6^2 from Figure 2. The corresponding element of ${}^2\mathrm{QSeq}_6$ is $a = (6,5,3,3)$, and the corresponding 2-slide polynomial equals

$$\mathfrak{F}^2_{6533} = x_3^2x_5x_6 + x_1x_3x_5x_6 + x_1^2x_5x_6 + x_1^2x_3x_6 + x_1^2x_3x_4.$$

Note that $\mathsf{T}_6^2\,\mathfrak{F}^2_{6533} = x_3^2x_5 + x_1x_3x_5 + x_1^2x_5 + x_1^2x_3 = \mathfrak{F}^2_{533}$ as predicted by Theorem 8.3.

We are now in position to identify a distinguished basis for ${}^m\mathrm{QSym}_n$.

Theorem 8.6. ${}^m\mathrm{QSym}_n$ has a $\mathbb{Z}$-basis $\{\mathfrak{P}_G \mid G \in \mathsf{ZigZag}_n^m\}$ of fundamental m-quasisymmetric polynomials.

Proof. Theorem 8.3 establishes that $\{\mathfrak{P}_G \mid G \in \mathsf{ZigZag}_n^m\}$ is the set of fundamental m-quasisymmetric polynomials. We have by Theorem 2.10 and Lemma 6.4 that

$$ {}^m\mathrm{QSym}_n = \mathrm{Pol}_n \cap \bigcap_{i=1}^{n-m} \ker(\mathsf{T}_i^m) = \bigcap_{i\notin\{n-m+1,\dots,n\}} \ker(\mathsf{T}_i^m). \tag{8.5}$$

By Proposition 6.9 this equals $\mathbb{Z}\{\mathfrak{P}_G : G \in \mathsf{LTFor}^m_{\{n-m+1,\dots,n\}}\} = \mathbb{Z}\{\mathfrak{P}_G : G \in \mathsf{ZigZag}_n^m\}$. □

In particular, using the T_G^m operators, for $f(x_1,\dots,x_n) \in {}^m\mathrm{QSym}_n$ we can directly extract the coefficients of the fundamental m-quasisymmetric expansion.

Corollary 8.7. If $f(x_1,\dots,x_n) \in {}^m\mathrm{QSym}_n$ is homogenous of degree k then

$$ f(x_1,\dots,x_n) = \sum_{\mathbf{a}\in {}^m\mathrm{QSeq}_n} (\mathsf{T}^m_{\mathbf{a}} f)\,\mathfrak{F}^m_{\mathbf{a}} \tag{8.6}$$

where we have denoted the reverse composition $\mathsf{T}^m_{\mathbf{a}} := \mathsf{T}^m_{a_k}\cdots\mathsf{T}^m_{a_1}$ for $\mathbf{a} = (a_1,\dots,a_k)$.

Proof. This follows from the formula in Proposition 6.8 and Theorem 8.3, since we have just shown that $f(x_1,\dots,x_n)$ is in the $\mathbb{Z}$-span of $\{\mathfrak{P}^m_G : G \in \mathsf{ZigZag}_n^m\}$. □

Example 8.8. Suppose we want to decompose the quasisymmetric polynomial $f(x_1,x_2,x_3) = 2x_1^2x_2 + 2x_1^2x_3 + 2x_2^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2x_3^2 \in \mathrm{QSym}_3$ into fundamental quasisymmetrics. We track in Figure 10 the nonzero applications $\mathsf{T}_{i_3}\mathsf{T}_{i_2}\mathsf{T}_{i_1}f$ where $(i_1,i_2,i_3) \in \mathrm{QSeq}_3$, and read off $f = \mathfrak{F}_{332} + 2\mathfrak{F}_{322} - 3\mathfrak{F}_{321}$.

$$\begin{tikzcd}
 & & x_1 + x_2 \arrow[r, "\mathsf{T}_2"] & 1 \\
f \arrow[r, "\mathsf{T}_3"] & 2x_1^2 + 2x_2^2 + x_1x_3 + x_2x_3 \arrow[ur, "\mathsf{T}_3"] \arrow[ddr, "\mathsf{T}_2"'] & & \\
 & & & 2 \\
 & & -x_1 + 2x_2 \arrow[ur, "\mathsf{T}_2"] \arrow[dr, "\mathsf{T}_1"'] & \\
 & & & -3
\end{tikzcd}$$

FIGURE 10. Trimming $f \in \mathrm{QSym}_3$

9. COINVARIANTS AND A QUASISYMMETRIC NIL-HECKE ALGEBRA

9.1. ***m*-quasisymmetric coinvariants.** One of the fundamental properties of the divided difference operators is that the operators $\partial_w : \mathrm{Pol}_n \to \mathrm{Pol}_n$ for $w \in S_n$ descend to the symmetric coinvariants $\partial_w : \mathrm{Coinv}_n \to \mathrm{Coinv}_n$. To show that ∂_w descends, one shows that ∂_i for $1 \leq i \leq n-1$ stabilizes Sym_n^+, a corollary of the fact that for $g \in \mathrm{Sym}_n$ and $f \in \mathrm{Pol}_n$ that

$$\partial_i(gf) = g\partial_w(f). \tag{9.1}$$

Although usually proved by an appeal to algebraic geometry, directly from these facts one can use the usual divided difference formalism to show that the images of Schubert polynomials $\{\mathfrak{S}_w \mid w \in S_n\}$ form a basis of Coinv_n and the images of Schubert polynomials $\{\mathfrak{S}_w \mid w \notin S_n \text{ and } \mathrm{Des}(w) \subset [n]\}$ forms a basis of Sym_n^+. Unable to find such a proof in extant literature we include it here, if only to emphasize the parallel picture for ${}^m\mathrm{QSCoinv}_n$ and m-forest polynomials.

Observation 9.1. $\{\mathfrak{S}_w : w \in S_n\}$ forms a basis of Coinv_n and $\{\mathfrak{S}_w \mid w \notin S_n \text{ and } \mathrm{Des}(w) \subset [n]\}$ forms a basis of Sym_n^+.

Proof. Because $\{\mathfrak{S}_w : \mathrm{Des}(w) \subset [n]\}$ forms a basis of Pol_n it suffices to show the basis statement for Sym_n^+. Consider the factorization $w = uv$ into $v \in S_n$ and $u \in \mathsf{Grass_n}$ with $\ell(w) = \ell(u) + \ell(v)$ from Observation 4.9. The key identity is that

$$\mathfrak{S}_u\mathfrak{S}_v = \mathfrak{S}_w + \sum a_{v'}\mathfrak{S}_{u'v'} \tag{9.2}$$

with $a_{v'} \in \mathbb{Z}$ where the sum is over pairs (u', v') with $u' \in \mathsf{Grass_n}$ and $v' \in S_n$ such that $\ell(u') > \ell(u)$ and $\ell(u'v') = \ell(u') + \ell(v') = \ell(w)$. This follows from noting that if $\ell(u') \leq \ell(u)$ then $\ell(v') \geq \ell(v)$ and we have

$$\partial_{u'v'}(\mathfrak{S}_u\mathfrak{S}_v) = \partial_{u'}\partial_{v'}(\mathfrak{S}_u\mathfrak{S}_v) = \partial_{u'}(\mathfrak{S}_u\partial_{v'}\mathfrak{S}_v) = \partial_{u'}\mathfrak{S}_u\delta_{v,v'} = \delta_{u,u'}\delta_{v,v'} \tag{9.3}$$

where in the second equality we used that $\mathfrak{S}_{u'} \in \mathrm{Sym}_n$ and $v' \in S_n$.

The identity (9.2) shows upper-triangularity between $\{\mathfrak{S}_u\mathfrak{S}_v : v \neq \mathrm{id}\} \subset \mathrm{Sym}_n^+$ and $\{\mathfrak{S}_w : w = uv \text{ with } v \neq \mathrm{id}\} = \{\mathfrak{S}_w : w \notin S_n, \mathrm{Des}(w) \subset [n]\}$ which implies $\{\mathfrak{S}_w : w \notin S_n \text{ and } \mathrm{Des}(w) \subset [n]\} \subset \mathrm{Sym}_n^+$.

It remains to show that $\mathrm{Sym}_n^+ \subset \mathbb{Z}\{\mathfrak{S}_w : w \notin S_n \text{ and } \mathrm{Des}(w) \subset [n]\}$. The identity (9.2) also establishes upper-triangularity between the set $\{\mathfrak{S}_u\mathfrak{S}_v \mid u \in \mathsf{Grass_n} \text{ and } v \in S_n\}$ and $\{\mathfrak{S}_w \mid \mathrm{Des}(w) \subset [n]\}$, which shows that Pol_n is generated by $\{\mathfrak{S}_v \mid v \in S_n\}$ as a Sym_n-module. Because we also know that $\{\mathfrak{S}_u : \mathrm{id} \neq u \in \mathsf{Grass_n}\}$ span the positive degree homogenous symmetric polynomials, we have reduced to showing that $\mathfrak{S}_u\mathfrak{S}_v \subset \mathbb{Z}\{\mathfrak{S}_w : w \notin S_n \text{ and } \mathrm{Des}(w) \subset [n]\}$ whenever $u \neq \mathrm{id}$ and $v \in S_n$. But this follows from (9.2). ☐

We further note that our argument is reminiscent of computations in the proofs of [45, Lemma 2.2 and Lemma 2.3]. Indeed the argument ibid. relies on a generalization of the factorization of a permutation used below to show that the corresponding Schubert polynomial lands in a certain ideal of Pol_n.

Using the quasisymmetric divided difference formalism, we can follow a similar route. We fix $n \geq 1$ for the rest of the section. Recall that ${}^m\mathrm{QSym}_n^+$ is the ideal in Pol_n generated by all polynomials $f \in {}^m\mathrm{QSym}_n$ with $\mathrm{ev}_0 f = 0$. We define the *m-quasisymmetric coinvariants* to be

$$ {}^m\mathrm{QSCoinv}_n := \mathrm{Pol}_n / {}^m\mathrm{QSym}_n^+. \tag{9.4}$$

We first establish the appropriate analogue of $\partial_i \in \mathrm{End}_{\mathrm{Sym}_n}(\mathrm{Pol}_n)$ for our purposes.

Proposition 9.2. If $H \in \mathsf{For}_n^m$ and $g \in {}^m\mathrm{QSym}_n$, then $\mathsf{T}_H^{\underline{m}}(gh) = \mathsf{R}_1^{m|H|}(g)\mathsf{T}_H^{\underline{m}}(h)$ for all $h \in \mathrm{Pol}$.

Proof. We proceed by induction on $|H|$. If $|H| = 0$ then there is nothing to prove, so suppose now the result is true for all smaller $|H|$. Let $i \in \mathrm{LTer}(H)$. As $H \in \mathsf{For}_n^m$ we have $1 \leq i \leq n-m$, so Theorem 2.6 implies $\mathsf{R}_i^m(g) = \mathsf{R}_1^m(g)$. Together with Lemma 2.12 and Theorem 2.10 this implies

$$\mathsf{T}_H^{\underline{m}}(gh) = \mathsf{T}_{H/i}^{\underline{m}}\mathsf{T}_i^{\underline{m}}(gh) = \mathsf{T}_{H/i}^{\underline{m}}(\mathsf{R}_i^m(g)\mathsf{T}_i^{\underline{m}}(h) + \mathsf{R}_{i+1}^m(h)\mathsf{T}_i^{\underline{m}}(g)) = \mathsf{T}_{H/i}^{\underline{m}}(\mathsf{R}_1^m(g)\mathsf{T}_i^{\underline{m}}(h)). \tag{9.5}$$

We know that $H/i \in \mathsf{For}_{n-m}^m$. Indeed, if there were a leaf $\geq n-m+1$ then as $i \leq n-m$ this would become a leaf $\geq n+1$ in $(H/i)\cdot i = H$. From the definition of ${}^m\mathrm{QSym}_n$ we see that $\mathsf{R}_1^m(g) \in {}^m\mathrm{QSym}_{n-m}$, and so by induction

$$\mathsf{T}_H^{\underline{m}}(gh) = \mathsf{R}_1^{m(|H|-1)}(\mathsf{R}_1^m(g))\mathsf{T}_{H/i}^{\underline{m}}(\mathsf{T}_i^{\underline{m}}(h)) = \mathsf{R}_1^{m|H|}(g)\mathsf{T}_H^{\underline{m}}(h). \tag{9.6}$$ ☐

Corollary 9.3. For $F \in \mathsf{For}_n^m$ we have $\mathsf{T}_F^{\underline{m}}({}^m\mathrm{QSym}_n^+) \subset {}^m\mathrm{QSym}_{n-m|F|}^+$, and so $\mathsf{T}_F^{\underline{m}}$ descends to a map

$$\mathsf{T}_F^{\underline{m}} : {}^m\mathrm{QSCoinv}_n \to {}^m\mathrm{QSCoinv}_{n-m|F|}.$$

In particular, $\mathsf{T}_1^{\underline{m}}, \dots, \mathsf{T}_{n-m}^{\underline{m}}$ descend to maps

$$\mathsf{T}_1^{\underline{m}}, \dots, \mathsf{T}_{n-m}^{\underline{m}} : {}^m\mathrm{QSCoinv}_n \to {}^m\mathrm{QSCoinv}_{n-m}.$$

Proof. We have $\mathsf{R}_1^{m|F|}({}^m\mathrm{QSym}_n) \subset {}^m\mathrm{QSym}_{n-m|F|}$ from the definition of ${}^m\mathrm{QSym}_n$, and $\mathsf{R}_1^{m|F|}$ preserves the property of being a positive degree homogenous polynomial, so we conclude by Proposition 9.2 that $\mathsf{T}_F^{\underline{m}}({}^m\mathrm{QSym}_n^+) \subset {}^m\mathrm{QSym}_{n-m|F|}^+$. ☐

To state our key result Theorem 9.7, it is useful to introduce the partial map $\star$ taking a pair (G, H) and returning the forest $(\Theta'_n)^{-1}(G, H)$, where Θ'_n is from Theorem 4.10:

Definition 9.4. Let $G \in \mathsf{ZigZag}^m_n$ and $H \in \mathsf{For}^m_n$. Then we define

$$G \star H := \begin{cases} (\tau^{-m|H|}G) \cdot H & \text{if } \min \operatorname{supp} G > m|H| \text{ or } G = \varnothing \\ \text{does not exist} & \text{otherwise.} \end{cases} \tag{9.7}$$

The second part of Theorem 4.10 can be then stated in the following equivalent form:

Corollary 9.5. Let $F \in \mathsf{For}^m$ and $n \geq 1$. Then $F \in \mathsf{LTFor}^m_n$ if and only if we can write $F = G \star H$ with $G \in \mathsf{ZigZag}^m_n$ and $H \in \mathsf{For}^m_n$. In that case the decomposition $F = G \star H$ is unique: $H \leq F$ is determined by having its set of internal nodes $\mathrm{IN}(H) \subset \mathrm{IN}(F)$ consist of all fully supported internal nodes of F, and $G = \tau^{m|H|}(F/H)$.

Rather than just describe a basis for ${}^m\mathrm{QSym}^+_n$, we also describe bases of the ideals generated by homogenous elements of ${}^m\mathrm{QSym}_n$ of degree $\geq k$, which will be important in the next subsection.

Definition 9.6. Let $\mathcal{I}^m_{k,n} \subset \mathrm{Pol}_n$ be the ideal generated by all homogenous polynomials $f \in {}^m\mathrm{QSym}_n$ with $\deg(f) \geq k$.

We also define a subspace $\mathcal{I}^{m,\star}_{k,n} \subset \mathrm{Pol}_n$ by

$$\mathcal{I}^{m,\star}_{k,n} = \bigoplus \mathbb{Z}\{\mathfrak{P}_{G\star H} \mid G \in \mathsf{ZigZag}^m_n, H \in \mathsf{For}^m_n, G \star H \text{ exists and } |G| \geq k\} \tag{9.8}$$

$$= \bigoplus \mathbb{Z}\{\mathfrak{P}_F \mid F \in \mathsf{LTFor}^m_n, \text{ and if } (G, H) = \Theta'_n(F) \text{ then } |G| \geq k\}. \tag{9.9}$$

Note that by Proposition 6.11 and Corollary 9.5 we have

$$\mathcal{I}^{m,\star}_{0,n} = \bigoplus \mathbb{Z}\{\mathfrak{P}_F \mid F \in \mathsf{LTFor}^m_n\} = \mathrm{Pol}_n = \mathcal{I}^m_{0,n}. \tag{9.10}$$

Directly from the definitions we also note

$$\mathcal{I}^m_{1,n} = {}^m\mathrm{QSym}^+_n, \text{ and} \tag{9.11}$$

$$\mathcal{I}^{m,\star}_{1,n} = \bigoplus \mathbb{Z}\{\mathfrak{P}_F \mid F \in \mathsf{LTFor}^m_n \setminus \mathsf{For}^m_n\}. \tag{9.12}$$

Theorem 9.7. We have $\mathcal{I}^m_{k,n} = \mathcal{I}^{m,\star}_{k,n}$ for all k, m, n. In particular for $k = 1$, we get

(1) ${}^m\mathrm{QSym}^+_n$ has a $\mathbb{Z}$-basis given by $\{\mathfrak{P}_F : F \in \mathsf{LTFor}^m_n \setminus \mathsf{For}^m_n\}$.
(2) ${}^m\mathrm{QSCoinv}_n$ has a $\mathbb{Z}$-basis given by $\{\mathfrak{P}_F : \mathsf{For}^m_n\}$. In particular its dimension is given by the Raney number in Lemma 3.4.

For the particular case $k = 1$, we note that when $m = 1$ the result was shown in [38, Corollary 4.3] via a more computational approach, while the dimension statement recovers [3, Theorem 5.1] by taking $n = pm$.

Lemma 9.8. If $G \in \mathsf{ZigZag}_n^m$ and $H \in \mathsf{For}_n^m$ then

$$\mathfrak{P}_G\mathfrak{P}_H - \delta_{G\star H \text{ exists}}\mathfrak{P}_{G\star H} \in \mathcal{I}^{m,\star}_{|G|+1,n}. \tag{9.13}$$

Proof. Since $\mathfrak{P}_G\mathfrak{P}_H$ is in Pol_n and has degree $|G|+|H|$, its m-forest expansion only contains terms $\mathfrak{P}_F$ where $F \in \mathsf{LTFor}_n^m$ and $|F| = |G|+|H|$. Given such F, use Corollary 9.5 to write

$$F = G' \star H' = (\tau^{-m|H'|}G')\cdot H' \tag{9.14}$$

with $H' \in \mathsf{For}_n^m$ and $G' \in \mathsf{ZigZag}_n^m$.

Then $\mathfrak{P}_{G'} \in {}^m\mathrm{QSym}_n$ by Theorem 9.7, so Proposition 9.2 yields

$$\mathsf{T}^{m}_{F}(\mathfrak{P}_G\mathfrak{P}_H) = \mathsf{T}^{m}_{\tau^{-m|H'|}G'}\mathsf{T}^{m}_{H'}(\mathfrak{P}_G\mathfrak{P}_H) = \mathsf{T}^{m}_{\tau^{-m|H'|}G'}(\mathsf{R}_1^{m|H'|}(\mathfrak{P}_G)\mathsf{T}^{m}_{H'}(\mathfrak{P}_H)) \tag{9.15}$$

By Proposition 6.12 this vanishes unless $\tau^{-m|H'|}G$ exists and in that case $\mathsf{R}_1^{m|H'|}(\mathfrak{P}_G) = \mathfrak{P}_{\tau^{-m|H'|}G}$. We thus get by Corollary 6.6

$$\mathsf{T}^{m}_{F}(\mathfrak{P}_G\mathfrak{P}_H) = \mathsf{T}^{m}_{\tau^{-m|H'|}G'}(\mathfrak{P}_{\tau^{-m|H'|}G}\mathfrak{P}_{H/H'}) \tag{9.16}$$

if $\tau^{-m|H'|}G$ exists and $H' \geq H$, and is 0 otherwise. If $H' = H$ then necessarily $|G'| = |G|$, and so

$$\mathsf{T}^{m}_{\tau^{-m|H'|}G'}(\mathfrak{P}_{\tau^{-m|H'|}G}\mathfrak{P}_{H/H'}) = \mathsf{T}^{m}_{\tau^{-m|H'|}G'}(\mathfrak{P}_{\tau^{-m|H'|}G}) = \delta_{G',G}. \tag{9.17}$$

$\square$

Proof of Theorem 9.7. Lemma 9.8 implies that for each fixed degree d, the $\mathbb{Z}$-linear transformation between the degree d homogenous component of $\mathcal{I}^{m,\star}_{k,n}$ to

$$\mathbb{Z}\{\mathfrak{P}_G\mathfrak{P}_H : G \in \mathsf{ZigZag}_n^m, H \in \mathsf{For}_n^m, G\star H \text{ exists}, |G| \geq k, \text{ and } |G|+|H| = d\} \tag{9.18}$$

taking $\mathfrak{P}_{G\star H}$ to $\mathfrak{P}_G\mathfrak{P}_H$ is strictly upper triangular and hence invertible. Therefore

$$\mathcal{I}^{m,\star}_{k,n} = \mathbb{Z}\{\mathfrak{P}_G\mathfrak{P}_H : G\star H \text{ exists and } |G| \geq k\} \tag{9.19}$$

and thus $\mathcal{I}^{m,\star}_{k,n} \subset \mathcal{I}^{m}_{k,n}$. As $\mathcal{I}^{m,\star}_{0,n} = \mathrm{Pol}_n$ by (9.10), this shows that Pol_n is spanned as a ${}^m\mathrm{QSym}_n$-module by $\{\mathfrak{P}_H : H \in \mathsf{For}_n^m\}$.

Now by Theorem 8.6 we also have $\{\mathfrak{P}_G : G \in \mathsf{ZigZag}_n^m \text{ and } |G| \geq k\}$ span the degree $\geq k$ homogenous components of ${}^m\mathrm{QSym}_n$ as a $\mathbb{Z}$-module. Thus to show the inclusion $\mathcal{I}^{m}_{k,n} \subset \mathcal{I}^{m,\star}_{k,n}$ it suffices to show that $\mathfrak{P}_G\mathfrak{P}_H \in \mathcal{I}^{m,\star}_{k,n}$ whenever $|G| \geq k$ and $H \in \mathsf{For}_n^m$. This final statement follows from Lemma 9.8. $\square$

9.2. **Endomorphisms of polynomials R_1-commuting with quasisymmetrics.** Recall (cf. [35]) that the ring $\mathrm{End}_{\mathrm{Sym}_n}(\mathrm{Pol}_n)$ is generated by the operations of (multiplication by) x_i and ∂_i, and in fact

$$\mathrm{End}_{\mathrm{Sym}_n}(\mathrm{Pol}_n) = \bigoplus_{w\in S_n} \mathrm{Pol}_n\,\partial_w. \tag{9.20}$$

Taking the limit of these algebras we obtain the subalgebra of $\mathrm{End}(\mathrm{Pol})$ generated by all x_i and ∂_i, which decomposes as $\bigoplus_{w\in S_\infty} \mathrm{Pol}_n\,\partial_w$. This may be informally thought of as those endomorphisms of Pol which modify only finitely many coordinates and commute (in an appropriate sense) with symmetric power series.

For quasisymmetrics we have in stark contrast the following observation.

Observation 9.9. $\mathrm{End}_{{}^m\mathrm{QSym}_n}(\mathrm{Pol}_n) = \mathrm{Pol}_n$.

Proof. If $\Phi \in \mathrm{End}_{{}^m\mathrm{QSym}_n}(\mathrm{Pol}_n)$ then because $(x_1 \cdots x_n)f$ is quasisymmetric for all $f \in \mathrm{Pol}_n$ we have

$$x_1 \cdots x_n \Phi(f) = \Phi((x_1 \cdots x_n)f) = x_1 \cdots x_n f \Phi(1), \tag{9.21}$$

implying that $\Phi(f) = f\Phi(1)$. □

To find the correct analogue we have to consider $\mathrm{Hom}_{{}^m\mathrm{QSym}_n}(\mathrm{Pol}_n, \mathrm{Pol}_{n-k})$ where Pol_{n-k} is considered as a ${}^m\mathrm{QSym}_n$-module by the map $\mathsf{R}_1^k|_{{}^m\mathrm{QSym}_n} : {}^m\mathrm{QSym}_n \to {}^m\mathrm{QSym}_{n-k}$, which is well-defined directly from the definition of ${}^m\mathrm{QSym}_n$. We note that if k is a multiple of m then $\mathsf{R}_1^k|_{{}^m\mathrm{QSym}_n} = \mathsf{R}_{n-k+1}^k|_{{}^m\mathrm{QSym}_n}$ by Theorem 2.6 which is the map setting $x_n = \cdots = x_{n-k+1} = 0$.

Theorem 9.10. We have

$$\mathrm{Hom}_{{}^m\mathrm{QSym}_n}(\mathrm{Pol}_n, \mathrm{Pol}_{n-k}) = \bigoplus_{H \in \mathsf{For}_n^m \text{ with } m|H| \le k} \mathrm{Pol}_{n-k}\, \mathsf{R}_1^{k-m|H|}\, \mathsf{T}_H^{\underline{m}}. \tag{9.22}$$

Remark 9.11. The limiting object

$$\bigoplus_k \lim_{n\to\infty} \mathrm{Hom}_{{}^m\mathrm{QSym}_n}(\mathrm{Pol}_n, \mathrm{Pol}_{n-k}) = \bigoplus_{F \in \mathsf{For}^m} \mathrm{Pol}\, \mathsf{R}_1^a\, \mathsf{T}_F^{\underline{m}} \tag{9.23}$$

is the subalgebra of $\mathrm{End}(\mathrm{Pol}_n)$ generated by all x_i, R_1, and $\mathsf{T}_i^{\underline{m}}$. This may be informally thought of as those endomorphisms of Pol which act on all but finitely many coordinates as $x_i \mapsto x_{i-k}$ and commute (in an appropriate sense) with m-quasisymmetric power series.

Proof of Theorem 9.10. First, we show that $\mathsf{R}_1^{k-m|H|}\mathsf{T}_H^{\underline{m}} \in \mathrm{Hom}_{{}^m\mathrm{QSym}_n}(\mathrm{Pol}_n, \mathrm{Pol}_{n-k})$. By Proposition 9.2 we have $\mathsf{T}_H^{\underline{m}} \in \mathrm{Hom}_{{}^m\mathrm{QSym}_n}(\mathrm{Pol}_n, \mathrm{Pol}_{n-m|H|})$ for any $H \in \mathsf{For}_n^m$. Then, we have $\mathsf{R}_1^{k-m|H|}$ lies in $\mathrm{Hom}_{{}^m\mathrm{QSym}_{n-m|H|}}(\mathrm{Pol}_{n-m|H|}, \mathrm{Pol}_{n-k})$ because for $f \in {}^m\mathrm{QSym}_n$ and $g \in \mathrm{Pol}_n$ we have $\mathsf{R}_1^{k-m|H|}(fg) = \mathsf{R}_1^{k-m|H|}(f)\mathsf{R}_1^{k-m|H|}(g)$.

We now show that there are functions $\{\Psi_H : H \in \mathsf{For}_n^m \text{ and } m|H| \le k\} \subset \mathrm{Hom}_{{}^m\mathrm{QSym}_n}(\mathrm{Pol}_n, \mathrm{Pol}_{n-k})$ such that $\Psi_H(\mathfrak{P}_{H'}) = \delta_{H,H'}$ for all $H' \in \mathsf{For}_n^m$ with $|H'| > |H|$, and

$$\Psi_H = \mathsf{R}_1^{k-m|H|}\, \mathsf{T}_H^{\underline{m}} - \sum_{|H'|>|H|,\ H' \in \mathsf{For}_n} b_{H,H'}(x_1, \ldots, x_{n-k})\, \Psi_{H',k}. \tag{9.24}$$

We do this by backwards induction on $|H|$. For $|H| = \lfloor k/m \rfloor$ we take $\Psi_H = \mathsf{R}_1^{k-m|H|}\, \mathsf{T}_H^{\underline{m}}$. Otherwise, $\mathsf{R}_1^{k-m|H|}\mathsf{T}_H^{\underline{m}}(\mathfrak{P}_H) = 1$ and $\mathsf{R}_1^{k-m|H|}\mathsf{T}_H^{\underline{m}}(\mathfrak{P}_{H'}) = 0$ when $H \neq H' \in \mathsf{For}_n^m$ and $|H'| \le |H|$, so we can take $b_{H,H'} = \mathsf{R}_1^{k-|H|}\mathsf{T}_H^{\underline{m}}\mathfrak{P}_{H'}$.

As the Pol_{n-k}-linear transformation expressing $\{\mathsf{R}_1^{k-m|H|}\mathsf{T}_H : H \in \mathsf{For}_n^m \text{ and } m|H| \leq k\}$ in terms of $\{\Psi_H : H \in \mathsf{For}_n \text{ and } m|H| \leq k\}$ is invertible by upper-triangularity, it suffices to show that $\{\Psi_H : H \in \mathsf{For}_n^m \text{ and } |H| \leq k\}$ is a Pol_{n-k}-basis for $\mathrm{Hom}_{^m\mathrm{QSym}_n}(\mathrm{Pol}_n, \mathrm{Pol}_{n-k})$. The Ψ_H are Pol_{n-k}-linearly independent because if $\sum f_H(x_1, \ldots, x_{n-k})\Psi_H = 0$ then for all H' applying the left hand side to $\mathfrak{P}_{H'}$ shows that $f_{H'} = 0$. It remains to show that the Ψ_H span.

Let $\Phi \in \mathrm{Hom}_{^m\mathrm{QSym}_n}(\mathrm{Pol}_n, \mathrm{Pol}_{n-k})$. Define

$$\Phi' = \Phi - \sum_{H \in \mathsf{For}_n^m} \Phi(H)\,\Psi_H. \tag{9.25}$$

We want to show $\Phi' = 0$. Already $\Phi'(\mathfrak{P}_H) = 0$ for all $H \in \mathsf{For}_n^m$ with $m|H| \leq k$ by the properties of the endomorphisms $\Psi_{H'}$. It remains to show $\Phi'(\mathfrak{P}_H) = 0$ for those $H \in \mathsf{For}_n^m$ with $|H| > \lfloor k/m \rfloor$: this is enough since the $\mathfrak{P}_H$ for $H \in \mathsf{For}_n^m$ generate Pol_n as an $^m\mathrm{QSym}_n$-module by Theorem 9.7.

We induct on $|H|$. We assume that $|H| > \lfloor k/m \rfloor$. Because $H \in \mathsf{For}_n^m$ we know that $|H| \leq \lfloor n/m \rfloor$ so we may assume that $k < n$. Choose any $G \in \mathsf{ZigZag}_n^m$ with $\min\operatorname{supp} G = k+1$: these always exist as is readily checked. Since $\min\operatorname{supp} G \leq m|H|$ we have that $G \star H$ does not exist and thus by Lemma 9.8 $\mathfrak{P}_G\mathfrak{P}_H \in \mathcal{I}^{\star}_{|G|+1,n}$. By Theorem 9.7 we know that $\mathcal{I}^{m,\star}_{|G|+1,n} = \mathcal{I}^m_{|G|+1,n}$ so we may write

$$\mathfrak{P}_G\mathfrak{P}_H = \sum g_i(x_1, \ldots, x_n)h_i(x_1, \ldots, x_n) \tag{9.26}$$

with $g_i(x_1, \ldots, x_n) \in \mathrm{QSym}_n$ with $\deg g_i \geq |G| + 1$ and therefore with $\deg h_i = |G| + |H| - \deg g_i < |H|$.

Since $\{\mathfrak{P}_H : H \in \mathsf{For}_n^m\}$ is a $\mathbb{Z}$-basis of $^m\mathrm{QSCoinv}_n$ by Theorem 9.7 we may further assume that each $h_i = \mathfrak{P}_{H_i'}$ for some $H_i' \in \mathsf{For}_n^m$, and $|H_i'| = \deg \mathfrak{P}_{H_i'} < |H|$. We therefore have

$$(\mathsf{R}_1^k\,\mathfrak{P}_G)\,\Phi'(\mathfrak{P}_H) = \Phi'(\mathfrak{P}_G\,\mathfrak{P}_H) = \sum \Phi'(g_i\,\mathfrak{P}_{H_i'}) = \sum \mathsf{R}_1^k(g_i)\,\Phi'(\mathfrak{P}_{H_i'}) = 0. \tag{9.27}$$

Because $\min\operatorname{supp} G = k+1$ we have $\mathsf{R}_1^k\,\mathfrak{P}_G = \mathfrak{P}_{\tau^{-k}G} \neq 0$ by Proposition 6.12, so $\Phi'(\mathfrak{P}_H) = 0$ as desired. $\square$

9.3. **Quasisymmetric nil-Hecke Algebra.** The nil-Hecke algebra is the noncommutative algebra with generators denoted $x_1, x_2, \ldots$ and $\partial_1, \partial_2, \ldots$, modulo the relations

- (Comm.) $x_ix_j = x_jx_i$ for all i, j, $\partial_i\partial_j = \partial_j\partial_i$ for $|i-j| \geq 2$, and $x_i\partial_j = \partial_jx_i$ for $j \notin \{i-1, i\}$.
- (Braid) $\partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1}$
- (Nil-Hecke) $\partial_i^2 = 0$.
- (Leibniz) $\partial_ix_i = x_{i+1}\partial_i + \mathrm{id}$ and $\partial_ix_{i+1} = x_i\partial_i - \mathrm{id}$.

Using these relations it is easy to straighten any combination of x_i and ∂_i into a Pol-linear combination of operators ∂_w for $w \in S_\infty$, and this can be used to show that the nil-Hecke algebra is isomorphic to $\mathrm{End}_{\mathrm{Sym}_n}(\mathrm{Pol}_n)$.

This also affords a "diagrammatic presentation", encoded by the additional relations needed to specify the presentation beyond the formal commutation relations coming from the fact that for $Z \in \{\partial, x\}$ ($x : \mathrm{Pol}_1 \to \mathrm{Pol}_1$ representing the "multiplication by x" map), we have $Z_i = \mathrm{id}^{\otimes i-1} \otimes Z \otimes \mathrm{id}^{\otimes\infty} : \mathrm{Pol} \to \mathrm{Pol}$, where we view $\mathrm{Pol} = \mathrm{Pol}_1^{\otimes\infty}$. This is given by

- (Braid) $(\partial \otimes \mathrm{id})(\mathrm{id} \otimes \partial)(\partial \otimes \mathrm{id}) = (\mathrm{id} \otimes \partial)(\partial \otimes \mathrm{id})(\mathrm{id} \otimes \partial)$
- (nil-Hecke) $\partial^2 = 0$
- (Leibniz) $\partial(x \otimes \mathrm{id}) = (\mathrm{id} \otimes x)\partial + \mathrm{id}^{\otimes 2}$ and $\partial(\mathrm{id} \otimes x) = (x \otimes \mathrm{id})\partial - \mathrm{id}^{\otimes 2}$.

If we represent x and ∂ by the following diagrams and representing $F \circ G$ by=s stacking the dia-

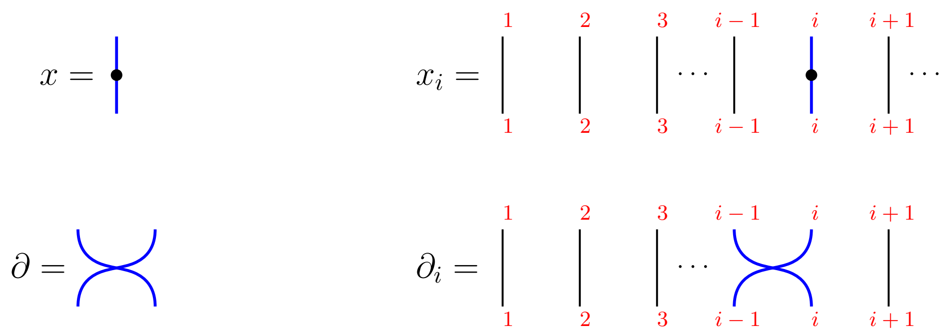

FIGURE 11. Diagram generators for the nil-Hecke algebra

gram for F on top of the diagram for G, the relations can be depicted as follows.

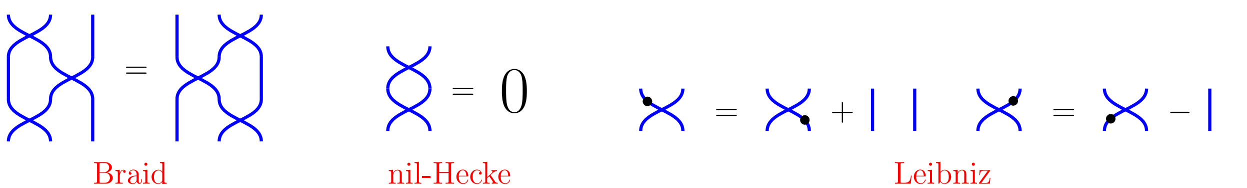

FIGURE 12. Diagram relations for the nil-Hecke algebra

As noted in Remark 9.11 the algebra in $\mathrm{End}(\mathrm{Pol})$ generated by R_1, T_i^m and x_i may be thought of as the m-quasisymmetric analogue of the nil-Hecke algebra.

Theorem 9.12. The algebra in $\mathrm{End}(\mathrm{Pol})$ generated by R_1, T_i^m and x_i has relations generated by

(i) (Comm.) $\mathsf{T}_i^m \mathsf{R}_1 = \mathsf{R}_1 \mathsf{T}_{i+1}^m$, for $i \geq 1$, $\mathsf{R}_1 x_i = x_{i-1}\mathsf{R}_1$ for $i > 1$, $x_i x_j = x_j x_i$ for all i, j
$\mathsf{T}_i^m x_j = x_j \mathsf{T}_i^m$ if $j < i$ and $\mathsf{T}_i^m x_j = x_{j-m}\mathsf{T}_i^m$ if $j > i + m$
$\mathsf{T}_i^m \mathsf{T}_j^m = \mathsf{T}_j^m \mathsf{T}_{i+m}^m$ for $i > j$,
(ii) $\mathsf{R}_1 x_1 = 0$ and $\mathsf{T}_i^m x_{i+j} = 0$ for $1 \leq j \leq m-1$
(iii) $\mathsf{T}_i^m x_i = \mathsf{R}_1^m + x_1\mathsf{T}_1^m + \cdots + x_i\mathsf{T}_i^m$ and $\mathsf{T}_i^m x_{i+m} = -(\mathsf{R}_1^m + x_1\mathsf{T}_1^m + \cdots + x_{i-1}\mathsf{T}_{i-1}^m)$

Proof. All of these relations are easy to verify directly. For (iii), we note by Lemma 2.12 that $\mathsf{T}_i^m(x_i f) = \mathsf{R}_{i+1}^m f$ and $\mathsf{T}_i^m(x_{i+m}f) = -\mathsf{R}_i^m f$, and then the expressions are obtained by telescoping the identity $x_j\mathsf{T}_j^m = \mathsf{R}_{j+1}^m - \mathsf{R}_j^m$.

Using these relations one can straighten any composition of $\mathsf{R}_1, \mathsf{T}_i^m$ and x_i into a Pol-linear combination of $\mathsf{R}_1^a \mathsf{T}_F^m$. It follows from Theorem 9.10 that there are no further relations. □

As the proof shows, the relations in 9.12(iii) could be simplified if we included redundant generators R_i^m in the presentation.

Let us now focus on the case $m = 1$. The quasisymmetric nil-Hecke algebra also admits a diagrammatic presentation. Note that for $Z \in \{\mathsf{T}, \mathsf{R}\}$ we again have $Z_i = \mathrm{id}^{\otimes i-1} \otimes Z \otimes \mathrm{id}^{\otimes \infty}$ where $\mathsf{T} : \mathrm{Pol}_2 \to \mathrm{Pol}_1$ is the operator introduced in Equation (5.3) and $\mathsf{R} : \mathrm{Pol}_1 \to \mathrm{Pol}_0$ is the operator $\mathsf{R}(f) = f(0)$.

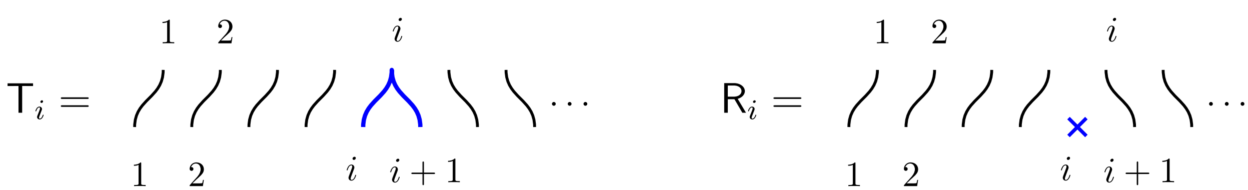

FIGURE 13. Diagram generators for the quasisymmetric nil-Hecke algebra

Corollary 9.13. The diagrammatic presentation of the quasisymmetric nil-Hecke algebra (for $m = 1$) is given by the following.

(i) $\mathsf{R}x = 0$
(ii) $\mathsf{T}(x \otimes \mathrm{id}) = \mathrm{id} \otimes \mathsf{R}$ and $\mathsf{T}(\mathrm{id} \otimes x) = -\mathsf{R} \otimes \mathrm{id}$
(iii) $x\mathsf{T} = \mathrm{id} \otimes \mathsf{R} - \mathsf{R} \otimes \mathrm{id}$

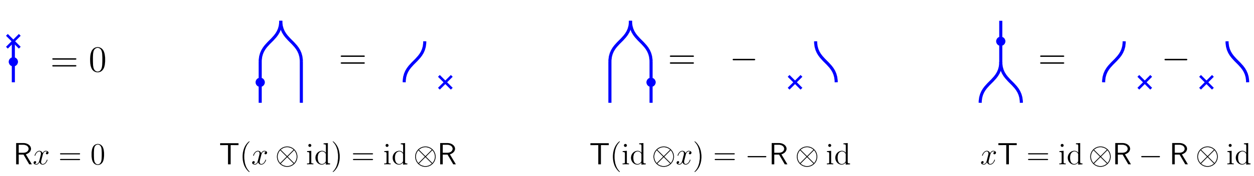

FIGURE 14. Diagram relations for the quasisymmetric nil-Hecke algebra

Proof. It is easy to verify (i)–(iii) are satisfied. Conversely, the relations in Theorem 9.12(i) are trivially satisfied, and (i) implies $\mathsf{R}_1 x_1 = 0$. It remains to show the relations in Theorem 9.12(iii). Using (ii) these amount to showing the relations $\mathsf{R}_{i+1} = \mathsf{R}_1 + x_1\mathsf{T}_1 + \cdots + x_i\mathsf{T}_i$ and $\mathsf{R}_i = \mathsf{R}_1 + x_1\mathsf{T}_1 + \cdots + x_{i-1}\mathsf{T}_{i-1}$. But (iii) implies $x_j\mathsf{T}_j = \mathsf{R}_{j+1} - \mathsf{R}_j$ so both equalities now follow. □

Example 9.14. The relation $\mathsf{T}_i\mathsf{R}_{i+1} = \mathsf{R}_{i+1}\mathsf{T}_i + \mathsf{R}_i\mathsf{T}_{i+1}$, written as $\mathsf{T}(\mathrm{id}\otimes\mathsf{R}\otimes\mathrm{id}) = \mathsf{T}\otimes\mathsf{R}+\mathsf{R}\otimes\mathsf{T}$, follows from the chain of equalities

$$\begin{aligned}\mathsf{T}(\mathrm{id}\otimes\mathsf{R}\otimes\mathrm{id}) =& \mathsf{T}(x\mathsf{T}\otimes\mathrm{id}) + \mathsf{T}(\mathsf{R}\otimes\mathrm{id}\otimes\mathrm{id})\\ =& \mathsf{T}(x\otimes\mathrm{id})(\mathsf{T}\otimes\mathrm{id}) + \mathsf{R}\otimes\mathsf{T}\\ =&(\mathrm{id}\otimes\mathsf{R})(\mathsf{T}\otimes\mathrm{id}) + \mathsf{R}\otimes\mathsf{T} = \mathsf{T}\otimes\mathsf{R}+\mathsf{R}\otimes\mathsf{T}\end{aligned}$$

where in the second equality we used the commutation relation $\mathsf{T}(\mathsf{R}\otimes\mathrm{id}\otimes\mathrm{id}) = \mathsf{R}\otimes\mathsf{T}$.

10. Harmonics

In this section we compute a basis for the m-quasisymmetric harmonics in terms of the volume polynomials $V_F(\lambda)$ of certain "forest polytopes" $\mathbf{C}_{F,\lambda}$ associated to a fully supported forest $F \in \mathsf{For}_n^m$ and a decreasing sequence $\lambda_1 \geq \cdots \geq \lambda_n$. We also show that the m-quasisymmetric harmonics are spanned by the derivatives of the top degree m-quasisymmetric harmonics. The special case $m = 1$ answers a question of Aval–Bergeron–Li [5].

Definition 10.1. The D-pairing $\langle,\rangle_D : \mathbb{Q}[x_1,\dots,x_n]\otimes\mathbb{Q}[\lambda_1,\dots,\lambda_n] \to \mathbb{Q}$ is the bilinear pairing

$$\langle f, g\rangle_D = \mathrm{ev}_0\, f(\mathrm{D}_1,\dots,\mathrm{D}_n)\, g(\lambda_1,\dots,\lambda_n), \tag{10.1}$$

where $\mathrm{D}_i := \frac{d}{d\lambda_i}$.

This pairing may be described alternatively as having $\langle \mathsf{x}^{\mathsf{c}}, \lambda^{\mathsf{d}}\rangle = \delta_{c,d}\,\mathsf{c}!$ where $\mathsf{c} = (c_1,\dots,c_n)$ and $\mathsf{d} = (d_1,\dots,d_n)$ are sequences of nonnegative integers, and $\mathsf{c}! := c_1!\cdots c_n!$.

Definition 10.2. The *m-quasisymmetric harmonics* are defined to be

$$\begin{aligned}\mathrm{HQSym}_n^m :=& \{f\in\mathbb{Q}[\lambda_1,\dots,\lambda_n] \mid \langle g, f\rangle_D = 0 \text{ for all } g\in {}^m\mathrm{QSym}_n^+\}\\ =&\{f\in\mathbb{Q}[\lambda_1,\dots,\lambda_n] \mid g(\mathrm{D}_1,\dots,\mathrm{D}_n)f = 0 \text{ for all } g\in {}^m\mathrm{QSym}_n \text{ with } \mathrm{ev}_0\, g = 0\}.\end{aligned}$$

The key insight is that we can translate the duality $\mathsf{T}_F^m\mathfrak{P}_G^m = \delta_{F,G}$ into a D-pairing duality $\langle \mathsf{V}_F^m(1), \mathfrak{P}_G^m\rangle_D = \delta_{F,G}$, where V_F^m is the D-pairing adjoint of T_F^m.

There are two main steps that we will carry out.

(1) We determine the adjoint V_i^m of individual T_i^m as an integration operator.
(2) We interpret composites of these V_i^m applied to 1 as recursively computing $V_F(\lambda)$ in terms of $V_{F/i}(\lambda)$.

For technical reasons we will have to carry out these steps using the D-pairing between polynomials rings $\mathbb{Q}[x_1,x_2,\dots]$ and $\mathbb{Q}[\lambda_1,\lambda_2,\dots]$ in infinitely many variables, and then return to finitely many variables case by truncating appropriately.

10.1. **The adjoint to trimming under the D-pairing.** All mentions of adjoints in what follows are meant to be with respect to the D-pairing. Note that if $X \in \mathrm{End}(\mathbb{Q}[x_1, x_2, \dots])$ then the adjoint $X^\vee \in \mathrm{End}(\mathbb{Q}[\lambda_1, \lambda_2, \dots])$ might not exist, but if it does then it is unique since $\langle f, g\rangle_D = 0$ for all f implies $g = 0$.

Definition 10.3. For $f \in \mathbb{Q}[\lambda_1, \lambda_2, \dots]$ we define

$$\mathsf{R}_i^\vee f := f(\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots) \tag{10.2}$$

$$\mathsf{V}_i f := \int_{\lambda_{i+1}}^{\lambda_i} f(\lambda_1, \dots, \lambda_{i-1}, z, \lambda_{i+2}, \dots) dz \tag{10.3}$$

$$\mathsf{V}_i^m f := \int_{\lambda_{i+m}}^{\lambda_i} f(\lambda_1, \dots, \lambda_{i-1}, z, \lambda_{i+m+1}, \dots) dz = (\mathsf{R}_{i+1}^\vee)^{m-1} \mathsf{V}_i f \tag{10.4}$$

Proposition 10.4. The operators R_i, T_i, and T_i^m are adjoint to $\mathsf{R}_i^\vee$, V_i, and V_i^m respectively. In symbols, for $g \in \mathbb{Q}[x_1, x_2, \dots]$ and $f \in \mathbb{Q}[\lambda_1, \lambda_2, \dots]$ we have

$$\langle g, \mathsf{R}_i^\vee f\rangle_D = \langle \mathsf{R}_i g, f\rangle_D, \qquad \langle g, \mathsf{V}_i f\rangle_D = \langle \mathsf{T}_i g, f\rangle_D, \qquad \langle g, \mathsf{V}_i^m f\rangle_D = \langle \mathsf{T}_i^m g, f\rangle_D. \tag{10.5}$$

Consequently, for $F \in \mathsf{For}^m$ we have a well-defined operator V_F^m adjoint to T_F^m defined by

$$\mathsf{V}_F^m = \mathsf{V}_{i_k}^m \cdots \mathsf{V}_{i_1}^m \text{ for any } (i_1, \dots, i_k) \in \mathrm{Trim}(F). \tag{10.6}$$

Proof. We prove the adjointness claims as the well-definedness of V_F^m follows by taking the adjoint of the equality $\mathsf{T}_F^m = \mathsf{T}_{i_1}^m \cdots \mathsf{T}_{i_k}^m$. We verify adjointness by checking them on monomials $f = \lambda^{\mathsf{c}}$ and $g = \mathsf{x}^{\mathsf{d}}$ for $\mathsf{c}, \mathsf{d} \in \mathsf{Codes}$.

For the adjointness of R_i and $\mathsf{R}_i^\vee$ we have $\langle \mathsf{R}_i \mathsf{x}^{\mathsf{d}}, \lambda^{\mathsf{c}}\rangle_D = \langle \mathsf{x}^{\mathsf{d}}, \mathsf{R}_i^\vee \lambda^{\mathsf{c}}\rangle_D = 0$ if $c_i \neq 0$ and if $c_i = 0$ then both are equal to $\mathsf{d}!\,\delta_{\mathsf{d},\mathsf{c}'}$ where $\mathsf{c}' = (c_1, \dots, c_{i-1}, c_{i+1}, \dots)$. For the adjointness of T_i and V_i we have on the one hand that $\langle \mathsf{x}^{\mathsf{d}}, \mathsf{V}_i \lambda^{\mathsf{c}}\rangle_D$ equals

$$\langle \mathsf{x}^{\mathsf{d}}, \lambda_1^{c_1} \cdots \lambda_{i-1}^{c_{i-1}} \frac{\lambda_i^{c_i+1} - \lambda_{i+1}^{c_i+1}}{c_i + 1} \lambda_{i+2}^{c_{i+1}} \cdots \rangle_D = \begin{cases} \mathsf{c}! & \text{if } \mathsf{d} = (c_1, \dots, c_{i-1}, c_i + 1, 0, c_{i+1}, \dots) \\ -\mathsf{c}! & \text{if } \mathsf{d} = (c_1, \dots, c_{i-1}, 0, c_i + 1, c_{i+1}, \dots) \\ 0 & \text{otherwise.} \end{cases} \tag{10.7}$$

On the other hand, $\mathsf{T}_i(\mathsf{x}^{\mathsf{d}})$ is always a monomial, and is a multiple of x^{c} exactly when $\mathsf{d} = (c_1, \dots, c_{i-1}, c_i + 1, 0, c_i, c_{i+1}, \dots)$ (in which case it is equal to x^{c}) or $\mathsf{d} = (c_1, \dots, c_{i-1}, 0, c_i + 1, c_{i+1}, \dots)$ (in which case it is equal to $-\mathsf{x}^{\mathsf{c}}$). Finally, by writing $\mathsf{T}_i^m = \mathsf{T}_i \mathsf{R}_{i+1}^{m-1}$ we have the adjoint of T_i^m is $(\mathsf{R}_{i+1}^\vee)^{m-1} \mathsf{V}_i = \mathsf{V}_i^m$. ☐

10.2. **Volume polynomials.** The following family of "forest polytopes" shall play a crucial role for us.

Definition 10.5. Let $F \in \mathsf{For}^m$ and let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a sequence with $\lambda_i \geq \lambda_{i+1}$ for all i. We define the *forest polytope* $\mathbf{C}_{F,\lambda} \subset \mathbb{R}^{\mathrm{IN}(F)}$ as the subset of assignments $\phi : \mathrm{IN}(F) \to \mathbb{R}$ satisfying the following constraints. Letting ϕ_λ be the extension of ϕ to $\mathrm{IN}(F) \sqcup \mathrm{supp}(F)$ by setting $\phi_\lambda(i) = \lambda_i$, we have for all $v \in \mathrm{IN}(F)$ the inequalities

$$\phi_\lambda(v_L) \geq \phi(v) \geq \phi_\lambda(v_R).$$

Thus the defining inequalities only involve edges in $\widehat{F}$ as defined in Section 5. Figure 15 shows an $F \in \mathsf{For}^2$ as well as the inequalities along the 'leftmost' and 'rightmost' edges cutting out the polytope $\mathbf{C}_{F,\lambda}$. In this case we have $\lambda_1 \geq \phi(a) \geq \lambda_3$, $\lambda_4 \geq \phi(c) \geq \lambda_6$, $\lambda_7 \geq \phi(e) \geq \lambda_9$, $\lambda_{13} \geq \phi(f) \geq \lambda_{15}$, $\phi(a) \geq \phi(b) \geq \lambda_{11}$, and $\phi(c) \geq \phi(d) \geq \lambda_{10}$.

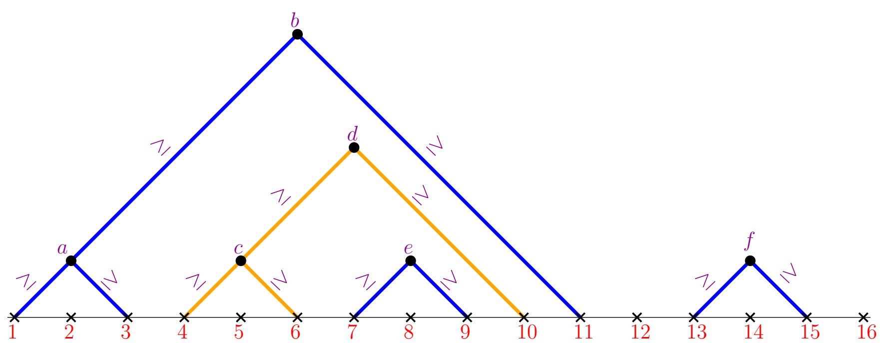

FIGURE 15. Inequalities defining $\mathbf{C}_{F,\lambda}$ for the F in Figure 7
(note the inequalities on the right edges are the opposite to Figure 7)

The following lemma casts the inherent recursive structure underlying F in the setting of forest polytopes. We omit the proof as it is straightforward.

Lemma 10.6. Let $F \in \mathsf{For}^m$. If $i \in \mathrm{LTer}(F)$ then the coordinate projection $\pi_v : \mathbf{C}_{F,\lambda} \to [\lambda_{i+m}, \lambda_i]$ has

$$\pi_v^{-1}(z) = \mathbf{C}_{F/i,\lambda'} \tag{10.8}$$

where $\lambda' = (\lambda_1, \ldots, \lambda_{i-1}, z, \lambda_{i+m+1}, \ldots)$. In particular,

$$\mathrm{Vol}(\mathbf{C}_{F,\lambda}) = \int_{\lambda_{i+m}}^{\lambda_i} \mathrm{Vol}(\mathbf{C}_{F/i,\lambda'}) = \mathsf{V}_i^m \,\mathrm{Vol}(\mathbf{C}_{F/i,\lambda}). \tag{10.9}$$

10.3. **Volumes as harmonics.**

Definition 10.7. For $F \in \mathsf{For}^m$, we define the volume polynomial associated to F to be

$$V_F(\lambda) = \mathrm{Vol}(\mathbf{C}_{F,\lambda}). \tag{10.10}$$

The following corollary verifies that this is indeed a polynomial.

Corollary 10.8. Let $F \in \mathsf{For}^m$. Then

$$V_F(\lambda) = \mathsf{V}_F^m(1), \tag{10.11}$$

and for $f \in \mathrm{Pol}$ we have $\langle f, V_F(\lambda) \rangle_D = \mathrm{ev}_0\, \mathsf{T}_F^m(f)$.

Proof. Iterating Lemma 10.6 and using that $V_{\varnothing}(\lambda) = 1$ shows the first statement. For the second, we note that because V^m_F is adjoint to T^m_F, we have

$$\langle f, V_F(\lambda)\rangle_D = \langle f, \mathsf{V}^m_F(1)\rangle_D = \langle \mathsf{T}^m_F f, 1\rangle_D = \mathrm{ev}_0\, \mathsf{T}^m_F(f). \tag{10.12}$$ □

Example 10.9. For the F in Figure 7 we have $V_F(\lambda) = \mathsf{V}^2_{13}\mathsf{V}^2_7\mathsf{V}^2_4\mathsf{V}^2_4\mathsf{V}^2_1\mathsf{V}^2_1(1)$, which equals

$$\left(\frac{\lambda_1^2-\lambda_3^2}{2} - \frac{\lambda_1-\lambda_3}{2}\lambda_{11}\right)\left(\frac{\lambda_4^2-\lambda_6^2}{2} - \frac{\lambda_4-\lambda_6}{2}\lambda_{10}\right)(\lambda_7-\lambda_9)(\lambda_{13}-\lambda_{15}). \tag{10.13}$$

The factorization of $V_F(\lambda)$ is explained because the defining inequalities of $\mathbf{C}_{F,\lambda}$ imply that we can express it as a product of forest polytopes for $G \in \mathsf{For}$ in shifted variable sets corresponding to the connected components of $\widehat{F}$.

For $n \in \mathbb{N}$ consider the *truncation operator* $P_n : \mathbb{Q}[\lambda_1, \lambda_2, \dots] \to \mathbb{Q}[\lambda_1, \dots, \lambda_n]$ defined by setting $\lambda_i = 0$ for all $i > n$. Note that for $f \in \mathbb{Q}[x_1, \dots, x_n]$ and $g \in \mathbb{Q}[\lambda_1, \lambda_2, \dots]$ we have

$$\langle f, g\rangle_D = \langle f, P_n(g)\rangle_D. \tag{10.14}$$

Given a basis of homogenous polynomials $\{g_i\}_{i\in\mathbb{N}}$ for $\mathbb{Q}[\lambda_1, \dots, \lambda_n]$, we say that a collection of homogenous polynomials $\{h_i\}_{i\in\mathbb{N}}$ in $\mathbb{Q}[x_1, \dots, x_n]$ is graded D-dual if $\langle h_i, g_j\rangle_D = \delta_{i,j}$. Because $\langle,\rangle_D$ is a perfect pairing when restricted to homogenous polynomials of degree d in $\mathbb{Q}[x_1, \dots, x_n]$, the graded D-dual set of polynomials always exists, is unique, and is a basis for $\mathbb{Q}[x_1, \dots, x_n]$.

Our next result, which is also a straightforward consequence of Proposition 10.4 and Corollary 10.8, shows that these volume polynomials $V_F(\lambda)$ for $F \in \mathsf{For}^m$ are graded duals to m-forest polynomials. The reader should compare this result with [43, Corollary 12.3(2)].[2]

Theorem 10.10. For all $F, G \in \mathsf{For}^m$ we have $\langle \mathfrak{P}_G, V_F(\lambda)\rangle_D = \delta_{F,G}$.

Furthermore, the family of projected volume polynomials $\{P_n V_F(\lambda)\}_{F\in\mathsf{LTFor}^m_n}$ in $\mathbb{Q}[\lambda_1, \dots, \lambda_n]$ is the graded D-dual basis to the homogenous basis $\{\mathfrak{P}_F\}_{F\in\mathsf{LTFor}^m_n}$ of $\mathbb{Q}[x_1, \dots, x_n]$.

Proof. By Corollary 10.8 and Corollary 6.6 we have

$$\langle \mathfrak{P}_G, V_F(\lambda)\rangle_D = \mathrm{ev}_0\, \mathsf{T}^m_F \mathfrak{P}_G = \delta_{F,G}. \tag{10.15}$$

For the second part, we have by Proposition 6.11 that $\{\mathfrak{P}_G : G \in \mathsf{LTFor}^m_n\}$ is a homogenous basis for $\mathbb{Q}[x_1, \dots, x_n]$, and $P_n V_F(\lambda)$ are homogenous polynomials in $\mathbb{Q}[\lambda_1, \dots, \lambda_n]$ which satisfy $\langle \mathfrak{P}_G, P_n V_F(\lambda))\rangle_D = \langle \mathfrak{P}_G, V_F(\lambda)\rangle_D = \delta_{F,G}$. □

We are ready to determine a basis for HQSym^m_n in terms of volume polynomials.

Theorem 10.11. A $\mathbb{Q}$-basis for HQSym^m_n is given by

$$\{V_F(\lambda) \mid F \in \mathsf{For}^m_n\}.$$

[2]Note that [43, Corollary 12.3(1)] is incorrect and the issue is highlighted in the footnote to [26, Theorem 1.1].

Proof. Recall by Proposition 6.11 that Pol_n has a homogenous basis $\{\mathfrak{P}_F : F \in \mathsf{LTFor}_n^m\}$, and by Theorem 9.7 ${}^m\mathrm{QSym}_n^+$ has a homogenous basis the subset $\{\mathfrak{P}_F : F \in \mathsf{LTFor}_n^m \setminus \mathsf{For}_n^m\}$. Because HQSym_n^m is the graded D-orthogonal complement to ${}^m\mathrm{QSym}_n^+$ in Pol_n and $\{P_n V_F(\lambda)\}$ is the graded D-dual basis to $\{\mathfrak{P}_F : F \in \mathsf{LTFor}_n^m\}$, we conclude that a $\mathbb{Q}$-basis for HQSym_n^m is given by $\{P_n V_F(\lambda) \mid F \in \mathsf{For}_n^m\}$. It remains to notice that $V_F(\lambda) \in \mathbb{Q}[\lambda_1, \ldots, \lambda_n]$ for $F \in \mathsf{For}_n^m$, so $P_n V_F(\lambda) = V_F(\lambda)$. $\square$

10.4. **A conjecture of Aval–Bergeron–Li.** We now proceed to establish a generalization of a conjecture of Aval–Bergeron–Li [5] that posited the existence of a family of Cat_{n-1}-many polynomials of degree $n-1$ the span of whose derivatives gave HQSym_n. We already know that the degree $\lfloor (n-1)/m \rfloor$ component of HQSym_n^m is the top degree component and this has a basis given by the polynomials $V_F(\lambda)$ with $F \in \mathsf{For}_n^m$ and $|F| = \lfloor (n-1)/m \rfloor$ (for $m = 1$ there are Cat_{n-1} many such polynomials). We will now show that the derivatives of this top degree component of HQSym_n^m span HQSym_n^m.

It turns out that the following proposition will formally imply the desired spanning.

Proposition 10.12. Let $f \in \mathrm{Pol}_n$ be homogenous of degree $d < \lfloor (n-1)/m \rfloor$, and assume that $x_1 f \in {}^m\mathrm{QSym}_n^+$. Then we have $f \in {}^m\mathrm{QSym}_n^+$.

Proof. We induct on d. If $d = 0$ then f is constant. The inequality for d implies $n \geq m+1$ and $x_1 = \mathfrak{P}_{\underline{1}}^m \notin {}^m\mathrm{QSym}_n^+$ by Theorem 9.7(2) since $\underline{1} \in \mathsf{For}_n^m$. Thus we must have $f = 0$.

Assume now $d > 0$, and write $f = \sum_F a_F \mathfrak{P}_F$ with $F \in \mathsf{LTFor}_n^m$, $|F| = d$ following Proposition 6.11. By Theorem 9.7(2) we can assume that $f = \sum_F a_F \mathfrak{P}_F$ with $F \in \mathsf{For}_n^m$, and we now want to show that f is zero. Fix any $2 \leq i \leq n-m$, so that $\mathsf{T}_i^m(x_1) = 0$ and $\mathsf{R}_{i+1}^m(x_1) = 1$. By Lemma 2.12 and Corollary 9.3 we have

$$\mathsf{T}_i^m(x_1 f) = x_1 \sum_{\substack{F \in \mathsf{For}_n^m \\ i \in \mathrm{LTer}(F)}} a_F \, \mathfrak{P}_{F/i} \in {}^m\mathrm{QSym}_{n-m}^+. \tag{10.16}$$

By induction, for this to happen the sum must vanish in ${}^m\mathrm{QSCoinv}_{n-m}$. But the F/i are distinct forests in For_{n-m}^m, so by Theorem 9.7 this implies that $a_F = 0$ for any F such that $i \in \mathrm{LTer}(F)$.

There remains the case where F satisfies $\mathrm{LTer}(F) = \{1\}$. There is a unique such $F \in \mathsf{For}_n^m$, namely $F = \underline{1}^d$, and $\mathfrak{P}_{\underline{1}^d}^m = x_1^d$. But then $x_1 \mathfrak{P}_{\underline{1}^d}^m = x_1^{d+1} = \mathfrak{P}_{\underline{1}^{d+1}}^m$ and $\underline{1}^{d+1} \in \mathsf{For}_n^m$ as $d+1 \leq \lfloor (n-1)/m \rfloor$, so does not lie in ${}^m\mathrm{QSym}_n^+$ by Theorem 9.7(1). $\square$

Lemma 10.13. Let $g_1, \ldots, g_r, h \in \mathbb{Q}[\lambda_1, \ldots, \lambda_n]$ be homogenous polynomials with $\deg(g_i) = k$ for $1 \leq i \leq r$ and $\deg(h) = d \leq k$. Assume that for any homogenous polynomial $f \in \mathbb{Q}[x_1, \ldots, x_n]$ of degree d such that

$$f(\mathrm{D}_1, \ldots, \mathrm{D}_n) g_1 = \cdots = f(\mathrm{D}_1, \ldots, \mathrm{D}_n) g_r = 0 \tag{10.17}$$

we have $\langle f, h \rangle_D = 0$. Then h lies in the span W of $\{\mathrm{D}_1^{c_1} \cdots \mathrm{D}_n^{c_n} g_i : c_1 + \cdots + c_n = k-d, 1 \leq i \leq r\}$.

Proof. First, we note that for any homogenous polynomials g of degree k and f of degree d

$$f(\mathrm{D}_1,\ldots,\mathrm{D}_n)g=0 \iff \langle f(x_1,\ldots,x_n),\mathrm{D}_1^{c_1}\cdots\mathrm{D}_n^{c_n}\,g\rangle_D=0 \text{ whenever } \sum c_i=k-d. \tag{10.18}$$

Indeed, this follows from the identity

$$\langle f,\mathrm{D}_1^{c_1}\cdots\mathrm{D}_n^{c_n}\,g\rangle_D=\langle \mathsf{x}^{\mathsf{c}},f(\mathrm{D}_1,\ldots,\mathrm{D}_n)g\rangle_D=\mathsf{c}!\,[\lambda^{\mathsf{c}}](f(\mathrm{D}_1,\ldots,\mathrm{D}_n)g). \tag{10.19}$$

Applying this equivalence to each g_i, we have reduced to showing that if for all homogenous degree d polynomials $f\in\mathbb{Q}[x_1,\ldots,x_n]$ we have $\langle f,W\rangle_D=0 \implies \langle f,h\rangle_D=0$, then $h\in W$. But this follows from the fact that the D-pairing on homogenous degree d polynomials is perfect. $\square$

Theorem 10.14. HQSym_n^m is spanned by the derivatives of the homogenous degree $\lfloor (n-1)/m\rfloor$ elements of HQSym_n^m.

Proof. Denote $N:=\lfloor (n-1)/m\rfloor$, and let $h\in\mathrm{HQSym}_n^m$ be of degree $d\le N$. By Lemma 10.13, it suffices to show that for all homogenous $f\in\mathbb{Q}[x_1,\ldots,x_n]$ of degree d such that $\langle f,h\rangle_D\neq 0$, there exists $g\in\mathrm{HQSym}_n^m$ of degree N such that $f(\mathrm{D}_1,\ldots,\mathrm{D}_n)g\neq 0$.

Fix such an f. Because $h\in\mathrm{HQSym}_n^m$ and $\langle f,h\rangle_D\neq 0$, we have $f\notin {}^m\mathrm{QSym}_n^+$. By Proposition 10.12 this implies $x_1^{N-d}f\notin {}^m\mathrm{QSym}_n^+$. Thus there exists $g\in\mathrm{HQSym}_n^m$ homogenous of degree N such that $\langle x_1^{N-d}f,g\rangle\neq 0$, and thus $f(\mathrm{D}_1,\ldots,\mathrm{D}_n)g\neq 0$. $\square$

10.5. **Volume polynomials into monomials and monomials into forests.** We now describe the explicit expansion for $V_F(\lambda)$ for $F\in\mathsf{For}^m$ in the basis of normalized monomials $\frac{\lambda^{\mathsf{c}}}{\mathsf{c}!}$, and the expansions of monomials x^{c} into the basis of m-forest polynomials. Finally, we show that $V_F(\lambda)$ has an expansion in monomials in the linear forms $\lambda_i-\lambda_{i+m}$ with nonnegative coefficients, which we interpret using our results on m-forest positivity of m-forest polynomial multiplication.

Let $\mathrm{Paths}(\widehat{F})$ denote the set of functions $\mathcal{P}:\mathrm{IN}(F)\to\{L,R\}$. By taking the union of edges $\bigcup_{v\in\mathrm{IN}(F)}\{v,v_{\mathcal{P}(v)}\}$ where $\{x,y\}$ denotes the edge joining x and y, we can encode $\mathcal{P}\in\mathrm{Paths}(\widehat{F})$ as a collection of vertex disjoint paths travelling up from the leaves of $\widehat{F}$ which cover every node in $\mathrm{IN}(F)$. For each $\mathcal{P}$, we let $d(\mathcal{P}):=(d_i)_{i\in\mathbb{N}}\in\mathsf{Codes}$ where d_i records the length of the path that has one endpoint at leaf i. It is easy to see that d is injective, and $\mathcal{P}$ the constant L-function we have $d(\mathcal{P})=\mathsf{c}(F)$.

For example, Figure 16 shows an $F\in\mathsf{For}^2$ with the corresponding $\widehat{F}$ obtained by omitting the dotted edges. If we take the collection $\mathcal{P}$ of paths determined by the edges highlighted in blue, then we get $d(\mathcal{P})=(2,1,0,\ldots)=\mathsf{c}(F)$.

Given $\mathsf{c}\in\mathsf{Codes}$ we define $\epsilon_F(\mathsf{c})$ as follows:

$$\epsilon_F(\mathsf{c})=\begin{cases}(-1)^{|\mathcal{P}^{-1}(R)|} & \text{if there exists } \mathcal{P}\in\mathrm{Paths}(\widehat{F}) \text{ such that } d(\mathcal{P})=\mathsf{c}\\ 0 & \text{otherwise.}\end{cases} \tag{10.20}$$

With this notation in hand we have

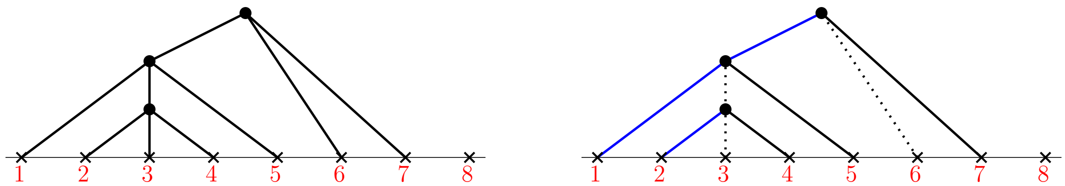

FIGURE 16. A forest $F \in \mathsf{For}^2$ and $\widehat{F}$, with the leftmost $\mathcal{P} \in \mathsf{Paths}(\widehat{F})$ colored in blue

Proposition 10.15. For $F \in \mathsf{For}^m$ we have

$$V_F(\lambda) = \sum_{\mathsf{c} \in \mathsf{Codes}} \epsilon_F(\mathsf{c}) \, \frac{\lambda^{\mathsf{c}}}{c!}.$$

Proof. We proceed by induction on $|F|$. If $F = \varnothing$ then $V_F(\lambda) = 1$ so the formula is true, and we may now assume $|F| \geq 1$.

By Lemma 10.6 we have

$$V_F(\lambda) = \mathsf{V}_i^m \, V_{F/i}(\lambda) = \sum_{\mathsf{d} \in \mathsf{Codes}} \epsilon_{F/i}(\mathsf{d}) \, \mathsf{V}_i^m \, \frac{\lambda^{\mathsf{d}}}{\mathsf{d}!}. \tag{10.21}$$

Given $d \in \mathsf{Codes}$ define compositions

$$\mathrm{left}(\mathsf{d}) = (d_1, \ldots, d_{i-1}, d_i + 1, 0^m, d_{i+1}, \ldots) \tag{10.22}$$

$$\mathrm{right}(\mathsf{d}) = (d_1, \ldots, d_{i-1}, 0^m, d_i + 1, d_{i+1}, \ldots). \tag{10.23}$$

Then the last term of (10.21) can be rewritten as

$$V_F(\lambda) = \sum_{\mathsf{d} \in \mathsf{Codes}} \epsilon_{F/i}(\mathsf{d}) \left(\frac{\lambda^{\mathrm{left}(\mathsf{d})}}{(\mathrm{left}(\mathsf{d}))!} - \frac{\lambda^{\mathrm{right}(\mathsf{d})}}{(\mathrm{right}(\mathsf{d}))!} \right), \tag{10.24}$$

It is then straightforward to check that the first summand (resp. second summand) on the right-hand side of (10.24) tracks the contribution of those paths $\mathcal{P} \in \mathsf{Paths}(\widehat{F})$ using the left (resp. right) leaf of the newly created internal node in F. $\square$

Going back to Figure 16 we see that the contribution of the collection $\mathcal{P}$ given by blue edges is $\frac{\lambda_1^2 \lambda_2}{2!1!}$ and the sign $\epsilon(\mathcal{P}) = (-1)^0 = 1$.

As an application we obtain the m-forest expansion of monomials.

Theorem 10.16. For $\mathsf{c} \in \mathsf{Codes}$ we have $\mathsf{x}^{\mathsf{c}} = \sum_{G \in \mathsf{For}^m} \epsilon_G(c) \, \mathfrak{P}_G$.

Proof. We have the sequence of equalities

$$\mathfrak{P}_G^m \mathsf{x}^{\mathsf{c}} = \langle \mathfrak{P}_G^m \mathsf{x}^{\mathsf{c}}, 1 \rangle_D = \langle \mathsf{x}^{\mathsf{c}}, \mathsf{V}_G^m(1) \rangle_D = \langle \mathsf{x}^{\mathsf{c}}, V_G(\lambda) \rangle_D = \epsilon_G(\mathsf{c}). \tag{10.25}$$

where the second is by Proposition 10.4, the third is by Corollary 10.8, and the fourth is by Proposition 10.15. $\square$

That the coefficients arising in Theorem 10.16 are 0 or ± 1 can alternatively be derived from Proposition 6.8 as the coefficient of $\mathfrak{P}_G$ equals $\mathrm{ev}_0\, \mathsf{T}^m_G(\mathsf{x}^{\mathsf{c}})$. Any T^m_i either maps x^{c} to 0 or $\pm\mathsf{x}^{\mathsf{c}'}$ where $|\mathsf{c}'| = |\mathsf{c}| - 1$. Tracking this sign carefully allows us to recover the preceding result. We omit further details.

Figure 17 shows the three indexed forests in the case $m = 1$ that contribute to the expansion of $x_2^2x_3$ as per Theorem 10.16. The leftmost tree has code precisely the exponent vector of this monomial. Explicitly we have $x_2^2x_3 = \mathfrak{P}_F - \mathfrak{P}_G - \mathfrak{P}_H$.

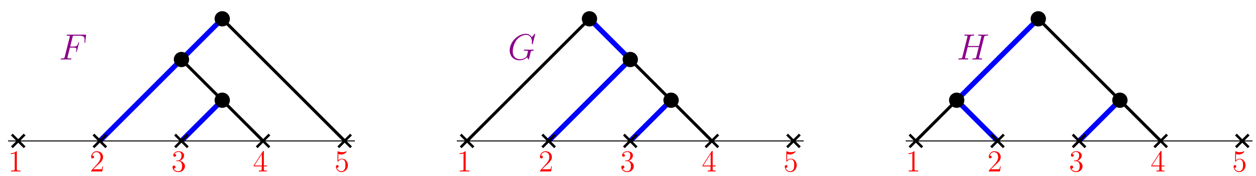

FIGURE 17. Three indexed forests that contribute to the monomial $x_2^2x_3$

Theorem 10.17. For $F \in \mathsf{For}^m$ we have $V_F(\lambda) \in \mathbb{Q}[\lambda_1 - \lambda_{m+1}, \lambda_2 - \lambda_{m+2}, \dots]$, and the coefficients in the expansion

$$V_F(\lambda) = \sum_{\mathsf{c}=(c_1,c_2,\dots)\in\mathsf{Codes}} b_{\mathsf{c}} \prod_{i\geq 1}(\lambda_i - \lambda_{i+m})^{c_i} \tag{10.26}$$

have $b_{\mathsf{c}} = \frac{1}{\mathsf{c}!}\mathsf{T}^m_F\left(\prod_{i\geq 1}(\mathfrak{P}^m_i)^{c_i}\right) \geq 0$.

Proof. The fact that $V_F(\lambda) \in \mathbb{Q}[\lambda_1 - \lambda_{m+1}, \lambda_2 - \lambda_{m+2}, \dots]$ can be verified inductively by checking that V^m_i preserves this ring and noting $V_F(\lambda) = \mathsf{V}^m_F(1)$ by Corollary 10.8. Noting that $\mathfrak{P}^m_i = x_i + x_{i-m} + \cdots + x_{i-m\lfloor i/m\rfloor}$, it is straightforward to check that

$$\left\langle \prod_{i\geq 1}(\mathfrak{P}^m_i)^{c_i}, \prod_{i\geq 1}(\lambda_i - \lambda_{i+m})^{d_i} \right\rangle_D = \delta_{\mathsf{c},\mathsf{d}}\, \mathsf{c}!, \tag{10.27}$$

and so

$$b_{\mathsf{c}} = \frac{1}{\mathsf{c}!}\left\langle \prod_{i\geq 1}(\mathfrak{P}^m_i)^{c_i}, V_F(\lambda) \right\rangle_D = \frac{1}{\mathsf{c}!}\,\mathrm{ev}_0\, \mathsf{T}^m_F\left(\prod_{i\geq 1}(\mathfrak{P}^m_i)^{c_i}\right). \tag{10.28}$$

Finally, $b_{\mathsf{c}} \geq 0$ since $\prod_{i\geq 1}(\mathfrak{P}^m_i)^{c_i}$ is m-forest positive by Theorem 7.3 or Theorem 7.4. $\square$

For $m = 1$, [38, §6.3] may be interpreted as giving a combinatorial interpretation for the coefficients $\mathsf{T}^1_F(x_1^{c_1}(x_1 + x_2)^{c_2}\cdots) = \mathsf{T}^1_F((\mathfrak{P}_1)^{c_1}(\mathfrak{P}_2)^{c_2}\cdots)$. It would be interesting to extend this interpretation to $m > 1$.

Appendix A. Proof of Theorem 6.1

For any k, let $C_k(F) = \{\kappa \in \mathcal{C}(F) \mid k, k+1, \dots, k+(m-1) \notin \mathrm{Im}(\kappa)\}$, and define $\Phi_k : \mathbb{N} \setminus \{k, \dots, k+(m-1)\} \to \mathbb{N}$ and its inverse $\Phi_k^{-1} : \mathbb{N} \to \{k, \dots, k+(m-1)\}$ by

$$\Phi_k(a) := \begin{cases} a & \text{if } a \le k-1 \\ a-m & \text{if } a \ge k+m \end{cases} \text{ and } \Phi_k^{-1}(a) = \begin{cases} a & \text{if } a \le k-1 \\ a+m & \text{if } a \ge k. \end{cases} \tag{A.1}$$

Then we have

$$\mathsf{R}_k^m(\mathfrak{P}_F) = \mathsf{R}_k^m\Big(\sum_{\kappa \in \mathcal{C}(F)} \prod_{v \in \mathrm{IN}(F)} x_{\kappa(v)}\Big) = \sum_{\kappa \in C_k(F)} \prod_{v \in \mathrm{IN}(F)} x_{\Phi_k \kappa(v)}. \tag{A.2}$$

Consider the map $f : \mathbb{N} \setminus \{i, \dots, i+(m-1)\} \to \mathbb{N} \setminus \{i+1, \dots, i+m\}$ and its inverse $f^{-1} : \mathbb{N} \setminus \{i+1, \dots, i+m\} \to \mathbb{N} \setminus \{i, \dots, i+(m-1)\}$ by

$$f(a) = \begin{cases} a-m & \text{if } a = i+m \\ a & \text{otherwise} \end{cases} \text{ and } f^{-1}(a) = \begin{cases} a+m & \text{if } a = i \\ a & \text{otherwise.} \end{cases} \tag{A.3}$$

We will use the following fact often to show that various compatible labellings retain the compatibility inequalities between internal children after being modified by one of the above functions.

Claim A.1. For g being any of the functions $\Phi_k, \Phi_k^{-1}, f, f^{-1}$, the following holds: if a, b are in the domain of g and $0 \le j \le m$ is such that $b \le a - j$ and $b \equiv a - j \bmod m$, then

$$g(a) - g(b) \ge j. \tag{A.4}$$

Proof. In all cases g is the unique increasing bijection from $\mathbb{N} \setminus A$ to $\mathbb{N} \setminus B$ for some finite sets A, B. It also satisfies $g(x) \equiv x \bmod m$ for all $x \notin A$. Thus $b \le a - j \le a$ implies $g(a) \ge g(b)$, while $b \equiv a - j \bmod m$ implies $g(a) - g(b) \equiv j \bmod m$. From there the conclusion follows immediately in all cases but one: if $j = m$, then we must forbid $g(a) = g(b)$, and indeed this cannot hold since $b \le a - m < a$ and g is a bijection. $\square$

We claim that for any $\kappa \in C_i(F)$ we have $f\kappa \in C_{i+1}(F)$. Because $i+1, \dots, i+m \notin \mathrm{Im}(f(\kappa))$ it remains to check that $f \in \mathcal{C}(F)$. Let $v \in \mathrm{IN}(F)$. Then $f(\kappa(v)) \le \kappa(v) \le \rho_F(v)$, $f(\kappa(v)) \equiv \kappa(v) \equiv \overline{\rho}_F(v)$, and by Claim A.1 for $v, v_j \in \mathrm{IN}(F)$ we have

$$f(\kappa(v_j)) - f(\kappa(v)) \ge j. \tag{A.5}$$

Additionally, the map $f^* : \kappa \mapsto f\kappa$ is injective as f is injective, and $\Phi_i \kappa = \Phi_{i+1} f \kappa$. It follows that

$$\mathsf{T}_i^m \mathfrak{P}_F = \frac{\mathsf{R}_{i+1}^m \mathfrak{P}_F - \mathsf{R}_i^m \mathfrak{P}_F}{x_i} = \sum_{\kappa' \in C_{i+1}(F) \setminus \mathrm{Im}(f^*)} \frac{1}{x_i} \prod_{v \in \mathrm{IN}(F)} x_{\Phi_{i+1}\kappa'(v)}. \tag{A.6}$$

Claim A.2. Let $\kappa' \in C_{i+1}(F)$. Then $\kappa' \notin C_{i+1}(F) \setminus \mathrm{Im}(f^*)$ if and only if $i \in \mathrm{LTer}(F)$ and the terminal node u with $\rho_F(u) = i$ has $\kappa'(u) = i$.

Proof. We have $\kappa' \notin C_{i+1}(F) \setminus \mathrm{Im}(f^*)$ if and only if $f^{-1}\kappa' \notin C_i(F)$. Note that if $i \in \mathrm{LTer}(F)$ and the terminal node u with $\rho_F(u) = i$ has $\kappa'(u) = i$ then $f^{-1}\kappa'(u) = i + m > \rho_F(u)$ and so $f^{-1}\kappa'(u) \notin \mathcal{C}(F) \supset C_i(F)$. Therefore it suffices to show that $f^{-1}\kappa' \notin C_i(F)$ implies there is a terminal node u with $\rho_F(u) = i$ and $\kappa'(u) = i$.

We first show that $f^{-1}\kappa' \notin C_i(F)$ implies there is some $v \in \mathrm{IN}(F)$ with $\kappa'(v) = i$ and $\rho_F(v) < i + m$. We do this by checking that all other conditions besides $f^{-1}\kappa'(v) \leq \rho_F(v)$ for $f^{-1}\kappa'(v)$ to lie in $C_i(F)$ are satisfied. Note that $i, \ldots, i+m \notin \mathrm{Im}(f^{-1}\kappa')$, for $v \in \mathrm{IN}(F)$ we have $f^{-1}(\kappa(v)) \equiv \kappa(v) \equiv \overline{\rho}_F(v)$, and by Claim A.1 we have for $v, v_j \in \mathrm{IN}(F)$ that

$$f^{-1}(\kappa'(v_j)) - f^{-1}(\kappa'(v)) \geq j. \tag{A.7}$$

Therefore as all other conditions for $f^{-1}\kappa' \in C_i(F)$ are met, we have $f^{-1}\kappa' \notin C_i(F)$ exactly if there is $v \in \mathrm{IN}(F)$ with $f^{-1}\kappa'(v) > \rho_F(v)$. Because $f^{-1}\kappa'(v) = \kappa'(v) \leq \rho_F(v)$ if $\kappa'(v) \neq i$, the inequality $f^{-1}\kappa'(v) > \rho_F(v)$ happens precisely if $\kappa'(v) = i$ and $f^{-1}\kappa'(v) = i + m > \rho_F(v)$.

Now from this v with $\kappa'(v) = i$ and $\rho_F(v) < i + m$, we construct the desired u. We have $i = \kappa'(v) \leq \kappa'(v_0) \leq \kappa'(v_{0^2}) \leq \cdots \leq \kappa'(v_{0^k}) \leq \rho_F(v) < i + m$ where $v_{0^k} \in \mathrm{IN}(F)$ is the last internal left descendant of v. Because $\rho_F(v) \equiv \kappa'(v) \equiv i \bmod m$ we must have $\rho_F(v) = i$ and so additionally $\kappa'(v_{0^k}) = i$. Therefore $u = v_{0^k}$ has $\kappa'(u) = \rho_F(u) = u_0 = i$.

We claim that u is terminal. If not, let $1 \leq j \leq m$ be the first index with $u_j \in \mathrm{IN}(F)$. Then

$$i + j = \kappa'(u) + j \leq \kappa'(u_j) \leq \rho_F(u_j) = i + j, \tag{A.8}$$

so $\kappa'(u_j) = i + j$, contradicting that $\kappa' \in C_{i+1}(F)$.

Therefore u is terminal with $\kappa'(u) = \rho_F(u) = i$ and in particular $i \in \mathrm{LTer}(F)$. □

Returning to the proof of Theorem 6.1, we may now conclude that if $i \notin \mathrm{LTer}(F)$ then $C_{i+1}(F) \setminus \mathrm{Im}(f^*) = \varnothing$ and so by Equation (A.6) we have $\mathsf{T}_i^m \mathfrak{P}_F = 0$. On the other hand, suppose $i \in \mathrm{LTer}(F)$ and let u be the terminal node with $\rho_F(u) = i$. Then by the claim we know that

$$C_{i+1}(F) \setminus \mathrm{Im}(f^*) = \{\kappa' \in C_{i+1}(F) : \kappa'(u) = i\}. \tag{A.9}$$

Claim A.3. If $i \in \mathrm{LTer}(F)$ then there is a bijection $\{\kappa' \in C_{i+1}(F) : \kappa'(u) = i\} \to \mathcal{C}(F/i)$ given by $\kappa' \mapsto \kappa'' = \Phi_{i+1}\kappa'|_{\mathrm{IN}(F)\setminus u}$ (identifying $\mathrm{IN}(F) \setminus u = \mathrm{IN}(F/i)$).

Proof. Before starting we show that for $v \in \mathrm{IN}(F) \setminus u$ we have $\Phi_{i+1}\rho_F(v) = \rho_{F/i}(v)$ (which explains the presence of Φ_{i+1} in the statement). This is because by definition of the monoid structure on For^m, for any $F \geq G$ we have $\rho_{F/G}(v) = \rho_G(w)$ for w the root of the $\rho_F(v)$'th tree of $G \in \mathsf{For}^m$. Taking $G = \underline{i}$ we directly see that $\rho_G(w) = \Phi_{i+1}(\rho_F(v))$.

First we check that the map from the claim is well-defined. Let $v \in \mathrm{IN}(F/i) = \mathrm{IN}(F) \setminus u$. Then $\Phi_{i+1}\kappa'(v) \leq \Phi_{i+1}\rho_F(v) = \rho_{F/i}(v)$, $\Phi_{i+1}\kappa'(v) \equiv \kappa'(v) \equiv \overline{\rho}_F(v) \equiv \overline{\rho}_{F/i}(v) \bmod m$, and by Claim A.1

we have for $v, v_j \in \mathrm{IN}(F/i)$ that

$$\Phi_{i+1}\kappa'(v_j) - \Phi_{i+1}\kappa'(v) \geq j. \tag{A.10}$$

This map is clearly injective, so it remains to check surjectivity. Given $\kappa'' \in \mathcal{C}(F/i)$ we claim that $\kappa' \in \mathcal{C}(F)$ where

$$\kappa'(v) = \begin{cases} \Phi_{i+1}^{-1}\kappa''(v) & \text{if } v \neq u \\ i & \text{if } v = u. \end{cases} \tag{A.11}$$

If this is the case then it is readily apparent that $\Phi_{i+1}\kappa'|_{\mathrm{IN}(F)\setminus u} = \kappa''$ so surjectivity will follow.

Let $v \in \mathrm{IN}(F)$. If $v = u$ then we have $\kappa'(v) = i = \rho_F(v)$ which shows $\kappa'(v) \leq \rho_F(v)$ and $\kappa'(v) \equiv \overline{\rho}_F(v) \bmod m$. If $v \neq u$ then $\kappa'(v) = \Phi_{i+1}^{-1}\kappa''(v) \leq \Phi_{i+1}^{-1}\rho_{F/i}(v) = \rho_F(v)$ and $\kappa'(v) = \Phi_{i+1}^{-1}\kappa''(v) \equiv \kappa'(v) \equiv \rho_{F/i}(v) \equiv \rho_F(v)$. Finally, if $v, v_j \in \mathrm{IN}(F)$ it remains to show that

$$\kappa'(v_j) - \kappa'(v) \geq j. \tag{A.12}$$

If $v_j = u$ then $\kappa'(v_j) - \kappa'(v) = i - \Phi_{i+1}^{-1}\kappa''(v) \geq i - \Phi_{i+1}^{-1}\rho_{F/i}(v) = i - \rho_F(v) \geq i - (\rho_F(v_j) - j) = j$ (where the last inequality is because $\rho_F(v) \leq \rho_F(v_j)$ and $\overline{\rho}_F(v) \equiv \overline{\rho}_F(v_j) - j \bmod m$). If $v_j \neq u$ then if follows from Claim A.1 that $\Phi_{i+1}^{-1}\kappa''(v_j) - \Phi_{i+1}^{-1}\kappa''(v) \geq j$. $\square$

Given this claim, we now conclude

$$\mathfrak{P}_{F/i} = \sum_{\kappa'' \in \mathcal{C}(F/i)} \prod_{v \in \mathrm{IN}(F/i)} x_{\kappa''(v)} = \sum_{\kappa' \in C_{i+1}(F)\setminus \mathrm{Im}(f^*)} \prod_{v \in \mathrm{IN}(F/i)} x_{\Phi_{i+1}\kappa'(v)} \tag{A.13}$$

$$= \sum_{\kappa' \in C_{i+1}(F)\setminus \mathrm{Im}(f^*)} \frac{1}{x_i} \prod_{v \in \mathrm{IN}(F)} x_{\Phi_{i+1}\kappa'(v)} = \mathsf{T}_i^m \mathfrak{P}_F. \tag{A.14}$$

where in the second last equality we used that $\Phi_{i+1}\kappa'(u) = \Phi_{i+1}(i) = i$.

References

[1] M. Aguiar, N. Bergeron, and F. Sottile. Combinatorial Hopf algebras and generalized Dehn-Sommerville relations. *Compos. Math.*, 142(1):1–30, 2006. 2

[2] S. Assaf and D. Searles. Schubert polynomials, slide polynomials, Stanley symmetric functions and quasi-Yamanouchi pipe dreams. *Adv. Math.*, 306:89–122, 2017. 30

[3] J.-C. Aval. Ideals and quotients of B-quasisymmetric polynomials. *Sém. Lothar. Combin.*, 54:Art. B54d, 13, 2005/07. 4, 6, 35

[4] J.-C. Aval, F. Bergeron, and N. Bergeron. Ideals of quasi-symmetric functions and super-covariant polynomials for S_n. *Adv. Math.*, 181(2):353–367, 2004. 2, 4, 12

[5] J.-C. Aval, Nantel Bergeron, and Huilan Li. Non-commutative combinatorial inverse systems. *Int. J. Algebra*, 4(21-24):1003–1020, 2010. 2, 5, 41, 45

[6] J.-C. Aval and F. Chapoton. Poset structures on $(m+2)$-angulations and polynomial bases of the quotient by G^m-quasisymmetric functions. *Sém. Lothar. Combin.*, 77:Art. B77b, 14, [2016–2018]. 4, 6, 30

[7] P. Baumann and C. Hohlweg. A Solomon descent theory for the wreath products $G \wr S_n$. *Trans. Amer. Math. Soc.*, 360(3):1475–1538, 2008. 6, 30

[8] J. M. Belk and K. S. Brown. Forest diagrams for elements of Thompson's group F. *Internat. J. Algebra Comput.*, 15(5-6):815–850, 2005. 4, 18

[9] N. Bergeron and L. Gagnon. The excedance quotient of the Bruhat order, Quasisymmetric Varieties and Temperley-Lieb algebras, 2023, 2302.10814. 4

[10] N. Bergeron and F. Sottile. Schubert polynomials, the Bruhat order, and the geometry of flag manifolds. *Duke Math. J.*, 95(2):373–423, 1998. 2, 8

[11] N. Bergeron and F. Sottile. Skew Schubert functions and the Pieri formula for flag manifolds. *Trans. Amer. Math. Soc.*, 354(2):651–673, 2002. 8

[12] I. N. Bernšteĭn, I. M. Gelfand, and S. I. Gelfand. Schubert cells, and the cohomology of the spaces G/P. *Uspehi Mat. Nauk*, 28(3(171)):3–26, 1973. 2, 3, 4

[13] S. C. Billey, W. Jockusch, and R. P. Stanley. Some combinatorial properties of Schubert polynomials. *J. Algebraic Combin.*, 2(4):345–374, 1993. 2, 30

[14] K. S. Brown. Finiteness properties of groups. In *Proceedings of the Northwestern conference on cohomology of groups (Evanston, Ill., 1985)*, volume 44, pages 45–75, 1987. 18

[15] A. S. Buch, A. Kresch, H. Tamvakis, and A. Yong. Schubert polynomials and quiver formulas. *Duke Math. J.*, 122(1):125–143, 2004. 2

[16] J. Burillo, S. Cleary, and M. I. Stein. Metrics and embeddings of generalizations of Thompson's group F. *Trans. Amer. Math. Soc.*, 353(4):1677–1689, 2001. 18

[17] J. W. Cannon, W. J. Floyd, and W. R. Parry. Introductory notes on Richard Thompson's groups. *Enseign. Math. (2)*, 42(3-4):215–256, 1996. 18

[18] S. Cleary. Restricted rotation distance between k-ary trees. *J. Graph Algorithms Appl.*, 27(1):19–33, 2023. 18

[19] P. Dehornoy and E. Tesson. Garside combinatorics for Thompson's monoid F^+ and a hybrid with the braid monoid B_∞^+. *Algebr. Comb.*, 2(4):683–709, 2019. 18

[20] N. Dershowitz and S. Zaks. The cycle lemma and some applications. *European J. Combin.*, 11(1):35–40, 1990. 12

[21] P. Duchon. On the enumeration and generation of generalized Dyck words. In *Formal power series and algebraic combinatorics (Toronto, ON, 1998)*, volume 225, pages 121–135. 2000. 12

[22] S. Fomin and A. N. Kirillov. The Yang-Baxter equation, symmetric functions, and Schubert polynomials. In *Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics (Florence, 1993)*, volume 153, pages 123–143, 1996. 2

[23] S. Fomin and R. P. Stanley. Schubert polynomials and the nil-Coxeter algebra. *Adv. Math.*, 103(2):196–207, 1994. 2

[24] I. M. Gessel. Multipartite P-partitions and inner products of skew Schur functions. In *Combinatorics and algebra (Boulder, Colo., 1983)*, volume 34 of *Contemp. Math.*, pages 289–317. Amer. Math. Soc., Providence, RI, 1984. 2, 29, 30

[25] D. Grinberg and V. Reiner. Hopf algebras in combinatorics, 2020, 1409.8356. 2, 9

[26] Z. Hamaker. Dual Schubert polynomials via a Cauchy identity, 2023, 2311.18099. 44

[27] Z. Hamaker, O. Pechenik, and A. Weigandt. Gröbner geometry of Schubert polynomials through ice. *Adv. Math.*, 398:Paper No. 108228, 29, 2022. 2

[28] F. Hivert. Hecke algebras, difference operators, and quasi-symmetric functions. *Adv. Math.*, 155(2):181–238, 2000. 4, 8

[29] S. K. Hsiao and T. K. Petersen. Colored posets and colored quasisymmetric functions. *Ann. Comb.*, 14(2):251–289, 2010. 7, 31

[30] A. Knutson and E. Miller. Gröbner geometry of Schubert polynomials. *Ann. of Math. (2)*, 161(3):1245–1318, 2005. 2

[31] A. Lascoux and M.-P. Schützenberger. Polynômes de Schubert. *C. R. Acad. Sci. Paris Sér. I Math.*, 294(13):447–450, 1982. 2, 3

[32] C. Lenart, S. Robinson, and F. Sottile. Grothendieck polynomials via permutation patterns and chains in the Bruhat order. *Amer. J. Math.*, 128(4):805–848, 2006. 8

[33] C. Lenart and F. Sottile. Skew Schubert polynomials. *Proc. Amer. Math. Soc.*, 131(11):3319–3328, 2003. 2

[34] I. G. Macdonald. Schubert polynomials. In *Surveys in combinatorics, 1991 (Guildford, 1991)*, volume 166 of *London Math. Soc. Lecture Note Ser.*, pages 73–99. Cambridge Univ. Press, Cambridge, 1991. 10

[35] L. Manivel. *Symmetric functions, Schubert polynomials and degeneracy loci*, volume 6 of *SMF/AMS Texts and Monographs*. American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2001. Translated from the 1998 French original by John R. Swallow, Cours Spécialisés [Specialized Courses], 3. 3, 10, 36

[36] P. Nadeau, H. Spink, and V. Tewari. Schubert expansions revisited, in preparation. 2

[37] P. Nadeau, H. Spink, and V. Tewari. Trimming operations and torus orbit closures, in preparation. 2

[38] P. Nadeau and V. Tewari. Forest polynomials and the class of the permutahedral variety, 2023, 2306.10939. 2, 4, 5, 10, 11, 14, 19, 20, 27, 35, 48

[39] P. Nadeau and V. Tewari. P-partitions with flags and back stable quasisymmetric functions, 2023, 2303.09019. 30

[40] O. Pechenik and M. Satriano. James reduced product schemes and double quasisymmetric functions, 2023, 2304.11508. 4

[41] O. Pechenik and M. Satriano. Quasisymmetric Schubert calculus, 2023, 2205.12415. 4, 8

[42] Stéphane Poirier. Cycle type and descent set in wreath products. In *Proceedings of the 7th Conference on Formal Power Series and Algebraic Combinatorics (Noisy-le-Grand, 1995)*, volume 180, pages 315–343, 1998. 6

[43] A. Postnikov and R. P. Stanley. Chains in the Bruhat order. *J. Algebraic Combin.*, 29(2):133–174, 2009. 44

[44] G. N. Raney. Functional composition patterns and power series reversion. *Trans. Amer. Math. Soc.*, 94:441–451, 1960. 12

[45] V. Reiner, A. Woo, and A. Yong. Presenting the cohomology of a Schubert variety. *Trans. Amer. Math. Soc.*, 363(1):521–543, 2011. 34

[46] F. Sottile. Pieri's formula for flag manifolds and Schubert polynomials. *Ann. Inst. Fourier (Grenoble)*, 46(1):89–110, 1996. 29

[47] R. P. Stanley. *Ordered structures and partitions*. Memoirs of the American Mathematical Society, No. 119. American Mathematical Society, Providence, R.I., 1972. 2, 29

[48] R. P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin. 10

[49] R. Steinberg. Differential equations invariant under finite reflection groups. *Trans. Amer. Math. Soc.*, 112:392–400, 1964. 4

[50] Z. Šunić. Tamari lattices, forests and Thompson monoids. *European J. Combin.*, 28(4):1216–1238, 2007. 4, 18

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