

THE GEOMETRY OF QUASISYMMETRIC COINVARIANTS

PHILIPPE NADEAU, HUNTER SPINK, AND VASU TEWARI

ABSTRACT. We develop a quasisymmetric analogue of the theory of Schubert cycles, building off of our previous work on a quasisymmetric analogue of Schubert polynomials and divided differences. Our constructions result in a natural geometric interpretation for the ring of quasisymmetric coinvariants.

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1. INTRODUCTION

Let $\text{Pol}_n := \mathbb{Z}[x_1, \dots, x_n]$, the ring of polynomials in n variables. A ubiquitous family of polynomials in algebraic combinatorics is the ring of quasisymmetric polynomials $\text{QSym}_n \subset \text{Pol}_n$, which

Key words and phrases. Forest polynomials, Schubert polynomials, toric Richardson varieties, quasisymmetric coinvariants, Thompson monoid.

PN was partially supported by French ANR grant ANR-19-CE48-0011 (COMBINÉ). HS and VT acknowledge the support of the Natural Sciences and Engineering Research Council of Canada (NSERC), respectively [RGPIN-2024-04181] and [RGPIN-2024-05433].

are variable truncations of the quasisymmetric functions pioneered by Gessel [20] and Stanley [55]. These polynomials are by definition those that satisfy a weak form of variable symmetry: for any sequence $a_1, \dots, a_k \geq 1$ and *increasing* sequence $1 \leq i_1 < \dots < i_k \leq n$, the coefficient of $x_1^{a_1} \dots x_k^{a_k}$ is the same as the coefficient of $x_{i_1}^{a_1} \dots x_{i_k}^{a_k}$. Note that without the increasing stipulation we recover the familiar notion of symmetric polynomials $\text{Sym}_n \subset \text{Pol}_n$, and so we have containments $\text{Sym}_n \subset \text{QSym}_n \subset \text{Pol}_n$.

A significant gap in our understanding of quasisymmetric polynomials is the subject of a research program [3, 5, 41, 50, 49] which seeks to answer the following question.

Question 1.1. What is the quasisymmetric analogue of Schubert calculus?

The combinatorial side of Schubert calculus consists of the interplay between Schubert polynomials $\{\mathfrak{S}_w \mid w \in S_n\}$, divided differences $\partial_i \in \text{End}(\text{Pol}_n)$ which recursively characterize \mathfrak{S}_w , and the symmetric coinvariants $\text{Coinv}_n := \text{Pol}_n / \text{Sym}_n^+$ where Sym_n^+ is the ideal generated by homogenous positive degree symmetric polynomials. These correspond on the geometric side to Schubert cycles $X^w \subset \text{Fl}_n$ in the complete flag variety, Bott–Samelson resolutions, and the cohomology ring $H^\bullet(\text{Fl}_n)$.

The ring of quasisymmetric coinvariants is defined to be $\text{QSCoinv}_n := \text{Pol}_n / \text{QSym}_n^+$, where QSym_n^+ is the ideal generated by homogenous positive degree quasisymmetric polynomials. An obstruction to answering Question 1.1 is that QSCoinv_n is less well-behaved than Coinv_n . For example, as shown by Aval–Bergeron–Bergeron [3] (see also [41]), the graded ranks are given by

$$\text{rank}(\text{QSCoinv}_n^{(i)}) = \frac{n-i}{n+i} \binom{n-i}{i},$$

for $i = 0, \dots, n-1$. This sequence is neither unimodal nor symmetric, so in particular implies that QSCoinv_n is not the cohomology ring of a smooth projective variety.

In a previous paper [41] we developed the combinatorial and algebraic sides of the quasisymmetric story, by describing *quasisymmetric divided differences* \mathbb{T}_i that interact with *forest polynomials* \mathfrak{P}_F and the quasisymmetric coinvariants QSCoinv_n in an analogous way to how the usual divided differences ∂_i interact with Schubert polynomials \mathfrak{S}_w and the symmetric coinvariants Coinv_n . One interesting feature of these quasisymmetric divided differences is that the role of the nil-Hecke relations $\partial_i^2 = 0$, $\partial_i \partial_j = \partial_j \partial_i$ for $|i-j| \geq 2$ and $\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$ is played by the Thompson monoid relations

$$\mathbb{T}_i \mathbb{T}_j = \mathbb{T}_j \mathbb{T}_{i+1}$$

for $i > j$. This implies that composites of \mathbb{T}_i operators are naturally indexed by certain leaf-labelled plane binary forests. The relevant parts of this combinatorial theory are recalled in Section 3.1.

1.1. The geometry of quasisymmetric coinvariants. Here we complete this story, and in particular give a natural answer to Question 1.1, by describing a quasisymmetric analogue of the

geometric theory of Schubert cycles. To obtain the clearest geometric picture, we have to extend in Section 3.2 the combinatorial considerations in [41] on composites of T_i operators to cover composites of both T_i operators and the *Bergeron–Sottile maps* R_j , so named in *ibid.* because of their relevance in the seminal paper [6] that developed Pieri rules for Schubert calculus. These lead us to a new combinatorial object we call the *augmented Thompson monoid* which governs the combinatorics of certain nested plane binary forests.

We will show that the distinct composite operations $\Pi_\Omega^n : \text{Pol}_n \rightarrow \mathbb{Z}$ of appropriately supported sequences Ω of n operations of the form T_i or R_j are naturally indexed by nested plane binary forests $\hat{F} \in \widehat{\text{For}}_n$. To such an \hat{F} we show that there is an associated toric variety $X(\hat{F}) \subset \text{Fl}_n$ which we call a **quasisymmetric Schubert cycle**, with the property that the degree map on $X(\hat{F})$ is computed using this composite operation. Our first main theorem (Theorem A) shows that the subcollection of $X(\hat{F})$ where \hat{F} does not involve any nesting are dual to the forest polynomials in the same way that Schubert cycles are dual to Schubert polynomials.

To the sequence Ω itself, we construct a toric Richardson variety $X(\Omega) \subset \text{Fl}_n$ we call a **quasisymmetric Bott–Samelson variety**. This toric variety is inductively constructed as an iterated \mathbb{P}^1 -bundle (i.e. as a “Bott manifold”) in such a way that the degree map on $X(\Omega)$ is computed as the composite operation Π_Ω via taking successive degrees of the \mathbb{P}^1 -bundle (Theorem 2.2). Furthermore, we show that there is a natural map $X(\Omega) \rightarrow X(\hat{F})$ (in fact an isomorphism) that geometrically computes the degree map on $X(\hat{F})$ via pullback to $X(\Omega)$, as Bott–Samelson varieties do for Schubert varieties.

The $X(\Omega)$ fit together into a toric complex

$$\text{HHMP}_n := \bigcup_{\Omega \in \text{RTSeq}_n} X(\Omega) \subset \text{Fl}_n$$

we call the **Ω -flag variety**, whose top-dimensional pieces were first studied by Haruda–Horiguchi–Masuda–Park [24] and later by Lian [36]. Our second main theorem (Theorem B) shows that under the restriction map $\psi^* : H^\bullet(\text{Fl}_n) \rightarrow H^\bullet(\text{HHMP}_n)$, we have

$$\text{QSCoinv}_n \cong \psi^*(H^\bullet(\text{Fl}_n)) \subset H^\bullet(\text{HHMP}_n),$$

which gives the desired geometric interpretation of the quasisymmetric coinvariants.

A curious feature of HHMP_n is that unlike in the classical story, each $X(\hat{F})$ appears isomorphically multiple times within HHMP_n (once for each $\Omega \in \text{Trim}(\hat{F})$). This is ultimately why the containment of QSCoinv_n in $H^\bullet(\text{HHMP}_n)$ is not an equality.

1.2. Applications. We give some applications of our theory beyond answering Question 1.1. One such application of our theory comes from the fact that the top dimensional $X(\Omega)$ show up in the study of the permutahedral toric variety in [24, 36]. Because we have a good understanding of the degree map on $X(\Omega)$ we can deduce new results about the degree map, which is combinatorially

the “divided symmetrization” of Postnikov [52, §3]. This also lets us better understand the q -divided symmetrization from [44] and explains the remarkable interactions with quasisymmetric polynomials.

Another application comes from the fact that the degree map on Richardson varieties such as our toric varieties $X(\Omega)$ computes generalized Littlewood–Richardson (LR) coefficients $c_{u,w}^v$, the structure coefficients for Schubert polynomial multiplication

$$\mathfrak{S}_u \mathfrak{S}_w = \sum c_{u,w}^v \mathfrak{S}_v.$$

By manufacturing operators via composites of T_i and R_j that extract interesting combinatorial invariants from Schubert polynomials, our theory implies that these quantities are themselves generalized LR coefficients, and hence have geometric significance.

As a final application, we give a geometric explanation for a well-known formula of Gessel [20, Theorem 3] for the coefficients in the expansion of a symmetric polynomial $f \in \text{Sym}_n$ into fundamental quasisymmetric polynomials. It makes use of the Hall inner product with skew Schur polynomials associated to ribbon shapes, and we relate these inner products to the degrees on some of our varieties $X(\Omega)$.

Outline of Paper. We detail the main results in the paper in Section 2, and only give a brief outline here. Section 3 contains the combinatorial description of the augmented Thompson monoid, which describes the composites of T and R operators. This extends the combinatorics of the Thompson monoid from [41] which dealt only with T operators. In Section 4 we describe a geometric interpretation of the T and R operators in the flag variety, and introduce the subvariety HHMP. In Section 5, using a pair of permutations associated to a sequence of T and R operators, we study the toric Richardson varieties $X(\Omega)$ which are the primary geometric objects we consider. In Section 6 we relate for $\Omega \in \text{Trim}(\hat{F})$ the toric varieties $X(\Omega)$ to torus-orbit closures $X(\hat{F}) \subset \text{Fl}_n$ we call quasisymmetric Schubert cycles, and show a subset of our quasisymmetric Schubert cycles are dual to the forest polynomials. In Section 7 we show that the $X(\Omega)$ assemble into a toric complex HHMP $_n$ such that its cohomology ring naturally contains QSCoinv_n .

In Section 8 we apply our results to generalized LR coefficients. In Section 9 we apply our results to general torus-orbit closures and divided symmetrization. Finally, in Section 10 we show how we can use the projection of our quasisymmetric Schubert cycles to Grassmannians to recover a result of Gessel on extracting the coefficients of a symmetric polynomial in the basis of fundamental quasisymmetric polynomials.

In Appendix A we collect some combinatorial proofs related to nested forests and in Appendix B we collect some combinatorial proofs related to the permutations defining the Richardson varieties $X(\Omega)$. Finally in Appendix C we compute the moment polytopes of these varieties and give a different perspective on the existence of the quasisymmetric Bott–Samelson resolutions.

Acknowledgements. We would like to thank Nantel Bergeron, Chris Eur, Lucas Gagnon, Kiumars Kaveh, Alexander Kupers, Daniel Litt, and Arun Ram for helpful conversations.

2. RESULTS

We start by recalling the classical story of Schubert cycles, Schubert polynomials, and divided differences. Let $B, B^-, T \subset GL_n$ denote the upper triangular, lower triangular, and diagonal matrices, and let $\text{Fl}_n = GL_n/B$ denote the complete flag variety. We will denote the Schubert cycles $X^v = \overline{BvB}$ and the opposite Schubert cycles $X_u = \overline{B^-uB}$. For permutations $u \leq v$ in the Bruhat order we denote the Richardson variety $X_u^v = X^v \cap X_u$. Recall the Borel presentation [12] of the cohomology ring of the complete flag variety as the symmetric coinvariants $H^\bullet(\text{Fl}_n) = \text{Coinv}_n := \mathbb{Z}[x_1, \dots, x_n]/\text{Sym}_n^+$, where x_1, \dots, x_n are the Chern roots of the tautological quotient flag, and Sym_n^+ is the ideal generated by positive degree symmetric polynomials in x_1, \dots, x_n . A basis of $H^\bullet(\text{Fl}_n)$ is given by the fundamental classes $[X^w]$ of the Schubert cycles.

We can compute the degree map on the Schubert cycles X^w by the following geometric observation of Bernstein–Gelfand–Gelfand [8] and Demazure [17]. Fix a reduced word decomposition $w = s_{i_1} \cdots s_{i_k}$ of w by adjacent transpositions $s_i = (i, i+1) \in S_n$, and denote $w_j = s_{i_1} \cdots s_{i_j}$. Then there is a commutative diagram

$$\begin{array}{ccccccc} \text{BS}(i_1, \dots, i_k) & \longrightarrow & \text{BS}(i_1, \dots, i_{k-1}) & \longrightarrow & \cdots & \longrightarrow & \text{BS}() = \{\text{pt}\} \\ \downarrow & & \downarrow & & & & \downarrow \\ X^{w_k} & \dashrightarrow & X^{w_{k-1}} & \dashrightarrow & \cdots & \dashrightarrow & X^{\text{id}} = \{\text{pt}\} \end{array}$$

where the bottom row are certain rational maps which are generically \mathbb{P}^1 -bundles, and the top row is a resolution of this sequence of rational maps by an iterated \mathbb{P}^1 -bundle $\text{BS}(i_1, \dots, i_k)$ known as the *Bott–Samelson resolution* [13, 23, 17]. The divided difference operators $\partial_1, \partial_2, \dots, \partial_{n-1} \in \text{End}(\text{Pol}_n)$ defined by

$$(2.1) \quad \partial_i f = \frac{f - f(x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_n)}{x_i - x_{i+1}}$$

interact in an important way with symmetric polynomials. For example

$$f \in \text{Sym}_n \iff \partial_1 f = \cdots \partial_{n-1} f = 0,$$

and they descend to $H^\bullet(\text{Fl}_n) = \text{Coinv}_n$ since for $g \in \text{Sym}_n$ we have $\partial_i(gh) = g\partial_i(h)$. The Bott–Samelson resolution has the property that $\deg_{\text{BS}(i_1, \dots, i_j)} f = \deg_{\text{BS}(i_1, \dots, i_{j-1})} \partial_{i_j} f$ for $f \in H^\bullet(\text{Fl}_n)$. These considerations then imply

$$\deg_{X^w} f = \deg_{\text{BS}(i_1, \dots, i_k)} f = \deg_{\text{BS}(i_1, \dots, i_{k-1})} \partial_{i_k} f = \cdots = \deg_{\text{BS}()} \partial_{i_1} \cdots \partial_{i_k} f = \text{ev}_0 \partial_w f,$$

where $\text{ev}_0 g = g(0, 0, \dots)$ is the constant term operator and $\partial_w = \partial_{i_1} \cdots \partial_{i_k}$ is the composite operator. For two choices of reduced word $w = s_{i_1} \cdots s_{i_k} = s_{i'_1} \cdots s_{i'_k}$, the fact that \deg_{X^w} can be

computed with the Bott–Samelson resolution associated to either sequence gives a geometric interpretation of the identity $\text{ev}_0 \partial_{i_1} \cdots \partial_{i_k} = \text{ev}_0 \partial_{i'_1} \cdots \partial_{i'_k}$. Algebraically, the divided difference operators satisfy the nil-Hecke relations $\partial_i^2 = 0$, $\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$, and $\partial_i \partial_j = \partial_j \partial_i$ for $|i - j| \geq 2$, and the identity can also be shown using these local relations via Coxeter word combinatorics.

The Schubert polynomials of Lascoux–Schützenberger [33] are a family of homogenous polynomials $\{\mathfrak{S}_w : w \in S_n\} \subset \text{Pol}_n$ with $\mathfrak{S}_{\text{id}} = 1$ that satisfy

$$\partial_i \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{ws_i} & i \in \text{Des}(w) \\ 0 & \text{otherwise.} \end{cases}$$

They also satisfy $\text{ev}_0 \partial_v \mathfrak{S}_w = \delta_{v,w}$, so they descend to the Kronecker dual basis to the cycles $\{X^w \mid w \in S_n\}$ in $H^\bullet(\text{Fl}_n)$ under the Poincaré pairing. In fact, as the fundamental classes of opposite Schubert cycles X_w are also a dual basis, we have $\mathfrak{S}_w = [X_w]$ in $H^\bullet(\text{Fl}_n)$.

2.1. The quasisymmetric operations R_i , and T_i . As in [41], we define the *Bergeron–Sottile maps* $R_1, \dots, R_n \in \text{End}(\text{Pol}_n)$ and the *quasisymmetric divided difference* $T_1, \dots, T_{n-1} \in \text{End}(\text{Pol}_n)$ by

$$(2.2) \quad R_i f := f(x_1, \dots, x_{i-1}, 0, x_i, x_{i+1}, \dots, x_{n-1})$$

$$(2.3) \quad T_i f := R_i \partial_i f = R_{i+1} \partial_i f = \frac{1}{x_i} (R_{i+1} f - R_i f).$$

These maps were shown to interact in an important way with quasisymmetric polynomials – for example, it was shown in [41, Theorems 2.6 and 2.10] that for $f \in \text{Pol}_n$ we have equivalences

$$f \in \text{QSym}_n \iff R_1 f = \cdots = R_n f \iff T_1 f = \cdots = T_{n-1} f = 0.$$

Unlike the ∂_i operators, the R_i and T_i operators naturally decrease the number of variables of a polynomial. For $X_n \in \{R_1, T_1, R_2, T_2, \dots, R_{n-1}, T_{n-1}, R_n\}$ we have $X_n(\text{Pol}_n) \subset \text{Pol}_{n-1}$. These descend to maps $X_n : \text{Coinv}_n \rightarrow \text{Coinv}_{n-1}$ and $X_n : \text{QSCoinv}_n \rightarrow \text{QSCoinv}_{n-1}$ since for $g \in \text{QSym}_n$ (or $g \in \text{Sym}_n$) we have $X_n(gh) = (R_1 g)X_n(h)$ and R_1 preserves symmetry and quasisymmetry.

As it is natural to compose the ∂_i operators in an order which makes a reduced word, this variable decreasing property of X_i leads to a natural class of composites of T_j and R_j to consider. We denote by RTSeq_n for the set of words $\Omega = x_1 x_2 \cdots x_n$ with letters

$$x_i \in \{r_1, t_1, r_2, t_2, \dots, r_{i-1}, t_{i-1}, r_i\}.$$

Note the slight redundancy that $x_1 = r_1$ necessarily; this will be useful from an algebraic viewpoint. Denote $\Pi_\Omega^n = X_1 X_2 \cdots X_n : \text{Pol}_n \rightarrow \mathbb{Z}$ for the composite linear functional under $r_i \mapsto R_i, t_i \mapsto T_i$, which we may view also as a linear functional on Coinv_n , or QSCoinv_n . While ∂_w and $\text{ev}_0 \partial_w$ are different operators, we have $\Pi_\Omega^n = \text{ev}_0 \Pi_\Omega^n$ for $\Omega \in \text{RTSeq}_n$ because the codomain is \mathbb{Z} .

Just as a reduced word $w = s_{i_1} \cdots s_{i_k}$ describes a way of reducing a permutation to the identity by applying adjacent transpositions, we will show that a sequence $\Omega = x_1 \cdots x_n \in \text{RTSeq}_n$ can

be viewed as describing a way of trimming a “plane nested binary forest” $\widehat{F} = \widehat{F}(\Omega) \in \widehat{\text{For}}_n$ (Definition 3.8) down to the empty forest via certain elementary transformations. We let

$$\text{Trim}(\widehat{F}) := \{\Omega \in \text{RTSeq}_n \mid \widehat{F}(\Omega) = \widehat{F}\}$$

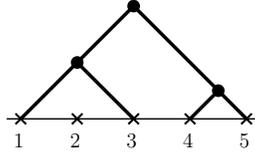
denote the set of all such sequences which trim the forest \widehat{F} (Definition 3.14), analogous to the set of reduced words of $w \in S_n$.

Using commutation relations between T_i and R_j we will show that the operators Π_Ω^n for $\Omega \in \text{Trim}(\widehat{F})$ are all equal to a common operator $\Pi_{\widehat{F}}^n$, with \widehat{F} representing a sequencing order for certain function compositions. This generalizes an analogous result in [41] describing composites $T_F = T_{i_1} T_{i_2} \cdots T_{i_k}$ in terms of certain plane binary forests $F \in \text{For}_n$.

Example 2.1. In our theory, we have $\Omega = r_1 t_1 t_2 t_1 r_2$ and $\Omega' = r_1 t_1 t_1 r_2 t_4$ both belong to $\text{Trim}(\widehat{F})$ with \widehat{F} depicted in the figure below. One can check directly for $f \in \mathbb{Z}[x_1, x_2, x_3, x_4, x_5]$ that

$$R_1 T_1 T_2 T_1 R_2 f = R_1 T_1 T_1 R_2 T_4 f,$$

i.e. $\Pi_\Omega^5 = \Pi_{\Omega'}^5$, which follows by applying the commutation relation $T_2 T_1 = T_1 T_3$ followed by the commutation relation $T_3 R_2 = R_2 T_4$.



2.2. A quasisymmetric Schubert cycle. Write $|\Omega|_t$ for the number of x_i which are equal to t_j for some j . To each element $\Omega \in \text{RTSeq}_n$ we will associate a pair of permutations $u(\Omega), v(\Omega)$ in S_n such that $u(\Omega) \leq v(\Omega)$ in Bruhat order and $|\Omega|_t = \ell(v(\Omega)) - \ell(u(\Omega))$ (Definition 5.1). We define the $|\Omega|_t$ -dimensional “ Ω -Richardson variety” by

$$X(\Omega) := X_{u(\Omega)}^{v(\Omega)} \subset \text{Fl}_n.$$

The maximal dimension $X(\Omega)$ come from sequences $\Omega = r_1 t_{i_1} \cdots t_{i_{n-1}} \in \text{RTSeq}_n$, and give the smooth toric Richardson varieties considered in [24, 36]:

$$\{X_u^{u c_n} \mid u \in S_n \text{ and } u(n) = n\}$$

where $c_n = n12 \cdots (n-1)$ is the backwards long cycle in S_n . The remaining $X(\Omega')$ are the torus-orbit closures contained in one of these maximal $X(\Omega)$.

The following is the quasisymmetric analogue of the BGG/Demazure geometric interpretation of divided differences.

Theorem 2.2 (Theorem 5.3). If $\Omega = \Omega' x \in \text{RTSeq}_n$ and $f \in H^\bullet(\text{Fl}_n)$ then the following are true.

- (1) If $x = r_i$ then $X(\Omega) \cong X(\Omega')$ and $\deg_{X(\Omega)} f = \deg_{X(\Omega')} R_i f$.

(2) If $x = t_i$ then there is a \mathbb{P}^1 -bundle $X(\Omega) \rightarrow X(\Omega')$ and $\deg_{X(\Omega)} f = \deg_{X(\Omega')} T_i f$.

In particular, we have $\deg_{X(\Omega)} f = \Pi_{\Omega}^n f$.

To each $\widehat{F} \in \widehat{\text{For}}_n$, we will show there is a torus-orbit closure $X(\widehat{F}) \subset \text{Fl}_n$ which has the property that for any $\Omega \in \text{Trim}(\widehat{F})$ we have

$$X(\widehat{F}) = u(\Omega)^{-1} \cdot X(\Omega).$$

We note that $X(\widehat{F})$ need not be a toric Richardson variety. Let $\pi_{\Omega} : X(\Omega) \rightarrow X(\widehat{F})$ be the isomorphism induced by multiplication by $u(\Omega)^{-1}$. Letting $\Omega = x_1 \cdots x_n \in \text{RTSeq}_n$, if we define $\Omega_i = x_1 \cdots x_i \in \text{RTSeq}_i$ and $\widehat{F}_i = \widehat{F}(\Omega_i)$, then there is a commutative diagram

$$\begin{array}{ccccccc} X(\Omega_n) & \longrightarrow & X(\Omega_{n-1}) & \longrightarrow & \cdots & \longrightarrow & X(\Omega_0) = \{\text{pt}\} \\ \downarrow \pi_{\Omega_n} & & \downarrow \pi_{\Omega_{n-1}} & & & & \downarrow \pi_{\Omega_0} \\ X(\widehat{F}_n) & \longrightarrow & X(\widehat{F}_{n-1}) & \longrightarrow & \cdots & \longrightarrow & X(\widehat{F}_0) = \{\text{pt}\} \end{array}$$

which for $f \in H^*(\text{Fl}_n)$ lets us compute

$$\deg_{X(\widehat{F})} f = \deg_{X(\Omega_n)} f = \deg_{X(\Omega_{n-1})} X_n f = \cdots = \deg_{X(\Omega_0)} X_1 \cdots X_n f = \Pi_{\Omega}^n f.$$

For $\Omega, \Omega' \in \text{Trim}(\widehat{F})$ this gives a geometric interpretation of the equality $\Pi_{\Omega}^n f = \Pi_{\Omega'}^n f$.

The analogies to the classical theory are therefore the following.

- (1) The varieties $X(\widehat{F}) \subset \text{Fl}_n$ correspond to the Schubert varieties $X^w \subset \text{Fl}_n$.
- (2) $\pi_{\Omega} : X(\Omega) \cong X(\widehat{F})$ for $\Omega \in \text{Trim}(\widehat{F})$ corresponds to a Bott–Samelson resolutions of X^w .

We note that in the classical case, the Bott–Samelson resolution rarely maps isomorphically onto the corresponding Schubert variety.

Example 2.3. Consider $\Omega = r_1 t_1 t_2 t_1 r_2$ and $\Omega' = r_1 t_1 t_1 r_2 t_4$, which both belong to $\text{Trim}(\widehat{F})$ as in Example 2.1. Writing permutations in one line notation we have the toric varieties $X(\Omega) = X_{21435}^{51243}$ and $X(\Omega') = X_{32415}^{52341}$, and $X(\Omega') = 23145 \cdot X(\Omega)$.

2.3. Duality with forest polynomials. We cannot find a Kronecker dual basis to the $X(\widehat{F})$ because there are linear relations between the fundamental classes, corresponding to nontrivial relations between $\Pi_{\widehat{F}}^n$ functionals arising from the identity

$$T_i R_{i+1} = R_{i+1} T_i + R_i T_{i+1}.$$

It turns out that we obtain nice duality statements if we restrict to the subset of indexed forests $\text{For}_n \subset \widehat{\text{For}}_n$ where nesting does not occur.

To see this, we note that Theorem 2.2 has particular significance in terms of the work from [41] on the interaction between the homogenous family of forest polynomials $\{\mathfrak{P}_F : F \in \text{For}\}$ and the

operators T_i . Similarly to the way that Schubert polynomials interact with the ∂_i operators, the forest polynomials satisfy

$$T_i \mathfrak{P}_F = \begin{cases} \mathfrak{P}_{F/i} & i \in \text{LTer}(F) \\ 0 & \text{otherwise,} \end{cases}$$

where $\text{LTer}(F)$ is a certain subset of the leaf labels of F and F/i is a certain “trimmed” forest (see Definition 3.5). Just as the Schubert polynomials $\{\mathfrak{S}_w \mid w \in S_n\}$ descend to a basis of Coinv_n , the forest polynomials $\{\mathfrak{P}_F \mid F \in \text{For}_n\}$ descend to a basis of QSCoinv_n .

Furthermore, the distinct composites T_G of the T_i operators are indexed by $G \in \text{For}$ and $\text{ev}_0 T_F \mathfrak{P}_G = \delta_{F,G}$. It is this observation that allows us to identify a subfamily of the $X(\widehat{F})$ as dual to the forest polynomials.

Theorem A. For $F \in \text{For}_n$ an indexed forest, we have $\text{deg}_{X(F)} f = \text{ev}_0 T_F f$, and there is a duality

$$\text{deg}_{X(F)} \mathfrak{P}_G = \delta_{F,G}.$$

Furthermore, for any $\Omega \in \text{RTSeq}_n$, the fundamental class $[X(\Omega)] \in H^\bullet(\text{Fl}_n)$ is uniquely a nonnegative linear combination of the fundamental classes $\{[X(F)] \mid F \in \text{For}_n\}$.

Consequently, the subset $\{X(F) \mid F \in \text{For}_n\}$ of the quasisymmetric Schubert cycles are to forest polynomials $\{\mathfrak{P}_F \mid F \in \text{For}_n\}$ as the Schubert cycles $\{X^w \mid w \in S_n\}$ are to Schubert polynomials $\{\mathfrak{S}_w \mid w \in S_n\}$.

2.4. The Ω -flag variety. We now describe the quasisymmetric analogue of the flag variety. The toric Richardson varieties $X(\Omega)$ fit together into a complex of smooth toric varieties

$$\text{HHMP}_n = \bigcup_{\Omega \in \text{RTSeq}_n} X(\Omega).$$

we call the Ω -flag variety. As HHMP_n is composed of isomorphic copies of $X(\widehat{F})$, one for each $\Omega \in \text{Trim}(\widehat{F})$, we call HHMP_n the Ω -flag variety. It can also be defined recursively as flags satisfying certain simple conditions (see Theorem 7.2). This complex of toric varieties was considered before in [24]. In [36] it was shown that HHMP arises as a toric degeneration of a general T -orbit closure in Fl_n (the permutahedral variety). The moment polytopes of the top-dimensional $X(\Omega)$ give a subdivision of the permutahedron

$$\text{Perm}_{n-1} = \text{conv}\{\sigma \cdot (n, n-1, \dots, 1) \mid \sigma \in S_n\}$$

into $(n-1)!$ combinatorial cubes [24], which we call the HHMP -subdivision, and which by what we have said earlier has the property that the lower dimensional faces are indexed by the moment polytopes of lower dimensional $X(\Omega')$ with $\Omega' \in \text{RTSeq}_n$. As we will see later, the facial structure of this subdivision (and hence the poset structure of the toric complex) is identified with the unit

cube subdivision of $[1, 2] \times [1, 3] \times \cdots \times [1, n - 1]$, where to a sequence $x_1 \cdots x_n \in \text{RTSeq}_n$ we associate the cuboid $Y_2 \times \cdots \times Y_n$ where $Y_i = \{j\}$ if $x_i = r_j$ and $Y_i = [j, j + 1]$ if $x_i = t_j$.

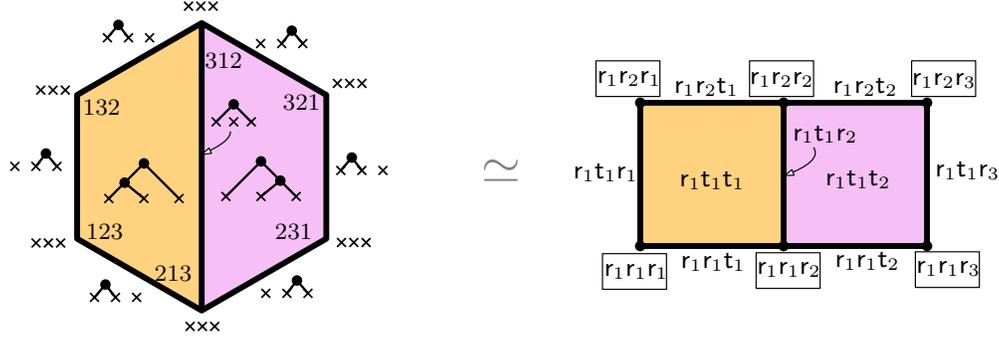


FIGURE 1. Unit cube subdivision for $n = 3$ where we have indicated the face labelings by RTSeq_3 on the right, and nested forests on the left.

Not only does HHMP_n contain the quasisymmetric Schubert cycles, but our next main theorem shows its cohomology ring naturally contains QSCoinv_n .

Theorem B. Under the inclusion $\psi : \text{HHMP}_n \rightarrow \text{Fl}_n$, we have

$$\psi^*(H^*(\text{Fl}_n)) \simeq \text{QSCoinv}_n.$$

2.5. Applications.

2.5.1. *Generalized Littlewood–Richardson coefficients.* The generalized LR coefficients $c_{u,w}^v$ are the structure coefficients for Schubert polynomial multiplication

$$\mathfrak{S}_u \mathfrak{S}_w = \sum c_{u,w}^v \mathfrak{S}_v.$$

Geometrically these can be realized as $c_{u,w}^v = \deg_{X_u^v} \mathfrak{S}_w$, so we can identify

$$c_{u(\Omega),w}^{v(\Omega)} = \Pi_{\Omega}^u \mathfrak{S}_w.$$

We will give a combinatorially nonnegative interpretation to these generalized LR coefficients for all Ω . By using well-chosen sequences Ω appearing in [41, 42], we arrive at the following results.

Theorem 2.4 (Theorem 8.6). Let $w \in S_n$. The coefficients of

- (1) A monomial $x_1^{c_1} \cdots x_n^{c_n}$ in the monomial expansion of \mathfrak{S}_w
- (2) A slide polynomial coefficient in the slide polynomial expansion $\mathfrak{S}_w = \sum a_i \mathfrak{S}_i$
- (3) A forest polynomial coefficient in the m -forest polynomial expansion $\mathfrak{S}_w = \sum a_F \mathfrak{P}_F^m$

are all generalized LR coefficients $c_{u(\Omega),w}^{v(\Omega)}$ for some $\Omega \in \text{RTSeq}_N$ with N possibly larger than n , with an explicit combinatorially nonnegative rule for computing them.

In [35] it was shown that the coefficient of a monomial arises in a different way as a generalized LR coefficient, and it would be interesting to compare these results. The second and third results are new, and as we will see also generalize to the m -forest polynomials of [41] and the m -slide polynomials of [42].

2.5.2. *Divided symmetrization.* Harada–Horiguchi–Masuda–Park [24] and Lian [36] showed that for a general $x \in \text{Fl}_n$ we have the equality

$$(2.4) \quad [\overline{T \cdot x}] = [\text{HHMP}_n].$$

On the other hand, by the work of Anderson–Tymoczko [1] we have

$$\deg_{\overline{T \cdot x}} f(x_1, \dots, x_n) = \sum_{\sigma \in S_n} \frac{f(x_{\sigma(1)}, \dots, x_{\sigma(n)})}{(x_{\sigma(1)} - x_{\sigma(2)}) \cdots (x_{\sigma(n-1)} - x_{\sigma(n)})},$$

realizing the degree map of classes from $H^\bullet(\text{Fl}_n)$ on $\overline{T \cdot x}$ as “divided symmetrization” (DS henceforth), first studied by Postnikov [52, §3]. We can combine these results with Theorem A to obtain a new factorized expression for DS.

Theorem 2.5. For $f \in \text{Pol}_n$ homogenous of degree $n - 1$, we have

$$\sum_{\sigma \in S_n} \frac{f(x_{\sigma(1)}, \dots, x_{\sigma(n)})}{(x_{\sigma(1)} - x_{\sigma(2)}) \cdots (x_{\sigma(n-1)} - x_{\sigma(n)})} = T_1(T_1 + T_2) \cdots (T_1 + \cdots + T_{n-1})f.$$

Expanding out the expression on the right fully, each term corresponds to the degree on a distinct maximal toric Richardson variety in HHMP_n . This factorized form of DS explains many of its interesting properties that have been observed in the past, particularly the interaction with quasisymmetric polynomials, and we show that the q -analogue $T_1(T_1 + qT_2) \cdots (T_1 + qT_2 + \cdots + q^{n-1}T_{n-1})$ recovers the q -divided symmetrization $\langle f \rangle_n^q$ considered in [44].

One corollary of the DS identity is that for any n -variable Schubert polynomial $\mathfrak{S}_w(x_1, \dots, x_n)$, the DS $\langle \mathfrak{S}_w \rangle_n$ is a polynomial all of whose coefficients are nonnegative. Through computer experimentation, we make the following conjecture.

Conjecture 2.6 (Conjecture 9.8). The q -divided symmetrization $\langle \mathfrak{S}_w \rangle_n^q$ of a Schubert polynomial $\mathfrak{S}_w(x_1, \dots, x_n)$ is Hall-Littlewood P-positive.

2.5.3. *A formula of Gessel.* Finally, we use our theory to give a geometric explanation for a formula of Gessel [20, Theorem 3] which, given a symmetric polynomial $f(x_1, \dots, x_n)$, determines the coefficients of the expansion

$$f(x_1, \dots, x_n) = \sum_{k=1}^n \sum_{i_k, \dots, i_n \geq 1} a_{i_k, \dots, i_n} \mathfrak{F}_{i_k, \dots, i_n}(x_1, \dots, x_n)$$

where $\mathfrak{F}_{i_k, \dots, i_n}(x_1, \dots, x_n)$ is the fundamental quasisymmetric polynomial whose reverse lexicographic leading term is $x_k^{i_k} \cdots x_n^{i_n}$. This formula is in terms of the Hall inner product with skew

Schur polynomials associated to ribbon shapes. We geometrically interpret it as a degree map on Grassmannian Richardson varieties associated to these shapes, and show that a certain subset of our $X(\Omega)$ project onto these Grassmannian Richardson varieties. Using our theory of degree maps on $X(\Omega)$ shows that Gessel's formula follows directly from a more general formula

$$f(x_1, \dots, x_n) = \sum_{k=1}^n \sum_{i_k, \dots, i_n \geq 1} (\text{ev}_0 T_k^{i_k} \cdots T_n^{i_n} f) \mathfrak{F}_{i_k, \dots, i_n}(x_1, \dots, x_n)$$

proved in [41] that works for $f \in \text{QSym}_n$ an arbitrary quasisymmetric polynomial.

3. COMPOSITES OF R AND T VIA NESTED FORESTS

We first recall the results of [41] about the combinatorics the Thompson monoid, its faithful polynomial representation given by $i \mapsto T_i$, and the forest polynomials \mathfrak{F}_F . We then generalize the combinatorics to what we call the augmented Thompson monoid and (marked) nested forests. Finally, we fix n and restrict the previous constructions to obtain operators $\Pi_{\hat{F}}^n$ on any of the spaces $\text{Pol}_n, \text{Coinv}_n, \text{QSCoinv}_n$ that depend on a certain forest $\hat{F} \in \widehat{\text{For}}_n$ in the collection of *nested forests supported on* $\{1, \dots, n\}$.

To avoid disrupting the exposition and keeping our overarching goal in mind we relegate the proofs of combinatorial results in this section to Appendix A.

3.1. Composites of T via indexed forests, and forest polynomials. In this section we recall the relationship between the T operators, plane binary forests, and forest polynomials from [41]. Let $\text{Pol} = \mathbb{Z}[x_1, x_2, \dots] = \bigcup_n \text{Pol}_n$. The *quasisymmetric divided differences* $T_1, T_2, \dots \in \text{End}(\text{Pol})$ defined as in (2.3) satisfy the elementary commutation relations

$$(3.1) \quad T_i T_j = T_j T_{i+1} \text{ for } i > j,$$

which imply that $i \mapsto T_i$ is a representation of the Thompson monoid

$$\text{ThMon} = \langle 1, 2, \dots \mid i \cdot j = j \cdot (i+1) \text{ for } i > j \rangle.$$

It is a classical fact in the theory of Thompson groups [4, 15, 16, 57] that ThMon is isomorphic to a monoid For of forests of plane binary trees. Hence one can define operators T_F for $F \in \text{For}$.

Definition 3.1. A *plane binary tree* is a binary tree where each internal (i.e. non-leaf) node v has a left child v_L and a right child v_R . An *indexed forest* F is a sequence of plane binary trees T_1, T_2, \dots where $T_i = \times$, the trivial plane binary tree, for all but finitely many i . We denote the set of indexed forests by For .

The leaves of a plane binary tree are naturally ordered, so there is a canonical identification of the leaves of $F \in \text{For}$ with \mathbb{N} . Denoting $\text{IN}(F)$ for the internal nodes of F , the totality of nodes of F may thus be identified with $\text{IN}(F) \sqcup \mathbb{N}$.

Definition 3.2. We define a monoid structure on For by taking the composition $F \cdot G$ to be the plane forest obtained by identifying the i 'th leaf of F with the i 'th root of G .

For \wedge the unique binary tree with one internal node, we denote

$$\underline{i} := \underbrace{\times \cdots \times}_{i-1} \wedge \times \cdots \times .$$

Theorem 3.3 ([41]). The map $i \mapsto \underline{i}$ induces a monoid isomorphism $\text{ThMon} \rightarrow \text{For}$.

This result allows us to tacitly identify i with \underline{i} from now on. Since $i \mapsto \text{T}_i$ is a representation of ThMon on Pol , the composite operation $\text{T}_{i_1} \cdots \text{T}_{i_k}$ only depends on the forest $i_1 \cdots i_k \in \text{For}$.

Definition 3.4. For $F \in \text{For}$, we define $\text{T}_F := \text{T}_{i_1} \cdots \text{T}_{i_k}$ for any sequence i_1, \dots, i_k with $F = i_1 \cdots i_k$.

An internal node of F is *terminal* if both of its children are leaves. The left leaves of terminal nodes form the set $\text{LTer}(F)$. Equivalently, $\text{LTer}(F)$ is the set of i such that $F = G \cdot i$ for some (necessarily unique) forest $G \in \text{For}$. For $i \in \text{LTer}(F)$ we write $F/i \in \text{For}$ for the unique forest with $F = (F/i) \cdot i$.

Definition 3.5 ([41, Theorem 6.1]). The forest polynomials $\{\mathfrak{P}_F \mid F \in \text{For}\}$ are the unique family of homogenous polynomials in Pol with $\mathfrak{P}_\emptyset = 1$ and

$$\text{T}_i \mathfrak{P}_F = \begin{cases} \mathfrak{P}_{F/i} & i \in \text{LTer}(F) \\ 0 & \text{otherwise.} \end{cases}$$

These polynomials were introduced combinatorially by the first and third authors in [45, 48] but we shall have no need for the explicit description. The recursive characterization in Definition 3.5 then implies [41, Corollary 6.6] that the forest polynomials are dual to the functionals $\text{ev}_0 \text{T}_G$ in the sense that

$$\text{ev}_0 \text{T}_G \mathfrak{P}_F = \delta_{G,F}.$$

As was shown in [41, Corollary 6.7], because the maps $\text{End}(\text{Pol}) \rightarrow \mathbb{Z}$ given by $\Phi \mapsto \text{ev}_0 \Phi \mathfrak{P}_F$ separate the linear functionals T_G , the map $\mathbb{Z}[\text{ThMon}] \rightarrow \text{End}(\text{Pol})$ is a faithful representation (i.e. the T_F operators are \mathbb{Z} -linearly independent).

We define For_n to be the subset of $F \in \text{For}$ where the leaves of all nontrivial trees lie in $\{1, \dots, n\}$.

Theorem 3.6 ([41, Proposition 6.8 and Theorem 9.7]). The forest polynomials are a \mathbb{Z} -basis for Pol , and every polynomial $f \in \text{Pol}$ can be uniquely written as

$$f = \sum_{F \in \text{For}} a_F \mathfrak{P}_F.$$

with $a_F = \text{ev}_0 \text{T}_F f$. Additionally, the forest polynomials $\{\mathfrak{P}_F \mid F \in \text{For}_n\}$ lie in $\mathbb{Z}[x_1, \dots, x_n]$ and descend to a \mathbb{Z} -basis of QSCoinv_n . Finally, for $f \in \text{Pol}_n$ we have $f \in \text{QSym}_n^+$ if and only if $\text{ev}_0 \text{T}_F f = 0$ for all $F \in \text{For}_n$.

3.2. Composites of R and T via marked nested forests. We now generalize the forest combinatorics of Section 3.1 to include the Bergeron–Sottile maps $R_1, R_2, \dots \in \text{End}(\text{Pol})$ defined as in (2.2). These satisfy the further commutation relations

$$(3.2) \quad T_i R_j = R_j T_{i+1} \text{ for } i \geq j$$

$$(3.3) \quad R_i T_j = T_j R_{i+1} \text{ for } i > j$$

$$(3.4) \quad R_i R_j = R_j R_{i+1} \text{ for } i \geq j.$$

Together with (3.1) these relations therefore define the following abstract monoid.

Definition 3.7. The *augmented Thompson monoid* $\widetilde{\text{ThMon}}$ is the quotient of the free monoid on the alphabet $\{1, 2, \dots\} \sqcup \{1_\circ, 2_\circ, \dots\}$ by the relations

$$(3.5) \quad i \cdot j = j \cdot (i + 1) \text{ if } i > j$$

$$(3.6) \quad i \cdot j_\circ = j_\circ \cdot (i + 1) \text{ if } i \geq j$$

$$(3.7) \quad i_\circ \cdot j = j \cdot (i + 1)_\circ \text{ for } i > j$$

$$(3.8) \quad i_\circ \cdot j_\circ = j_\circ \cdot (i + 1)_\circ \text{ for } i \geq j.$$

We define the polynomial representation of the augmented Thompson monoid to be the representation Pol given by $i \mapsto T_i$ and $i_\circ \mapsto R_i$.

We define here the monoid $\widetilde{\text{For}}$ of “marked nested binary forests”, which extends the monoid of indexed forests defined in Section 3.1. We start by defining nested binary forests, which are already an extension of indexed forests.

A *finite partition* π of \mathbb{N} is a set partition of \mathbb{N} such that all blocks have finite size, and all but finitely many blocks are singletons; equivalently, $\{i\}$ is a block for all i large enough. We say π is *noncrossing* if for any two blocks $B, B' \in \pi$ we never have $a < c < b < d$ with $a, b \in B$ and $c, d \in B'$.

Definition 3.8. A *(plane, binary) nested forest* is a family of plane binary trees $(T_B)_{B \in \pi}$ where π is a finite noncrossing partition of \mathbb{N} , and each tree T_B for $B \in \pi$ has $|B|$ leaves.

We denote by $\widehat{\text{For}}$ the set of nested forests. The subset B in the definition is called the *support* of T_B . An example of nested forest is given in Figure 2; the non-singleton supports are $\{2, 3\}$, $\{4, 5, 9, 11\}$, $\{6, 7, 8\}$ and $\{13, 14, 15\}$. If all supports of $F \in \widehat{\text{For}}$ are intervals $\{a, a + 1, \dots, b\}$, then we retrieve the notion of indexed forests: these intervals are ordered from left to right, giving trees T_1, T_2, \dots as in [41, Definition 3.1].

This notion will mostly be sufficient for the rest of the article. In order to build a monoid isomorphic to the one generated by the T_i and R_i however, we need to enrich this structure with a notion of *marks* on the roots of these trees.

This requires some extra notions on noncrossing partitions. Two blocks B, B' in a noncrossing partition π are either:

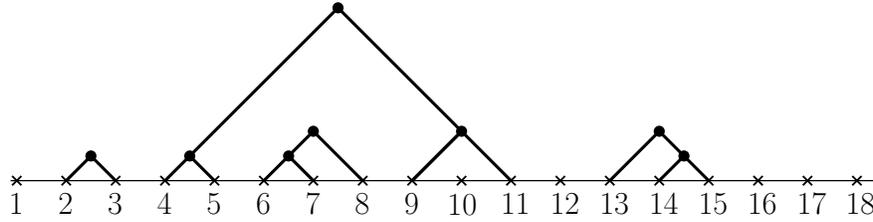


FIGURE 2. A nested forest.

- *aligned*: the elements of B are smaller than the elements of B' (or the opposite), or
- *nested*: there are two elements $a < b$ of B such all elements of B' are between a and b .

The *outer blocks* of π are those that are not nested under any other block. Outer blocks are pairwise aligned, and thus are totally ordered from left to right. We call a tree in a forest an outer (resp. nested) tree if its support is an outer (resp. nested) block.

Definition 3.9. A *marked nested forest* is a nested forest where a finite number of tree roots are marked, and these include the roots of all nested trees. We write $\widetilde{\text{For}}$ for the set of all marked nested forests.

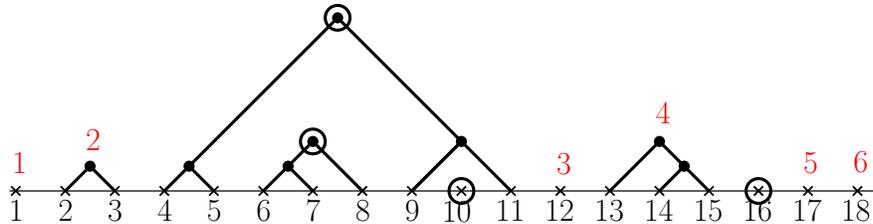


FIGURE 3. A marked nested forest.

Figure 3 depicts a marked nested forest with circles denoting the marking on the roots. Note that the roots of the trees with supports $\{6, 7, 8\}$ and $\{10\}$ are necessarily marked since the trees are nested, while the marks on all other roots can be picked arbitrarily.

For $\widetilde{F} \in \widetilde{\text{For}}$ we write $\widehat{F} \in \widehat{\text{For}}$ for the nested forest obtained by forgetting the markings. As an example, note that the nested forest in Figure 2 is the \widehat{F} for the \widetilde{F} in Figure 3.

Remark 3.10 (Indexed forests as marked nested forests). If a marked nested forest has no marks, then all blocks are outer blocks, and this entails that they are integer intervals $\{a, a + 1, \dots, b\}$. As we saw above this is naturally equivalent to the notion of indexed forests from [41].

Note that the *unmarked* roots of (outer) trees are naturally ordered from left to right, and so we may canonically identify the unmarked roots by \mathbb{N} and talk about the “ i ’th unmarked root”. We then have the following generalization of [41, Definition 4.1], recalled in Definition 3.2.

Definition 3.11. Let $\tilde{F}, \tilde{G} \in \widetilde{\text{For}}$. Define a monoid structure on $\widetilde{\text{For}}$ by letting $\tilde{F} \cdot \tilde{G}$ be obtained by identifying the i 'th leaf of \tilde{F} with the i 'th unmarked root of \tilde{G} .

We can describe $\widetilde{\text{For}}$ recursively using this product. The “empty forest” $\emptyset := \times \times \times \cdots$ is made of unmarked trivial trees \times . For \wedge the unmarked binary tree with one root and two leaves and \otimes the trivial binary tree whose only node is marked, consider the elementary forests

$$\underline{i} = \underbrace{\times \cdots \times}_{i-1} \wedge \times \times \cdots \quad \text{and} \quad \underline{i}_\circ = \underbrace{\times \cdots \times}_{i-1} \otimes \times \times \cdots$$

Then, given $\tilde{F} \in \widetilde{\text{For}}, i \in \mathbb{N}$, we obtain

- $\tilde{F} \cdot \underline{i}$ by taking the i 'th leaf and giving it two children.
- $\tilde{F} \cdot \underline{i}_\circ$ by inserting a trivial marked tree \otimes between the $(i-1)$ 'st and the i 'th leaves of \tilde{F} (if $i = 1$ then we add \otimes before the first leaf of \tilde{F}).

Theorem 3.12. The map $i \mapsto \underline{i}, i_\circ \mapsto \underline{i}_\circ$ induces an isomorphism $\widetilde{\text{ThMon}} \cong \widetilde{\text{For}}$.

The proof is in Appendix A. This result allows us to tacitly identify i, i_\circ with $\underline{i}, \underline{i}_\circ$ from now on.

3.3. RTSeq and composites Π_Ω of T and R operators. We now study the polynomial representation of the augmented Thompson monoid on Pol given by $i \mapsto T_i$ and $i_\circ \mapsto R_i$. The following definition will be useful for indexing sequences of R_i and T_i operators, and to avoid confusing such sequences with composite operators.

Definition 3.13. Let RTSeq denote the set of sequences $\Omega = x_1 \cdots x_k$ with $k \geq 0$ and

$$x_i \in \{r_j, t_j \mid j \in \mathbb{Z}_{>0}\}.$$

We write $|\Omega| = k$ and let $|\Omega|_t$ be the number of x_i equal to t_j for some j . We write $\text{up}(r_i) = R_i$, $\text{up}(t_i) = T_i$, and the composite operator associated to the sequence $\Omega \in \text{RTSeq}$ to be

$$\Pi_\Omega := \text{up}(x_1) \cdots \text{up}(x_n) \in \text{End}(\text{Pol}).$$

Viewing RTSeq as the free monoid on the alphabet $\{r_j, t_j \mid i \in \mathbb{N}\}$, there is a natural monoid surjection $\text{RTSeq} \rightarrow \widetilde{\text{ThMon}} \cong \widetilde{\text{For}}$ sending $t_i \mapsto i$ and $r_i \mapsto i_\circ$, and hence to each $\Omega \in \text{RTSeq}$ we can associate an element $\tilde{F}(\Omega) \in \widetilde{\text{For}}$. Recursively, $\tilde{F}(\emptyset) = \emptyset$ and for $\Omega = \Omega'x$ we have

$$\tilde{F}(\Omega) = \begin{cases} \tilde{F}(\Omega') \cdot j & \text{if } x = t_j \\ \tilde{F}(\Omega') \cdot j_\circ & \text{if } x = r_j. \end{cases}$$

Definition 3.14. For $\tilde{F} \in \widetilde{\text{For}}$, we define

$$\text{Trim}(\tilde{F}) = \{\Omega \in \text{RTSeq} \mid \tilde{F}(\Omega) = \tilde{F}\} \subset \text{RTSeq}.$$

Proposition 3.15. Under the polynomial representation, the image of Ω in $\widetilde{\text{ThMon}}$ is Π_Ω . In particular, for $\Omega, \Omega' \in \text{Trim}(\widetilde{F})$, we have $\Pi_\Omega = \Pi_{\Omega'}$.

Proof. The first part follows by definition of the polynomial representation on Ω , and the second follows from the fact that $\widetilde{F}(\Omega)$ is by construction the image of Ω under the composite $\text{RTSeq} \rightarrow \widetilde{\text{ThMon}} \cong \widetilde{\text{For}}$. \square

Definition 3.16. For $\widetilde{F} \in \widetilde{\text{For}}$, we define $\Pi_{\widetilde{F}} := \Pi_{\widetilde{F}(\Omega)}$ for any $\Omega \in \text{Trim}(\widetilde{F})$.

Remark 3.17. A more conceptual proof of the commutation relations (3.1) and (3.2)–(3.4) follows from the observations in [41]: if we define

- (1) $T : \text{Pol}_1^{\otimes 2} \rightarrow \text{Pol}_1$ by $Tf(x, y) = \frac{f(x,0) - f(0,x)}{x}$, and
- (2) $R : \text{Pol}_1 \rightarrow \text{Pol}_0$ by $R(f(x)) = f(0)$,

then under the identification $\text{Pol} = \text{Pol}_1^{\otimes \infty}$ we have

$$T_i = \text{id}^{\otimes i-1} \otimes T \otimes \text{id}^{\otimes \infty} \quad \text{and} \quad R_i = \text{id}^{\otimes i-1} \otimes R \otimes \text{id}^{\otimes \infty}.$$

The commutation relations are then the universal relations satisfied by compositions of the shifts of any fixed operators $\text{Pol}_1^{\otimes 2} \rightarrow \text{Pol}_1$ and $\text{Pol}_1 \rightarrow \text{Pol}_0$, irrespective of the particular definitions. The marked nested forest \widetilde{F} may be seen as a composition tree for two-to-one and one-to-zero operations acting on consecutive variables, with a marking indicating the application of a one-to-zero operation, and then $\Pi_{\widetilde{F}}$ is the associated composition of these operations.

The polynomial representation of the augmented Thompson monoid is not faithful as a linear representation $\mathbb{Z}[\widetilde{\text{ThMon}}] \rightarrow \text{End}(\text{Pol})$, because of the presence of the nontrivial relation

$$(3.9) \quad T_i R_{i+1} = R_i T_{i+1} + R_{i+1} T_i$$

which will play an important role later on. As the first part of the next theorem shows however, it is however faithful as a monoid representation.

Theorem 3.18. Let $\widetilde{F}, \widetilde{F}' \in \widetilde{\text{For}}$. Then

- (1) $\widetilde{F} = \widetilde{F}'$ if and only if $\Pi_{\widetilde{F}} = \Pi_{\widetilde{F}'}$, and
- (2) $\widehat{F} = \widehat{F}'$ if and only if $\text{ev}_0 \Pi_{\widetilde{F}} = \text{ev}_0 \Pi_{\widetilde{F}'}$.

Corollary 3.19. The distinct operators Π_Ω for $\Omega \in \text{RTSeq}$ are indexed by $\widetilde{F} \in \widetilde{\text{For}}$ and we can index the distinct operators $\text{ev}_0 \Pi_\Omega$ by $\widehat{F} \in \widehat{\text{For}}$.

The proof is given in Appendix A.

3.4. Fully supported forests and RTSeq_n . To conclude this section, we classify the nested and marked nested forests arising from a subset $\text{RTSeq}_n \subset \text{RTSeq}$.

Definition 3.20. We denote by $\text{RTSeq}_n \subset \text{RTSeq}$ for the subset of sequences $\Omega = x_1 \cdots x_n$ with the restriction that $x_i \in \{r_1, t_1, r_2, t_2, \dots, r_{i-1}, t_{i-1}, r_i\}$.

Note that for $\Omega \in \text{RTSeq}_n$ we always have $x_1 = r_1$.

Let us write

$$\Pi_\Omega^n : \text{Pol}_n \rightarrow \mathbb{Z}$$

for the restriction of Π_Ω to Pol_n . This functional descends to Coinv_n and QSCoinv_n by the discussion in Section 2, and we write by abuse of notation $\Pi_\Omega^n : \text{Coinv}_n, \text{QSCoinv}_n \rightarrow \mathbb{Z}$.

Proposition 3.21. Let $\Omega \in \text{RTSeq}_n$, and write $\tilde{F} = \tilde{F}(\Omega)$. Then

- (1) the $(n+1)$ 'st leaf of \tilde{F} onwards belong to trivial trees, and
- (2) all root nodes of the trees of \tilde{F} supported on $\{1, \dots, n\}$ are marked.

Conversely, if $\tilde{F} \in \tilde{\text{For}}$ satisfies these two properties then $\tilde{F} = \tilde{F}(\Omega)$ for $\Omega \in \text{RTSeq}_n$.

This is proved in Appendix A. It follows that for $\Omega \in \text{RTSeq}_n$ we have $\hat{F}(\Omega) = \hat{F}(\Omega')$ if and only if $\tilde{F}(\Omega) = \tilde{F}(\Omega')$, so there is no need to remember marked roots. Let $\widehat{\text{For}}_n$ denote the set of nested forests where the $(n+1)$ 'st leaf onwards belong to trivial trees.

Definition 3.22. For $\hat{F} \in \widehat{\text{For}}_n$, we define

$$\text{Trim}(\hat{F}) = \{\Omega \in \text{RTSeq}_n \mid \hat{F}(\Omega) = \hat{F}\} \subset \text{RTSeq}_n,$$

and we define $\Pi_{\hat{F}}^n = \Pi_\Omega^n$ for any $\Omega \in \text{Trim}(\hat{F})$. As with Π_Ω^n , by abuse of notation, we also denote by $\Pi_{\hat{F}}^n$ the induced maps $\text{Coinv}_n \rightarrow \mathbb{Z}$ and $\text{QSCoinv}_n \rightarrow \mathbb{Z}$.

This notion $\Pi_{\hat{F}}^n$ naturally encompasses the operations $\text{ev}_0 \text{T}_F$ used to extract the coefficients of forest polynomials as discussed in Section 3.1.

Corollary 3.23. For $F \in \text{For}_n$ we have $\text{ev}_0 \text{T}_F = \Pi_F^n$.

Proof. Suppose $F = i_1 \cdots i_k$. For $f \in \text{Pol}_n$, we have $\text{T}_F \in \text{Pol}_{n-|F|}$, and we can write $\text{ev}_0 \text{T}_F = \Pi_\Omega^n$ where $\Omega = r_1^{n-|F|} t_{i_1} \cdots t_{i_k}$. The nested forest associated to Ω is F , so $\Pi_\Omega = \Pi_F^n$. \square

We conclude by giving a visual depiction of $\Omega \in \text{Trim}(\hat{F})$. For a sequence $\Omega = x_1 \cdots x_n \in \text{RTSeq}_n$, define the *trimming diagram* of Ω as follows: replace a letter r_i (resp. t_i) in position j in Ω by the elementary diagram on the left (resp. right) below, and concatenate such diagrams from top to bottom.



We note that the initial r_1 of Ω does not contribute any edges. A trimming diagram for a particular $\Omega \in \text{RTSeq}_{13}$ is given on the left in Figure 4. The highlighted elementary diagram corresponds to the letter in the fifth position being a t_2 . It is easy to see that the nested forest \hat{F} such $\Omega \in \text{Trim}(\hat{F})$ is the blue forest obtained after contracting all red edges. This is illustrated on the right.

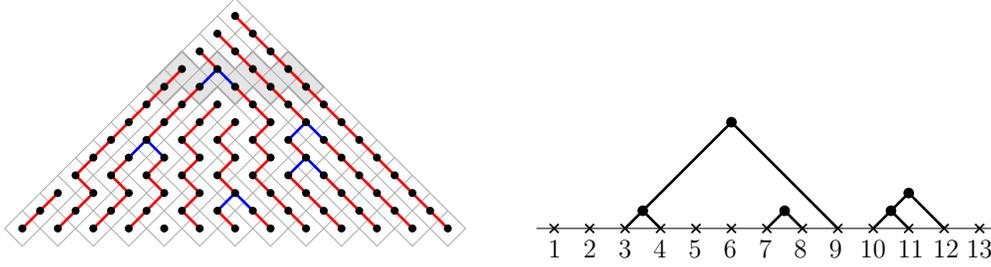


FIGURE 4. Trimming diagram for $r_1^4 t_2 r_3 r_4 t_6 t_2 t_7 r_1 t_6 r_5 \in \text{RTSeq}_{13}$ and the associated nested forest.

In Appendix C we will show that removing the blue edges from the trimming diagram yields a GZ diagram whose associated polytope is the moment polytope for the Ω -Richardson varieties $X(\Omega)$ we will be discussing in subsequent sections.

4. GEOMETRIC REALIZATIONS OF R AND T

Denote the tautological flag subbundle on Fl_n by $\{0\} \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \subset \mathcal{V}_n = \underline{\mathbb{C}}^n$. Consider the projection map

$$\pi_i : \text{Fl}_n \rightarrow \text{Fl}_{1, \dots, i-1, i+1, \dots, n}$$

forgetting the i 'th subspace, which realizes Fl_n as the \mathbb{P}^1 -bundle given $\mathbb{P}(\mathcal{V}_{i+1}/\mathcal{V}_{i-1})$ on the partial flag variety. The following folklore facts describe how pullback and pushforward along π_i interacts with cohomology.

Fact 4.1.

- (1) Pullback $(\pi_i)^*$ induces an injection $H^\bullet(\text{Fl}_{1, \dots, i-1, i+1, \dots, n}) \hookrightarrow H^\bullet(\text{Fl}_n) = \text{Coinv}_n$ whose image is the subring generated by $x_1, \dots, x_{i-1}, x_i + x_{i+1}, x_i x_{i+1}, x_{i+2}, \dots, x_n$, i.e. the subring generated by polynomials symmetric in x_i, x_{i+1} ,
- (2) Pushforward $(\pi_i)_* : H^\bullet(\text{Fl}_n) \rightarrow H^\bullet(\text{Fl}_{1, \dots, i-1, i+1, \dots, n})$ is given by $f \mapsto \partial_i f$.

We recall the thrust of the BGG computation in the following geometric fact.

Fact 4.2. If $w \in S_n$ and $i \notin \text{Des}(w)$, then $X^{ws_i} \rightarrow \pi_i(X^{ws_i})$ is birational onto its image, and $X^w = \pi_i^{-1} \pi_i(X^{ws_i})$ is a \mathbb{P}^1 -bundle over $\pi_i(X^w) = \pi_i(X^{ws_i})$.

From this fact we see that

$$\deg_{X^w} f(x_1, \dots, x_n) = \deg_{\pi_i(X^w)} \partial_i f = \deg_{X^{ws_i}} \partial_i f.$$

In this section we carry out this computation with certain Richardson varieties that will later allow us to geometrically interpret the composite operators Π_Ω^n for $\Omega \in \text{RTSeq}_n$. For $1 \leq i \leq n$, define the map $\varepsilon_i : S_{n-1} \rightarrow S_n$ on permutations by

$$\varepsilon_i(w)(j) = \begin{cases} w(j) + 1 & j < i \\ 1 & j = i \\ w(j-1) + 1 & j > i. \end{cases}$$

Concretely this inserts a 1 in the i 'th available position in the one-line notation for w and increases all other values by 1, so e.g. $\varepsilon_3(143652) = 2514763$.

4.1. A geometric realization of R_i . The geometric description of the Bergeron–Sottile maps R_i comes from the seminal paper [7] of the Pieri rule for Schubert varieties, in particular the map Ψ_i – we make no claims of originality. For $1 \leq i \leq n$, let $\Psi_i : \text{Fl}_{n-1} \rightarrow \text{Fl}_n$ denote the map

$$(\Psi_i(\mathcal{V}))_j = \begin{cases} \{0\} \oplus \mathcal{V}_j & \text{if } j < i \\ \underline{\mathbb{C}} \oplus \mathcal{V}_{j-1} & \text{if } j \geq i. \end{cases}$$

Theorem 4.3. For $f \in \text{Pol}_n$ we have

$$\Psi_i^* f = R_i f.$$

Proof. We have

$$\Psi_i^*(\mathcal{V}_j) = \begin{cases} \mathcal{V}_j & \text{if } j < i \\ \underline{\mathbb{C}} \oplus \mathcal{V}_{j-1} & \text{if } j \geq i. \end{cases}$$

Therefore

$$\Psi_i^*(x_1 + \dots + x_j) = c_1(\Psi_i^*(\mathcal{V}_j)^\vee) = c_1((\mathcal{V}_{j-\delta_{j \geq i}})^\vee) = x_1 + \dots + x_{j-\delta_{j \geq i}} = R_i(x_1 + \dots + x_j).$$

Since Ψ_i^* and R_i agree on the generators x_1, \dots, x_n of $H^\bullet(\text{Fl}_n)$, they are equal. \square

Theorem 4.4. For $u \leq v$ in S_{n-1} , the map Ψ_i is an isomorphism $X_u^v \rightarrow X_{\varepsilon_i(u)}^{\varepsilon_i(v)}$. In particular,

$$\deg_{X_{\varepsilon_i(u)}^{\varepsilon_i(v)}} f = \deg_{X_u^v} R_i f.$$

Proof. X_u^v and $X_{\varepsilon_i(u)}^{\varepsilon_i(v)}$ are irreducible of the same dimension $\ell(v) - \ell(u) = \ell(\varepsilon_i(v)) - \ell(\varepsilon_i(u))$. Since Ψ_i is a closed embedding, it suffices to show that $\Psi_i(X_u^v) \subset X_{\varepsilon_i(u)}^{\varepsilon_i(v)}$.

For $\mathcal{V} \in X_u^v$, let w_1, \dots, w_{n-1} be a basis of \mathbb{C}^{n-1} such that for all j we have $\langle w_1, \dots, w_j \rangle = \mathcal{V}_j$ and $\langle w_{(v')^{-1}(1)}, \dots, w_{(v')^{-1}(j)} \rangle = \langle e_1, \dots, e_j \rangle$ for all j for some $v' \leq v$. Then the vectors $\tilde{w}_1, \dots, \tilde{w}_n \in \mathbb{C}^n$ with

$$\tilde{w}_j = \begin{cases} \{0\} \oplus w_j & j < i \\ e_1 & j = i \\ \{0\} \oplus w_{j-1} & j > i \end{cases}$$

are a basis of \mathbb{C}^n such that $\langle \tilde{w}_1, \dots, \tilde{w}_j \rangle = \Psi_i(\mathcal{V})_j$ and $\langle \tilde{w}_{\varepsilon_i(v')^{-1}(1)}, \dots, \tilde{w}_{\varepsilon_i(v')^{-1}(j)} \rangle = \langle e_1, \dots, e_j \rangle$ so $\Psi_i(\mathcal{V}) \in X_{\varepsilon_i(v')} \subset X_{\varepsilon_i(v)}$. Similarly $\Psi_i(\mathcal{V}) \in X_{\varepsilon_i(u)}$ and the result follows since $X_{\varepsilon_i(v)} = X_{\varepsilon_i(v)} \cap X_{\varepsilon_i(u)}$. \square

Recall that a *toric Richardson variety* in Fl_n is a Richardson variety which is also a torus-orbit closure under the left action of torus of diagonal matrices $(\mathbb{C}^*)^n$ on $GL_n/B = \text{Fl}_n$.

Corollary 4.5. If X_u^v is a toric Richardson variety, then so is $X_{\varepsilon_i(u)}^{\varepsilon_i(v)}$.

Proof. For $(t_1, \dots, t_n) \in (\mathbb{C}^*)^n$ this follows because $(t_1, \dots, t_n) \cdot \Psi_i(\mathcal{V}) = \Psi_i((t_2, \dots, t_n)\mathcal{V})$. \square

4.2. A geometric realization of \mathbb{T} . To realize \mathbb{T}_i geometrically we will compose the geometric realization of \mathbb{R}_i from the previous subsection with $\pi_i^{-1}\pi_i$, as in the BGG realization of ∂_i . This happens to interact well with the Richardson varieties produced in the previous subsection.

Proposition 4.6. $X_{\varepsilon_i(u)}^{\varepsilon_{i+1}(v)} = \pi_i^{-1}\pi_i(X_{\varepsilon_i(u)}^{\varepsilon_{i+1}(v)})$ and $\pi_i(X_{\varepsilon_i(u)}^{\varepsilon_{i+1}(v)}) = \pi_i(X_{\varepsilon_i(u)}^{\varepsilon_i(v)})$

Proof. By Fact 4.2 we have $X_{\varepsilon_{i+1}(v)} = X_{\varepsilon_i(v)s_i} = \pi_i^{-1}\pi_i X_{\varepsilon_i(v)}$ because $i \notin \text{Des}(\varepsilon_i(v))$. Since $X_{\varepsilon_i(u)} = w_{0,n} X^{w_{0,n}\varepsilon_i(u)}$, where $w_{0,n}$ denotes the longest permutation in S_n , we may apply Fact 4.2 again to conclude that $X_{\varepsilon_i(u)} = \pi_i^{-1}\pi_i X_{\varepsilon_i(u)}$ because $i \notin \text{Des}(\varepsilon_i(u))$.

Combining these, we obtain the equality

$$\pi_i^{-1}\pi_i X_{\varepsilon_i(u)}^{\varepsilon_i(v)} = X_{\varepsilon_i(u)} \cap \pi_i^{-1}\pi_i X_{\varepsilon_i(v)} = X_{\varepsilon_i(u)} \cap X_{\varepsilon_{i+1}(v)} = X_{\varepsilon_i(u)}^{\varepsilon_{i+1}(v)},$$

where we note that the first equality follows just from the π_i -saturatedness $\pi_i^{-1}\pi_i X_{\varepsilon_i(u)} = X_{\varepsilon_i(u)}$. This implies both desired statements. \square

Theorem 4.7. For $f \in \text{Pol}_n$ and $1 \leq i \leq n-1$ we have

$$(4.1) \quad \deg_{X_{\varepsilon_i(u)}^{\varepsilon_{i+1}(v)}} f = \deg_{X_u^v} \mathbb{T}_i f.$$

Proof. We have

$$\deg_{X_{\varepsilon_i(u)}^{\varepsilon_{i+1}(v)}} f = \deg_{\pi_i(X_{\varepsilon_i(u)}^{\varepsilon_{i+1}(v)})} \partial_i f = \deg_{\pi_i(X_{\varepsilon_i(u)}^{\varepsilon_i(v)})} \partial_i f = \deg_{X_{\varepsilon_i(u)}^{\varepsilon_i(v)}} \partial_i f = \deg_{X_u^v} \mathbb{R}_i \partial_i f = \deg_{X_u^v} \mathbb{T}_i f.$$

Here the first and third equalities follow from Fact 4.1, the second equality follows from Proposition 4.6, the fourth equality follows from Theorem 4.4, and the fifth equality follows from the identity $\mathbb{R}_i \partial_i = \mathbb{T}_i$. \square

Note that $\pi_i \left(X_{\varepsilon_i(u)}^{\varepsilon_{i+1}(v)} \right) = \pi_i \left(X_{\varepsilon_i(u)}^{\varepsilon_i(v)} \right) = (\pi_i \circ \Psi_i)(X_u^v)$, and $\pi_i \circ \Psi_i|_{X_u^v}$ is an isomorphism onto its image since Ψ_i is a closed embedding and $\pi_i : X_u^v \rightarrow \pi_i \left(X_{\varepsilon_i(u)}^{\varepsilon_i(v)} \right)$ is an isomorphism. This allows us to make the following definition.

Definition 4.8. For $u \leq v$ permutations in S_{n-1} and $1 \leq i \leq n-1$, let $\Phi_i : X_{\varepsilon_i(u)}^{\varepsilon_{i+1}(v)} \rightarrow X_u^v$ denote the composite map $(\pi_i \circ \Psi_i)^{-1} \circ \pi_i$.

Theorem 4.9. For $u \leq v$ permutations in S_{n-1} and $1 \leq i \leq n-1$, the map $\Phi_i : X_{\varepsilon_i(u)}^{\varepsilon_{i+1}(v)} \rightarrow X_u^v$ realizes $X_{\varepsilon_i(u)}^{\varepsilon_{i+1}(v)}$ as the \mathbb{P}^1 -bundle $\mathbb{P}(\underline{\mathbb{C}} \oplus (\mathcal{V}_i^{(n-1)} / \mathcal{V}_{i-1}^{(n-1)}))$ on X_u^v .

Proof. The map π_i makes $X_{\varepsilon_i(v)}^{\varepsilon_{i+1}(u)}$ a \mathbb{P}^1 -bundle $\mathbb{P}(\mathcal{V}_{i+1}^{(n)} / \mathcal{V}_{i-1}^{(n)})$ over $\pi_i \left(X_{\varepsilon_i(v)}^{\varepsilon_i(v)} \right)$. Under the isomorphism $\pi_i \circ \Psi_i : X_u^v \cong \pi_i \left(X_{\varepsilon_i(u)}^{\varepsilon_i(v)} \right)$, we see that $\mathcal{V}_{i+1}^{(n)} / \mathcal{V}_{i-1}^{(n)}$ corresponds to the vector bundle

$$\Psi_i^* \pi_i^* \left(\mathcal{V}_{i+1}^{(n)} / \mathcal{V}_{i-1}^{(n)} \right) = \Psi_i^* \left(\mathcal{V}_{i+1}^{(n)} / \mathcal{V}_{i-1}^{(n)} \right) = \left(\underline{\mathbb{C}} \oplus \mathcal{V}_i^{(n-1)} \right) / \left(\{0\} \oplus \mathcal{V}_{i-1}^{(n-1)} \right) = \underline{\mathbb{C}} \oplus \left(\mathcal{V}_i^{(n-1)} / \mathcal{V}_{i-1}^{(n-1)} \right). \quad \square$$

Corollary 4.10. If X_u^v is a toric Richardson variety, then so is $X_{\varepsilon_i(u)}^{\varepsilon_{i+1}(v)}$.

Proof. The action of $(\mathbb{C}^*)^n$ on a triple (x, a, s) where $x \in X_{\varepsilon_i(u)}^{\varepsilon_i(v)}$, $a \in \mathbb{C}$ and $s \in \left(\mathcal{V}_i^{(n-1)} / \mathcal{V}_{i-1}^{(n-1)} \right)_x$ is given by

$$(t_1, \dots, t_n) \cdot (x, a, s) = ((t_2, \dots, t_n) \cdot x, t_1 \cdot a, (t_2, \dots, t_n) \cdot s).$$

For generic x , we can use t_2, \dots, t_n to move x to any point in the base and then for $a, s \neq 0$ we can use t_1 to change the fiber to any $a', s' \neq 0$. \square

5. Ω -RICHARDSON VARIETIES

We will want to iterate the results of the previous section, which will involve iteratively applying the maps $(u, v) \mapsto (\varepsilon_j(u), \varepsilon_j(v))$ and $(u, v) \mapsto (\varepsilon_j(u), \varepsilon_{j+1}(v))$ for varying j . Let $S_\infty = \bigcup S_n$ denote the group of permutations of $\{1, 2, \dots\}$ that fix all but finitely many elements, with S_n identified with the subgroup fixing all $i \geq n+1$. Then we may view $\varepsilon_i : S_\infty \rightarrow S_\infty$.

Definition 5.1. For $\Omega \in \text{RTSeq}$ we define $u(\Omega), v(\Omega) \in S_\infty$ recursively as follows. If $|\Omega| = 0$ then we set $u = v = \text{id}$. If $|\Omega| \geq 1$ then writing $\Omega = \Omega'x$, we define

$$(u(\Omega), v(\Omega)) = \begin{cases} (\varepsilon_j u(\Omega'), \varepsilon_j v(\Omega')) & \text{if } x = r_j \\ (\varepsilon_j u(\Omega'), \varepsilon_{j+1} v(\Omega')) & \text{if } x = t_j. \end{cases}$$

Appendix B collects relevant combinatorial results about $u(\Omega)$ and $v(\Omega)$. We shall recall them wherever necessary.

It is immediate that for $\Omega \in \text{RTSeq}_n$ the permutations $u(\Omega)$ and $v(\Omega)$ are in S_n . We are now ready to introduce a class of Richardson varieties $X(\Omega)$ which will play the quasisymmetric analogue of the Bott–Samelson varieties.

Definition 5.2. For $\Omega \in \text{RTSeq}_n$ we define the Ω -Richardson variety to be

$$X(\Omega) := X_{u(\Omega)}^{v(\Omega)} \subset \text{Fl}_n.$$

The dimension of this variety is given by

$$(5.1) \quad \dim X(\Omega) = \ell(v(\Omega)) - \ell(u(\Omega)) = |\Omega|_{\mathfrak{t}}.$$

By Proposition B.4, the minimal dimension $X(\Omega)$ are in bijection with the T -fixed points X_u^u of Fl_n , and the maximal dimension $X(\Omega)$, i.e. those with $|\Omega|_{\mathfrak{t}} = n - 1$, are exactly those of the form $X_u^{uc_n}$ for $u \in S_n$ satisfying $u(n) = n$ and $c_n = n1 \cdots (n - 1)$ the backwards long cycle, which are precisely the toric Richardson varieties considered in [24, 36].

Recall that for any $f \in H^\bullet(\text{Fl}_n)$, the Bott–Samelson resolution $\text{BS}(i_1, \dots, i_k) \rightarrow X^w$ has the property that $\deg_{\text{BS}(i_1, \dots, i_k)} f = \deg_{\text{BS}(i_1, \dots, i_{k-1})} \partial_{i_k} f$, and

$$\deg_{X^w} f = \deg_{\text{BS}(i_1, \dots, i_k)} f = \deg_{\text{BS}(i_1, \dots, i_{k-1})} \partial_{i_k} f = \cdots = \deg_{\text{BS}(\cdot)} \partial_{i_1} \cdots \partial_{i_k} f = \text{ev}_0 \partial_w f.$$

Theorem 5.3 (Theorem 2.2). Let $f(x_1, \dots, x_n) \in H^\bullet(\text{Fl}_n)$ and $\Omega \in \text{RTSeq}_n$. Then writing $\Omega = \Omega'/x$, the following are true.

- (1) If $x = r_i$ then $X(\Omega) \cong X(\Omega')$ and $\deg_{X(\Omega)} f = \deg_{X(\Omega')} R_i f$.
- (2) If $x = \mathfrak{t}_i$ then $X(\Omega) \rightarrow X(\Omega')$ is a \mathbb{P}^1 -bundle given as a projectivization $\mathbb{P}(\mathbb{C} \oplus \mathcal{L})$, and $\deg_{X(\Omega)} f = \deg_{X(\Omega')} T_i f$.

Furthermore

$$\deg_{X(\Omega)} f(x_1, \dots, x_n) = \Pi_\Omega^n f.$$

Proof. The result follows from Theorem 4.4 and the second result follows from Theorem 4.9 and Theorem 4.7. The final result follows from iteratively applying these first two results. \square

Recall that an algebraic variety X is called a *Bott manifold* if there is a sequence of maps (called a *Bott tower*)

$$X_m \rightarrow X_{m-1} \rightarrow \cdots \rightarrow X_0 = \{\text{pt}\}$$

where $X_i = \mathbb{P}^1(\underline{\mathbb{C}} \oplus \mathcal{L})$ for some line bundle \mathcal{L} on X_{i-1} [21, §2]. Bott manifolds are always smooth toric varieties whose moment polytopes are combinatorial cubes [40] (i.e. a polytope whose face poset is isomorphic to the face poset of a cube). We will now see that $X(\Omega)$ are in fact Bott manifolds, and are smooth torus-orbit closures under the T -action on Fl_n . The following result may also be deduced from [34] on smooth toric Bruhat interval polytopes, but it will be important for us to identify explicitly the way the Bott manifold structure is realized on $X(\Omega)$.

Theorem 5.4. The Ω -Richardson variety $X(\Omega)$ is a Bott manifold and a smooth toric Richardson variety.

Proof. Let $\Omega = x_1 \cdots x_n \in \text{RTSeq}_n$ and $\Omega_i = x_1 \cdots x_i$. Then by Theorem 5.3 there are maps

$$X(\Omega) = X(\Omega_n) \rightarrow X(\Omega_{n-1}) \rightarrow \cdots \rightarrow X(\Omega_0) = \{\text{pt}\}$$

such that each map is either an isomorphism or realizes $X(\Omega_i) = \mathbb{P}(\underline{\mathbb{C}} \oplus \mathcal{L})$ for some line bundle \mathcal{L} on $X(\Omega_{i-1})$, which shows that it is a Bott manifold. Furthermore, iteratively applying Corollary 4.5 and Corollary 4.10 to this sequence shows that $X(\Omega)$ is in fact a smooth torus-orbit under the T -action on Fl_n . \square

We now show that the sub-torus-orbit closures in $X(\Omega)$ are themselves of the form $X(\Omega')$.

Theorem 5.5. The torus-orbit closures contained inside $X(\Omega)$ are exactly $X(\Omega')$ with Ω' obtained by replacing some subset of the t_i appearing in Ω with either r_i or r_{i+1} .

Proof. Because $X(\Omega)$ is toric, by [56] the sub-torus-orbit closures are the Richardson varieties contained in $X(\Omega)$. These are the $X(\Omega')$ as described in Corollary B.7. Each of these is smooth since $X(\Omega)$ is smooth, and the moment polytope of $X(\Omega')$ is a face of the moment polytope of $X(\Omega)$ which is therefore a combinatorial cube, so the result follows. \square

We now give a more concrete interpretation of the Ω -Richardson varieties by describing a parametrization of the open torus orbit in $X(\Omega)$, which for the maximal dimension $X(\Omega)$ recovers the description from [36, Section 4.2].

Definition 5.6. For $\Omega \in \text{RTSeq}_n$, let $M(\Omega)$ be an $n \times n$ matrix filled with $0, 1, *$ defined recursively as follows. Set $M(\emptyset)$ to be the unique 0×0 matrix, and for $M(\Omega') = \begin{bmatrix} v_1 & \cdots & v_{n-1} \end{bmatrix}$ let

$$M(\Omega' r_i) = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ v_1 & \cdots & v_{i-1} & \mathbf{0} & v_i & \cdots & v_{n-1} \end{bmatrix},$$

$$M(\Omega' t_i) = \begin{bmatrix} 0 & \cdots & * & 1 & 0 & \cdots & 0 \\ v_1 & \cdots & v_i & \mathbf{0} & v_{i+1} & \cdots & v_{n-1} \end{bmatrix}.$$

Example 5.7. The matrices for $r_1, r_1 t_1, r_1 t_1 t_1, r_1 t_1 t_1 r_2$ and $r_1 t_1 t_1 r_2 t_4$ are

$$\begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} * & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} * & 1 & 0 \\ * & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ * & 0 & 1 & 0 \\ * & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & * & 1 \\ 0 & 1 & 0 & 0 & 0 \\ * & 0 & 1 & 0 & 0 \\ * & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Theorem 5.8. The open torus in $X(\Omega)$ is parametrized by replacing the $|\Omega|_i$ -many $*$ that appear in $M(\Omega)$ with elements of \mathbb{C}^* .

Proof. Assume by induction that the open torus in $X(\Omega')$ is given by this description, and let $A = \begin{bmatrix} w_1 & \cdots & w_{n-1} \end{bmatrix}$ be a matrix representing a point in this open torus-orbit. Then

$$\begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ w_1 & \cdots & w_{i-1} & \mathbf{0} & w_i & w_{i+1} & \cdots & w_{n-1} \end{bmatrix}$$

represents $\Psi_i(A)$ so applying this for all points in the open torus of $X(\Omega')$ shows that the new open torus in $X(\Omega'_i)$ is $M(\Omega'_i)$ as desired. Applying $\pi_i^{-1}\pi_i$ to $\Psi_i(A)$ corresponds to replacing the i 'th and $(i + 1)$ 'st column of this matrix with two linear combinations. In GL_n/B two matrices are the same if we can obtain one from the other by scaling columns and adding multiples of one column to a future column. Doing this we can put every such matrix either into the form

$$\begin{bmatrix} 0 & \cdots & 0 & * & 1 & 0 & \cdots & 0 \\ w_1 & \cdots & w_{i-1} & w_i & \mathbf{0} & w_{i+1} & \cdots & w_{n-1} \end{bmatrix} \text{ or } \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ w_1 & \cdots & w_{i-1} & \mathbf{0} & w_i & w_{i+1} & \cdots & w_{n-1} \end{bmatrix}$$

with $* \in \mathbb{C}$. The first matrix with $* = 0$ gives a point in the boundary torus-orbit closure $X(\Omega'_{i+1})$, the second matrix gives a point in the boundary torus-orbit closure $X(\Omega'_i)$, so the remaining points where $* \in \mathbb{C}^*$ in the first matrix lie in the open torus-orbit of $X(\Omega'_i)$. \square

From $M(\Omega)$ one can read off the sequence in $\Omega \in \text{RTSeq}_n$ by reversing the recursive process used to build $M(\Omega)$. The following non-recursive characterization of the matrices $M(\Omega)$ is straightforward to show.

Theorem 5.9. A square matrix filled with 0, 1, * is of the form $M(\Omega)$ if and only if

- The 1's form a permutation matrix
- There is at most one * per row.
- Every * appears above the 1 in its column and to the left of the 1 in its row.
- For every 0 between * and 1 in a row, all the entries below this 0 are also 0.

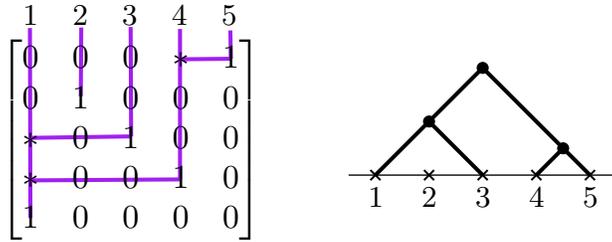


FIGURE 5. Recovering $\widehat{F}(\Omega)$ from $M(\Omega)$

We conclude by describing how to directly recover the nested forest $\widehat{F}(\Omega)$ from $M(\Omega)$. The forest $\widehat{F}(\Omega)$ has internal nodes the nonzero entries of $M(\Omega)$ and leaves $\{1, \dots, n\}$. For each row containing an * and a 1, we do the following. Connect the 1 and *, and then connect each of these

to the nearest $*$ above them in the same column provided it exists; otherwise connect to the leaf i where i is the column number. This is illustrated in Figure 5 using the last matrix in Example 5.7.

6. THE QUASISYMMETRIC SCHUBERT CYCLES $X(\widehat{F})$ AND BOTT MANIFOLD STRUCTURES

We now define the quasisymmetric Schubert cycles associated to $\widehat{F} \in \widehat{\text{For}}_n$. In order to define them, we will need to prove the following theorem.

Theorem 6.1. If $\widehat{F} \in \widehat{\text{For}}_n$ and $\Omega, \Omega' \in \text{Trim}(\widehat{F})$, then

$$u(\Omega)^{-1}X(\Omega) = u(\Omega')^{-1}X(\Omega).$$

Given this theorem, the following is well-defined.

Definition 6.2. We define the *quasisymmetric Schubert cycle* associated to $\widehat{F} \in \widehat{\text{For}}_n$ to be $X(\widehat{F}) = u(\Omega)^{-1}X(\Omega)$ for any $\Omega \in \text{Trim}(\widehat{F})$.

Proof of Theorem 6.1. First, note that the set of row vectors in $M(\Omega)$ is determined by \widehat{F} . Indeed, there is a row with exactly one 1 in position i exactly when there is a tree in \widehat{F} whose leftmost leaf is i , and there is a row with a $*$ in position i and a 1 in position j if there is an internal node v with the property that the leftmost leaf descendent of its left child v_L is i and the leftmost leaf descendent of its right child v_R is j .

By induction, $v(\Omega)$ is the permutation matrix obtained by selecting the rightmost entry from each row of $M(\Omega)$. Therefore $v(\Omega)^{-1}X(\Omega)$ is obtained by permuting the rows in the matrix model in the unique way so that the rightmost entries lie on the main diagonal, which shows $v(\Omega)^{-1}X(\Omega) = v(\Omega')^{-1}X(\Omega')$.

By induction, $u(\Omega)^{-1}v(\Omega)$ is the product of the backwards cycles on the support sets of the trees in \widehat{F} . Therefore $u(\Omega)^{-1}v(\Omega) = u(\Omega')^{-1}v(\Omega')$, and so $u(\Omega)^{-1}X(\Omega) = u(\Omega')^{-1}X(\Omega')$ as desired. \square

The following is analogous to the fact that for $f \in H^\bullet(\text{Fl}_n)$ we have $\deg_{\mathcal{G}_{X^w}} f = \text{ev}_0 \partial_w f$.

Corollary 6.3. For $\widehat{F} \in \widehat{\text{For}}_n$, we have

$$\deg_{X(\widehat{F})} f = \Pi_{\widehat{F}}^n f.$$

Proof. If $\Omega \in \text{Trim}(\widehat{F})$, then for any $f \in H^\bullet(\text{Fl}_n)$ we have $\deg_{X(\Omega)} f = \Pi_{\Omega}^n f = \Pi_{\widehat{F}}^n f$. \square

Fix some $\Omega = x_1 \cdots x_n \in \text{Trim}(\widehat{F})$, and let $\Omega_i = x_1 \cdots x_i$. Then under the isomorphism $X(\Omega) \rightarrow X(\widehat{F})$, if $f \in H^\bullet(\text{Fl}_n)$ we can compute using the Bott manifold structure on $X(\Omega)$

$$\deg_{X(\widehat{F})} f = \deg_{X(\Omega_n)} f = \deg_{X(\Omega_{n-1})} \text{up}(x_n) f = \cdots = \deg_{X(\Omega_0)} \text{up}(x_1) \cdots \text{up}(x_n) f = \Pi_{\widehat{F}}^n f.$$

In this way, the isomorphisms $X(\Omega) \rightarrow X(\widehat{F})$ play an analogous role as the Bott–Samelson resolutions play for Schubert varieties outlined in Section 2.

The fundamental classes of the quasisymmetric Schubert cycles $X(\widehat{F})$ satisfy linear relations among them, but by passing to the subset indexed by indexed forests $\text{For}_n \subset \widehat{\text{For}}_n$ we obtain a duality with the family of forest polynomials.

Theorem 6.4. For $F \in \text{For}_n \subset \widehat{\text{For}}_n$ and $f \in \text{Pol}_n$ we have $\deg_{\mathfrak{g}_{X(F)}} f = \text{ev}_0 \mathbb{T}_F f$. In particular,

$$\deg_{X(F)} \mathfrak{P}_G = \delta_{F,G},$$

i.e. the forest polynomials $\{\mathfrak{P}_G \mid G \in \text{For}_n\} \subset H^\bullet(\text{Fl}_n)$ are Kronecker dual to the fundamental classes $\{[X(F)] \mid F \in \text{For}_n\} \subset H^\bullet(\text{Fl}_n)$.

Proof. Because $F \in \text{For}_n$, we can express $F = \widehat{F}(\Omega)$ for $\Omega = r_1^{n-\sum c_i} t_1^{c_1} \cdots t_k^{c_k}$ where $F = 1^{c_1} 2^{c_2} \cdots k^{c_k}$. This means that $\text{ev}_0 \Pi_\Omega = \text{ev}_0 r_1^{n-\sum c_i} t_1^{c_1} \cdots t_k^{c_k} = \text{ev}_0 \mathbb{T}_F$. Therefore

$$\deg_{X(F)} \mathfrak{P}_G = \text{ev}_0 \mathbb{T}_F \mathfrak{P}_G = \delta_{F,G}. \quad \square$$

We now show that the remaining fundamental classes are nonnegative linear combinations of the fundamental classes associated to the non-nested forests.

Theorem 6.5. For $\widehat{H} \in \widehat{\text{For}}_n$, we have

$$[X(\widehat{H})] = \sum_{F \in \text{For}_n} a_F [X(F)] \in H^\bullet(\text{Fl}_n)$$

for nonnegative integers a_F .

Proof. Let $\Omega \in \text{RTSeq}_n$ have $\widehat{F}(\Omega) = \widehat{H}$ so that $[X(\Omega)] = [X(\widehat{H})]$. Note that

$$\mathbb{T}_i \mathbb{R}_{i+1} = \mathbb{R}_i \mathbb{T}_{i+1} + \mathbb{R}_{i+1} \mathbb{T}_i.$$

If $t_i r_{i+1}$ in Ω and Ω_1, Ω_2 are the sequences where these two letters are replaced with $r_i t_{i+1}$ and $r_{i+1} t_i$ respectively, then $\Omega_1, \Omega_2 \in \text{RTSeq}_n$ and $\Pi_\Omega^n = \Pi_{\Omega_1}^n + \Pi_{\Omega_2}^n$. This implies $[X(\Omega)] = [X(\Omega_1)] + [X(\Omega_2)]$ by Poincaré duality. Applying this repeatedly allows us to move all r_i to the left of all t_j to express

$$[X(\Omega)] = \sum a_{\Omega'} [X(\Omega')]$$

where each $\Omega' = r_{i_1} \cdots r_{i_k} t_{j_1} \cdots t_{j_\ell}$, and so $\widehat{F}(\Omega') \in \text{For}_n$. □

7. THE Ω -FLAG MANIFOLD

We are now ready to define our quasisymmetric analogue of the flag variety. Rather than take the union of the $X(\widehat{F})$, which turns out to be a much more combinatorially opaque object as a toric complex, we instead take the union of the $X(\Omega)$ with $\Omega \in \text{RTSeq}_n$, each of which as we have already shown is isomorphic to some $X(\widehat{F})$.

Definition 7.1. We define the Ω -flag variety to be

$$\text{HHMP}_n = \bigcup_{\Omega \in \text{RTSeq}_n} X(\Omega) \subset \text{Fl}_n.$$

This also has a natural intrinsic recursive characterization.

Theorem 7.2. We have $\text{HHMP}_1 = \text{Fl}_1$, and for $n > 1$ we have HHMP_n is the set of all $\mathcal{V} \in \text{HHMP}_n$ such that there is $\mathcal{W} \in \text{HHMP}_{n-1}$ and $i \in \{1, \dots, n-1\}$ such that $\mathcal{V}_j = \{0\} \oplus \mathcal{W}_j$ for $j < i$ and $\mathcal{V}_j = \mathbb{C} \oplus \mathcal{W}_{j-1}$ for $j > i$.

Proof. This recursive description claims that $\text{HHMP}_n = \Phi_1(\text{HHMP}_{n-1}) \cup \dots \cup \Phi_{n-1}(\text{HHMP}_{n-1})$ where Φ_i is the operator from Definition 4.8. By induction, it remains to prove

$$\bigcup_{1 \leq i_j \leq j} \Phi_{i_{n-1}} \dots \Phi_{i_1}(\text{Fl}_1) = \text{HHMP}_n.$$

Because $\Psi_1(\text{Fl}_0) = \text{Fl}_1$, by the recursive construction of the $X(\Omega)$, we see that the left hand side is the union of all $X(\Omega)$ with $\Omega \in \text{RTSeq}_n$ of the form $r_1 t_{i_1} \dots t_{i_{n-1}}$. But by Theorem 5.5, every other $X(\Omega')$ with $\Omega' \in \text{RTSeq}_n$ is a subvariety of the $X(\Omega)$ with $\Omega \in \text{RTSeq}_n$ obtained by replacing each $x_i = r_j$ beyond the first one with either t_j or t_{j-1} so that the index lies in $\{1, \dots, i\}$, so we conclude. \square

Example 7.3. $\text{HHMP}_1 = \text{Fl}_1$ is a point, and $\text{HHMP}_2 = \text{Fl}_2$.

For $\mathcal{V} = (0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \mathbb{C}^3) \in \text{Fl}_3$, we have $\mathcal{V} \in \text{HHMP}_3$ if \mathcal{V}_2 contains $\mathbb{C}e_1$ or if \mathcal{V}_1 is contained in $\mathbb{C}e_2 \oplus \mathbb{C}e_3$. Note that both can be true, and this occurs precisely for flags with $\mathcal{V}_1 \in \mathbb{C}e_2 \oplus \mathbb{C}e_3$ and $\mathcal{V}_2 = \mathbb{C}e_1 \oplus \mathcal{V}_1$.

7.1. A cubical complex. We now describe the structure of the toric complex HHMP_n . In [24] it was shown that the images of the top-dimensional $X(\Omega)$ appearing in HHMP_n under the moment map $\text{Fl}_n \rightarrow \mathbb{R}^n / \langle (1, \dots, 1) \rangle$ give a subdivision of the permutahedron

$$\text{Perm}_n := \text{conv}\{w \cdot (n, \dots, 1) \mid w \in S_n\}$$

into combinatorial cubes.

As a consequence of the RTSeq_n -encoding for the combinatorial cubes appearing in this subdivision, this subdivision is combinatorially isomorphic to a unit cube subdivision of a particular cuboid (a fact that does not appear to have been previously observed).

Definition 7.4. For $\Omega = x_1 \dots x_n \in \text{RTSeq}_n$, let \square_Ω be defined as the box $Y_2 \times \dots \times Y_n$ where

$$Y_i = \begin{cases} [j, j+1] & x_i = t_j \\ \{j\} & x_i = r_j. \end{cases}$$

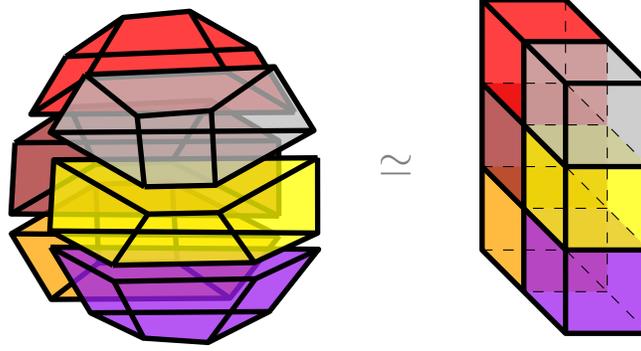


FIGURE 6. The HHMP subdivision and the unit cube subdivision for $n = 4$

Theorem 7.5. There is a face preserving bijection from the image of HHMP_n under the moment map to the unit cube subdivision of the $(n - 1)$ -dimensional cuboid $[1, 2] \times [1, 3] \times \cdots \times [1, n]$, mapping the polytope $P_{u(\Omega)}^{\nu(\Omega)}$ to the box \square_{Ω} .

Proof. For a toric variety X , the faces of the moment polytope are the images of the sub-torus-orbit closures of X . For any $X(\Omega)$ with $\Omega \in \text{RTSeq}_n$, these sub-torus-orbit closures were identified in Theorem 5.5 to be the $X(\Omega')$ with Ω' obtained by replacing some subset of the t_j with either r_j or r_{j+1} . By construction this happens precisely when $\square_{\Omega'} \subset \square_{\Omega}$, and we conclude. \square

See Figure 6 demonstrating the HHMP-subdivision and the unit cube subdivision for $n = 4$.

7.2. Quasisymmetric coinvariants in $H^*(\text{HHMP}_n)$. We now prove the following.

Theorem 7.6. [Theorem A] The image of $H^*(\text{Fl}_n)$ in $H^*(\text{HHMP}_n)$ under the natural restriction map is isomorphic to QSCoinv_n .

First, we show that the normalization $\widetilde{\text{HHMP}}_n$ of HHMP_n knows QSCoinv_n . Because $X(\Omega)$ are smooth, $\widetilde{\text{HHMP}}_n$ is the disjoint union

$$\bigsqcup_{\substack{\Omega \in \text{RTSeq}_n \\ |\Omega|_t = n-1}} X(\Omega).$$

Proposition 7.7. The image of $H^*(\text{Fl}_n)$ in $H^*(\widetilde{\text{HHMP}}_n)$ is isomorphic to QSCoinv_n .

Proof. The statement is equivalent to showing that QSym_n^+ is the kernel of the map

$$\mathbb{Z}[x_1, \dots, x_n] \rightarrow \prod_{\substack{\Omega \in \text{RTSeq}_n \\ |\Omega|_t = n-1}} H^*(X(\Omega)).$$

By the Minkowski weight description of the cohomology ring of smooth toric varieties [18], a cohomology class of degree k vanishes if and only if its restriction to every sub-torus-orbit closure

is 0. Indeed, [18, Proposition 1.1] tells us that this cohomology ring is generated by the sub-torus-orbit closures. Since the sub-torus-orbit closures occurring in some $X(\Omega)$ are of the form $X(\Omega')$ with $\Omega' \in \text{RTSeq}_n$ and $\deg_{X(\Omega')} f = \Pi_{\Omega'} f$ for $f \in \mathbb{Z}[x_1, \dots, x_n]$, we have reduced to showing that

$$f \in \text{QSym}_n^+ \iff \Pi_{\Omega'}^n f = 0 \text{ for all } \Omega' \in \text{RTSeq}_n.$$

The forward direction follows because, as mentioned in Section 2, the operator $\Pi_{\Omega'}$ descends to QSCoinv_n . For the reverse direction we note that $\Pi_{\widehat{F}} f = 0$ for all $\widehat{F} \in \widehat{\text{For}}_n$, so we apply this fact to the subset of indexed forests $\text{For}_n \subset \widehat{\text{For}}_n$. By Corollary 3.23 we have $\text{ev}_0 \mathbb{T}_F = \Pi_F^n$ for $F \in \text{For}_n$ and we conclude. \square

To show that $H^\bullet(\text{HHMP}_n)$ contains the quasisymmetric coinvariants, we will show that

$$H^\bullet(\text{HHMP}_n) \rightarrow H^\bullet(\widetilde{\text{HHMP}}_n)$$

is an injection. This is a topological statement, and we show this using a spectral sequence.

Definition 7.8. Let F_n be the topological space obtained by gluing n copies of the sphere \mathbb{P}^1 in a line. Formally, if we define points s_i for $1 \leq i \leq n-1$ belonging to the i 'th sphere and t_i for $1 \leq i \leq n-1$ belonging to the $(i+1)$ 'st sphere, with $s_{i+1} \neq t_i$ for $2 \leq i \leq n-1$, then we define F to be the quotient

$$F = \underbrace{\mathbb{P}^1 \sqcup \dots \sqcup \mathbb{P}^1}_n / \sim$$

by the equivalence relation \sim identifying s_i with t_i .

Proposition 7.9. Suppose that X is a (not necessarily reducible) algebraic variety with trivial first fundamental group $\pi_1(X)$ and all even dimensional cohomology groups. Suppose $Y_1, \dots, Y_k \rightarrow X$ are \mathbb{P}^1 -bundles with distinguished sections $s_i : X \rightarrow Y_i$ for $i = 1, \dots, k-1$ and $t_i : X \rightarrow Y_{k+1}$ for $i = 1, \dots, k-1$ such that s_{i+1} and t_i are disjoint for $2 \leq i \leq n-1$. Let Y be the space obtained by identifying the sections s_i and t_i for each $1 \leq i \leq n-1$. Then the pullback map

$$H^\bullet(Y) \rightarrow H^\bullet(Y_1 \sqcup \dots \sqcup Y_k)$$

is an injection.

Proof. $Y \rightarrow X$ is a fiber bundle with fiber F_k , which has $H^0(F) = \mathbb{Z}$ and $H^2(F) = \mathbb{Z}^k$. Because $\pi_1(X) = 0$ there are no nontrivial local systems on X , so $H^i(X; H^j(F)) = 0$ if i, j are not both even, so the first page of the Serre spectral sequence degenerates and we have as abelian groups

$$\bigoplus_{i,j} H^i(X; H^j(F)) \cong H^\bullet(Y).$$

The same reasoning for $Y_1 \sqcup \dots \sqcup Y_k \rightarrow X$ shows that

$$\bigoplus_{i,j} H^i(X; H^j(S^2 \sqcup \dots \sqcup S^2)) \cong H^\bullet(Y).$$

Finally, the map $H^\bullet(Y) \rightarrow H^\bullet(Y_1 \sqcup \dots \sqcup Y_k)$ is induced by the natural maps $H^i(X; H^j(F)) \rightarrow H^i(X; H^j(\mathbb{P}^1 \sqcup \dots \sqcup \mathbb{P}^1))$. Because $H^j(F)$ and $H^j(\mathbb{P}^1 \sqcup \dots \sqcup \mathbb{P}^1)$ are free abelian groups, this map is given by the composite

$$H^i(X; H^j(F)) \cong H^i(X) \otimes H^j(F) \rightarrow H^i(X) \otimes H^j(S^2 \sqcup \dots \sqcup S^2) \cong H^i(X; H^j(S^2 \sqcup \dots \sqcup S^2)),$$

and the middle map is an injection because $H^j(F) \hookrightarrow H^j(S^2 \sqcup \dots \sqcup S^2)$ is an injection of free abelian groups. \square

We are now ready to prove Theorem 7.6.

Proof of Theorem 7.6. By Proposition 7.7, it suffices to show that the pullback map $H^\bullet(\text{HHMP}_n) \rightarrow H^\bullet(\widetilde{\text{HHMP}}_n)$ is an injection. First we show that $\pi_1(\text{HHMP}_n)$ is trivial by induction. For $1 \leq i \leq n-1$ let

$$Y_i = \bigcup_{\Omega = x_1 \cdots x_n \text{ with } x_n \in \{r_i, t_i, r_{i+1}\}} X(\Omega),$$

and note that this is a \mathbb{P}^1 -bundle over HHMP_{n-1} by the map $(\Psi_i \circ \pi_i)^{-1} \pi_i$, which has distinguished sections coming from Ψ_i and Ψ_{i+1} . This realizes HHMP_n as an F_{n-1} -bundle over HHMP_{n-1} . As $\pi_1(F_{n-1})$ is trivial we have by the long exact sequence on homotopy groups that $\pi_1(\text{HHMP}_n) = \pi_1(\text{HHMP}_{n-1})$, which is trivial by the inductive hypothesis.

Now we prove that $H^\bullet(\text{HHMP}_n) \rightarrow H^\bullet(\widetilde{\text{HHMP}}_n)$ is an injection by induction. By the inductive hypothesis we have an injection $H^\bullet(\text{HHMP}_{n-1}) \hookrightarrow H^\bullet(\widetilde{\text{HHMP}}_{n-1})$, so because HHMP_{n-1} is a disjoint union of smooth projective varieties we have that the even-dimensional cohomology of HHMP_{n-1} vanishes.

The variety HHMP_n arises as the Y from Proposition 7.9 applied to the \mathbb{P}^1 -bundles $Y_1, \dots, Y_{n-1} \rightarrow Y$, so we conclude that $H^\bullet(\text{HHMP}_n) \rightarrow H^\bullet(Y_1 \sqcup \dots \sqcup Y_{n-1})$ is an injection. Let

$$\widetilde{Y}_i = \bigsqcup_{\Omega = x_1 \cdots x_n \text{ with } x_n \in \{r_i, t_i, r_{i+1}\}} X(\Omega).$$

Note that \widetilde{Y}_i is the \mathbb{P}^1 -bundle $Y_i \rightarrow \text{HHMP}_{n-1}$ pulled back along the normalization $\widetilde{\text{HHMP}}_{n-1} \rightarrow \text{HHMP}_{n-1}$. By induction $H^\bullet(\text{HHMP}) \hookrightarrow H^\bullet(\widetilde{\text{HHMP}}_{n-1})$ is an injection on cohomology, so by the projective bundle formula we have the pullback map $H^\bullet(Y_i) \rightarrow H^\bullet(\widetilde{Y}_i)$ is an injection. Therefore we have the composite pullback map

$$H^\bullet(\text{HHMP}_n) \hookrightarrow H^\bullet(Y_1 \sqcup \dots \sqcup Y_n) \hookrightarrow H^\bullet(\widetilde{Y}_1 \sqcup \dots \sqcup \widetilde{Y}_n) \hookrightarrow H^\bullet(\widetilde{\text{HHMP}}_n),$$

is an injection as desired. \square

8. APPLICATIONS TO GENERALIZED LITTLEWOOD–RICHARDSON COEFFICIENTS

In this section we show the combinatorial aspects of Theorem 2.2 and its application to generalized LR coefficients $c_{u,w}^v$ by recasting various coefficients of interest in algebraic combinatorics as generalized LR coefficients. Recall that the $c_{u,w}^v$ are defined by the equation

$$\mathfrak{S}_u \mathfrak{S}_w = \sum_v c_{u,w}^v \mathfrak{S}_v,$$

or equivalently as

$$c_{u,w}^v = \text{ev}_0 \partial_v(\mathfrak{S}_u \mathfrak{S}_w).$$

We have $\text{ev}_0 \partial_v(\mathfrak{S}_u f) = 0$ if $u \not\leq v$ in the Bruhat order on S_∞ , so in particular in this case $c_{u,w}^v = 0$ for all w . Coming up with a combinatorial rule for the $c_{u,w}^v$ is a major open problem in combinatorial algebraic geometry; see [19, 25, 26, 32, 29, 30, 51] for various results in special cases.

If we want to show combinatorially for fixed $u \leq v$ that $c_{u,w}^v$ is nonnegative for all w , then this is equivalent to showing combinatorially that the operator

$$f \mapsto \text{ev}_0 \partial_v(\mathfrak{S}_u f)$$

is nonnegative on Schubert polynomials. We will show for $(u, v) = (u(\Omega), v(\Omega))$ for $\Omega \in \text{RTSeq}$ that we can combinatorially realize this nonnegativity by a recursive procedure, and then give certain pairs of (u, v) where $c_{u,w}^v$ computes interesting combinatorial invariants of \mathfrak{S}_w . As we shall need them we record here the *twisted Leibniz relations* satisfied by the divided difference operators:

$$\partial_i(fg) = f\partial_i(g) + \partial_i(f)(s_i \cdot g)$$

for all $f, g \in \text{Pol}$, so in particular if f is symmetric in variables $\{x_i, x_{i+1}\}$ then $\partial_i(fg) = f\partial_i(g)$.

The results in this section rest on the following proposition. We shall give two proofs, one geometric and the other combinatorial. The latter is to emphasize the fact that the generalized LR coefficient computation of this section can be made entirely combinatorial.

Proposition 8.1. For $u, v \in S_\infty$ we have

$$(8.1) \quad \text{ev}_0 \partial_{\varepsilon_j(v)}(\mathfrak{S}_{\varepsilon_j(u)} f) = \text{ev}_0 \partial_v(\mathfrak{S}_u R_j f),$$

$$(8.2) \quad \text{ev}_0 \partial_{\varepsilon_{i+1}(v)}(\mathfrak{S}_{\varepsilon_i(u)} f) = \text{ev}_0 \partial_v(\mathfrak{S}_u T_i f).$$

Furthermore, for $\Omega \in \text{RTSeq}$ we have

$$(8.3) \quad \text{ev}_0 \partial_{v(\Omega)}(\mathfrak{S}_{u(\Omega)} f) = \text{ev}_0 \Pi_\Omega f.$$

Geometric Proof of Proposition 8.1. Because $\text{ev}_0 \partial_v(\mathfrak{S}_u f) = \text{deg}_{X_u^v} f$, these results follow directly from Theorem 5.3. \square

Combinatorial Proof of Proposition 8.1. First, we establish (8.1). For $j = 1$ we first note that if $v = s_{i_1} \cdots s_{i_k}$ is a reduced word then $\varepsilon_1(v) = s_{i_{1+1}} \cdots s_{i_{k+1}}$ is a reduced word for $\varepsilon_1(v)$. Therefore because $\partial_i R_1 = R_1 \partial_{i+1}$, we have $\partial_v R_1 = R_1 \partial_{\varepsilon_1(v)}$. Furthermore, because \mathfrak{S}_u is characterized by the property that $\text{ev}_0 \partial_v \mathfrak{S}_u = \delta_{u,v}$ and $\text{ev}_0 \partial_v R_1 \mathfrak{S}_{\varepsilon_1(u)} = \text{ev}_0 R_1 \partial_{\varepsilon_1(v)} \mathfrak{S}_{\varepsilon_1(u)} = \delta_{\varepsilon_1(u), \varepsilon_1(v)} = \delta_{u,v}$ we conclude that $R_1 \mathfrak{S}_{\varepsilon_1(u)} = \mathfrak{S}_u$. Therefore,

$$\text{ev}_0 \partial_v (\mathfrak{S}_u (R_1 f)) = \text{ev}_0 \partial_v (R_1 (\mathfrak{S}_{\varepsilon_1(u)} f)) = \text{ev}_0 R_1 \partial_{\varepsilon_1(v)} (\mathfrak{S}_{\varepsilon_1(u)} f) = \text{ev}_0 \partial_{\varepsilon_1(v)} (\mathfrak{S}_{\varepsilon_1(u)} f).$$

Now, assume the result for j , we will show the result for $j+1 \geq 2$. Since $\varepsilon_{j+1}(v) = \varepsilon_j(v) s_j$ we know that $\partial_{\varepsilon_{j+1}(v)} = \partial_{\varepsilon_j(v)} \partial_j$. Thus we compute first that

$$\text{ev}_0 \partial_{\varepsilon_{j+1}(v)} (\mathfrak{S}_{\varepsilon_{j+1}(u)} f) = \text{ev}_0 \partial_{\varepsilon_j(v)} \partial_j (\mathfrak{S}_{\varepsilon_{j+1}(u)} f).$$

Now applying the twisted Leibniz rule to the right-hand side we get

$$\text{ev}_0 \partial_{\varepsilon_{j+1}(v)} (\mathfrak{S}_{\varepsilon_{j+1}(u)} f) = \text{ev}_0 \partial_{\varepsilon_j(v)} ((\partial_j \mathfrak{S}_{\varepsilon_{j+1}(u)}) s_j f) + \text{ev}_0 \partial_{\varepsilon_j(v)} (\mathfrak{S}_{\varepsilon_{j+1}(u)} \partial_j f).$$

Now consider the second term on the right-hand side. Since $\varepsilon_j(v) \not\geq \varepsilon_{j+1}(u)$ (as $j = (\varepsilon_j(v))^{-1}(1) < (\varepsilon_{j+1}(u))^{-1}(1) = j+1$) we have the vanishing

$$\text{ev}_0 \partial_{\varepsilon_j(v)} (\mathfrak{S}_{\varepsilon_{j+1}(u)} \partial_j f) = 0.$$

For the first term, using $\varepsilon_{j+1}(u) = \varepsilon_j(u) s_j$, we know that $\partial_j \mathfrak{S}_{\varepsilon_{j+1}(u)} = \mathfrak{S}_{\varepsilon_j(u)}$. Thus it remains to compute $\text{ev}_0 \partial_{\varepsilon_j(v)} (\mathfrak{S}_{\varepsilon_j(u)} s_j f)$. By our inductive hypothesis we have

$$\text{ev}_0 \partial_{\varepsilon_j(v)} (\mathfrak{S}_{\varepsilon_j(u)} s_j f) = \text{ev}_0 \partial_v (\mathfrak{S}_u R_j s_j f) = \text{ev}_0 \partial_v (\mathfrak{S}_u R_{j+1} f),$$

as desired.

Now we establish (8.2). Like before we have $\partial_{\varepsilon_{i+1}(v)} = \partial_{\varepsilon_i(v)} \partial_i$ and so we get

$$\text{ev}_0 \partial_{\varepsilon_{i+1}(v)} (\mathfrak{S}_{\varepsilon_i(u)} f) = \text{ev}_0 \partial_{\varepsilon_i(v)} \partial_i (\mathfrak{S}_{\varepsilon_i(u)} f)$$

Since $\mathfrak{S}_{\varepsilon_i(u)}$ is symmetric in variables $\{x_i, x_{i+1}\}$, the twisted Leibniz rule simplifies to give

$$\text{ev}_0 \partial_{\varepsilon_{i+1}(v)} (\mathfrak{S}_{\varepsilon_i(u)} f) = \text{ev}_0 \partial_{\varepsilon_i(v)} (\mathfrak{S}_{\varepsilon_i(u)} \partial_i f) = \text{ev}_0 \partial_v (\mathfrak{S}_u R_i \partial_i f) = \text{ev}_0 \partial_v (\mathfrak{S}_u T_i f),$$

where the last two equalities use (8.1) and $T_i = R_i \partial_i$ respectively.

Finally, iterating (8.1) and (8.2) yields (8.3), thereby concluding the proof. \square

Theorem 8.2. For $\Omega \in \text{RTSeq}$ we have

$$c_{u(\Omega), w}^{v(\Omega)} = \text{ev}_0 \Pi_{\Omega} \mathfrak{S}_w$$

Proof. This follows from Proposition 8.1 as $c_{u, w}^{v, w} = \text{ev}_0 \partial_v (\mathfrak{S}_u \mathfrak{S}_w)$. \square

We now show that these LR coefficients are combinatorially nonnegative with an explicit combinatorial rule for computing them. To do this, we remind the reader of Sottile’s Pieri rule for the Schubert polynomial expansion of $x_1 \cdots x_k \mathfrak{S}_w$ [53, Theorem I] phrased in terms of the k -Bruhat order introduced by Bergeron–Sottile [6, Section 3].

Fix k a positive integer. We say that u is covered by v in k -Bruhat order if $v = us_{ij}$ if $i \leq k < j$ and $\ell(v) = \ell(u) + 1$, in which case we write $u \prec_{v(j)} v$. Here s_{ij} denotes the transposition swapping i and j . A saturated chain $u \prec_{i_1} u_1 \prec \cdots \prec_{i_p} u_p = v$ in k -Bruhat order is said to be decreasing if $i_1 > \cdots > i_p$. If a decreasing chain from u to v exists then it is unique – we write $u \xrightarrow{c_k} v$ and write c_k as a shorthand for $c_{k,k}$. Sottile [53] established that

$$(8.4) \quad c_{s_1 \cdots s_k, w}^v = \begin{cases} 1 & w \xrightarrow{c_k} v \\ 0 & \text{otherwise.} \end{cases}$$

Example 8.3. Let $w = 15243$ and $k = 3$, which means $s_1 s_2 s_3 = 2341$ in one line notation. One can check that

$$\mathfrak{S}_{2341} \mathfrak{S}_{15243} = \mathfrak{S}_{263415} + \mathfrak{S}_{264135},$$

and the terms on the right-hand come from the following decreasing chains in 3-Bruhat order: $152436 \prec_6 162435 \prec_4 164235 \prec_2 264135$ and $152436 \prec_6 162435 \prec_3 163425 \prec_2 263415$.

Equation 8.4 was used [6] in the computation of the Schubert expansion of $R_i \mathfrak{S}_w$ – we rederive this result emphasizing that by following the combinatorial proof above one could avoid all geometric considerations. From this result we also compute the Schubert expansion of $T_i \mathfrak{S}_w$.

Proposition 8.4. Given a positive integer i and $w \in S_\infty$ we have

$$(8.5) \quad R_i \mathfrak{S}_w = \sum_{w \xrightarrow{c_i} \varepsilon_i(v)} \mathfrak{S}_v$$

$$(8.6) \quad T_i \mathfrak{S}_w = \delta_{i \in \text{Des}(w)} \sum_{ws_i \xrightarrow{c_i} \varepsilon_i(v)} \mathfrak{S}_v$$

Proof. Noting that $\mathfrak{S}_{\text{id}} = 1$ and $\varepsilon_i(\text{id}) = s_1 \cdots s_{i-1}$, we compute the coefficient of \mathfrak{S}_v in the Schubert expansion of $R_i \mathfrak{S}_w$ as

$$\text{ev}_0 \partial_v R_i \mathfrak{S}_w = \text{ev}_0 \partial_{\varepsilon_i(v)} (\mathfrak{S}_{\varepsilon_i(\text{id})} \mathfrak{S}_w) = c_{s_1 \cdots s_{i-1}, w}^{\varepsilon_i(v)}.$$

where the first equality follows by Proposition 8.1. Since $T_i = R_i \partial_i$ we infer (8.6) from (8.5). \square

Example 8.5. Take $w = 146352$ and $i = 4$. We get the following expansions

$$R_4 \mathfrak{S}_w = \mathfrak{S}_{346215}, \quad T_4 \mathfrak{S}_w = \mathfrak{S}_{246315}$$

from the decreasing chains $1465327 \prec_7 1475326 \prec_5 1574326 \prec_4 4571326 = \varepsilon_4(346215)$ and $1463527 \prec_7 1473526 \prec_5 1573426 \prec_3 3571426 = \varepsilon_4(246315)$ respectively.

Before stating the central result in this section, we recall here that the basis of slide polynomials was introduced and studied in Assaf–Searles [2]. This family of polynomials is precisely the family that appears in the seminal work of Billey–Jockusch–Stanley [10] describing Schubert polynomials combinatorially via reduced pipe dreams. One consequence of our next result is a new proof of the nonnegativity of the slide expansion of Schubert polynomials, which can be considered either geometric or combinatorial depending on how one reads Proposition 8.1.

Theorem 8.6. [Theorem 2.4] Let $w \in S_n$. The coefficients of

- (1) A monomial $x_1^{c_1} \cdots x_n^{c_n}$ in the monomial expansion of \mathfrak{S}_w
- (2) A slide polynomial coefficient in the slide polynomial expansion $\mathfrak{S}_w = \sum a_i \mathfrak{S}_i$
- (3) A forest polynomial coefficient in the m -forest polynomial expansion $\mathfrak{S}_w = \sum a_F \mathfrak{P}_F^m$

are all generalized LR coefficients $c_{u(\Omega),w}^{v(\Omega)}$ for some $\Omega \in \text{RTSeq}_N$ with N possibly larger than n , with an explicit combinatorially nonnegative rule for computing them.

Proof. We note that if $\Omega \in \text{RTSeq}$, if N is sufficiently large then $r_1^{N-|\Omega|}\Omega \in \text{RTSeq}_N$ and

$$\text{ev}_0 \Pi_{r_1^{N-|\Omega|}\Omega} = \text{ev}_0 R_1^{N-|\Omega|} \Pi_\Omega = \text{ev}_0 \Pi_\Omega$$

so it suffices to find such an $\Omega \in \text{RTSeq}$ rather than RTSeq_N . Having established the explicit combinatorial rules for $R_i \mathfrak{S}_w$ and $T_i \mathfrak{S}_w$, we conclude that $\Pi_\Omega \mathfrak{S}_w$, and hence $\text{ev}_0 \Pi_\Omega \mathfrak{S}_w$, has an explicit combinatorial rule for computing it. Therefore it remains to show that each of these quantities can be computed as $\text{ev}_0 \Pi_\Omega \mathfrak{S}_w$ for some Ω . But this follows from results [42] wherein we find *extractors* Ω for the families under consideration here. \square

9. DIVIDED SYMMETRIZATION

Divided symmetrization (henceforth DS) is the map $\langle \cdot \rangle_n : \text{Pol}_n \rightarrow \text{Sym}_n$ given by

$$(9.1) \quad \langle f \rangle_n = \sum_{\pi \in S_n} \pi \left(\frac{f}{\prod_{1 \leq i \leq n-1} (x_i - x_{i+1})} \right).$$

It was introduced by Postnikov [52, Section 3] in the context of volume polynomials of permutahedra. This operator was then shown [43, 46] to come up naturally when expressing the class of the permutahedral toric variety, i.e. the closure $\overline{T \cdot x}$ of a generic torus orbit in the flag variety Fl_n , in terms of Schubert classes. Indeed, by work of Anderson–Tymoczko [1] we know that the constant term of divided symmetrization computes $\deg_{\overline{T \cdot x}} f$, so by the Kronecker duality between Schubert polynomials \mathfrak{S}_w and Schubert cycles X^w we have

$$(9.2) \quad \overline{T \cdot x} = \sum_{\substack{w \in S_n \\ \ell(w) = n-1}} \langle \mathfrak{S}_w \rangle_n [X^w].$$

The main result of [43] is that $\langle f \rangle_n$ for any homogenous polynomial f of degree $n - 1$ can be computed from the knowledge of the representative in QSCoinv_n of f expressed in the distinguished

monomial basis of Aval–Bergeron–Bergeron [3]. For this purpose forest polynomials were introduced in [48]. We now come full circle and demonstrate how our T operators arise from grouping terms on the right-hand side of (9.1) appropriately.

In contrast to previous work that has primarily focused on understanding DS in degree $n - 1$, we give a formula that works in all degrees and is amenable to combinatorics.

9.1. DS via a generalized trim. For $1 \leq i \leq n$ define the map $R_{i,n} : \text{Pol}_n \rightarrow \text{Pol}_n$ by

$$f(x_1, \dots, x_n) \mapsto f(x_1, \dots, x_{i-1}, x_n, x_i, \dots, x_{n-1}).$$

It is simply the action of the permutation with a single nontrivial cycle $(n, n - 1, \dots, i)$. Note that after the specialization $x_n = 0$, we obtain the operator R_i on Pol_n . The following analogue of T_i is now natural: For $1 \leq i \leq n - 1$ define $T_{i,n} : \text{Pol}_n \rightarrow \text{Pol}_n$ by

$$T_{i,n}(f) = \frac{R_{i+1,n}f - R_{i,n}f}{x_i - x_n} = R_{i+1,n}\partial_i f = R_{i,n}\partial_i f.$$

We recall the divided symmetrization operator $f \mapsto \langle f \rangle_n$ on Pol_n defined by

$$\langle f(x_1, \dots, x_n) \rangle_n = \sum_{\sigma \in S_n} \frac{f(x_{\sigma(1)}, \dots, x_{\sigma(n)})}{(x_{\sigma(1)} - x_{\sigma(2)}) \cdots (x_{\sigma(n-1)} - x_{\sigma(n)})}.$$

For $1 \leq j \leq n$ let $\text{cyc}_{j,n} \in S_n$ be the cycle $(n, n - 1, \dots, n - j + 1)$, and subsequently define the group algebra element $\tau_n := \sum_{1 \leq j \leq n} \text{cyc}_{j,n}$. We then have the following factorization in the group algebra (over \mathbb{Z}) of S_n :

$$(9.3) \quad \sum_{\sigma \in S_n} \sigma = \tau_2 \tau_3 \cdots \tau_n.$$

Theorem 9.1. For $f \in \text{Pol}_n$ we have

$$(9.4) \quad \langle f \rangle_n = T_{1,2}(T_{1,3} + T_{2,3}) \cdots (T_{1,n} + \cdots + T_{n-1,n})f.$$

In particular, if f is homogenous of degree $n - 1$ then

$$\langle f \rangle_n = T_1(T_1 + T_2) \cdots (T_1 + \cdots + T_{n-1})f = \sum_{\substack{F \in \text{For}_n \\ |F|=n-1}} |\text{Trim}(F)| T_F f.$$

Proof. Let $D_n := (x_1 - x_2)(x_2 - x_3) \cdots (x_{n-1} - x_n)$. Begin by noting thanks to (9.3) that

$$(9.5) \quad \langle f \rangle_n = \tau_2 \tau_3 \cdots \tau_n \left(\frac{f}{D_n} \right).$$

Note that $\tau_2 \cdots \tau_{n-1}$ only acts on variables x_1 through x_{n-1} . Let us consider $\tau_n(f/D_n)$.

Writing τ_n as $\sum_{1 \leq j \leq n} \text{cyc}_{j,n}$, we have

$$(9.6) \quad \tau_n \left(\frac{f}{D_n} \right) = \frac{1}{D_{n-1}} \left(-\frac{R_{1,n}f}{(x_1 - x_n)} + \sum_{2 \leq i \leq n-1} \frac{(x_i - x_{i-1})R_{i,n}f}{(x_{i-1} - x_n)(x_i - x_n)} + \frac{R_{n,n}f}{(x_{n-1} - x_n)} \right).$$

Rewriting $x_i - x_{i-1}$ as $(x_i - x_n) - (x_{i-1} - x_n)$ for $2 \leq i \leq n-1$ then yields

$$(9.7) \quad \tau_n \left(\frac{f}{D_n} \right) = \frac{1}{D_{n-1}} \left(-\frac{R_{1,n}f}{(x_1 - x_n)} + \sum_{i=2}^{n-1} \frac{R_{i,n}f}{(x_{i-1} - x_n)} - \sum_{i=2}^{n-1} \frac{R_{i,n}f}{(x_i - x_n)} + \frac{R_{n,n}f}{(x_{n-1} - x_n)} \right),$$

which given the definition of the operator $\mathbb{T}_{i,n}$ translates to

$$(9.8) \quad \tau_n \left(\frac{f}{D_n} \right) = \frac{1}{D_{n-1}} \left(\sum_{1 \leq i \leq n-1} \mathbb{T}_{i,n}(f) \right).$$

The desired identity (9.4) is obtained by immediate recursion using (9.5).

When $\deg(f) \leq n-1$, then $\langle f \rangle_n \in \mathbb{Z}$. In particular, setting $x_n = 0$ in (9.8) does not impact the final result, and so we get

$$(9.9) \quad \langle f \rangle_n = \mathbb{T}_1(\mathbb{T}_1 + \mathbb{T}_2) \cdots (\mathbb{T}_1 + \cdots + \mathbb{T}_{n-1})f.$$

To conclude, observe that

$$(9.10) \quad \mathbb{T}_1(\mathbb{T}_1 + \mathbb{T}_2) \cdots (\mathbb{T}_1 + \cdots + \mathbb{T}_{n-1}) = \sum_{\substack{(i_1, \dots, i_{n-1}) \\ i_j \leq j}} \mathbb{T}_{i_1} \cdots \mathbb{T}_{i_{n-1}}.$$

The set of sequences over which the sum ranges is closed under the relation $\mathbb{T}_a \mathbb{T}_b = \mathbb{T}_b \mathbb{T}_{a+1}$ where $b < a$. Furthermore the subset of weakly increasing sequences (i_1, \dots, i_k) subject to $i_j \leq j$ contains precisely the trimming sequences for indexed forests F in For_n with $|F| = n-1$. It then follows

$$(9.11) \quad \mathbb{T}_1(\mathbb{T}_1 + \mathbb{T}_2) \cdots (\mathbb{T}_1 + \cdots + \mathbb{T}_{n-1}) = \sum_{\substack{F \in \text{For}_n \\ |F|=n-1}} |\text{Trim}(F)| \mathbb{T}_F. \quad \square$$

Remark 9.2. Setting $t_i = x_{n-i}$, then the part of the above computation showing

$$\langle f \rangle_n = \langle (\mathbb{T}_{1,n} + \cdots + \mathbb{T}_{n-1,n})f \rangle_{n-1}$$

is in fact exactly the computation one would do using the Atiyah–Bott localization formula to show that under the projection map $\pi : X_{\text{Perm}_{n-1}} \rightarrow X_{\text{Perm}_{n-2}}$ induced by the map on normal fans obtained by forgetting the first coordinate, we have $\pi_* f = (\mathbb{T}_{1,n} + \cdots + \mathbb{T}_{n-1,n})f$.¹

The following corollary generalizes the nonnegativity of a_w to higher degree Schuberts.

Corollary 9.3. For any Schubert polynomial $\mathfrak{S}_w \in \text{Pol}_n$ we have $\langle \mathfrak{S}_w \rangle_n$ is a polynomial with coefficients that are combinatorially nonnegative.

Proof. It suffices to show that $\mathbb{T}_{i,n}$ applied to \mathfrak{S}_w is a combinatorially nonnegative combination of Schubert polynomials. To do this, note that $\mathbb{T}_{i,n} = R_{i,n} \partial_i$ so it suffices to show this result for $R_{i,n}$. But this follows from one of the main results of Bergeron–Sottile [7, Theorem 5.1], which expresses

$$\mathfrak{S}_w(x_1, \dots, x_{i-1}, z, x_i, \dots) = \sum a_{i,v} z^i \mathfrak{S}_v$$

for combinatorially nonnegative coefficients $a_{i,v}$ (computed using chains in k -Bruhat order for various k). \square

9.2. Strict positivity of $\langle \mathfrak{S}_w \rangle_n$ via trims. We use Theorem 9.1 to give a straightforward combinatorial proof of the *strict* positivity $\langle \mathfrak{S}_w \rangle_n > 0$ for $w \in S_n$, answering [24, Problem 6.6]. This strict positivity was established earlier in [46] by expressing $\langle \mathfrak{S}_w \rangle_n$ as a sum of normalized mixed Eulerian numbers [52, §16] and using positivity of mixed volumes. A combinatorial proof of nonnegativity was then given in [48] via a parking procedure applied to reduced words of w , but strict positivity via this method is unclear.

Fix $w \in S_n$ with $\ell(w) = n - 1$. By Theorem 9.1 we have

$$(9.12) \quad \langle \mathfrak{S}_w \rangle_n = \sum_{i_j \leq j} \mathsf{T}_{i_1} \cdots \mathsf{T}_{i_{n-1}} \mathfrak{S}_w.$$

It suffices to construct a sequence of positive integers $\mathbf{i} = (i_1, \dots, i_{n-1})$ satisfying $i_j \leq j$ for $1 \leq j \leq n - 1$ such that $\mathsf{T}_{\mathbf{i}} \mathfrak{S}_w > 0$.

Lemma 9.4. For $w \in S_n$, if there is $1 \leq i \leq n - 1$ such that $w^{-1}(1) < i \leq w^{-1}(n)$, then the Schubert expansion of $R_i \mathfrak{S}_w$ contains some \mathfrak{S}_v with $v \in S_{n-1}$.

Proof. By Proposition 8.4, it suffices to show there is a decreasing chain in the $(i - 1)$ -Bruhat order for some $v \in S_{n-1}$ as follows:

$$w = w_0 \prec_{b_1} w_1 \prec \cdots \prec_{b_{i-1}} w_{i-1} = \varepsilon_i(v).$$

We describe how to create this chain.

Let a_1, \dots, a_{i-1} be the numbers $w(1), \dots, w(i - 1)$ written in decreasing order. Since $w^{-1}(1) < i$, we must have $a_{i-1} = 1$. For $1 \leq j \leq i - 1$, take

$$(9.13) \quad w_j := w_{j-1} s_{a_j b_j}$$

where b_j is the first number in the list $w_{j-1}(i), \dots, w_{j-1}(n)$ such that $a_j \leq b_j$. The existence of b_1 follows because $n \in \{w(i), \dots, w(n)\}$ and the existence of b_j is clear for $j \geq 2$ because $a_{j-1} \in \{w_{j-1}(i), \dots, w_{j-1}(n)\}$. We also have $w_{j-1} \prec_{b_j} w_j$ in the $(i - 1)$ -Bruhat order by construction. The numbers b_1, b_2, \dots are a decreasing sequence since if $b_j < b_{j+1}$ then b_{j+1} must have appeared later than b_j in the list $w_{j-1}(i), \dots, w_{j-1}(n)$, so in the list $w_j(i), \dots, w_j(n)$ we know that a_j appears before b_{j+1} which contradicts that b_{j+1} is the first number in this list larger than a_{j-1} .

All transpositions $s_{a_j b_j}$ in (9.13) lie in S_n , so $w_{i-1} \in S_n$. Additionally, $a_{i-1} = 1$ implies $w_{i-1} = w_{i-2} s_{(1, w_{i-1}(i))}$, which means $w_{i-1}(i) = 1$. Therefore $w_{i-1} = \varepsilon_i(v)$ for some $v \in S_{n-1}$. \square

Theorem 9.5. Let $w \in S_n$ with $\ell(w) = n - 1$. Then there exists a sequence $\mathbf{i} = (i_1, \dots, i_{n-1})$ of positive integers satisfying $i_j \leq j$ and $\mathsf{T}_{\mathbf{i}} \mathfrak{S}_w > 0$. In particular $a_w > 0$.

Proof. We claim that there is a descent $i \in \text{Des}(w)$ such that $(ws_i)^{-1}(1) \leq i \leq (ws_i)^{-1}(n)$. Indeed, because $\ell(w) = n - 1$ we know that $w^{-1}(1) \leq w^{-1}(n) - 1$. If this is an equality then $i = w^{-1}(1)$ works. Otherwise $w^{-1}(1) < w^{-1}(n)$ and we can take

- $i = w^{-1}(1) - 1$ if $w^{-1}(1) \neq 1$
- $i = w^{-1}(n)$ if $w^{-1}(n) \neq n$
- Any $i \in \text{Des}(w)$ if $w^{-1}(1) = 1$ and $w^{-1}(n) = n$.

By Lemma 9.4 this implies that $T_i \mathfrak{S}_w = R_i \partial_i \mathfrak{S}_w$ contains a summand S_v with $v \in S_{n-1}$. Iterating this produces the desired sequence. \square

9.3. q -divided symmetrization. Let $\Delta_n := \prod_{1 \leq i < j \leq n} (x_i - x_j)$ and $\widehat{\Delta}_n^q := \prod_{1 \leq i+1 < j \leq n} (qx_i - x_j)$. Recall from [44, Definition 4.1] the q -DS operator $f \mapsto \langle f \rangle_n^q$ on Pol_n defined by

$$\langle f(x_1, \dots, x_n) \rangle_n^q = \sum_{\sigma \in S_n} \sigma \left(\frac{f(x_1, \dots, x_n) \cdot \widehat{\Delta}_n^q}{\Delta_n} \right).$$

This clearly recovers ordinary DS at $q = 1$. The first and third authors, by leveraging a connection [44, Theorem 4.11] between q -DS and coefficient extraction in the q -Klyachko algebra, showed that several known results involving DS q -deformed rather nicely.

We then have the following result that helps us completely understand q -DS in degree $n - 1$. Compare this with the analogous statement in Theorem 9.1.

Theorem 9.6. For $f \in \text{Pol}_n$ homogenous of degree $n - 1$ we have

$$\langle f \rangle_n^q = T_1(T_1 + qT_2) \cdots (T_1 + qT_2 + \cdots + q^{n-2}T_{n-1})f.$$

Proof. Using (9.3) again we may write

$$(9.14) \quad \langle f \rangle_n^q = \tau_2 \tau_3 \cdots \tau_n \left(\frac{f \cdot \widehat{\Delta}_n^q}{\Delta_n} \right)$$

We have

$$(9.15) \quad \tau_n \left(\frac{f \cdot \widehat{\Delta}_n^q}{\Delta_n} \right) = \frac{\widehat{\Delta}_{n-1}^q}{\Delta_{n-1}} \left(\frac{R_{1,n}f}{x_n - x_1} \prod_{i=2}^{n-1} \frac{qx_n - x_i}{x_n - x_i} + \sum_{j=2}^{n-1} \frac{(qx_{j-1} - x_j)R_{j,n}f}{(x_{j-1} - x_n)(x_n - x_j)} \prod_{i=1}^{j-2} \frac{qx_i - x_n}{x_i - x_n} \prod_{i=j+1}^{n-1} \frac{qx_n - x_i}{x_n - x_i} + \frac{R_{n,n}f}{x_{n-1} - x_n} \prod_{i=1}^{n-2} \frac{qx_i - x_n}{x_i - x_n} \right).$$

Since $\langle f \rangle_n^q \in \mathbb{Z}[q]$ given that $\deg(f) = n - 1$ we may set $x_n = 0$ throughout in the preceding equality without impacting the eventual result. We then obtain

$$(9.16) \quad \tau_n \left(\frac{f \cdot \widehat{\Delta}_n^q}{\Delta_n} \right) \Big|_{x_n=0} = \frac{\widehat{\Delta}_{n-1}^q}{\Delta_{n-1}} \left(-\frac{R_1 f}{x_1} + \sum_{j=2}^{n-1} q^{j-2} \frac{R_j f}{x_{j-1}} - \sum_{j=2}^{n-1} q^{j-1} \frac{R_j f}{x_j} + q^{n-2} \frac{R_n f}{x_{n-1}} \right)$$

which upon rearranging terms and using the definition of T operators becomes

$$(9.17) \quad \tau_n \left(\frac{f \cdot \widehat{\Delta}_n^q}{\Delta_n} \right) \Big|_{x_n=0} = \frac{\widehat{\Delta}_{n-1}^q}{\Delta_{n-1}} \left(\sum_{1 \leq j \leq n-1} q^{j-1} T_j f \right).$$

Recurring using (9.14) concludes the proof. \square

Observe that our statement of Theorem 9.6 only concerns the case $\deg(f) = n - 1$, even though it is largely in the same spirit as the proof of Theorem 9.1. Indeed, in contrast to the fact that the term within parentheses on the right-hand side in (9.7) is already a polynomial, we see that the analogous term in (9.15) is not necessarily a polynomial. This deficit is precisely what the specialization $x_n = 0$ fixes as the resulting expression is then a polynomial.

Remark 9.7. As in Theorem 9.1 we may express the operator $T_1(T_1 + qT_2) \cdots (T_1 + qT_2 + \cdots + q^{n-2}T_{n-1})$ as a q -weighted sum of T_F for $F \in \text{For}_n$ with $|F| = n - 1$. One writes

$$(9.18) \quad T_1(T_1 + qT_2) \cdots (T_1 + qT_2 + \cdots + q^{n-2}T_{n-1}) = \sum_{\substack{\mathbf{i}=(i_1, \dots, i_{n-1}) \\ i_j \leq j}} q^{i_1 + \cdots + i_{n-1} - (n-1)} T_{i_1} \cdots T_{i_{n-1}}.$$

Collecting terms according to the indexing sequences (i_1, \dots, i_k) that trim a common indexed forest F , we get one term for each decreasing labeling of F . Let us denote the set of such labelings by $\text{Dec}(F)$. By reading such a labeling κ in in-order one obtains a permutation in S_{n-1} that we shall continue to call κ . Tracking the q -weight tells us that

$$(9.19) \quad \sum_{\substack{\mathbf{i}=(i_1, \dots, i_{n-1}) \\ i_j \leq j}} q^{i_1 + \cdots + i_{n-1} - (n-1)} T_{i_1} \cdots T_{i_{n-1}} = \sum_{\substack{F \in \text{For}_n \\ |F|=n-1}} \left(\sum_{\kappa \in \text{Dec}(F)} q^{\text{inv}(\kappa)} \right) T_F,$$

where $\text{inv}(\kappa)$ is the number of inversions of κ . To complete this remark we note that when $F \in \text{For}_n$ with $|F| = n - 1$ then $|\text{Trim}(F)|$ in (9.11) is the specialization at $x_i = 1$ for all i [48, Proposition 3.14], whereas $\sum_{\kappa \in \text{Dec}(F)} q^{\text{inv}(\kappa)}$ in (9.19) is the specialization of \mathfrak{P}_F at $x_i = q^{i-1}$ for all i .

We conclude this section by speculating a bit more on the case where $\deg(f)$ exceeds $n - 1$. Recall that the P -Hall–Littlewood polynomial $P_\lambda(x_1, \dots, x_n; q)$ [38] for a partition $(\lambda_1, \dots, \lambda_n)$ is obtained by normalizing the following symmetric polynomial

$$\sum_{\sigma \in S_n} \sigma \left(x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{1 \leq i < j \leq n} \frac{x_i - qx_j}{x_i - x_j} \right).$$

Thus it is a common generalization of monomial symmetric polynomials and Schur polynomials, recovering the former at $q = 1$ and the latter at $q = 0$. We conjecture the following generalization of Corollary 9.3 which we have verified on all permutations up until S_8 .

Conjecture 9.8. For any Schubert polynomial $\mathfrak{S}_w \in \text{Pol}_n$ we have

$$\langle \mathfrak{S}_w \rangle_n^q = \sum_{\lambda=(\lambda_1 \geq \dots \geq \lambda_n)} b_{\lambda,w}(q) P_\lambda(x_1, \dots, x_n; q^{-1}),$$

where the $b_{\lambda,w}(q)$ are Laurent polynomials in q with nonnegative integer coefficients.

At $q = 1$ this recovers the nonnegative expansion involving monomial symmetric polynomials. We also note that the appearance of q^{-1} is simply a reflection of the fact that our definition of q -DS employed factors $qx_i - x_j$ (keeping with the choice in [44]) rather than $x_i - qx_j$ used in the definition of the P -Hall–Littlewoods. With this latter choice, Conjecture 9.8 becomes a P -Hall–Littlewood positivity statement with coefficients in $\mathbb{N}[q]$. The conjecture is already nontrivial in the simplest case where \mathfrak{S}_w is a dominant monomial $x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ where $\lambda_1 \geq \dots \geq \lambda_n$, i.e. when w is a 132-avoiding permutation. The $q = 1$ case is due to Postnikov [52, Theorem 4.3] and is a simple consequence of a famous result of Brion [14] on integer point transforms of integral polytopes. We are unaware of a generalization that incorporates the parameter q .

10. A GEOMETRIC PERSPECTIVE ON A FORMULA OF GESSEL

We now give a geometric interpretation for a result of Gessel on extracting the coefficients for the expansion of a symmetric polynomial in Sym_n into fundamental quasisymmetric polynomials in QSym_n .

In [41, Corollary 8.6] the present authors showed that for any polynomial $f \in \text{Pol}_n$, there is a decomposition into fundamental quasisymmetric polynomials given by

$$f(x_1, \dots, x_n) = \sum_k \sum_{a_k, \dots, a_n \geq 1} (\text{ev}_0 \tau_k^{a_k} \tau_{k+1}^{a_{k+1}} \cdots \tau_n^{a_n} f) \mathfrak{F}_{a_k, \dots, a_n}(x_1, \dots, x_n)$$

where $\mathfrak{F}_{a_k, \dots, a_n}(x_1, \dots, x_n)$ for $a_k, \dots, a_n \geq 1$ is the fundamental quasisymmetric polynomial² whose reverse lexicographic leading monomial is $x_k^{a_k} \cdots x_n^{a_n}$.

Recall that the Schubert cells Y^λ and the opposite Schubert cells Y_μ of $\text{Gr}(n; N)$ are indexed by partitions λ, μ of integers into at most n parts with the largest part of size at most $N - n$. If $\lambda \geq \mu$ coordinate-wise, then there is an associated Grassmannian Richardson variety $Y_\mu^\lambda \subset \text{Gr}(n; N)$. We show that certain Ω -Richardson varieties with $\Omega \in \text{RTSeq}_N$ push forward to the Grassmannian Richardson varieties associated to ribbon shapes.

Definition 10.1. The ribbon shape λ/μ indexed by (a_k, \dots, a_n) for $a_k, \dots, a_n \geq 1$ is the one which associated to partitions

$$\begin{aligned} \lambda &= (a_n + \cdots + a_k - (n - k), \dots, a_n + a_{n-1} - 1, a_n) \text{ of length } n - k + 1, \text{ and} \\ \mu &= (a_n + \cdots + a_{k+1} - (n - k), \dots, a_n + a_{n-1} - 2, a_n - 1) \text{ of length } n - k. \end{aligned}$$

²The indexing on fundamental quasisymmetric polynomials is different from [41].

These are the skew shapes whose rows reading from top to bottom are of lengths a_k, \dots, a_n , and such that there is exactly one square in the same column in two consecutive rows. For instance take $k = 2$, $n = 5$, and suppose that $(a_2, \dots, a_5) = (2, 1, 1, 3)$. Then the corresponding λ and μ equal $(7 - 3, 5 - 2, 4 - 1, 3) = (4, 3, 3, 3)$ and $(5 - 3, 4 - 2, 3 - 1) = (2, 2, 2)$ respectively. See Figure 7 which demonstrates the ribbon λ/μ with shaded cells corresponding to μ .

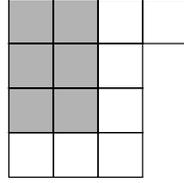


FIGURE 7. The ribbon corresponding to $(a_2, \dots, a_5) = (2, 1, 1, 3)$ according to Definition 10.1

For fixed $N \geq n$ denote the forgetful map $\pi : \text{Fl}_N \rightarrow \text{Gr}(n; N)$. We will show the following theorem.

Theorem 10.2. For any $N \geq k + \sum a_i$, let $\Omega = r_1^{N-k-\sum a_i} t_1^{a_k} \dots t_{n-(k-1)}^{a_n} r_1^{k-1} \in \text{RTSeq}_N$. Then

$$\pi_*(X(\Omega)) = Y_\mu^\lambda \subset \text{Gr}(n; N)$$

where λ/μ is the ribbon indexed by (a_k, \dots, a_n) .

As a corollary of this geometric fact we show how to recover a formula of Gessel for the coefficient of the fundamental quasisymmetric polynomial $\mathfrak{F}_{a_k, \dots, a_n}(x_1, \dots, x_n)$ in the fundamental quasisymmetric polynomial expansion of f .

Corollary 10.3 ([20, Theorem 3]). For $a_k, \dots, a_n \geq 1$, the coefficient of $\mathfrak{F}_{a_k, \dots, a_n}(x_1, \dots, x_n)$ in $f \in \text{Sym}_n$ is $\langle s_{\lambda/\mu}, f \rangle_{\text{Hall}}$ where λ/μ is the ribbon shape for (a_k, \dots, a_n) and $\langle \cdot, \cdot \rangle_{\text{Hall}}$ is the Hall inner product on Sym_n .

Proof. The Hall inner product with a skew Schur function $s_{\lambda/\mu}$ computes the degree of f on the Grassmannian Richardson variety $Y_\mu^\lambda \subset \text{Gr}(n; N)$. The pullback $\pi^* : H^\bullet(\text{Gr}(n; N)) \rightarrow H^\bullet(\text{Fl}_N)$ identifies $H^\bullet(\text{Gr}(n; N))$ with the subalgebra of $H^\bullet(\text{Fl}_N) = \text{Coinv}_N$ generated by Sym_n . Therefore

$$\langle s_{\lambda/\mu}, f \rangle_{\text{Hall}} = \deg_{Y_\mu^\lambda} f = \deg_{\text{Gr}(n; N)} \pi_*[X(\Omega)]f = \deg_{\text{Fl}_N} [X(\Omega)]f = \Pi_\Omega^N f = \text{ev}_0 T_k^{a_k} \dots T_n^{a_n} f,$$

where in the last step we used that $T_{i-(k-1)}^{a_i} R_1^{k-1} = R_1^{k-1} T_i^{a_i}$ and $\text{ev}_0 R_1^{N-1-\sum a_i} = \text{ev}_0$. \square

Remark 10.4. To emphasize the contrast to the geometric approach here, we briefly remark on Gessel's proof³ in [20]. Given a symmetric function f , the coefficient of a fixed monomial symmetric function m_λ equals $\langle f, h_\lambda \rangle_{\text{Hall}}$ where h_λ is the complete homogenous symmetric function

³which while phrased in the language of (quasi)symmetric functions, restricts to the finite variable case stated earlier

associated with λ . Observing that the coefficient of m_λ is in fact equal to the coefficient of the monomial quasisymmetric function M_α for *any* sequence α of positive integers that rearranges to λ , we see that the monomial quasisymmetric expansion may thus be easily obtained. The punchline then relies on the observation that the expansion of a monomial quasisymmetric function in terms of fundamental quasisymmetric functions is the same as an expansion (going back to MacMahon [39]) of skew ribbon Schurs in terms of homogenous symmetric functions.

We prove Theorem 10.2 in the rest of this section. For $w \in S_N$, define $g(w)$ to be the n -Grassmannian permutation obtained by sorting the first n and the last $N - n$ entries in the one-line notation of w . Then we always have $\pi(X_u^v) \subset Y_{g(u)}^{g(v)}$. Note that $\pi(X_u^v)$ is a *projected Richardson variety* in the sense of [28], or *projection variety* in the sense of [9]. In fact, since π is a map into $\text{Gr}(n; N)$, we obtain *positroid varieties*.

Proposition 10.5. $\pi|_{X_u^v} : X_u^v \rightarrow Y_{g(u)}^{g(v)}$ is birational if and only if $u \leq_n v$ and $\ell(v) - \ell(u) = \ell(g(v)) - \ell(g(u))$. (Here \leq_n stands for comparison in the n -Bruhat order.)

Proof. As recalled in [54, Proposition 1.24], the map $\pi|_{X_u^v} : X_u^v \rightarrow \pi(X_u^v)$ is birational if and only if $u \leq_n v$. The result follows because X_u^v and $Y_{g(u)}^{g(v)}$ are irreducible and we have $\dim X_u^v = \ell(v) - \ell(u)$ and $\dim Y_{g(u)}^{g(v)} = \ell(g(v)) - \ell(g(u))$. \square

Lemma 10.6. Letting $\Omega = r_1^{N-k-\sum a_i} t_1^{a_k} \cdots t_{n-(k-1)}^{a_n} r_1^{k-1} \in \text{RTSeq}_N$ as in the statement of Theorem 10.2 we have

$$\begin{aligned} u(\Omega) &= 1 \cdots (k-1) p_{k+1} \cdots p_n k \cdots \\ v(\Omega) &= 1 \cdots (k-1) p_k \cdots p_n k \cdots \end{aligned}$$

where $p_i = k + \sum_{j=i}^n a_j$ and the tails of unlisted numbers are in increasing order. Furthermore we have $u(\Omega) \leq_n v(\Omega)$.

Proof. The first half of the claim rests on the elementary observation that for a sequence $c_2, \dots, c_m \geq 1$ we have

$$\varepsilon_m^{c_m} \cdots \varepsilon_2^{c_2}(\text{id}) = b_2 b_3 \cdots b_m 1 \cdots$$

where $b_i = 1 + \sum_{j=i}^m c_j$ and the tail of unlisted numbers is given by the numbers in $\mathbb{N} \setminus \{b_2, \dots, b_m\}$ listed in increasing order.

For the second half, it is not hard to see that $u(\Omega) \leq_n v(\Omega)$ from the characterization [6, Section 3.1] that for u, v permutations, we have $u \leq_n v$ if

- (1) $i \leq n < j$ implies $u(i) \leq w(i)$ and $u(j) \geq w(j)$, and
- (2) if $i < j$, $u(i) < u(j)$, and $v(i) > v(j)$, then $i \leq n < j$.

We leave the details to the reader. \square

Proof of Theorem 10.2. We set $u := u(\Omega)$ and $v := v(\Omega)$ for brevity. Lemma 10.6 implies that

$$\begin{aligned} g(u) &= 1 \cdots (k-1)kp_n \cdots p_{k+1} \cdots, \\ g(v) &= 1 \cdots (k-1)p_n \cdots p_k k \cdots. \end{aligned}$$

We have $\ell(g(u)) = \ell(u) - (n - k + 1)$ and $\ell(g(v)) = \ell(v) - (n - k + 1)$, so we conclude by Proposition 10.5 that $\pi|_{X_u^v} : X_u^v \rightarrow Y_{g(u)}^{g(v)}$ is birational, and in particular $\pi_*(X_u^v) = Y_{g(u)}^{g(v)}$. It remains to identify $Y_{g(u)}^{g(v)}$.

The Lehmer codes of $g(u)$ and $g(v)$ are

$$\begin{aligned} \text{lcode}(u(\Omega)) &= (0^{k-1}, 0, a_n - 1, a_n + a_{n-1} - 2, \dots, a_n + \cdots + a_{k+1} - (n - k)) \\ \text{lcode}(v(\Omega)) &= (0^{k-1}, a_n, a_n + a_{n-1} - 1, \dots, a_n + \cdots + a_k - (n - k), 0). \end{aligned}$$

This implies that the skew shape λ/μ associated to u and v is the ribbon indexed by $(a_k, a_{k+1}, \dots, a_n)$, and we conclude. \square

Example 10.7. Consider the fundamental quasisymmetric polynomial $\mathfrak{F}_{2,1,1,3}(x_1, x_2, x_3, x_4)$. Then for $N = 9$ we have $\Omega = t_1^2 t_2 t_3 t_4^3 \in \text{RTSeq}_9$. The permutations $u(\Omega), v(\Omega)$ are equal to $65412378 \cdots$ and $86541237 \cdots$. On sorting the first four letters in each, we get the 4-Grassmannian permutations $g(u)$ and $g(v)$ equaling $14562378 \cdots$ and $45681237 \cdots$ with Lehmer codes $(0, 2, 2, 2, 0, \dots)$ and $(3, 3, 3, 4, 0, \dots)$, which yields the ribbon shape indexed by $(2, 1, 1, 3)$.

APPENDIX A. NESTED FOREST COMBINATORICS

Here we gather the proofs of various statements of Section 3.

Lemma A.1. The forests of the form \underline{i} and \underline{i}_\circ generate the monoid $\widetilde{\text{For}}$.

Proof of Lemma A.1. Let \widetilde{G} be a nontrivial forest. If the i 'th leaf is the trivial marked tree \otimes , then $\widetilde{G} = \widetilde{F} \cdot \underline{i}_\circ$ where \widetilde{F} is obtained by deleting \otimes . Otherwise, consider a tree in \widetilde{G} which has no forests nested underneath it, so its support is a contiguous discrete interval $\{a, a + 1, \dots, b\}$, and let v be an internal node of this tree which is farthest from the root. Then its children are two consecutive leaves $i, i + 1$, and we can write $\widetilde{G} = \widetilde{F} \cdot i$ where \widetilde{F} is obtained by deleting the two children of v (making v a leaf).

We conclude by induction on the sum of the number of marked nodes and the number of internal nodes of \widetilde{G} . \square

We recall the usual code map on plane forests.

Definition A.2. Let Codes denote finite supported sequences of natural numbers. For each plane forest F , define the *code of F* , $c(F) = (c_1, c_2, \dots) \in \text{Codes}$ to be sequence defined by taking c_i to be the number of internal nodes of F whose leftmost leaf descendent is labeled $i \in \mathbb{N}$.

For marked nested plane forests, we will need to augment the code map. Let $\widetilde{\text{Codes}}$ denote the set of infinite sequences indexed by \mathbb{N} whose elements are pairs (ε, i) with $\varepsilon \in \{0, 1\}$ and $i \in \{0, 1, 2, \dots\}$, such that only finitely many terms are not $(0, 0)$.

We define the augmented code map

$$\tilde{c} : \widetilde{\text{For}} \rightarrow \widetilde{\text{Codes}}.$$

as follows: $\tilde{c}(F)_i = (\varepsilon, c)$ where c is the number of non-leaf nodes such that iteratively taking left children leads to the i 'th leaf, and $\varepsilon = 1$ if the highest such node is the root of its connected component and is marked, and $\varepsilon = 0$ otherwise.

We note that by its construction, we have

$$(A.1) \quad \tilde{c}(\underline{1}_o^{\varepsilon_1} \cdot \underline{1}^{a_1} \cdot \underline{2}_o^{\varepsilon_2} \cdot \underline{2}^{a_2} \dots) = ((\varepsilon_1, a_1), (\varepsilon_2, a_2), \dots).$$

Proof of Theorem 3.12. We directly check the relations

$$(A.2) \quad \underline{i} \cdot \underline{j} = \underbrace{\times \cdots \times}_{j-1} \wedge \underbrace{\times \cdots \times}_{i-j-1} \wedge \times \cdots = \underline{j} \cdot \underline{i+1} \text{ for } i > j$$

$$(A.3) \quad \underline{i} \cdot \underline{j}_o = \underbrace{\times \cdots \times}_{j-1} \otimes \underbrace{\times \cdots \times}_{i-j} \wedge \times \cdots = \underline{j}_o \cdot \underline{i+1} \text{ for } i \geq j,$$

$$(A.4) \quad \underline{i}_o \cdot \underline{j} = \underbrace{\times \cdots \times}_{j-1} \wedge \underbrace{\times \cdots \times}_{i-j-1} \otimes \times \cdots = \underline{j} \cdot \underline{(i+1)}_o \text{ for } i > j,$$

$$(A.5) \quad \underline{i}_o \cdot \underline{j}_o = \underbrace{\times \cdots \times}_{j-1} \otimes \underbrace{\times \cdots \times}_{i-j} \otimes \times \cdots = \underline{j}_o \cdot \underline{(i+1)}_o \text{ for } i \geq j.$$

so there is a monoid morphism $\widetilde{\text{ThMon}} \rightarrow \widetilde{\text{For}}$ given by $i \mapsto \underline{i}$ and $i_o \mapsto \underline{i}_o$. Because $\widetilde{\text{For}}$ is generated by the elements \underline{i} and \underline{i}_o , the morphism $\widetilde{\text{ThMon}} \rightarrow \widetilde{\text{For}}$ is surjective.

Using the relations in $\widetilde{\text{ThMon}}$, we may write any element as $1_o^{\varepsilon_1} 1^{a_1} 2_o^{\varepsilon_2} 2^{a_2} \dots$ with each $\varepsilon_i \in \{0, 1\}$. Indeed, let $m \in \widetilde{\text{ThMon}}$, and consider any factorization of m in the generators. The relations of Definition 3.7 for $j = 1$ and $i > 1$ are $i \cdot 1 = 1 \cdot (i+1)$, $i \cdot 1_o = 1_o \cdot (i+1)$, $i_o \cdot 1 = 1 \cdot (i+1)_o$, $i_o \cdot 1_o = 1_o \cdot (i+1)_o$. Using those we can obtain a factorization of m where the occurrences of 1 and 1_o form a prefix. Using the extra relations $1 \cdot 1_o = 1_o \cdot 2$, $1_o \cdot 1_o = 1_o \cdot 2_o$, we can in fact have this prefix be of the form $1_o^{\varepsilon_1} 1^{a_1}$. It is then easily concluded by induction that we can have the desired factorization, as the relations $j > 1$ are just shifts of the ones for $j = 1$.

Now these special factorizations map to distinct elements of $\widetilde{\text{For}}$ via the morphism, because of (A.1). We conclude that the morphism is injective, and thus $\widetilde{\text{ThMon}} \cong \widetilde{\text{For}}$.

Finally, the augmented code map is surjective by (A.1), and injective because $\widetilde{\text{ThMon}} \rightarrow \widetilde{\text{For}}$ is surjective and we have just shown that the composite $\widetilde{\text{ThMon}} \rightarrow \widetilde{\text{For}} \rightarrow \widetilde{\text{Codes}}$ is injective. \square

Proof of forward implications of Theorem 3.18. The first forward implication was already proved. Now note that if $\tilde{F} = \tilde{F}(\Omega)$, then the forest $\tilde{F}(r_1^N \Omega)$ is obtained by marking the first N unmarked outer trees in \tilde{F} . It follows that if $\hat{F}(\Omega) = \hat{F}(\Omega')$ then there are $N, M \gg 1$ so that $\tilde{F}(r_1^N \Omega) = \tilde{F}(r_1^M \Omega')$; by the first part, we get $R_1^N \Pi_\Omega = R_1^M \Pi_{\Omega'}$. Now for any operator A and polynomial f we have $R_1^N Af = R_1^M f = \text{ev}_0 Af$ for N, M large enough, so we can conclude $\text{ev}_0 \Pi_\Omega = \text{ev}_0 \Pi_{\Omega'}$. \square

We now prove the reverse implications. For this we make use of the following proposition of independent interest:

Proposition A.3. Let $c \in \text{Codes}$. For each leaf i , consider the unique path P_i of length c_i towards the root of the tree containing i . If any of these paths does not exist, or if the paths do not partition all internal nodes of $\hat{F}(\Omega)$ then $\text{ev}_0 \Pi_{\tilde{F}} x^c = 0$. Otherwise, $\text{ev}_0 \Pi_{\tilde{F}} x^c = (-1)^m$ where m is the number of edges connecting a node to a right child in $\sqcup P_i$.

Proof. We note that $\text{ev}_0 \emptyset = \text{ev}_0$, so the description holds for \emptyset . Write $\tilde{F} = \tilde{F}(\Omega)$ for some $\Omega \in \text{RTSeq}$, and assume the result holds for all smaller length $\Omega \in \text{RTSeq}$. We write $\Omega = (\Omega', X)$, and let $\tilde{F}' = \tilde{F}(\Omega')$. There are two cases.

If $X = r_i$ then

$$\text{ev}_0 \Pi_\Omega = \text{ev}_0 \Pi_{\Omega'} R_i x^c = \begin{cases} \text{ev}_0 \Pi_{\Omega'} x_1^{c_1} \cdots x_{i-1}^{c_{i-1}} x_i^{c_i+1} \cdots & c_i = 0 \\ 0 & c_i \neq 0. \end{cases}$$

The description holds by induction, since \tilde{F} has a trivial tree with support $\{i\}$ in this case and \tilde{F}' is obtained by removing this trivial tree.

If $X = t_i$ then

$$\text{ev}_0 \Pi_\Omega = \text{ev}_0 \Pi_{\Omega'} T_i x^c = \begin{cases} \text{ev}_0 \Pi_{\Omega'} x_1^{c_1} \cdots x_{i-1}^{c_{i-1}} x_i^{c_i-1} x_{i+1}^{c_{i+2}} \cdots & c_i \neq 0 \text{ and } c_{i+1} = 0 \\ -\text{ev}_0 \Pi_{\Omega'} x_1^{c_1} \cdots x_{i-1}^{c_{i-1}} x_i^{c_{i+1}-1} x_{i+1}^{c_{i+2}} \cdots & c_i = 0 \text{ and } c_{i+1} \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Here \tilde{F} has a terminal node with support $\{i, i+1\}$ and \tilde{F}' is obtained by transforming it into a leaf. The description holds by induction. \square

Proof of the reverse implication of Theorem 3.18. We first show how to reconstruct \hat{F} from $\text{ev}_0 \Pi_{\tilde{F}}$. For each i , let b_i be maximal such that $f \mapsto \phi(x_i^{b_i} f)$ is not identically zero; then b_i is the distance in $\hat{F}(\Omega)$ from leaf i to the root of the tree it lies in. Now for $i \neq j$, $f \mapsto \phi(x_i^{b_i} x_j^{b_j} f)$ is identically zero if and only if i, j lie in the same tree of $\hat{F}(\Omega)$. Finally, if i, j lie in the same tree in $\hat{F}(\Omega)$ then there is a smallest integer $c_{i,j}$ such that $\text{ev}_0 \Pi_\Omega x_i^{b_i} x_j^{c_{i,j}} f$ is identically zero, namely the distance from leaf j to the nearest common parent of i and j . This information is enough to reconstruct the nested forest.

Now, we show how to recover \tilde{F} from $\Pi_{\tilde{F}}$. We already know that $\hat{F} = \hat{F}'$ by what was said above, so we need to show that the markings coincide. Now for any i we have $\Pi_{i, \tilde{F}} = T_i \Pi_{\tilde{F}} =$

$\tau_i \Pi_{\tilde{F}'} = \Pi_{i \cdot \tilde{F}'}$ so the unmarked forests associated to $i \cdot \tilde{F}$ and $i \cdot \tilde{F}'$ have to coincide. Because the unmarked forests for \tilde{F} and \tilde{F}' coincide this implies that i 'th unmarked outer roots of $\tilde{F}(\Omega)$ and $\tilde{F}(\Omega')$ are the same, so $\tilde{F} = \tilde{F}'$ as desired. \square

Proof of Proposition 3.21. The proposition holds for $n = 1$ as $\Omega = r_1$ in this case and $\tilde{F}(\Omega) = \otimes \times \times \cdots$. For $n > 1$, the fact that $\tilde{F}(\Omega)$ satisfies the properties is then immediate by induction. For the converse statement, by the proof of Lemma A.1 we have that $\tilde{F} = \tilde{F}' \cdot i$ for some $i < n$ or $\tilde{F} = \tilde{F}' \cdot i_\circ$ for some $i \leq n$. Then \tilde{F}' satisfies the properties for $n - 1$, so we have $\tilde{F}' = \tilde{F}'(\Omega')$ for some $\Omega' \in \text{RTSeq}_{n-1}$ by immediate induction, and thus $\tilde{F} = \tilde{F}'(\Omega' \times)$ with $\times = \tau_i$ with $i < n$ or $\times = r_i$ for $i \geq n$. We have thus $\Omega' \times \in \text{RTSeq}_n$ which concludes the proof. \square

APPENDIX B. COMBINATORICS ASSOCIATED TO THE PERMUTATIONS $u(\Omega), v(\Omega)$

Given a permutation $u \in S_\infty$ and $a \geq 1$, let $\ell_a(u)$ be the number of $b \geq a$ such that $u^{-1}(b) \leq u^{-1}(a)$. In words, $\ell_a(u)$ counts numbers larger than a that occur before a in the one-line notation of u . For instance if $w = 426153789 \cdots$ then $\ell_a(w)$ is given by $4, 2, 4, 1, 2, 1, 1, 1, 1, \dots$ for $a = 1, 2, \dots$

This gives a bijection between S_∞ and sequences of positive integers eventually constant equal to 1. It is clear that the permutations in S_n are characterized as those with $\ell_i(u) \leq n + 1 - i$ for $1 \leq i \leq n$ and $\ell_i(u) = 1$ for $i \geq n + 1$. For any sequence j_1, \dots, j_k , the permutation

$$w = \varepsilon_{j_1} \cdots \varepsilon_{j_k}(1)$$

is then characterized by $\ell_a(w) = j_a$ for $a = 1, \dots, k$ and $\ell_a(w) = 1$ for $a \geq k + 1$.

Corollary B.1. If $|\Omega| \leq |\Omega'|$, then $(u(\Omega), v(\Omega)) = (u(\Omega'), v(\Omega'))$ if and only if $\Omega' = r_1^{|\Omega'| - |\Omega|} \Omega$.

Proposition B.2. For fixed m , the map $\Omega \rightarrow (u(\Omega), v(\Omega))$ is a bijection between length m sequences in RTSeq and pairs of permutations $(u, v) \in S_\infty \times S_\infty$ such that $\ell_a(v) - \ell_a(u) \in \{0, 1\}$ for $1 \leq a \leq m$ and $\ell_a(v) = \ell_a(u) = 1$ for $a \geq m + 1$.

Proof. For a length m sequence $\Omega = x_1 \cdots x_m \in \text{RTSeq}$, we have $\ell_i(u) = \ell_i(v) = 1$ for $i \geq m + 1$ and for $1 \leq i \leq m$ we have

$$(\ell_i(u), \ell_i(v)) = \begin{cases} (k, k) & x_{m+1-i} = r_k \\ (k, k+1) & x_{m+1-i} = \tau_k, \end{cases}$$

which clearly establishes the bijection. \square

We now describe what pairs $(u(\Omega), v(\Omega))$ are produced for $\Omega \in \text{RTSeq}_n$. This is not particularly a restriction by Corollary B.1, since $r_1^{N-1-|\Omega|} \Omega \in \text{RTSeq}_N$ for N sufficiently large.

Proposition B.3. The map $\text{RTSeq}_n \rightarrow S_n \times S_n$ given by $\Omega \mapsto (u(\Omega), v(\Omega))$ is an injective map with image those pairs $(u, v) \in S_n \times S_n$ such that $\ell_a(v) - \ell_a(u) \in \{0, 1\}$ for $1 \leq a \leq n$.

Proof. For $\Omega \in \text{RTSeq}_n$, the way that $u(\Omega), v(\Omega)$ are both of the form $\varepsilon_{j_1} \cdots \varepsilon_{j_{n-1}}(\text{id})$ with $j_i \leq n+1-i$ which shows that $(u(\Omega), v(\Omega)) \in S_n \times S_n$. Conversely, if $(u, v) \in S_n \times S_n$ with $\ell_a(v) - \ell_a(u) \in \{0, 1\}$ for $1 \leq a \leq n$ then setting $\Omega = x_1 \cdots x_n$ where

$$x_i = \begin{cases} r_k & (\ell_{n+1-i}(u), \ell_{n+1-i}(v)) = (k, k) \\ t_k & (\ell_{n+1-i}(u), \ell_{n+1-i}(v)) = (k, k+1) \end{cases}$$

achieves $(u(\Omega), v(\Omega)) = (u, v)$. \square

We have the following special cases with $|\Omega|_t = 0$ minimal and $|\Omega|_t = n-1$ maximal.

Proposition B.4.

- (1) The pairs of permutations $(u(\Omega), v(\Omega)) \in S_n \times S_n$ for $\Omega \in \text{RTSeq}_n$ and $|\Omega|_t = 0$ are precisely those with $u = v$.
- (2) Let $c_n = n12 \cdots (n-1) = s_{n-1}s_{n-2} \cdots s_1$ be the reverse long cycle in S_n . The pairs of permutations $(u(\Omega), v(\Omega)) \in S_n \times S_n$ with $\Omega \in \text{RTSeq}_n$ and $|\Omega|_t = n-1$ are exactly the pairs (u, uc_n) with $u \in S_n$ and $u(n) = n$.

We now describe the Bruhat intervals associated to Ω .

Proposition B.5. For x, y permutations, if $j, k \in \{i, i+1\}$ then $\varepsilon_j(x) \leq \varepsilon_k(y)$ if and only if $j \leq k$ and $x \leq y$.

Proof. Recall the *tableau criterion* [11, Theorem 2.6.3], which says that $a \leq b$ in the Bruhat order if for all t , the first t entries in the one-line permutation of a when sorted are elementwise less than the first t entries in the one-line permutation of b when sorted.

In particular, if we have $\varepsilon_r(a) \leq \varepsilon_s(b)$, then the tableau criterion applied to the first s entries implies that 1 appears in the first r entries so $r \leq s$. Therefore it remains to show that if $i \leq j \leq k \leq i+1$ then $\varepsilon_j(x) \leq \varepsilon_k(y)$ if and only if $x \leq y$.

If $i \leq j \leq k \leq i+1$ and $x \leq y$ then the tableau criterion shows directly that $\varepsilon_j(x) \leq \varepsilon_j(y) \leq \varepsilon_k(y)$. Conversely, suppose $\varepsilon_j(x) \leq \varepsilon_k(y)$. Then for $1 \leq r \leq j-1$, the first r entries of x and y agree respectively with the first i entries of $\varepsilon_j(x)$ and $\varepsilon_k(y)$ decremented by 1, while for $r \geq j$ they agree respectively with the first $r+1$ entries of $\varepsilon_j(x)$ and $\varepsilon_k(y)$ after removing 1 and decrementing the remaining entries by 1. Hence the tableau criterion for $\varepsilon_j(x) \leq \varepsilon_k(y)$ implies the tableau criterion for $x \leq y$. \square

Corollary B.6. If $u \leq v$ in the Bruhat order, then the following are true.

- (1) The map $w \mapsto \varepsilon_i(w)$ is a poset isomorphism between $[u, v]$ and $[\varepsilon_i(u), \varepsilon_i(v)]$.
- (2) $[\varepsilon_i(u), \varepsilon_{i+1}(v)] = [\varepsilon_i(u), \varepsilon_i(v)] \sqcup [\varepsilon_{i+1}(u), \varepsilon_{i+1}(v)]$. Furthermore for $w, w' \in [u, v]$ and $j, k \in \{i, i+1\}$ we have $\varepsilon_j(w) \leq \varepsilon_k(w')$ if and only if $w \leq w'$ and $j \leq k$.

Corollary B.7. The Bruhat intervals contained in $[u(\Omega), v(\Omega)]$ are those of the form $[u(\Omega'), v(\Omega')]$ with Ω' obtained by replacing some t_i in Ω with either r_i or r_{i+1} .

APPENDIX C. MOMENT POLYTOPES OF Ω -RICHARDSON VARIETIES

For $\lambda = (\lambda_1 > \cdots > \lambda_n)$ a decreasing sequence of integers, let $P_{u(\Omega)}^{v(\Omega)}(\lambda)$ be the moment polytope of $X(\Omega)$ under the generalized Plücker embedding $\text{Pl}_\lambda : \text{Fl}_n \rightarrow \mathbb{P}^{n!-1}$. We now show that

- (1) $P_{u(\Omega)}^{v(\Omega)}(\lambda) \cong \text{GZ}(\lambda; \Omega)$ where $\text{GZ}(\lambda; \Omega)$ is a face associated to Ω of the Gelfand–Zetlin polytope determined by λ .
- (2) The moment polytopes for $\Omega \in \text{RTSeq}_n$ are exactly the faces of a subdivision of the permutahedron

$$\text{Perm}(\lambda) := \{w \cdot (\lambda_1, \dots, \lambda_n) \mid w \in S_n\}$$

into combinatorial cubes.

- (3) $P_{u(\Omega)}^{v(\Omega)}(\lambda) \cong \mathbf{C}(\widehat{F}; \lambda)$ where $\mathbf{C}(\widehat{F}; \lambda)$ is a polytope intrinsically associated to the nested forest $\widehat{F} = \widehat{F}(\Omega)$.

The first two points slightly generalize [24, Lemma 6.3] which in our terminology shows the first point for those $\Omega \in \text{RTSeq}_n$ with $|\Omega|_t = n - 1$ maximal, and shows that these are the top dimensional faces of a dissection of $\text{Perm}(\lambda)$ into combinatorial cubes. In Remark C.19 we explain how we can see the different Bott manifold structures on the toric variety associated to $\mathbf{C}(\widehat{F}; \lambda)$ (which is naturally isomorphic to every $X(\Omega)$ with $\widehat{F}(\Omega) = \widehat{F}$) from the recursive structure of the combinatorial cube $\mathbf{C}(\widehat{F}; \lambda)$, giving a different perspective to the identifications from Section 6.

C.1. The generalized Plücker embedding and moment polytopes of Richardson varieties. For $\lambda_1 > \cdots > \lambda_n$ a strictly decreasing sequence, the generalized Plücker embedding $\text{Pl}_\lambda : \text{Fl}_n \rightarrow \mathbb{P}^{n!-1}$ associated to λ realizes Fl_n as a projective variety, and takes

$$M \mapsto \left[\prod_{i=1}^N (\det M_{\{\sigma(1), \dots, \sigma(i)\}, \{1, \dots, i\}})^{\lambda_i - \lambda_{i+1}} \right]_{\sigma \in S_n}$$

where $M_{A,B}$ is the submatrix of M determined by the rows indexed by A and the columns indexed by B , and we set $\lambda_{n+1} = 0$ by convention.

The T -fixed points of Fl_n are the permutation matrices $\{P_\sigma \mid \sigma \in S_n\}$, and we identify P_σ with the permutation σ itself. Recall that S_n acts on \mathbb{R}^n by $\sigma \cdot e_i = e_{\sigma(i)}$, or equivalently

$$(C.1) \quad \sigma \cdot (\lambda_1, \dots, \lambda_n) = (\lambda_{\sigma^{-1}(1)}, \dots, \lambda_{\sigma^{-1}(n)}).$$

Let the i 'th standard character of T be denoted by t_i . If we take the action of T on $\mathbb{P}^{n!-1}$ to be obtained by scaling the coordinate indexed by $\sigma \in S_n$ by $t_{\sigma(1)}^{\lambda_1} t_{\sigma(2)}^{\lambda_2} \cdots t_{\sigma(n)}^{\lambda_n} = t^{\sigma \cdot (\lambda_1, \dots, \lambda_n)}$, then the generalized Plücker embedding is a T -equivariant embedding $\text{Fl}_n \hookrightarrow \mathbb{P}^{n!-1}$, and $\sigma \in \text{Fl}_n$ is mapped to the standard basis vector e_σ .

For a torus T' acting on \mathbb{P}^N by scaling the i 'th coordinate by $t_i^{\mu_i}$, with $\mu_i \in \mathbb{Z}^n$, the moment map is the map $\mathbb{P}^N \rightarrow \mathbb{R}^n$ given by $\mu(\{x_i\}) := \sum_{i=1}^n \frac{|x_i|^2}{\sum_{j=0}^N |x_j|^2} \mu_i$. By the convexity theorem for moment maps [22], if the characters μ_i are distinct (so the only T' -fixed points are the e_i), then for a T' -invariant subvariety $Y \subset \mathbb{P}^N$ we have $\mu(Y) = \text{conv}\{\mu(e_i) : e_i \in Y\}$.

For $X \subset \text{Fl}_n$ the image of a T -invariant subvariety of Fl_n , the moment map $\mu_\lambda : X \rightarrow \mathbb{R}^n$ is obtained by composing the embedding $X \rightarrow \mathbb{P}^{n-1}$ with the moment map on \mathbb{P}^{n-1} under the torus action mentioned earlier. In particular, this implies

$$\mu_\lambda(X) = \text{conv}\{(\sigma \cdot (\lambda_1, \dots, \lambda_n) \mid \sigma \in X)\}.$$

Hence for example the moment polytope of Fl_n is the permutahedron

$$\text{Perm}(\lambda) := \text{conv}\{\sigma \cdot (\lambda_1, \dots, \lambda_n) \mid \sigma \in S_n\} \subset \mathbb{R}^n.$$

For the Richardson variety X_u^v , we have $\sigma \in X_u^v$ if and only if $u \leq \sigma \leq v$ in the Bruhat order.

Definition C.1. If $u \leq v$ in Bruhat order on S_n , then we define the *twisted inverse Bruhat interval polytope* by

$$P_u^v(\lambda) = \text{conv}(\{w \cdot (\lambda_1, \dots, \lambda_n) \mid w \in [u, v]\}).$$

We therefore have the following theorem of Tsukerman–Williams.

Theorem C.2 ([56, Proposition 2.9]). The moment polytope of X_u^v under the generalized Plücker embedding Pl_λ is the twisted Bruhat interval polytope $P_u^v(\lambda)$.

Remark C.3. We note the relation to the *Bruhat interval polytope* [56] $Q_u^v = \text{conv}(\{(w(1), \dots, w(n)) : w \in [u, v]\})$ is that

$$P_u^v(n, \dots, 1) = Q_{w_{0,n}v^{-1}}^{w_{0,n}u^{-1}}.$$

C.2. Recalling GZ polytopes. We now introduce the Gelfand–Zetlin polytopes. To keep exposition brief we quickly recall material that has appeared elsewhere (cf. [47, Section 7.1] for instance, which builds upon [24, 27]).

Definition C.4. For $\lambda = (\lambda_1 > \dots > \lambda_n) \in \mathbb{R}^n$ the *Gelfand–Zetlin polytope* $\text{GZ}(\lambda)$ is the polytope in $\mathbb{R}^{n(n-1)/2}$ containing points $(p_{i,j})_{1 \leq i \leq j \leq n}$ such that $p_{i,1} = \lambda_i$ for $1 \leq i \leq n$ and

$$p_{i,j} \geq p_{i+1,j+1} \geq p_{i+1,j}.$$

We shall think of points in $\text{GZ}(\lambda)$ as fillings of triangular/staircase shape as shown in Figure 8 where the bottom row reads λ_1 through λ_n left to right. Such a filling is often called a *GZ pattern*.

We will specify a face of $\text{GZ}(\lambda)$ by a “face diagram”, consisting of a graph whose underlying vertex set is the positions of entries in a GZ pattern, and whose edges record defining hyperplanes whose intersection is the face. We represent the facet $p_{i,j} = p_{i+1,j+1}$ by a *left edge* connecting vertices (i, j) and $(i+1, j+1)$, and the facet $p_{i,j} = p_{i,j-1}$ by a *right edge* between vertices (i, j) and $(i, j-1)$.

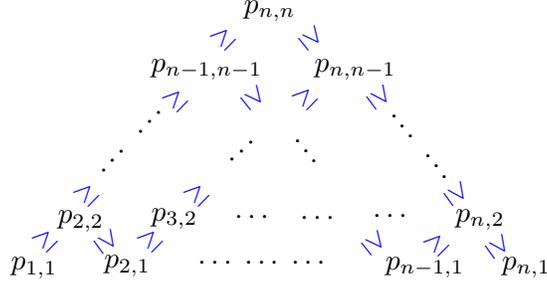


FIGURE 8. A GZ pattern determining a point in $\text{GZ}(\lambda)$

The face diagram associated to a face is not necessarily unique. We remark that it is typical in the literature, after [31], to call a *Kogan* (respectively *dual Kogan*) face of $\text{GZ}(\lambda)$ to be one determined by a collection of left (respectively right) edges, so that every face is the intersection of a Kogan and a dual Kogan face.

The following linear projection (cf. [24, §5]) will be used to relate certain faces of $\text{GZ}(\lambda)$ and Bruhat interval polytopes.

Definition C.5. The linear projection $\mu : \text{GZ}(\lambda) \rightarrow \text{Perm}(\lambda)$ is defined by

$$(C.2) \quad \mu(p) = (y_1 - y_2, y_2 - y_3, \dots, y_n - y_{n+1}),$$

where $y_i := \sum_{k \leq i} p_{k,i}$ for $1 \leq i \leq n+1$ (so $y_{n+1} = 0$).

Referring to Figure 8 the y_i s are produced by summing the entries in each row, going bottom to top, and then $\mu(p)$ records the differences in successive rows.

C.3. Ω -Richardson varieties and GZ-polytopes. Recall the notion of trimming diagram given $\Omega \in \text{RTSeq}_n$ that was introduced at the end of Section 3. We now derive face diagrams from these trimming diagrams.

Definition C.6. For $\Omega \in \text{RTSeq}_n$ we define a face $\text{GZ}(\lambda; \Omega)$ of $\text{GZ}(\lambda)$ by removing the blue edges from the trimming diagram associated to Ω .

See Figure 9 for the face diagram obtained from the trimming diagram in Figure 4.

Proposition C.7. The polytope $\text{GZ}(\lambda; \Omega)$ is cut out by the $p_{i,1} = \lambda_i$, and GZ equalities along the red edges and the inequalities corresponding to the blue edges. Furthermore, in $\text{GZ}(\lambda; \Omega)$ we have $p_{i,j} > \dots > p_{n,i}$ for all i .

Proof. First, note that every GZ inequality is clearly cut out by the red equalities, the blue inequalities, and the inequalities $p_{i,j} \geq p_{i+1,j}$ for all i, j . For $i = 1$ the totality of inequalities $p_{1,1} > \dots > p_{n,1}$ follow because $\lambda_1 > \dots > \lambda_n$. It is straightforward to see that the totality

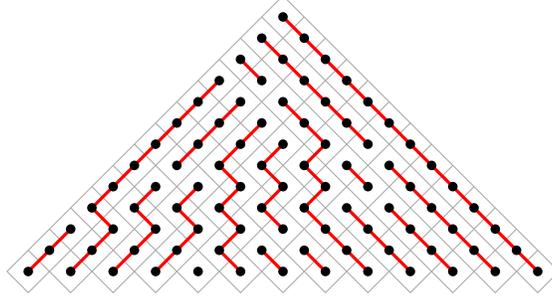


FIGURE 9. The sequence $r_1^4 t_2 r_3 r_4 t_6 t_2 t_7 r_1 t_6 r_5 \in \text{RTSeq}_{13}$

of inequalities $p_{i,i} > \dots > p_{n,i}$ together with the red/blue restrictions between $p_{i,j}$ and $p_{i+1,k}$ for various i and k then imply the totality of inequalities $p_{i+1,1} > \dots > p_{n,i+1}$ and we conclude. \square

Corollary C.8. The faces of $\text{GZ}(\lambda; \Omega)$ are exactly those of the form $\text{GZ}(\lambda; \Omega')$ where Ω' is obtained by changing some subset of the t_i to r_i or r_{i+1} .

Proof. The faces of $\text{GZ}(\lambda; \Omega)$ are obtained by setting some of the defining inequalities to equalities. Since there are only blue edge inequalities, we can either replace two blue edges at an internal vertex with a right red edge or a left red edge, and these correspond to replacing a t_i with either r_i or r_{i+1} respectively (noting that we cannot add in both edges since then that would force an equality $p_{i,j} = p_{i,j+1}$). \square

Figure 10 depicts the three-dimensional $\text{GZ}(\lambda_1, \lambda_2, \lambda_3)$. The two-dimensional faces $\text{GZ}(\lambda; r_1 t_1 t_1)$ and $\text{GZ}(\lambda; r_1 t_1 t_2)$ are shaded. In accordance with the preceding corollary we note, for instance, that the vertices of these faces may be obtained by replacing the t_i with r_i or r_{i+1} . Observe that the unique vertex of degree 4 does not appear as a vertex of either $\text{GZ}(\lambda; r_1 t_1 t_1)$ or $\text{GZ}(\lambda; r_1 t_1 t_2)$.

Definition C.9 ([31, Section 2.2.2]). A *nondegenerate (or simple) vertex* of $\text{GZ}(\lambda)$ is a vertex of the form $\text{GZ}(\lambda; \Omega)$ where $\Omega = r_{i_1} \dots r_{i_n} \in \text{RTSeq}_n$. We define the associated permutation $\pi(v) \in S_n$ to be obtained by setting $\pi(v)(i)$ to be the number of vertices in the path for v starting at $(i, 1)$.

Proposition C.10. If $\Omega = r_{i_1} \dots r_{i_n} \in \text{RTSeq}_n$ then for $v = \text{GZ}(\lambda; \Omega)$ the associated vertex we have $\pi(v) = \varepsilon_{i_n} \dots \varepsilon_{i_1}(\text{id})$. Furthermore, $\mu_\lambda(v) = \pi(v) \cdot (\lambda_1, \dots, \lambda_n)$.

Proof. We induct on n . If $v' = \text{GZ}(\lambda; r_{i_1} \dots r_{i_{n-1}})$ then we can check directly that

$$\pi(v)_j = \begin{cases} \pi(v')_j & j < i_n \\ 1 & j = i_n \\ \pi(v')_{j-1} & j \geq i_n + 1 \end{cases}$$

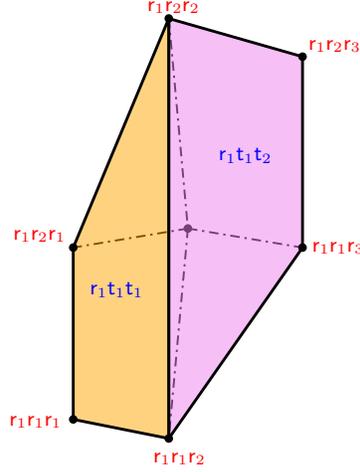


FIGURE 10. $GZ(\lambda_1, \lambda_2, \lambda_3) \subset \mathbb{R}^3$ with maximal dimension faces $GZ(\lambda; \Omega)$ shaded

which shows that $\pi(v) = \varepsilon_{i_n}(\pi(v'))$. Finally, the index α_i of the path which stop at the i 'th row from the bottom is given by $\alpha_i = \pi(v)^{-1}(i)$ so

$$\mu(v) = (\lambda_{\pi(v)^{-1}(1)}, \dots, \lambda_{\pi(v)^{-1}(n)}) = \pi(v) \cdot (\lambda_1, \dots, \lambda_n). \quad \square$$

The similarity between Figure 10 and Figures 1 is explained by our next result.

Theorem C.11. For $\Omega \in \text{RTSeq}_n$, the map μ linearly identifies

$$GZ(\lambda; \Omega) \cong P_{v(\Omega)}^{\mu(\Omega)}(\lambda).$$

Proof. All vertices of $GZ(\lambda; \Omega)$ are nondegenerate, so letting

$$S(\Omega) = \{\pi(v) \mid v \text{ vertex of } GZ(\lambda; \Omega)\},$$

it suffices by Proposition C.10 to check that $S(\Omega) = [u(\Omega), v(\Omega)]$.

By Proposition C.10 again we have

$$S(\Omega_x) = \begin{cases} \{\varepsilon_i(w) \mid w \in S(\Omega)\} & x = r_i \\ \{\varepsilon_i(w) \mid w \in S(\Omega)\} \sqcup \{\varepsilon_{i+1}(w) \mid w \in S(\Omega)\} & x = t_i. \end{cases}$$

On the other hand, by Corollary B.6 we have

$$[u(\Omega_x), v(\Omega_x)] = \begin{cases} [\varepsilon_i(u), \varepsilon_i(v)] = \{\varepsilon_i(w) \mid w \in [u, v]\} & x = r_i \\ [\varepsilon_i(u), \varepsilon_{i+1}(v)] = \{\varepsilon_i(w) \mid w \in [u, v]\} \sqcup \{\varepsilon_{i+1}(w) \mid w \in [u, v]\} & x = t_i, \end{cases}$$

so we conclude by induction. \square

C.4. The HHMP subdivision of the permutahedron. It was shown in [24] that the moment polytopes of the top-dimensional $X(\Omega)$ give a subdivision of the permutahedron into $(n-1)!$ combinatorial cubes [24], which we call the HHMP-subdivision. We reprove this here for the convenience of the reader.

Theorem C.12 ([24]). The moment polytopes $P_{u(\Omega)}^{v(\Omega)}(\lambda)$ are the faces of a subdivision (the HHMP-subdivision) of $\text{Perm}(\lambda)$ into combinatorial cubes.

Proof. The faces of $P_{u(\Omega)}^{v(\Omega)}(\lambda)$ are of the form $P_{u(\Omega')}^{v(\Omega')}(\lambda)$, so it remains to show that the relative interiors $P_{u(\Omega')}^{v(\Omega')}(\lambda)^\circ$ partition $\text{Perm}(\lambda)$. For $z \in [\lambda_i, \lambda_{i+1}]$ we have (see [37, Proposition 3.7])

$$\text{Perm}(\lambda_1, \dots, \lambda_n) \cap \{x_1 = z\} = \{z\} \times \text{Perm}(\lambda_1, \dots, \lambda_{i-1}, \lambda_i + \lambda_{i+1} - z, \lambda_{i+2}, \dots, \lambda_n).$$

For such a z , we let $\lambda(z) := (\lambda_1, \dots, \lambda_{i-1}, \lambda_i + \lambda_{i+1} - z, \lambda_{i+2}, \dots, \lambda_n)$. We recursively record a sequence in RTSeq_n by

$$S(z_1, \dots, z_n; \lambda) = \begin{cases} S(z_2, \dots, z_n; \lambda(z_1))r_i & z_1 = \lambda_i \\ S(z_2, \dots, z_n; \lambda(z_1))t_i & z_1 \in (\lambda_i, \lambda_{i+1}). \end{cases}$$

Then it is straightforward to check inductively that $S(z_1, \dots, z_n; \lambda)$ records the unique relative interior $\mu(\text{GZ}(\lambda; \Omega)^\circ) = P_{u(\Omega)}^{v(\Omega)}(\lambda)^\circ$ that z belongs to. \square

C.5. Ω -Richardson varieties and Nested Forest polytopes.

Definition C.13. For $\widehat{F} \in \widehat{\text{For}}$ we define the nested forest polytope $\mathbf{C}(\widehat{F}; \lambda)$ as the polytope of functions $\phi \in \mathbb{R}^{\text{IN}(\widehat{F})}$ such that for the extension $\phi_\lambda : \text{IN}(\widehat{F}) \sqcup \mathbb{N} \rightarrow \mathbb{R}$ taking $i \mapsto \lambda_i$, we have for each $v \in \text{IN}(\widehat{F})$ the inequalities

$$\phi_\lambda(v_L) \geq \phi_\lambda(v) \geq \phi_\lambda(v_R).$$

Figure 11 demonstrates a nested forest \widehat{F} as well as the inequalities cutting out the associated polytope (in \mathbb{R}^3).

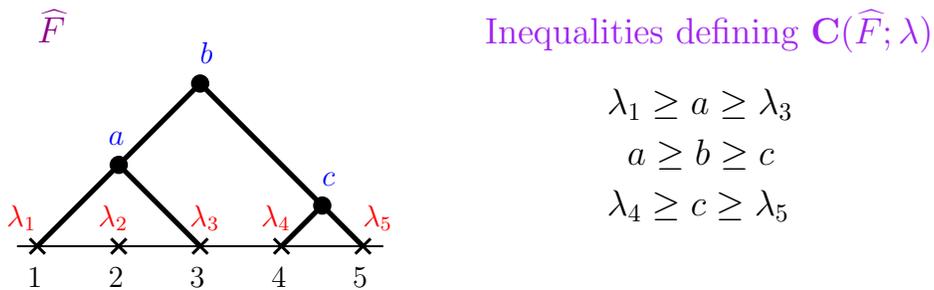


FIGURE 11. A nested forest \widehat{F} and the corresponding $\mathbf{C}(\widehat{F}; \lambda)$

Theorem C.14. There is a linear isomorphism $\text{GZ}(\lambda; \Omega) \cong \mathbf{C}(\widehat{F}(\Omega); \lambda)$ given by assigning a GZ pattern to the function which takes an internal node v to the corresponding $p_{i,j}$ in the trimming diagram of \widehat{F} .

Proof. This immediately follows from Proposition C.7 after identifying equal $p_{i,j}$. \square

Corollary C.15. For a strictly decreasing λ , $\mathbf{C}(\widehat{F}; \lambda)$ is a combinatorial cube.

Corollary C.16. $X(\widehat{F})$ is the toric variety associated to the polytope $\mathbf{C}(\widehat{F}; \lambda)$.

Proof. By definition, $X(\widehat{F})$ is isomorphic to $X(\Omega)$ for any $\Omega \in \text{Trim}(\widehat{F})$, and we have just shown that $X(\Omega)$ is the toric variety associated to $\text{GZ}(\lambda; \Omega) \cong \mathbf{C}(\widehat{F}(\Omega); \lambda) = \mathbf{C}(\widehat{F}; \lambda)$. \square

Remark C.17. A tedious verification which we omit shows that the isomorphism $\pi_{\Omega'}^{-1} \pi_{\Omega} : X(\Omega) \rightarrow X(\Omega')$ is induced by the composite isomorphism on moment polytopes $\text{GZ}(\lambda; \Omega) \cong \mathbf{C}(\widehat{F}; \lambda) \cong \text{GZ}(\lambda; \Omega')$, which shows that the identification of $X(\widehat{F})$ with the toric variety associated to $\mathbf{C}(\widehat{F}; \lambda)$ can be done in such a way that the isomorphism $\pi_{\Omega'} : X(\Omega') \rightarrow X(\widehat{F})$ for any $\Omega' \in \text{Trim}(\widehat{F})$ is induced by the linear isomorphism $\text{GZ}(\lambda; \Omega') \cong \mathbf{C}(\widehat{F}; \lambda)$.

One way of creating a combinatorial cube is to take a linear family of combinatorial cubes $C(\lambda) \subset \mathbb{R}^n$ (i.e. for two strongly equivalent combinatorial cubes $C(a)$ and $C(b)$ we define $C(\lambda)$ for $\lambda \in [a, b]$ by $C(ta + (1-t)b) = tC(a) + (1-t)C(b)$) and take

$$C' = \{(x, z) \in \mathbb{R}^n \times [a, b] \mid x \in C(z)\}.$$

It turns out that $\mathbf{C}(\widehat{F}; \lambda)$ has this recursive structure that makes it into a combinatorial cube. Furthermore different ways of expressing \widehat{F} as a product of \underline{i} and \underline{i}_o give different realizations of $\mathbf{C}(\widehat{F}; \lambda)$ as a combinatorial cube.

Theorem C.18. Let λ be a decreasing sequence, and let $\lambda' = (\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots)$. Then we have

$$\mathbf{C}(\widehat{F} \cdot i_o; \lambda) = \mathbf{C}(\widehat{F}; \lambda').$$

For $z \in [\lambda_{i-1}, \lambda_i]$, let $\lambda(z) = (\lambda_1, \dots, \lambda_{i-1}, z, \lambda_{i+1}, \dots)$. Then

$$\mathbf{C}(\widehat{F} \cdot i; \lambda) = \{(y, z) \mid z \in [\lambda_i, \lambda_{i+1}], \text{ and } y \in \mathbf{C}(\widehat{F}; \lambda(z))\} \subset \mathbb{R}^{\text{IN}(\widehat{F})} \times \mathbb{R} \cong \mathbb{R}^{\text{IN}(\widehat{F} \cdot i)}.$$

Remark C.19. The toric variety associated to a P' arising as the total family of a linear family of polytopes $P(z)$ strongly equivalent to a fixed polytope P realizes the toric variety of P' as $\mathbb{P}(\underline{\mathbb{C}} \oplus \mathcal{L})$ for a toric line bundle \mathcal{L} on the toric variety associated to P . The different ways of realizing the nested forest polytope therefore correspond to different Bott manifold structures on $X(\Omega)$, giving an alternate perspective on the computations in Section 6.

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UNIVERSITE CLAUDE BERNARD LYON 1, CNRS, ECOLE CENTRALE DE LYON, INSA LYON, UNIVERSITÉ JEAN MONNET, ICJ UMR5208, 69622 VILLEURBANNE, FRANCE

Email address: `nadeau@math.univ-lyon1.fr`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, ON M5S 2E4, CANADA

Email address: `hunter.spink@utoronto.ca`

DEPARTMENT OF MATHEMATICAL AND COMPUTATIONAL SCIENCES, UNIVERSITY OF TORONTO MISSISSAUGA, MISSISSAUGA, ON L5L 1C6, CANADA

Email address: `vasu.tewari@utoronto.ca`