# THE COXETER FLAG VARIETY

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ABSTRACT. For a Coxeter element $c$ in a Weyl group $W$, we define the $c$-Coxeter flag variety $\mathrm{CFl}_c \subset G/B$ as the union of left-translated Richardson varieties $w^{-1}X_w^{wc}$. This is a complex of toric varieties whose geometry is governed by the lattice $\mathrm{NC}(W,c)$ of $c$-noncrossing partitions. We show that $\mathrm{CFl}_c$ is the common vanishing locus of the generalized Plücker coordinates indexed by $W\setminus\mathrm{NC}(W,c)$. We also construct an explicit affine paving of $\mathrm{CFl}_c$ and identify the $T$-weights of each cell in terms of $c$-clusters. This paving gives a GKM description of $H^\bullet(\mathrm{CFl}_c)$ and $H^\bullet_{T_{ad}}(\mathrm{CFl}_c)$ in terms of the induced Cayley subgraph on $\mathrm{NC}(W,c)$, and we show these rings are naturally isomorphic for different choices of $c$. In type A, this recovers the quasisymmetric flag variety for a special $c$, and for general $c$ we show the cohomology ring has a presentation as permuted quasisymmetric coinvariants.

CONTENTS

# 1. Introduction

Let $G$ be a reductive group with opposite Borel subgroups $B, B^- \subseteq G$ and maximal torus $T = B \cap B^-$. The *generalized flag variety* is the coset space $G/B$, which is a projective algebraic variety. The Weyl group $W = N_G(T)/T$ permutes the $T$-stable subvarieties in $G/B$ and indexes the cells in the Bruhat decomposition

$$G/B = \bigsqcup_{w \in W} \mathring{X}^w \qquad \text{with} \qquad \mathring{X}^w \coloneqq BwB/B \cong \mathbb{A}^{\ell(w)}.$$

One of the main ways in which the geometry of $G/B$ connects with the combinatorics of $W$ and the associated Bruhat order $\leqslant_B$ is via Schubert, opposite Schubert, and Richardson varieties

$$X^w \coloneqq \overline{BwB/B}, \qquad X_w \coloneqq \overline{B^- wB/B}, \qquad \text{and} \qquad X_u^v \coloneqq X^v \cap X_u \text{ for } u \leqslant_B v.$$

Let $c \in W$ be a *Coxeter element*, by which we mean the product, in any order, of the simple reflections of $W$ determined by $B$. We will show later that $w \leqslant_B wc$ is equivalent to $\ell(w) + \ell(c) = \ell(wc)$ and we will write $w \cdot c$ to represent the product $wc$ and the assertion that the product is length additive. We define a *$c$-Coxeter Richardson variety* to be a Richardson variety of the form

$$X_w^{w \cdot c} \subseteq G/B,$$

which we show is a toric variety of dimension $\ell(c)$. In this article we introduce an equidimensional complex of toric varieties that we call the *$c$-Coxeter flag variety*

$$\mathrm{CFl}_c \coloneqq \bigcup w^{-1} X_w^{w \cdot c} \subseteq G/B.$$

Surprisingly, many $w \in W$ produce exactly the same $w^{-1}X_w^{w \cdot c}$, so we can view this translation process as removing certain redundancies from the set of Coxeter Richardson varieties. We will show that $\mathrm{CFl}_c$ is a geometric realization of Coxeter–Catalan phenomena in algebraic combinatorics.

Beginning with Reiner's 1997 paper [55], developments in algebraic combinatorics have demonstrated that the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$ are the "Type $A$" member in the family of *$W$-Catalan numbers* $\mathrm{Cat}_W$; see e.g. [62]. These numbers enumerate interesting combinatorial structures for each $W$, many of which vary nontrivially over the choice of a Coxeter element $c \in W$. We will need three examples of this "Coxeter–Catalan" phenomenon.

(A) The lattice of *$c$-noncrossing partitions* $\mathrm{NC}(W, c) \subseteq W$ comprises all prefixes $\tau_1 \cdots \tau_i$ of minimal length reflection factorizations $c = \tau_1 \cdots \tau_n$; this type-independent definition comes from the theory of Artin groups and the $K(\pi, 1)$ conjecture, see [11, 22] and [3, Chapter 1].

(B) The *(combinatorial) $c$-clusters* $\mathrm{Cl}_c$ are subsets of the almost positive roots corresponding to $g$-vectors for a cluster algebra; Fomin–Zelevinsky [30, 31] introduced combinatorial clusters to record cluster algebraic data, and Reading [52] generalized this notion to arbitrary $c$.

(C) The *c-Cambrian congruence* $\equiv_c$ on $W$ is a lattice quotient of the right weak order $\leqslant_R$; introduced by Reading [51] as a generalization of the Tamari lattice and the associahedron, it provides an interface between noncrossing partitions and clusters [52, 54].

For a fixed $W$ the size of each set above is independent of $c$, and we have

$$\mathrm{Cat}_W := |\,\mathrm{NC}(W,c)| = |\mathrm{Cl}_c| = |W/\equiv_c|,$$

which we take as our definition of the $W$-Catalan number. There is also a well-known formula for $\mathrm{Cat}_W$ using the fundamental degrees of $W$-invariants [26].

Our results show that the families in (A)–(C) anticipate the structure of $\mathrm{CFl}_c$. We begin with a description of $\mathrm{CFl}_c$ in terms of the generalized Plücker coordinates on $G/B$, among which we denote by $\mathrm{Pl}_w$ the extremal coordinate indexed by $w \in W$.

**Theorem A** (Theorem 9.3)**.** We have $\mathrm{CFl}_c = \bigcap_{w \in W \setminus \mathrm{NC}(W,c)} \{\mathrm{Pl}_w = 0\}$.

The "thick matroid strata" where extremal Plücker coordinates vanish was first studied in [32, 33] and plays an important role in combinatorial algebraic geometry, but these strata exhibit notorious combinatorial complexity. In fact, Mnëv universality [49] shows that even in type $A$ Grassmannians the vanishing loci for extremal Plücker coordinates can realize arbitrary singularities, so the tractability of $\mathrm{CFl}_c$ is noteworthy.

Theorem A implies that the $T$-fixed points $(\mathrm{CFl}_c)^T$ are the $c$-noncrossing partitions $\mathrm{NC}(W,c)$, and $\mathrm{CFl}_c$ is the union of all $T$-invariant subvarieties $X$ with $X^T \subset \mathrm{NC}(W,c)$. In particular, the intersection of $\mathrm{CFl}_c$ with a Schubert cell $\mathring{X}^w$ is nonempty if and only if $w \in \mathrm{NC}(W,c)$. We call these *Coxeter Schubert cells*

$$\mathring{X}^u_{\mathrm{NC}} := \mathrm{CFl}_c \cap \mathring{X}^u \qquad \text{for } u \in \mathrm{NC}(W,c).$$

In the following result, we use a bijection $\mathrm{Clust} : \mathrm{NC}(W,c) \to \mathrm{Cl}_c$ of Biane–Josuat-Vergès [13], and write $\mathrm{Clust}^+(u)$ for the subset of positive roots in $\mathrm{Clust}(u)$; see Section 3.5.

**Theorem B** (Theorem 9.3)**.** The Coxeter Schubert cells give a $T$-stable affine paving of $\mathrm{CFl}_c$,

$$\mathrm{CFl}_c = \bigsqcup_{u \in \mathrm{NC}(W,c)} \mathring{X}^u_{\mathrm{NC}}, \qquad \text{and as a } T\text{-representation} \qquad \mathring{X}^u_{\mathrm{NC}} \cong \bigoplus_{\alpha \in \mathrm{Clust}^+(u)} \mathbb{C}_{-u\cdot\alpha}.$$

An immediate consequence of Theorem B is that $H_\bullet(\mathrm{CFl}_c)$ has a free basis indexed by the closure of each Coxeter Schubert cell, which we call a *Coxeter Schubert variety*

$$X^u_{\mathrm{NC}} = \overline{\mathring{X}^u_{\mathrm{NC}}} \qquad \text{for } u \in \mathrm{NC}(W,c).$$

Therefore we have

$$\dim H_{2i}(\mathrm{CFl}_c) = \#\{C \in \mathrm{Cl}_c \mid C \text{ contains } i \text{ positive roots}\},$$

see also Remark 9.5. The irreducible components are those $X^u_{\mathrm{NC}}$ indexed by $u \in \mathrm{NC}(W,c)$ where $\mathrm{Clust}(u) = \mathrm{Clust}^+(u)$. These are the *fully supported c-noncrossing partitions* $\mathrm{NC}(W,c)^+ \subset \mathrm{NC}(W,c)$, those which do not lie in any subgroup generated by a strict subset of the simple reflections, enumerated by the lesser-known *positive $W$-Catalan numbers* $\mathrm{Cat}^+_W := |\,\mathrm{NC}(W,c)^+|$.

The Coxeter Schubert varieties are $W$-translates of toric Richardson varieties; to specify which ones, we use Reading's $c$-sortable combinatorics [52]; see Section 13. In particular there is a canonical way to assign each $x \in W$ a noncrossing partition $\mathrm{nc}_c(\pi_\downarrow(x)) \in \mathrm{NC}(W,c)$ which is bijectively determined by the Cambrian class of $x$.

**Theorem C** (Section 15)**.** For each $u \in \mathrm{NC}(W,c)$ there is a unique $c' \leqslant_B c$ (the subproduct of simple reflections in the minimal standard parabolic subgroup containing $u$) such that

$$X^u_{\mathrm{NC}} = w^{-1}X^{w\cdot c'}_w \qquad \text{for every } w \text{ with } \mathrm{nc}_c(\pi_\downarrow(w^{-1}w_\circ)) = u.$$

In particular, $w, w' \in W$ correspond to the same $X^u_{\mathrm{NC}}$ if and only if $w^{-1}w_\circ \equiv_c (w')^{-1}w_\circ$.

Further aspects of the combinatorics are elaborated in Section 2, where we describe the complex of moment polytopes associated to $\mathrm{CFl}_c$.

We conclude this portion of the introduction by sketching some results and questions about the cohomology ring of $\mathrm{CFl}_c$. Each of the following generalizes to—and depends upon—similar statements about torus-equivariant cohomology, where in all cases the torus in question is the adjoint torus $T_{ad} := T/Z(G)$. We omit these results for brevity.

First recall that in addition to the classical $W$-coinvariant presentation of $H^\bullet(G/B)$ due to Borel, there is a presentation of $H^\bullet(G/B)$ as a GKM-type graph cohomology ring for the Cayley graph of $W$. Moreover, Billey's formula [14] defines a distinguished basis of Schubert classes

$$\mathfrak{S}_w \in H^\bullet(G/B) \qquad \text{for } w \in W$$

which are dual to the homology classes of the Schubert varieties.

We find similar results for $\mathrm{CFl}_c$, presenting $H^\bullet(\mathrm{CFl}_c)$ as the GKM ring for the Hasse diagram of $\mathrm{NC}(W,c)$, which is a subgraph of the Cayley graph. This allows us to prove the following result, which makes use of the fact [42, Proposition 3.16] that all Coxeter elements are conjugate.

**Theorem D** (Corollary 10.7)**.** The cohomology ring of the Coxeter flag variety is independent of the choice of Coxeter element: if $c$ and $c' = wcw^{-1}$ are both Coxeter elements then there is an associated isomorphism

$$\Psi_{c,w} : H^\bullet(\mathrm{CFl}_c) \cong H^\bullet(\mathrm{CFl}_{c'}).$$

Furthermore $\Psi_{c',v} \circ \Psi_{c,w} = \Psi_{c,vw}$ when these maps are well-defined.

We also show, using a combinatorial argument, that $H^\bullet(\mathrm{CFl}_c)$ has a distinguished basis—dual to a homology basis of translated Richardson varieties described in Theorem C—which we call *Coxeter Schubert classes* and denote by

$$\mathfrak{S}_u^{\mathrm{NC}} \in H^\bullet(\mathrm{CFl}_c) \qquad \text{for } u \in \mathrm{NC}(W,c).$$

Surprisingly, the maps $\Psi_{c,w}$ in Theorem D do not map Coxeter Schubert classes to Coxeter Schubert classes, making this basis heavily dependent on $c$.

For a specific $c$ in type A we have a complete combinatorial understanding of the $\mathfrak{S}_u^{\mathrm{NC}}$ [7, 8] (see the discussion after the statement of Theorem E). We understand very little about the combinatorics of $\mathfrak{S}_u^{\mathrm{NC}}$ outside of this case, and leave the reader with a list of questions to generalize results in [7, 8] for what we might aspirationally call "Coxeter Schubert calculus".

**Question 1.1.** Is there an analogue of Billey's formula for the $\mathfrak{S}_u^{\mathrm{NC}}$ in the graph cohomology ring?

**Question 1.2.** For $\iota : \mathrm{CFl}_c \hookrightarrow G/B$ the inclusion map, is there a combinatorially positive interpretation of the coefficients $a_v^u$ in

$$\iota^* \mathfrak{S}_v = \sum_{u \in \mathrm{NC}(W,c)} a_v^u \mathfrak{S}_u^{\mathrm{NC}} \in H^\bullet(\mathrm{CFl}_c)?$$

These numbers equivalently decompose the classes $[X_{\mathrm{NC}}^u] = \sum a_v^u [X^v]$ in the Schubert homology basis $[X^v]$ of $H_\bullet(G/B)$, and are therefore positive for geometric reasons.

**Question 1.3.** How do Coxeter Schubert polynomials multiply? Are the coefficients $c_{u,v}^{w,\mathrm{NC}}$ in

$$\mathfrak{S}_u^{\mathrm{NC}} \mathfrak{S}_v^{\mathrm{NC}} = \sum_{w \in \mathrm{NC}(W,c)} c_{u,v}^{w,\mathrm{NC}} \mathfrak{S}_w^{\mathrm{NC}} \in H^\bullet(\mathrm{CFl}_c)$$

positive, and is there a combinatorial witness to this fact? Because it is a reducible variety, $\mathrm{CFl}_c$ lacks Poincaré duality and to our knowledge does not have a geometric reason to be true.

As remarked above, there are $T_{ad}$-equivariant versions of these latter questions involving the notion of Graham-positivity [37].

1.1. **Type** A**.** We now highlight some special features that appear when $W = S_{n+1}$, e.g. when $G = \mathrm{GL}_{n+1}$. In type A, $G/B$ is isomorphic to the variety of complete flags in $n$-space. Denoting $\mathrm{Sym}_{n+1} = \mathbb{Z}[x_1, \ldots, x_{n+1}]^{S_{n+1}}$, Borel's theorem [18] gives an isomorphism

$$H^\bullet(G/B) \cong \mathbb{Z}[x_1, \ldots, x_{n+1}]/\langle f(x_1, \ldots, x_{n+1}) - f(0, \ldots, 0) \mid f \in \mathrm{Sym}_{n+1} \rangle, \tag{1.1}$$

by identifying $x_1, \ldots, x_{n+1}$ with the negative Chern roots of the tautological flag.

We first recall our prior work with P. Nadeau in [7, 8]. Here, the Coxeter flag variety for $c = (n+1 \, \cdots \, 2\, 1)$ was studied under the name *quasisymmetric flag variety*, due to the following result.

**Theorem 1.4** ([8, Theorem A]). For $G = \mathrm{GL}_{n+1}$, $W = S_{n+1}$, and $c = (n+1\ \cdots\ 2\,1)$, we have

$$H^\bullet(\mathrm{CFl}_c) \cong \mathbb{Z}[x_1, \ldots, x_{n+1}]/\langle f(x_1, \ldots, x_{n+1}) - f(0, \ldots, 0) \mid f \in \mathrm{QSym}_{n+1}\rangle$$

where $\mathrm{QSym}_n$ is the ring of quasisymmetric polynomials of Gessel [35] and Stanley [59].

Combining Theorem 1.4 with our results, particularly Theorem D, we see that each Coxeter flag variety in type A corresponds to a nonstandard definition of quasisymmetry.

**Theorem E** (Theorem 12.2). For $G/B = \mathrm{Fl}_{n+1}$ and any Coxeter element $c \in S_{n+1}$, the restriction map $H^\bullet(\mathrm{Fl}_{n+1}) \to H^\bullet(\mathrm{CFl}_c)$ is surjective. Moreover, if $c = (w(n+1)\,w(n)\,\cdots\,w(1))$ as a cycle, then this map realizes $H^\bullet(\mathrm{CFl}_c)$ as the further quotient of (1.1) as

$$H^\bullet(\mathrm{CFl}_c) \cong \mathbb{Z}[x_1, \ldots, x_{n+1}]\Big/\Big\langle f(x_{w(1)}, \ldots, x_{w(n+1)}) - f(0, \ldots, 0) \Bigm| f \in \mathrm{QSym}_{n+1}\Big\rangle,$$

the ring of *permuted quasisymmetric coinvariants*.

For $c = (n+1\ \cdots\ 2\,1)$, we were able to answer Question 1.1 very concretely and to give a (highly recursive) combinatorial solution to Questions 1.2 and 1.3 that confirmed positivity. Moreover, it was possible to realize each Coxeter Schubert class in the Borel presentation by defining a family of *forest polynomials*, paralleling the Lascoux–Schutzenberger model of Schubert polynomials for the Schubert classes [47].

**Question 1.5.** For $G = \mathrm{GL}_{n+1}$, do the Coxeter Schubert classes $\mathfrak{S}_u^{\mathrm{NC}}$ have nice polynomial representatives in $x_1, \ldots, x_{n+1}$?

In other types, we can be certain the analogous question has a negative answer: the 2nd Betti numbers of $\mathrm{CFl}_c$ exceed those of $G/B$ in types $B$ and $C$, so the canonical map $\iota^*$ from Borel's model to $H^\bullet(\mathrm{CFl}_c)$ cannot be surjective; see Remark 9.5.

1.2. **Toric complexes.** The class of $c$-Coxeter Richardson varieties has been considered elsewhere owing to its connection to the permutohedral toric variety, which is the closure $\overline{T\cdot x}$ of a generic $T$-orbit in $G/B$.

For $W = S_{n+1}$ and $c = (n+1\ \cdots\ 2\,1)$, this connection was first observed in purely cohomological terms by [2], and then later as a regular toric degeneration of $\overline{T\cdot x}$ into $c$-Coxeter Richardson varieties by the authors of [40, 48]. This result was greatly generalized in [44], where it was shown that in any type and for any Coxeter element $c$ there is a degeneration of $\overline{T\cdot x}$ into the union

$$\mathrm{HHMP}_c := \bigcup X_w^{w\cdot c}.$$

Each of these results relies on the fact that the moment polytopes for the $c$-Coxeter Richardson varieties form a regular subdivision of the generalized flag permutahedron

$$\mathrm{Perm}_W := \mathrm{conv}\{w\cdot\rho \mid w \in W\},$$

where $\rho \in \mathrm{Char}(T)$ is the Weyl vector. Figure 1 illustrates this in type $\mathrm{A}_3$: for $c = s_3 s_2 s_1$, the two polytopes of the same color are translates of one another. Defant–Sherman-Bennett–Williams [28] studied a triangulation refining this subdivision which is determined by the combinatorics of noncrossing partitions and $c$-sortability.

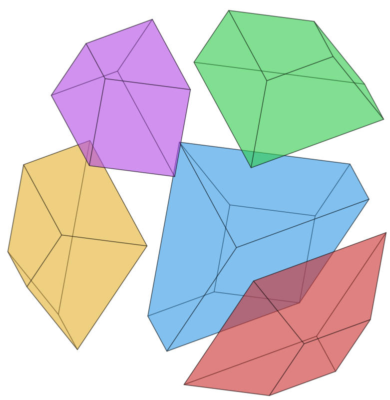
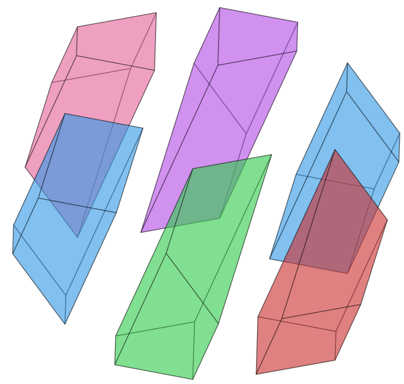

FIGURE 1. HHMP decomposition of the type $\mathrm{A}_3$ permutahedron corresponding to $c = s_1 s_3 s_2$ (left) and $c = s_3 s_2 s_1$ (right). The corresponding Coxeter flag varieties are each unions of five 3-dimensional toric varieties, whose moment polytopes are translates of the depicted polytopes (the two blue polytopes on the right are identified under this translation). See Figure 3 for how the polytopes on the left assemble to a polytopal complex whose face lattice is $\mathrm{CFl}_{s_1 s_3 s_2}$.

Our approach also makes use of moment polytopes, and in particular the "moment complex" comprising all moment polytopes for translated $c$-Coxeter Richardson varieties. This necessitates some combinatorial overlap between our results and [28], and in particular Theorem 14.3 is proved independently in [28, Theorem 1.9 and Corollary 1.10].

1.3. **Paper structure.** After Section 2, which reviews the moment complex of $\mathrm{CFl}_c$, we recall preliminary material on posets and Coxeter groups (Section 3), reductive groups (Section 4) and the geometry of $G/B$ (Section 5). Section 6 concerns Plücker coordinates, including a mix of new and folklore results that we present in a single exposition. The study of $\mathrm{CFl}_c$ begins in earnest in Section 7, where we relate the $T$-fixed points of $c$-Coxeter Richardson varieties to noncrossing partitions, and in Section 8 where we study $\mathrm{CFl}_c$ as a toric complex using these combinatorics. In

Section 9 we introduce cluster charts and prove both Theorems A and B. Sections 10, 11, and 12 contain a full account of our cohomological results, including Theorems D and E. Finally, Sections 13, 14, and 15 resolve the problem of determining when two $c$-Coxeter Richardson varieties are equal after translation, proving Theorem C. In Appendices A, B, C, and D, we apply Sections 1–9 to the classical groups of the corresponding type and construct explicit examples of a $c$-Coxeter flag variety in each case.

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## 2. Detailed structure of the moment complex

In this section we describe the moment complex of $\mathrm{CFl}_c$, which underpins the combinatorics. We will consider $\mathrm{CFl}_c$ as a *toric complex*, which we take to mean a union of projective toric varieties $\bigcup Y_i$ in a common projectivized $T$-representation, each of which determines a moment polytope $\mu(Y_i)$ as the convex hull of the characters associated to the $T$-fixed points $Y_i^T$.

**Remark 2.1.** Similar terms and ideas are present in the literature; see for example [23, 25, 63]. As no source gives both a clear definition and a satisfactory name, we take the above out of expediency.

The generalized Plücker embedding Pl realizes $\mathrm{CFl}_c$ as a toric complex. The $T$-fixed points of a $c$-Coxeter Richardson variety $X_w^{w\cdot c}$ come from the Bruhat interval $[w, w\cdot c]$. The moment polytope is the $\ell(c)$-dimensional twisted Bruhat interval polytope

$$P_{[w,w\cdot c]} = \mathrm{conv}\left([w, w\cdot c]\cdot\rho\right) \qquad \text{where } \rho \text{ is the Weyl vector,}$$

see Section 6.4. Translation by $w^{-1}$ in $G/B$ also translates the moment polytope by $w^{-1}$, so the moment polytopes determined by $\mathrm{CFl}_c$ are all $w^{-1}P_{[w,w\cdot c]}$ and their lower-dimensional faces. These translated twisted Bruhat interval polytopes and their faces have remarkably special combinatorics.

**Theorem 2.2** (Theorem 8.5)**.** Each $w^{-1}P_{[w,w\cdot c]}$ (and its faces) are $c$-polypositroids in the sense of Lam–Postnikov [46, §13].

The remainder of this section will consider how we can glue these polypositroids together to create a polytopal complex that encodes the structure of $\mathrm{CFl}_c$.

Our results show that the set of vertices and edges in the totality of the moment polytopes of $\mathrm{CFl}_c$ are the Hasse diagram of $\mathrm{NC}(W, c)$ under the Kreweras order, corresponding to a decomposition of $\mathrm{NC}(W, c)$ (Theorem 7.1) into translates of certain Bruhat intervals:

$$\mathrm{NC}(W, c) = \bigcup w^{-1}[w, w \cdot c].$$

We would like to upgrade this statement about the 1-skeleton of the "union" of the moment polytopes to describe a higher dimensional polytopal "moment complex" that faithfully encodes the combinatorics of how the $T$-orbit closures fit together in $\mathrm{CFl}_c$. We cannot do this by simply taking the union of the polytopes directly as there are spurious overlaps. In parallel with the face–orbit correspondence for toric varieties, we would like

(1) The dimension $k$ faces of the polytopal complex biject with $k$-dimensional $T$-orbits.
(2) Face containment corresponds to $T$-orbit closure containment.

One issue that could arise in creating such a complex is that there could be two toric varieties in the complex with the exact same moment polytope, such as if the toric complex was the union of two general $T$-orbit closures in a projectivized $T$-representation. $\mathrm{CFl}_c$ has a remarkable property that allows us to carefully glue the overlapping polytopes together in a way that achieves the intended goal. In Theorem 8.3 we show if $Y, Z \subset \mathrm{CFl}_c$ then

$$Y^T \subset Z^T \iff Y \subset Z.$$

In particular, the facial structure we are attempting to achieve is detected entirely at the level of vertex sets. We can therefore define an abstract polytopal complex whose vertex set is $\mathrm{NC}(W, c)$ and edge set is $\mathcal{E}(\mathrm{NC}(W, c))$ which encodes the combinatorial structure of $\mathrm{CFl}_c$:

$$\mathrm{Complex}(\mathrm{CFl}_c) = \Big(\bigsqcup w^{-1} P_{[w, w \cdot c]}\Big) / \sim,$$

where $\sim$ glues two faces when they share the same vertex set. To clarify our intentions, consider the right panel of Figure 2, consisting of the three polytopes which glue to form $\mathrm{Complex}(\mathrm{CFl}_c)$. Even though the polytopes overlap in a diamond shape, the above gluing process says we should only identify the two bolded edges. See Figure 3 for another example.

The top-dimensional faces of $\mathrm{Complex}(\mathrm{CFl}_c)$ correspond to the $\mathrm{Cat}^+_W$-many irreducible components of $\mathrm{CFl}_c$, which does not depend on the choice of $c$. This is in contrast to the HHMP subdivisions of $\mathrm{Perm}_W$ as illustrated in Figure 1.

Finally, we note that the affine paving of $\mathrm{CFl}_c$ in Theorem B induces a decomposition of the moment complex $\mathrm{Complex}(\mathrm{CFl}_c)$ into "open half-cubes" around simple vertices. In fact, the more careful statement Theorem 9.3 shows that we can build $\mathrm{Complex}(\mathrm{CFl}_c)$ one half-cube at a time, in such a way that each intermediate complex is closed. By collapsing these half-cubes in reverse order we show that $\mathrm{Complex}(\mathrm{CFl}_c)$ is contractible.

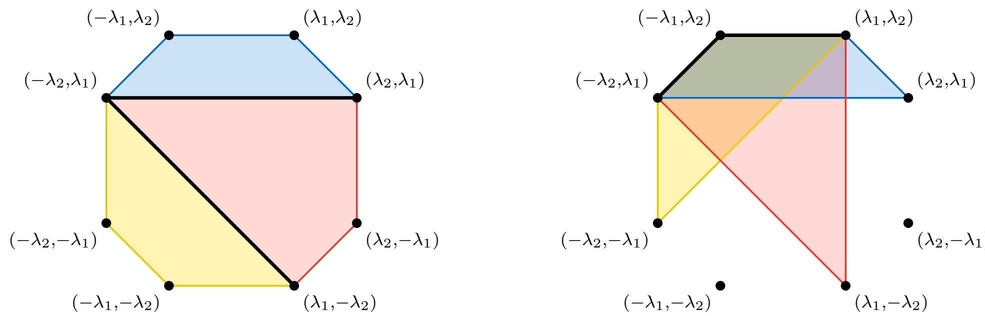


FIGURE 2. The blue, red, and yellow trapezoids $P_{[\mathrm{id},c]}$, $P_{[s_1,s_0s_1s_0]}$, and $P_{[c,w_\circ]}$ comprising the HHMP subdivision of the type $\mathrm{B}_2$-permutahedron for $c = s_0s_1$ (left) and the intersecting polytopes $P_{[\mathrm{id},c]}$, $s_1P_{[s_1,s_0s_1s_0]}$, and $c^{-1}P_{[c,w_\circ]}$ (right), where we only glue the bold edges. The elements $s_1s_0$ and $w_\circ$ outside of $\mathrm{NC}(\mathrm{B}_2,c)$ correspond to the vertices $s_1s_0 \cdot (\lambda_1,\lambda_2) = (\lambda_2,-\lambda_1)$ and $w_\circ \cdot (\lambda_1,\lambda_2) = (-\lambda_1,-\lambda_2)$

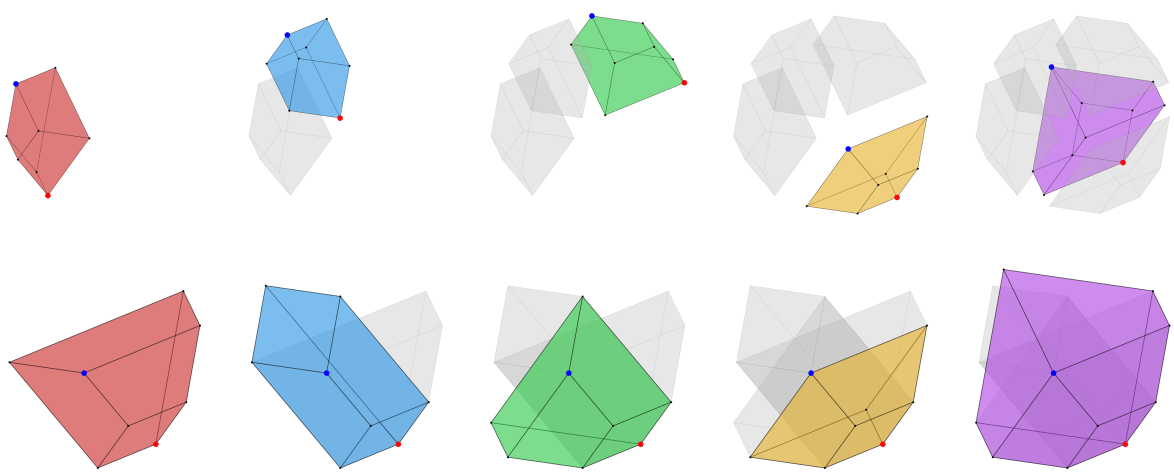

FIGURE 3. Cumulative overlay of the moment polytopes for $c = s_1s_3s_2$; The top row is an exploded view of the "HHMP" decomposition and the bottom is the polytopal complex. The moment polytopes $P_{[w,wc]}$ from left to right come from $w$ equal to 2314, 2413, 1423, 1234, and 1324. The first four have the same face lattice as a cube, and the last has the face lattice of the 10 vertex tetragonal trapezohedron. The red and blue dots indicate respectively the images of $w$ and $w \cdot c$ either before or after translation by $w^{-1}$.

## 3. Combinatorial Preliminaries

We recall standard Coxeter group notation and refer to [16, 21, 42] for background. Let $\mathbb{E}$ be a Euclidean space with inner product $(\cdot,\cdot)$, and $\Phi \subseteq \mathbb{E}$ a finite, crystallographic root system with a fixed choice of positive roots $\Phi^+$ and simple base $\Delta \subseteq \Phi^+$. Let $W$ be the group generated by the set of reflections $\mathsf{T} = \{s_\beta \mid \beta \in \Phi^+\}$, which is a finite, crystallographic Coxeter group.

The association between positive roots and reflections is a bijection, and for $\tau \in \mathsf{T}$ we denote the corresponding positive root by $r(\tau) \in \Phi^+$. Let $\mathsf{S} = \{s_\alpha \mid \alpha \in \Delta\}$ be the *simple reflections* in $W$ associated to $\Delta$, so that $\mathsf{T} = \{wsw^{-1} \mid s \in \mathsf{S}, w \in W\}$. Denote the rank of $W$ by

$$n = |\mathsf{S}| = |\Delta|.$$

Note that we do not require $n$ to be the dimension of the ambient space $\mathbb{E}$.

A *reflection subgroup* $W' \subseteq W$ is a subgroup generated by a subset of $\mathsf{T}$. Each reflection subgroup is the Coxeter group for a sub-root system $\Phi' \subseteq \Phi$. A reflection subgroup $W' \subseteq W$ is a *standard parabolic subgroup* if it can be generated by a subset $\mathsf{S}' \subseteq \mathsf{S}$; in this case each Coxeter element $c \in W$ determines a Coxeter element $c' \in W'$ by multiplying the elements of $\mathsf{S}'$ in the same order as in $c$.

All posets $P$ we consider in this article are finite and graded, with unique minimal and maximal elements. We denote the partial order on $P$ by $<_P$. We refer the reader to standard combinatorial texts (cf. [60]) for any undefined terminology in the context of posets. As is standard in combinatorics, we will identify a poset with its Hasse diagram whenever necessary. The underlying partial order will be clear from context.

3.1. **Bruhat order.** A *reduced word* $\boldsymbol{w} = (s_1^{\boldsymbol{w}}, \ldots, s_\ell^{\boldsymbol{w}})$ for $w \in W$ is any minimal-length factorization

$$w = s_1^{\boldsymbol{w}} \cdots s_\ell^{\boldsymbol{w}} \text{ with } s_1^{\boldsymbol{w}}, \ldots, s_\ell^{\boldsymbol{w}} \in \mathsf{S}.$$

The *length* of $w$ is $\ell = \ell(w)$, the number of simple reflections in any such factorization. We shall call $t \in \mathsf{T}$ an *inversion* if $\ell(tw) < \ell(w)$, and denote the set of inversions of $w$ by $\mathrm{Inv}(w)$. We have $|\mathrm{Inv}(w)| = \ell(w)$, and for any reduced word $\boldsymbol{w}$ a complete set of inversions for $w$ may be produced by $\tau_j^{\boldsymbol{w}} = s_1^{\boldsymbol{w}} \cdots s_j^{\boldsymbol{w}} \cdots s_1^{\boldsymbol{w}}$ for $1 \leqslant j \leqslant \ell(w)$, so that

$$\tau_j^{\boldsymbol{w}} w = s_1^{\boldsymbol{w}} \cdots \widehat{s_j^{\boldsymbol{w}}} \cdots s_\ell^{\boldsymbol{w}}, \tag{3.1}$$

where $\hat{\cdot}$ denotes omission from the product. Accordingly, $w$ has an *inversion factorization*

$$w = \tau_\ell^{\boldsymbol{w}} \cdots \tau_1^{\boldsymbol{w}}.$$

The positive roots associated to the inversions of $w$ can be described as $r(\mathrm{Inv}(w)) = \Phi^+ \cap w(\Phi^-)$.

The *Bruhat order* $\leqslant_B$ on $W$ is the transitive closure of the cover relations

$$tw \lessdot_B w \text{ for } t \in \mathrm{Inv}(w).$$

Equivalently $v \leqslant_B w$ provided that a reduced word for $v$ appears as a subsequence inside any (equivalently, all) reduced words for $w$.

The Bruhat order has a unique maximum element $w_\circ \in W$, and $\mathrm{Inv}(w_\circ) = \mathsf{T}$. We write $[u, v]$ for the Bruhat interval between $u$ and $v$, considered as a subposet, and note that $[u, v]$ is isomorphic to $[u^{-1}, v^{-1}]$ and anti-isomorphic to both $[w_\circ v, w_\circ u]$ and $[vw_\circ, uw_\circ]$ under the obvious identifications.

3.2. **Right weak order.** We call $s \in \mathsf{S}$ a *descent* of $w \in W$ if $\ell(ws) < \ell(w)$. Equivalently, $s$ is a descent of $w$ if there exists a reduced word for $w$ whose last letter is $s$. We denote the set of descents of $w$ by $\mathrm{Des}(w)$.

The *(right) weak order* $\leqslant_R$ on $W$ is the transitive closure of the cover relations $ws \lessdot_R w$ for $s \in \mathrm{Des}(w)$. In terms of reduced words, we have $u \leqslant_R v$ in weak order if some reduced word of $u$ is a prefix of some reduced word for $v$, or $\ell(v) = \ell(u) + \ell(u^{-1}v)$. Note that $u \leqslant_R v$ implies $u \leqslant_B v$. If $a \leqslant_R ab$ then $\ell(ab) = \ell(a) + \ell(b)$ – we write $a \cdot b$ both for the product $ab$ and the assertion that the product is length additive.

3.3. **Absolute order, Coxeter elements, and noncrossing partitions.** A *minimal reflection factorization* of $w \in W$ is any factorization

$$w = \tau_{i_1} \cdots \tau_{i_k} \text{ with } \tau_{i_1}, \ldots, \tau_{i_k} \in \mathsf{T}$$

with as few terms as possible. The *absolute length* of $w \in W$ is $\ell_{\mathsf{T}}(w) = k$, the number of reflections in any such factorization, which is also,

$$\ell_{\mathsf{T}}(w) = n - \dim(\mathrm{Fix}(w)),$$

where $\mathrm{Fix}(w)$ is the subspace in the defining representation for $W$ fixed by $w$. The *absolute order* $\leqslant_{\mathsf{T}}$ on $W$ is defined by:

$$v \leqslant_{\mathsf{T}} w \iff \ell_{\mathsf{T}}(w) = \ell_{\mathsf{T}}(v) + \ell_{\mathsf{T}}(v^{-1}w).$$

A *Coxeter element* $c$ of $W$ is any product of all the simple reflections taken in any order

$$c = s_1^{\mathbf{c}} \cdots s_n^{\mathbf{c}} \text{ with } \mathsf{S} = \{s_1^{\mathbf{c}}, \ldots, s_n^{\mathbf{c}}\}.$$

Then $\ell_{\mathsf{T}}(c) = n$ and the *c-noncrossing partitions* are defined to be the elements

$$\mathrm{NC}(W, c) = \{u \mid u \leqslant_T c\}.$$

Two equivalent definitions of $\mathrm{NC}(W, c)$ that we will use later are:

- the set of prefixes $\tau_1 \cdots \tau_\ell$ of minimal length factorizations $\tau_1 \cdots \tau_n$ of $c$, or
- the set of subproducts $\tau_{i_1} \cdots \tau_{i_\ell}$ of minimal length factorizations $\tau_1 \cdots \tau_n$ of $c$,

with the equivalence following from applying Hurwitz moves $ab = b(b^{-1}ab) = (aba^{-1})a$.

The absolute order gives $\mathrm{NC}(W, c)$ the lattice structure known as the *Kreweras lattice* [12, 45]. This lattice is, up to isomorphism, independent of the choice of Coxeter element. Indeed, one need only observe that all Coxeter elements are conjugate [42, Proposition 3.16] and that absolute order is invariant under conjugation.

The subset $\mathrm{NC}(W,c)^+ \subset \mathrm{NC}(W,c)$ of *fully supported c-noncrossing partitions* are those which do not lie in the subgroup generated by a strict subset of the simple reflections $\mathsf{S}$. The number $|\mathrm{NC}(W,c)^+| = \mathrm{Cat}_W^+$ is called the positive $W$-Catalan number, and is again independent of $c$.

3.4. **EL-labelings and Reflection orders.** We let $\mathcal{E}(P)$ denote the set of edges in the Hasse diagram of $P$ and $\mathcal{M}(P)$ denote the set of maximally refined chains. An *edge-labeling* of $P$ is a labeling of $\mathcal{E}(P)$ by a totally ordered set. The following foundational notion is due to Björner [15].

**Definition 3.1.** An *EL-labeling* is an edge-labeling of $P$ such that there is a unique increasing chain in $\mathcal{M}([s,t])$ for each interval $[s,t]$ in $P$ and, furthermore, this chain is lexicographically smaller than all other chains in the interval.

The following result shows that EL-labeling carries topological information about $P$[1].

**Fact 3.2** ([15, Theorem 2.7])**.** For $P$ a finite graded EL-labeled poset with $\hat{0}$ and $\hat{1}$, the number of decreasing maximal chains is the absolute value $|\mu_P(\hat{0},\hat{1})|$ of the Möbius function.

We will be interested in the EL-labelings of posets where the edge labels are reflections in $\mathsf{T}$.

**Definition 3.3** ([29, §2.2])**.** A total ordering $\prec$ of $\mathsf{T}$ is a *reflection ordering* if whenever $\alpha,\beta,\gamma \in \Phi^+$ are such that $\alpha$ is a positive linear combination of $\beta$ and $\gamma$ then either $t_\beta \prec t_\alpha \prec t_\gamma$ or $t_\gamma \prec t_\alpha \prec t_\beta$.

In Dyer's study of Kazhdan–Lusztig polynomials, he showed [29, Proposition 2.13] that every reflection order corresponds to a reduced word $\mathbf{w}_\circ$ for the longest element $w_\circ \in W$ by setting $\tau_1^{\mathbf{w}_\circ} \prec \tau_2^{\mathbf{w}_\circ} \prec \cdots \prec \tau_N^{\mathbf{w}_\circ}$, where $\tau_i^{\mathbf{w}_\circ}$ is the reflection defined in (3.1). Furthermore, he produced the following natural EL-labeling of Bruhat intervals[2].

**Theorem 3.4** ([29, Proposition 4.3])**.** The edge labeling of a Bruhat interval $[u,v]$ where we label $w \lessdot_B wt$ with the reflection $t$ is an EL-labeling for any reflection order $\prec$. Consequently, there is a unique increasing (and decreasing) chain in $[u,v]$ with respect to any reflection ordering.

One distinguished reflection order which we will need is the *c-reflection order*. The choice of Coxeter element $c$ determines a unique word for $w_\circ$, the *c-sorting word*, which we recall in Section 13. Athanasiadis–Brady–Watt [4] show that the reflection order determined by the $c$-sorting word for $w_\circ$ gives an EL-labeling of $\mathrm{NC}(W,c)$.[3]

**Theorem 3.5** ([4])**.** The edge labeling of $\mathrm{NC}(W,c)$ where we label $w \lessdot_{\mathsf{T}} wt$ with the reflection $t$ is an EL-labeling for the $c$-reflection order.

[1] In fact, this result was first discovered by Richard Stanley under weaker hypotheses.

[2] The existence of EL-labelings for Bruhat intervals is originally due to Björner and Wachs [17, Theorem 4.2], but they did not employ reflection orders.

[3] The paper [4] only establishes this labeling for bipartite Coxeter elements, but the more general claim holds. See, for instance, [43, Appendix A].

The Möbius value of $\mathrm{NC}(W, c)$ is given by $\mu_{\mathrm{NC}(W,c)}(\mathrm{id}, c) = \pm \mathrm{Cat}_W^+$; see [27, Proposition 9]. We say that a maximal chain in $\mathrm{NC}(W, c)$ is *c-decreasing* if it is decreasing under the edge labeling determined by the $c$-reflection order. Fact 3.2 then implies the following.

**Corollary 3.6.** In $\mathrm{NC}(W, c)$ there are $\mathrm{Cat}_W^+$-many $c$-decreasing maximal chains.

See Figure 4 which shows the $c$-decreasing maximal chains in the Hasse diagram of $\mathrm{NC}(S_4, s_1 s_3 s_2)$.

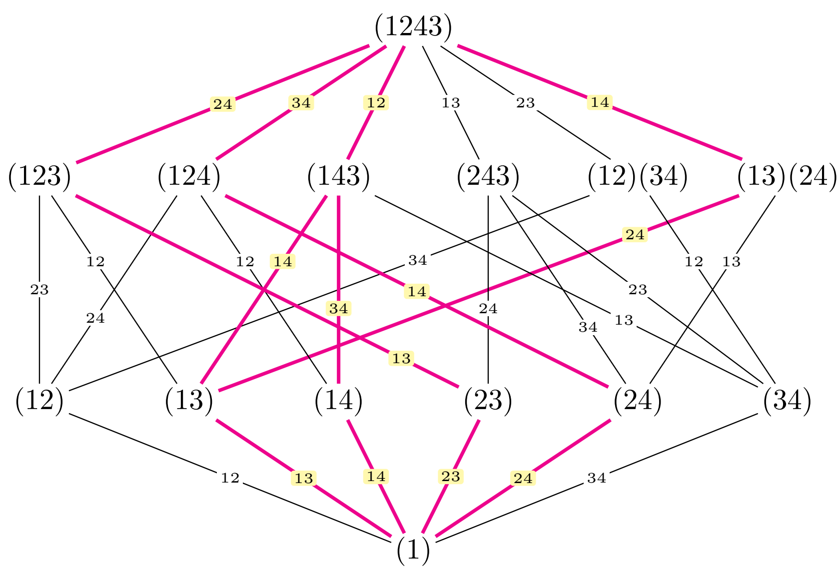


FIGURE 4. The Hasse diagram of $\mathrm{NC}(S_4, s_1 s_3 s_2)$ with the five decreasing chains in the $c$-reflection order $(1\,2) \prec (3\,4) \prec (1\,4) \prec (2\,4) \prec (1\,3) \prec (2\,3)$ bolded.

3.5. **Clusters and noncrossing inversions.** In [30, Theorems 1.9 and 1.13], Fomin and Zelevinsky show that the cluster variables of a finite-type cluster algebra are in bijection with the almost-positive roots $\Phi_{\geqslant -1} = \Phi^+ \cup \{-\alpha \mid \alpha \in \Delta\}$ for the corresponding root system. In [52, Section 7], Reading extends Fomin and Zelevinsky's result into a family of bijections parameterized by any pair $(W, c)$ for $W$ a finite Coxeter group and $c \in W$ a Coxeter element. Let

$$\mathrm{Cl}_c = \{\mathcal{F} \subseteq \Phi_{\geqslant -1} \mid \mathcal{F} \text{ is the image of a cluster under Reading's } c\text{-bijection}\}.$$

We refer to the elements of $\mathrm{Cl}_c$ as *c-clusters*, and define $\mathrm{Cone}(\mathcal{F})$ to be the cone determined by the positive real span of the roots in $\mathcal{F} \in \mathrm{Cl}_c$. The *c-cluster fan* is the simplicial fan whose cones are $\mathrm{Cone}(\mathcal{F})$ for $\mathcal{F} \in \mathrm{Cl}_c$. This generalizes, and is combinatorially isomorphic to, the cluster fan constructed by Fomin–Zelevinsky [31, Theorem 1.10], but has additional orientation data $c$.

In what follows, we restrict our attention to the positive roots in each $c$-cluster, as in

$$\mathrm{Cl}_c^+ = \{\mathcal{F} \cap \Phi^+ \mid \mathcal{F} \in \mathrm{Cl}_c\}.$$

Each $c$-cluster $\mathcal{F}$ is determined by the positive roots that it contains, so this change sacrifices no combinatorial data. In fact, unlike $\mathrm{Cl}_c$, the sizes of the sets in $\mathrm{Cl}_c^+$ vary from $0$ to $n$. The cones generated by size $n$ elements of $\mathrm{Cl}_c^+$, together with all of their faces, give a simplicial fan, the *positive c-cluster fan*, which subdivides the positive root cone $\mathrm{Cone}(\Delta)$ so that every positive root is the ray generator of a cone.

**Remark 3.7.** Not every face of the positive $c$-cluster fan corresponds to an element of $\mathrm{Cl}_c^+$. Equivalently, $\mathrm{Cl}_c^+$ is not closed under taking subsets. See Example 3.9.

**Definition 3.8.** Given $u \in \mathrm{NC}(W, c)$, define the *noncrossing inversion set* and *right noncrossing inversion set* to be

$$\mathrm{Inv_{NC}}(u) := \{\tau \in \mathrm{Inv}(u) \mid \tau u \in \mathrm{NC}(W, c)\} \text{ and } \mathrm{Inv}^R_{\mathrm{NC}}(u) := \{u^{-1}\tau u \mid \tau \in \mathrm{Inv_{NC}}(u)\}.$$

We can equivalently define the right noncrossing inversions as $\mathrm{Inv}^R_{\mathrm{NC}}(u) = \{\tau \in \mathsf{T} \mid u\tau \lessdot_B u \text{ and } u\tau \in \mathrm{NC}(W, c)\}$. In [13, Theorem 8.2, Lemma 8.3], Biane and Josuat-Vergès define a bijection

$$\begin{aligned} \mathrm{Clust}^+ : \mathrm{NC}(W, c) &\to \mathrm{Cl}_c^+ \\ u &\mapsto \{r(\tau) \mid \tau \in \mathrm{Inv}^R_{\mathrm{NC}}(u)\}. \end{aligned}$$

Clearly, restricting to $\mathrm{NC}(W, c)^+$ gives a bijection to the maximal cones in $\mathrm{Cl}_c^+$.

**Example 3.9.** Let us consider $W = \mathrm{B}_2$ with $c = s_0 s_1$. Choose $\Delta = \{\alpha_0 = \epsilon_1, \alpha_1 = \epsilon_2 - \epsilon_1\}$, so that the subset $\Phi^+$ of positive roots given by $\{\alpha_0, \alpha_1, \alpha_0 + \alpha_1, 2\alpha_0 + \alpha_1\}$ corresponds respectively to the set $\mathsf{T}$ of reflections $\{s_0, s_1, s_1 s_0 s_1, s_0 s_1 s_0\}$. The table below lists the image of the map $\mathrm{Clust}^+$ in the rightmost column. See Figure 5 for the root system of $\mathrm{B}_2$ with the maximal cones in the positive $c$-cluster fan highlighted. These correspond to the last three rows in the preceding table. Note that the rays $\alpha_0 + \alpha_1$ and $2\alpha_0 + \alpha_1$ do not correspond to positive $c$-clusters.

| $\mathrm{Sort}(W, c)$ | $\mathrm{NC}(W, c)$ | $\mathrm{Inv_{NC}}$ | $\mathrm{Clust}^+$ |
|---|---|---|---|
| id | id | $\varnothing$ | $\varnothing$ |
| $s_0$ | $s_0$ | $\{ s_0 \}$ | $\{ \alpha_0 \}$ |
| $s_1$ | $s_1$ | $\{ s_1 \}$ | $\{ \alpha_1 \}$ |
| $s_0 s_1$ | $s_0 s_1 s_0$ | $\{ s_0 s_1 s_0,\ s_1 s_0 s_1 \}$ | $\{ 2\alpha_0 + \alpha_1,\ \alpha_0 \}$ |
| $s_0 s_1 s_0$ | $s_1 s_0 s_1$ | $\{ s_1,\ s_1 s_0 s_1 \}$ | $\{ 2\alpha_0 + \alpha_1,\ \alpha_0 + \alpha_1 \}$ |
| $s_0 s_1 s_0 s_1$ | $s_0 s_1$ | $\{ s_0,\ s_0 s_1 s_0 \}$ | $\{ \alpha_0 + \alpha_1,\ \alpha_1 \}$ |

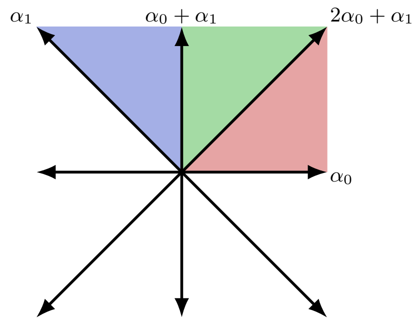


FIGURE 5. The $B_2$ root system with its positive $c$-cluster fan highlighted, for $c = s_0 s_1$. Note that the rays $\alpha_0 + \alpha_1$ and $2\alpha_0 + \alpha_1$ do not correspond to positive $c$-clusters.

## 4. REPRESENTATIONS OF REDUCTIVE GROUPS

We now review facts about reductive groups and their representations; further details and definitions can be found in standard texts such as [19, 41, 58]. Let $G$ be a complex reductive group, so that $G$ is an algebraic group with a faithful completely reducible representation $V_{def}$. Let $T \subseteq G$ be a maximal algebraic torus and $B \subseteq G$ a Borel subgroup. The Weyl group is $W = N_G(T)/T$, where $N_G(T)$ denotes the normalizer of $T$ in $G$, and the opposite Borel $B^-$ is $w_\circ B w_\circ$.

4.1. **Cartan decomposition.** Let $\mathrm{Char}(T)$ be the group of algebraic characters of $T$. The Weyl group $W$ acts on $\mathbb{E} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Char}(T)$, and we fix once and for all a $W$-invariant inner product $(\cdot,\cdot)$, making $\mathbb{E}$ a Euclidean space.

Every finite-dimensional representation $V$ of $G$ determines a weight space decomposition

$$V = \bigoplus_{\lambda \in \mathrm{Char}(T)} V_\lambda \qquad \text{where} \qquad V_\lambda = \{v \in V \mid h.v = \lambda(h)v \text{ for all } h \in T\}.$$

Let $\mathfrak{g}$ be the Lie algebra of $G$. The weight space decomposition of the adjoint representation on $\mathfrak{g}$ is the Cartan decomposition

$$\mathfrak{g} = \mathfrak{g}^T \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

The set $\Phi$ in the Cartan decomposition is a root system under an appropriate generalization of the Killing form on $\mathrm{Char}(T)$, see [58, §7.4], and the corresponding Coxeter group is isomorphic to $W$. The Lie algebras $\mathfrak{b}, \mathfrak{b}^- \subset \mathfrak{g}$ of $B$ and $B^-$ are

$$\mathfrak{b} = \mathfrak{g}^T \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \qquad \text{and} \qquad \mathfrak{b}^- = \mathfrak{g}^T \oplus \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha,$$

which induces a decomposition $\Phi = \Phi^+ \sqcup \Phi^-$ into the *positive roots* and the *negative roots*.

Going forward, we fix a *Chevalley basis* $\{H_{\lambda_1}, \ldots, H_{\lambda_n}\} \cup \{E_\alpha \mid \alpha \in \Phi\} \subset \mathfrak{g}$ with respect to $V_{\mathrm{def}}$ with each $H_{\lambda_i} \in \mathfrak{g}^T$ and $E_\alpha \in \mathfrak{g}_\alpha$.

4.2. **Chevalley presentation.** We now describe some key generators and relations of $G$.

We first identify the set $\mathrm{Char}^\vee(T) = \{\mu^\vee \in \mathbb{E} \mid (\lambda, \mu^\vee) \in \mathbb{Z} \text{ for all } \lambda \in \mathrm{Char}(T)\}$ with the cocharacters of $T$ by associating to each $\mu^\vee \in \mathrm{Char}^\vee(T)$ a *one-parameter cocharacter subgroup*

$$h_{\mu^\vee} : \mathbb{C}^\times \to T \qquad \text{such that for all } \lambda \in \mathrm{Char}(T), \qquad \lambda(h_{\mu^\vee}(x)) = x^{(\lambda,\mu^\vee)}. \tag{4.1}$$

In particular, we have one-parameter subgroups $h_{\alpha^\vee}$ for each *coroot* $\alpha^\vee = 2\alpha/(\alpha,\alpha)$, $\alpha \in \Phi$.

In order to define our second family of one-parameter subgroups, we note that the derivative of a rational representation of $G$ gives a representation of $\mathfrak{g}$ on the same space. In particular, we can identify each Chevalley generator $E_\alpha \in \mathfrak{g}_\alpha$ with its image in $\mathrm{End}_{\mathbb{C}}(V_{\mathrm{def}})$ under this representation. Define now the *one-parameter root subgroup* for $\alpha \in \Phi$ by

$$e_\alpha : \mathbb{C} \to G \qquad e_\alpha(x) = \exp(xE_\alpha) = 1 + xE_\alpha + \frac{x^2}{2}E_\alpha^2 + \cdots . \tag{4.2}$$

The group generated by the $e_\alpha(\mathbb{C})$ is a semisimple subgroup of $G$, and together with $T$ these elements generate $G$. Before giving a presentation on generators and relations, we define one new family of elements: for $\alpha \in \Phi$, let

$$s_\alpha(x) = e_\alpha(x)e_{-\alpha}(-x^{-1})e_\alpha(x), \qquad \text{for } x \in \mathbb{C}^\times. \tag{4.3}$$

Then each $s_\alpha(x)$ is a representative for $s_\alpha$ in $N_G(T)$. Moreover, with the one-parameter subgroups defined in (4.1), the following relations hold for all $\alpha, \beta \in \Phi$ and $x, y \in \mathbb{C}$:

$$e_\alpha(x)e_\alpha(y) = e_\alpha(x+y), \tag{4.4}$$

$$e_\alpha(x)^{-1}e_\beta(y)^{-1}e_\alpha(x)e_\beta(y) = \prod_{i\alpha+j\beta\in\Phi^+} e_{i\alpha+j\beta}(c_{i,j}x^iy^j), \quad \text{for } c_{i,j} \in \mathbb{Z} \text{ independent of } x, y, \tag{4.5}$$

$$s_\alpha(x)s_\alpha(-1) = h_{\alpha^\vee}(x), \qquad \text{for } x \in \mathbb{C}^\times, \tag{4.6}$$

$$s_\alpha(x)h_{\lambda^\vee}(y)s_\alpha(-x) = h_{s_\alpha\lambda^\vee}(y), \qquad \text{for all } \lambda^\vee \in \mathrm{Char}^\vee(T), \tag{4.7}$$

$$s_\alpha(x)e_\beta(y)s_\alpha(-x) = e_{s_\alpha\beta}(\pm x^{-(\alpha^\vee,\beta)}y), \qquad \text{for } \pm \text{ independent of } x, y, \tag{4.8}$$

$$he_\alpha(x)h^{-1} = e_\alpha(\alpha(h)\, x), \qquad \text{for all } h \in T. \tag{4.9}$$

We omit relations between different one-parameter subgroups of $T$, so the above is not a complete presentation for $G$.

4.3. **Weights and representations.** A *dominant weight* for $\mathfrak{g}$ is an element of

$$\Lambda^+ = \{\lambda \in \mathbb{E} \mid (\lambda, \alpha^\vee) \in \mathbb{Z}_{\geqslant 0} \text{ for all } \alpha \in \Phi^+\}.$$

Say that $\lambda \in \Lambda^+$ is *regular* if $(\lambda, \alpha^\vee) > 0$ for each $\alpha \in \Phi^+$.

Every dominant weight $\lambda \in \Lambda^+$ determines a finite-dimensional, irreducible $\mathfrak{g}$-module. When $\lambda \in \Lambda^+ \cap \mathrm{Char}(T)$, this $\mathfrak{g}$-module integrates to a simple $G$-module $V^\lambda$. The following facts, paired with the Chevalley presentation, will allow us to compute in these representations.

**Fact 4.1.** For any dominant integral weight $\lambda$ and $\mu \in \mathrm{Char}(T)$ we have:

(1) if $\dot{w} \in N_G(T)$ represents $w \in W$, $v \mapsto \dot{w}v$ is a vector space isomorphism from $V^\lambda_\mu$ to $V^\lambda_{w\cdot\mu}$,
(2) for $\alpha \in \Phi$, $E_\alpha$ maps $V^\lambda_\mu$ to $V^\lambda_{\mu+\alpha}$, and
(3) if $\mu \in \lambda + \mathbb{Z}_{\geqslant 0}\Phi^+$ then $\dim(V^\lambda_\mu) = \delta_{\lambda,\mu}$.

The *extremal weights* of $V^\lambda$ are those of the form $w\lambda$ for $w \in W$. The above properties imply that all nonzero weight spaces come from weights $\mu$ contained in the convex hull of the extremal weights, and the dimension of each extremal weight space is 1.

## 5. Geometric preliminaries

We now review important facts about the *generalized flag variety* $G/B$. First recall that the group $W$ permutes the $T$-orbits in $G/B$ by having $w \in W$ act as left multiplication by any representative in $N_G(T)$, and that the $T$-fixed point set $(G/B)^T$ is the $W$-orbit of $B$.

For $u \in W$, we define the *Schubert cell* $\mathring{X}^u := BuB \subset G/B$ and *Schubert variety* $X^u := \overline{BuB} \subset G/B$. Similarly we define the *opposite Schubert cell* $\mathring{X}_u := B^-uB \subset G/B$ and *opposite Schubert variety* $X_u := \overline{B^-uB} \subset G/B$. These varieties are related by the identity $w_\circ X^u = X_{w_\circ u}$. For $u \leqslant_B v$, we then define the *Richardson variety* $X^v_u := X^v \cap X_u$. The $T$-fixed points of $X^v_u$ are given by the Bruhat interval $[u, v]$.

### 5.1. The adjoint group.

In this section, we explain why the space $G/B$ depends only on the root system $\Phi$ for $G$. It is nonetheless useful to allow varying choices of $G$, for example in the appendices. However, a few results in Section 6.2 rely on using the *adjoint group* $G_{ad} := G/Z(G)$, where $Z(G)$ denotes the center. This is a centerless semisimple reductive group with the same root system and Weyl group as $G$. As $Z(G) \subseteq T$, $G_{ad}$ has Borel subgroup $B_{ad} = B/Z(G)$ and torus $T_{ad} = T/Z(G)$.

The *adjoint torus* $T_{ad}$ has a rank $n$ character lattice

$$\mathrm{Char}(T_{ad}) = \mathbb{Z}\Phi \subseteq \mathbb{E}.$$

As $Z(G)$ acts trivially on $G/B$, the adjoint torus acts on $G/B$, and

$$G/B = G_{ad}/B_{ad} \qquad \text{as } T_{ad}\text{-varieties.}$$

Hence every root system $\Phi$ has a unique generalized flag variety $G/B$ associated to it.

### 5.2. Closed subsets of roots and the Bruhat decomposition.

For any $u \in W$ we have a decomposition $X^u = \bigsqcup_{w \leqslant u} \mathring{X}^w$, and applying this for $u = w_\circ$ we obtain the *Bruhat decomposition*

$$G/B = \bigsqcup_{w \in W} \mathring{X}^w.$$

This decomposes $G/B$ into Schubert cells, with each $\mathring{X}^w$ $T$-equivariantly isomorphic to the $T$-representation $\bigoplus_{\tau \in \mathrm{Inv}(w)} \mathfrak{g}_{r(\tau)}$. This isomorphism is non-canonical and we recall how it arises.

Recall that in (4.2) we defined a family of one-parameter subgroups $e_\alpha(\mathbb{C})$, $\alpha \in \Phi^+$, by

$$e_\alpha(x) = \exp(xE_\alpha).$$

Say that a subset $C \subseteq \Phi^+$ is *closed* if $\alpha, \beta \in C$ and $\alpha + \beta \in \Phi$ implies that $\alpha + \beta \in C$. For a closed set $C$, choose an order $C = \{\beta_1, \beta_2, \cdots, \beta_{|C|}\}$ and define

$$U_C = \left\{ e_{\beta_1}(a_1) \cdots e_{\beta_{|C|}}(a_{|C|}) \mid a_i \in \mathbb{C} \right\} \qquad \text{and} \qquad N_C = \bigoplus_{i=1}^{|C|} \mathfrak{g}_{\beta_i}.$$

We will apply this construction with

$$C \subseteq r(\mathrm{Inv}(w)) = \Phi^+ \cap w(\Phi^-),$$

the set of positive roots corresponding to inversions of $w$. By [61, Lemma 17, Lemma 34], $U_C$ is a unipotent subgroup of $B$ which is independent of the chosen order on $C$. Moreover there is an (order-dependent) isomorphism of varieties

$$\begin{array}{rcl} N_C & \to & U_C wB/B \\ a_1 E_{\beta_1} + \cdots + a_{|C|} E_{\beta_{|C|}} & \mapsto & e_{\beta_1}(a_1) \cdots e_{\beta_{|C|}}(a_{|C|}) wB/B \end{array} \tag{5.1}$$

By relation (4.9), this isomorphism is $T$-equivariant with respect to the adjoint action on $N_C$ and left multiplication by $T$ on $U_C wB/B$.

In the extreme case that $C = r(\mathrm{Inv}(w))$, we write $U_w = U_{r(\mathrm{Inv}(w))}$ and $N_w = N_{r(\mathrm{Inv}(w))}$, so that (5.1) gives a $T$-equivariant isomorphism

$$N_w \cong U_w wB = \mathring{X}^w.$$

**Example 5.1.** We take notation as in Appendix A. For $\mathrm{GL}_3 / B$ with $\alpha = \epsilon_1 - \epsilon_2, \beta = \epsilon_2 - \epsilon_3, \gamma = \epsilon_1 - \epsilon_3$ we take bases of the corresponding weight spaces of $\mathfrak{g}$ to be the elementary matrices $E_{1,2}$, $E_{2,3}$, $E_{1,3}$ where $(E_{i,j})_{k,l} = \delta_{i,k}\delta_{j,l}$. Then

$$e_\alpha(a) = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad e_\beta(b) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}, \quad e_\gamma(c) = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then two different parametrizations of $U_{w_\circ}$ are

$$e_\beta(b)e_\alpha(a)e_\gamma(c) = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } e_\alpha(a)e_\beta(b)e_\gamma(c) = \begin{bmatrix} 1 & a & c+ab \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}.$$

We caution the reader that the $T$-equivariant automorphism $(a, b, c) \mapsto (a, b, c + ab)$ between these two parametrizations does not preserve the linear structure on the $T$-representations.

5.3. $T$**-invariant curves in** $G/B$**.** As a consequence of the Bruhat decomposition, the $T$-invariant curves in $G/B$ are given by the closures of the $T$-invariant lines $\{e_\beta(t)wB \mid t \in \mathbb{C}\}$ for $s_\beta \in \operatorname{Inv}(w)$. We will need the following fact.

**Fact 5.2.** We have $\{e_\beta(t)wB \mid t \in \mathbb{C}\} \cong \mathbb{A}^1$, its closure contains the unique additional point $s_\beta wB$, and $\{e_\beta(t)wB \mid t \in \mathbb{C}\} \cup \{s_\beta wB\} \cong \mathbb{P}^1$.

Therefore, the $T$-invariant curves are the unique $T$-invariant $\mathbb{P}^1$'s which "connect" $T$-fixed points $u, v$ with $v = s_\beta u$ for some $\beta \in \operatorname{Inv}(v)$. We present this more symmetrically as follows.

**Definition 5.3.** When $v = \tau u$ for some reflection $\tau$, we denote by $\mathbb{P}_{u,v} \subset G/B$ the unique $T$-invariant $\mathbb{P}^1$ which has $T$-fixed points $u$ and $v$.

Thus if we draw the Cayley graph $\operatorname{Cayley}(W, \mathsf{T})$ generated by reflections, every vertex corresponds to a $T$-fixed point and every edge $uv$ corresponds to $\mathbb{P}_{u,v}$. See for example Figure 6.

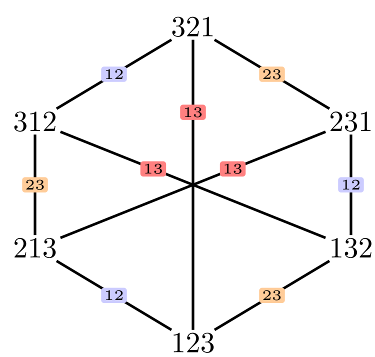


FIGURE 6. The Cayley graph of $\operatorname{Cayley}(S_3, \{(12), (23), (13)\})$ generated by reflections. The vertices correspond to the six $T$-fixed points in $\mathrm{GL}_3/B$ and the edges correspond to the nine $T$-invariant $\mathbb{P}_{u,v}$'s in $\mathrm{GL}_3/B$.

## 6. Plücker functions and Plücker vanishing varieties

In this section we record facts about Plücker functions that will be used for the remainder of the paper. Most of the relevant literature focuses on the special case of type A, or the related case of Grassmannians associated to minuscule roots – to avoid any confusion we gather what is true in the fullest generality for $G/B$.

Fix, now and for the remainder of the paper, a regular dominant character

$$\lambda_{\mathrm{reg}} \in \Lambda^+ \cap \operatorname{Char}(T).$$

6.1. **The Plücker embedding.** For $\lambda \in \Lambda^+ \cap \mathrm{Char}(T)$, recall the simple $G$-module $V^\lambda$ from Section 4.2. The *$\lambda$-Plücker map* is the map

$$\begin{array}{rcl} \mathrm{Pl}^\lambda : G/B & \hookrightarrow & \mathbb{P}(V^\lambda) \\ gB & \mapsto & \langle g v_\lambda \rangle. \end{array}$$

When $\lambda = \lambda_{\mathrm{reg}}$, our fixed regular dominant character, the Plücker map is an embedding, which we call the *Plücker embedding*, and we write

$$\mathrm{Pl} := \mathrm{Pl}^{\lambda_{\mathrm{reg}}}.$$

This embedding depends on the choice of $\lambda_{\mathrm{reg}}$, but our results are independent of this choice unless otherwise stated, so we will suppress it from the notation.

We now fix generators for each 1-dimensional extremal weight space $V^\lambda_{w\lambda}$: for $w \in W$, let

$$0 \neq v_{w\lambda} \in V^\lambda_{w\lambda}. \tag{6.1}$$

For any representative $\dot{w} \in N_G(T)$ of $w$, we have $\dot{w} v_\lambda \in \langle v_{w\lambda} \rangle$ by Fact 4.1.

**Remark 6.1.** If the Weyl group $W$ embeds into $G$, as is the case with $G = \mathrm{GL}_n$, we can take the more straightforward definition $v_{w\lambda} = w v_\lambda$ without ambiguity.

Now define the *$w$-Plücker function* $\mathrm{Pl}^\lambda_w$ as the coordinate function for $v_{w\lambda}$ with respect to the weight space decomposition of $V^\lambda$. We will abuse notation and also write $\mathrm{Pl}^\lambda_w$ for the projective coordinate of $G/B$ under $\mathrm{Pl}^\lambda$; as we will only consider the vanishing of this coordinate, the arbitrary choices made in Equation (6.1) do not cause any issues. For $\lambda = \lambda_{\mathrm{reg}}$, we write $\mathrm{Pl}_w$.

We now summarize some basic properties of Plücker functions for the Plücker embedding and their connection to $T$-fixed points.

**Fact 6.2.**

(1) For $t \in T$ we have $\mathrm{Pl}_w(tgB) = (w\lambda_{\mathrm{reg}})(t)\,\mathrm{Pl}_w(gB)$.
(2) For a $T$-invariant variety $X \subseteq G/B$, we have $X^T = \{wB \mid w \in W,\ \mathrm{Pl}_w|_X \not\equiv 0\}$.
(3) We have $\mathrm{Pl}_w(gB) \neq 0 \iff wB \in \overline{T \cdot gB}$. In particular the vanishing set $\{\mathrm{Pl}_w = 0\}$ is independent of the choice of $\lambda_{\mathrm{reg}}$.

*Proof.* (1) follows as $v_{w\lambda_{\mathrm{reg}}}$ belongs to the weight space for $w\lambda_{\mathrm{reg}}$. For (2) note that if $\mathrm{Pl}_w(x) \neq 0$, then because the extremal weights $w'\lambda_{\mathrm{reg}}$ are in convex position we are able to find a cocharacter $\psi$ of $T$ which limits $\lim_{t\to 0} \psi(t)x$ to a point where all Plücker functions vanish except for $\mathrm{Pl}_w$. We claim that only $w \in W$ has this Plücker vanishing property – indeed, any $T$-orbit closure in this vanishing set could only have $wB$ as a fixed point. But a projective toric variety with a single fixed point is a singleton, so this $T$-orbit closure must be $\{wB\}$ itself. (3) follows from (2) once we note that because $\mathrm{Pl}_w$ is $T$-equivariant we have $\mathrm{Pl}_w(gB) = 0 \iff \mathrm{Pl}_w(\overline{T \cdot gB}) = 0$. $\square$

**Remark 6.3.** Even though $\{\mathrm{Pl}_w\}_{w\in W}$ is basepoint-free, and hence induces a map $G/B \to \mathbb{P}^{|W|-1}$, this map is not necessarily injective even for a well-chosen $\lambda_{\mathrm{reg}}$, see Appendix A.

6.2. **Relation to Grassmannian Plücker coordinates.** This section is logically independent of the remainder of the paper. Here we relate Pl to the Grassmannian Plücker coordinates that appear more commonly in the literature, assuming for ease of exposition in this section only that $G$ is semisimple so that $\mathrm{Char}(T) = \Lambda$ and $\mathbb{E}$ is spanned by the simple roots (see the discussion in Section 5.1).

Enumerate the simple roots in $\Phi$ as $\Delta = \{\alpha_1, \dots, \alpha_n\}$, and let the *fundamental weights* $\omega_1, \dots, \omega_n \in \Lambda^+$ be defined by $(\omega_i, \alpha_j^\vee) = \delta_{i,j}$ where $\delta_{i,j}$ is the Kronecker delta. Then $\Lambda^+$ is the nonnegative integral span of $\omega_1, \dots, \omega_n$ so we can write

$$\lambda_{\mathrm{reg}} = \sum k_i \omega_i \text{ with } k_i \in \mathbb{Z}_{>0}.$$

The *Grassmannian Plücker map* $\mathrm{Pl}^{\omega_i} : G/B \to \mathbb{P}(V^{\omega_i})$ is associated to $\omega_i$, and factors as

$$G/B \to G/P_i \hookrightarrow \mathbb{P}(V^{\omega_i})$$

where $P_i$ is associated to the maximal parabolic subgroup associated to $\omega_i$. We now observe that the Plücker coordinates associated to any weight, in particular $\lambda_{\mathrm{reg}}$, decompose as a product of Grassmannian Plücker coordinates.

**Observation 6.4.** If $a, b \in \Lambda^+$ then

$$\mathrm{Pl}^a_w \, \mathrm{Pl}^b_w \text{ is a constant nonzero multiple of } \mathrm{Pl}^{a+b}_w.$$

*Proof.* The map $G/B \to \mathbb{P}(V^a) \times \mathbb{P}(V^b) \to \mathbb{P}(V^a \otimes V^b)$ obtained by composing $(\mathrm{Pl}^a, \mathrm{Pl}^b)$ with the Segre embedding can also be written as $G/B \to \mathbb{P}(V^{a+b}) \hookrightarrow \mathbb{P}(V^a \otimes V^b)$ where the first map is $\mathrm{Pl}^{a+b}$ and the second map is the inclusion of $V^{a+b} \hookrightarrow V^a \otimes V^b$ with the one-dimensional weight space $V^{a+b}_{a+b}$ mapping isomorphically onto $(V^a \otimes V^b)_{a+b}$. □

**Corollary 6.5.** We have $\mathrm{Pl}^{\lambda_{\mathrm{reg}}}_w = \prod (\mathrm{Pl}^{\omega_i}_w)^{k_i}$.

In particular by Fact 6.2(3), the vanishing locus $\{\mathrm{Pl}_w = 0\} \subset G/B$ decomposes as

$$\{\mathrm{Pl}_w = 0\} = \bigcup_{i=1}^{n} \{\mathrm{Pl}^{\omega_i}_w = 0\} \subset G/B. \tag{6.2}$$

6.3. **Plücker vanishing subvarieties.** For any $\mathcal{A} \subset W$, we define the *Plücker vanishing subvariety*

$$\mathrm{PV}_{\mathcal{A}} := \bigcap_{w \in W \setminus \mathcal{A}} \{\mathrm{Pl}_w = 0\} \subset G/B.$$

We note that this variety is not necessarily irreducible, even when $|W \setminus \mathcal{A}| = 1$ (see e.g. (6.2)).

In the following result, say that a $T$-invariant subvariety $X \subset G/B$ is *rigid* if

$$Y^T \subset X^T \implies Y \subset X \text{ for any } T\text{-invariant subvariety } Y \subset G/B.$$

**Theorem 6.6.** Let $\mathcal{A} \subseteq W$.

(1) $\mathrm{PV}_{\mathcal{A}}$ is the union of all $T$-orbit closures $Y \subset G/B$ with $Y^T \subset \mathcal{A}$.
(2) The variety $\mathrm{PV}_{\mathcal{A}}$ is the unique rigid variety with $T$-fixed points given by $\mathcal{A} \subset W$.

*Proof.* For (1), every $T$-invariant variety is the union of the $T$-orbit closures contained within, so it suffices to show that if $\overline{T \cdot y}^T \subset \mathcal{A}$ then $\mathrm{Pl}_w(y) = 0$ for all $w \notin \mathcal{A}$. But

$$\mathrm{Pl}_w(y) = 0 \iff \mathrm{Pl}_w|_{\overline{T\cdot y}} \equiv 0 \iff wB \notin \overline{T \cdot y}.$$

For (2) we show that $\mathrm{PV}_{\mathcal{A}}$ is rigid; uniqueness then follows from rigidity. Indeed, if $Y^T \subset X^T$, then for $y \in Y$ we have $\overline{T \cdot y}^T \subset X^T$ and hence by (1) we have $y \in \overline{T \cdot y} \subset X$. □

The next fact will be essential in our study of Plücker vanishing varieties in later sections.

**Fact 6.7.** For $u, v \in W$, we have $X_u^v = \mathrm{PV}_{[u,v]}$. More generally, for any $z \in W$ the translated Richardson variety

$$zX_u^v = \mathrm{PV}_{z[u,v]}$$

is the Plücker vanishing variety associated to the translated Bruhat interval $\mathcal{A} = z[u, v]$, and is therefore rigid.

*Proof.* We first show that our claim about translated Richardson varieties $zX_u^v$ reduces to the claim about un-translated $X_u^v$. By Fact 4.1(1), left translating any Plücker vanishing variety $\mathrm{PV}_{\mathcal{A}}$ by any representative of $w \in W$ gives the Plücker vanishing variety $\mathrm{PV}_{w\mathcal{A}}$.

Next we note that our claim about un-translated Richardson varieties $X_u^v$ need only be verified for each Schubert variety $X^v = X_{\mathrm{id}}^v$. Indeed, this is an immediate consequence of the equalities $X_u^v = X^v \cap X_u$, $X_u = w_\circ X^{w_\circ u}$, and $[u, v] = [\mathrm{id}, v] \cap w_\circ[\mathrm{id}, w_\circ u]$, once we note that for subsets $\mathcal{A}, \mathcal{B} \subseteq W$ we have $\mathrm{PV}_{\mathcal{A}} \cap \mathrm{PV}_{\mathcal{B}} = \mathrm{PV}_{\mathcal{A}\cap\mathcal{B}}$.

Finally, we prove the claim for un-translated Schubert varieties. As the $T$-fixed points of $X^v$ are by definition elements of $[\mathrm{id}, v]$, we need only prove that $X^v$ is rigid and apply Theorem 6.6(2) in order to complete the proof. To this end suppose that $Y^T \subseteq [\mathrm{id}, v]$ for some $T$-invariant subvariety $Y$. Each $x \in Y$ belongs to a Schubert cell $\mathring{X}^w$, and because $\mathring{X}^w \cong N_w$ any strictly antidominant cocharacter $\psi : \mathbb{C}^* \to T$ (i.e. one which pairs negatively with the roots in $\Phi^-$) has the property that $\lim_{t\to 0} \psi(t) \cdot x = wB$. It follows that $wB \in \overline{T \cdot x} \subseteq Y$. By assumption on $Y$ we have $w \in [\mathrm{id}, v]$, and therefore $x \in X^v$. □

Going forward, say that two Bruhat intervals $[u, v]$ and $[u', v']$ are *shape equivalent* if $u'u^{-1}[u, v] = [u', v']$. An immediate application of Fact 6.7 is to geometrically interpret shape equivalence.

**Definition 6.8.** Given a Bruhat interval $[u, v]$, we denote $\overleftarrow{[u, v]} := u^{-1}[u, v]$.

**Corollary 6.9.** For $u, v, u', v' \in W$, the following are equivalent.

(1) $[u, v]$ and $[u', v']$ are shape equivalent.

(2) $\overleftarrow{[u,v]} = \overleftarrow{[u',v']}$
(3) $u^{-1}X_u^v = (u')^{-1}X_{u'}^{v'}$

*Proof.* The first two are clearly equivalent, and the equivalence of (2) and (3) follows immediately from Fact 6.7. ☐

6.4. **Coxeter matroids and moment polytopes.** Because the $T$-fixed points $wB \in G/B$ map to the coordinate lines $\langle v_{w\cdot\lambda_{\mathrm{reg}}}\rangle$ in $\mathbb{P}(V^{\lambda_{\mathrm{reg}}})$, for an irreducible $T$-invariant subvariety $X \subset G/B$ the *moment polytope* [5, 38] is by definition

$$\mu(X) = \mathrm{conv}(\{w \cdot \lambda_{\mathrm{reg}} \mid w \in X^T\}).$$

In particular $\mu(G/B)$ is the *$W$-permutahedron*

$$\mathrm{Perm}_W := \mathrm{conv}(\{w \cdot \lambda_{\mathrm{reg}} \mid w \in W\}).$$

Since $(X_u^v)^T = [u,v]$, the moment polytope of a Richardson variety is the *twisted Bruhat interval polytope* [64]

$$\mu(X_u^v) = P_{[u,v]} := \mathrm{conv}(\{w \cdot \lambda_{\mathrm{reg}} \mid w \in [u,v]\}) \subset \mathrm{Perm}_W .$$

A *Coxeter matroid* $\mathcal{M} \subset W$ is a subset of $W$ such that $w\mathcal{M}$ has a unique Bruhat-maximum element for every $w \in W$, and a *Coxeter matroid polytope* is a polytope $P \subset \mathrm{Perm}_W$ whose vertices are a subset of the vertices of $\mathrm{Perm}_W$ and whose edges are all parallel to roots of $W$. For example, the set $W$ is a Coxeter matroid, and the corresponding Coxeter matroid polytope is $\mathrm{Perm}_W$.

The Gelfand–Serganova theorem (see [20, Theorem 6.3.1] for a textbook treatment) gives a bijection between Coxeter matroids and Coxeter matroid polytopes given by

$$\mathcal{M} \mapsto P_{\mathcal{M}} := \mathrm{conv}(\{w \cdot \lambda_{\mathrm{reg}} \mid w \in \mathcal{M}\}) \subset \mathrm{Perm}_W .$$

**Fact 6.10** ([33, §7 Theorem 1])**.** The moment polytope $\mu(X)$ of an irreducible $T$-invariant variety $X \subset G/B$ is a Coxeter matroid polytope.

Since this result is not stated in this precise way in loc. cit. we include a sketch of the proof.

*Proof Sketch.* $\mu(X) = \mu(\overline{T\cdot x})$ for generic $x \in X$. Each edge of the moment polytope corresponds to a $T$-invariant curve in $\overline{T\cdot x}$, which is of the form $\mathbb{P}_{u,\tau u}$ for a reflection $\tau$, and the moment polytope of such a curve is a line segment in the direction $r(\tau) \in \Phi^+$. ☐

Because $(zX_u^v)^T = z[u,v]$, we deduce the following combinatorial corollary.

**Corollary 6.11.** For $u \leqslant_B v$, the translated Bruhat interval $z[u,v]$ is a Coxeter matroid.

## 7. Noncrossing partitions via translated Bruhat intervals

In this section we introduce the central combinatorial observation that will connect noncrossing partitions and Bruhat combinatorics. Given a Bruhat interval $[u, v]$ recall that $\overleftarrow{[u, v]} := u^{-1}[u, v]$. We show the following.

**Theorem 7.1.** Let $w \in W$ be such that $w \leqslant_B wc$. Then $\overleftarrow{[w, wc]} \subseteq \mathrm{NC}(W, c)$, and considered as an induced subposet of $(\mathrm{NC}(W, c), \leqslant_{\mathsf{T}})$, it is poset-isomorphic to the Bruhat interval $([w, wc], \leqslant_B)$ via left multiplication by $w^{-1}$:

$$[w, wc] \xrightarrow{\sim} \overleftarrow{[w, wc]} \qquad x \longmapsto w^{-1}x.$$

Both intervals have rank $n$, and in particular $\ell(w) + \ell(c) = \ell(wc)$.

Moreover, every maximal chain in $\mathrm{NC}(W, c)$ lies in some translated Bruhat interval: for each

$$\mathrm{id} \lessdot_{\mathsf{T}} \tau_1 \lessdot_{\mathsf{T}} \tau_1\tau_2 \lessdot_{\mathsf{T}} \cdots \lessdot_{\mathsf{T}} \tau_1\tau_2\cdots\tau_n = c,$$

there exists a $w \in W$ such that $w \lessdot_B w\tau_1 \lessdot_B w\tau_1\tau_2 \lessdot_B \cdots \lessdot_B w\tau_1\tau_2\cdots\tau_n = wc$.

**Corollary 7.2.** $w \leqslant_B wc$ is equivalent to $\ell(wc) = \ell(w) + \ell(c)$. In particular, in either of these two equivalent situations we may write the product $wc$ in the length additive notation $w \cdot c$.

*Proof of Theorem 7.1.* We deal with the two halves of the statement separately. For the first half, fix $w \in W$ with $w \leqslant_B wc$.

**Step 1: containment and induced-subposet identification.** Since Bruhat order is graded by $\ell$, every $x \in [w, wc]$ lies on some maximal chain from $w$ to $wc$. Moreover, the product of the labels of each chain give a factorization of $c$ into reflections. Thus the length of this chain must be exactly $n$: it cannot be larger as $\ell(wc) \leqslant \ell(w) + n$, and it cannot be smaller as $\ell_{\mathsf{T}}(c) = n$. Thus translating such a chain by $w^{-1}$ yields a maximal chain in absolute order from id to $c$ passing through $w^{-1}x$. Hence $\overleftarrow{[w, wc]} \subseteq \mathrm{NC}(W, c)$, and the length of this chain must be $\ell_{\mathsf{T}}(c) = n$.

Moreover, along any maximal Bruhat chain $w = x_0 \lessdot_B x_1 \lessdot_B \cdots \lessdot_B x_{\ell(c)} = wc$, $w^{-1}$-translation gives a maximal absolute-order chain $\mathrm{id} \lessdot_{\mathsf{T}} w^{-1}x_1 \lessdot_{\mathsf{T}} \cdots \lessdot_{\mathsf{T}} c$. In particular, for $x \in [w, wc]$:

$$\ell_{\mathsf{T}}(w^{-1}x) = \ell(x) - \ell(w). \tag{7.1}$$

Since both posets are graded (by $\ell$ on $[w, wc]$ and by $\ell_{\mathsf{T}}$ on $\mathrm{NC}(W, c)$), the equality in (7.1) forces the translation map $x \mapsto w^{-1}x$ to have the property that for $x, y \in [w, wc]$,

$$x \lessdot_B y \iff w^{-1}x \lessdot_{\mathsf{T}} w^{-1}y.$$

This is exactly the statement that $\overleftarrow{[w, wc]}$ with the induced order from $\mathrm{NC}(W, c)$, is poset isomorphic to $[w, wc]$ via $x \mapsto w^{-1}x$.

**Step 2: every absolute chain comes from a translated Bruhat chain.** Let $\mathrm{id} \lessdot_{\mathsf{T}} \tau_1 \lessdot_{\mathsf{T}} \tau_1\tau_2 \lessdot_{\mathsf{T}} \cdots \lessdot_{\mathsf{T}} \tau_1\tau_2\cdots\tau_n = c$ be a maximal chain in $\mathrm{NC}(W, c)$. Define roots

$$\beta_i := \tau_1 \cdots \tau_{i-1}(r(\tau_i)) \qquad (1 \leqslant i \leqslant n).$$

Since $s_{\beta_1} \cdots s_{\beta_n} = c$ is a minimal reflection factorization we have that $\beta_1, \ldots, \beta_n$ are linearly independent. In particular there exists $w \in W$ so that $w(\beta_i) \in \Phi^+$ for $1 \leqslant i \leqslant n$. We claim that for this choice of $w$ we get a maximal chain in Bruhat order that we seek. Setting $y_i := w\tau_1 \cdots \tau_i$ we have

$$y_{i-1}(r(\tau_i)) = w(\beta_i) \in \Phi^+,$$

so $y_{i-1} <_B y_i$ for all $i$. As there are $n = \ell(c)$ steps, the chain $w <_B y_1 <_B \cdots <_B wc$ is maximal, and every consecutive relation corresponds to a cover. In particular $wu \lessdot_B wv$. Also, the same length count gives $\ell(wc) = \ell(w) + \ell(c)$.

This proves the $(\Rightarrow)$ direction in the final equivalence of the theorem. The converse $(\Leftarrow)$ direction is exactly the cover correspondence established in the first half. ☐

Recall that there is an EL-labeling of $\mathrm{NC}(W, c)$ where the edge $xy$ is labeled with the reflection $x^{-1}y = y^{-1}x$. Decreasing chains under the $c$-reflection order are called $c$-decreasing chains, and there are $\mathrm{Cat}_W^+$-many $c$-decreasing maximal chains in $\mathrm{NC}(W, c)$ (Corollary 3.6).

**Corollary 7.3.** A translated interval $\overleftarrow{[w, w \cdot c]} \subset \mathrm{NC}(W, c)$ contains a unique $c$-decreasing maximal chain. Furthermore each $c$-decreasing maximal chain in $\mathrm{NC}(W, c)$ is contained in $\overleftarrow{[w, w \cdot c]}$.

*Proof.* By Theorem 3.4 there is a unique $c$-decreasing maximal chain in $[w, wc]$ under the edge labeling of $xy$ by $x^{-1}y = y^{-1}x$. By Theorem 7.1, left-multiplication by $w^{-1}$ gives a poset isomorphism between $\overleftarrow{[w, wc]}$ and $[w, wc]$, which in view of $(wx)^{-1}wy = x^{-1}y$, induces an edge labeling on $\overleftarrow{[w, wc]}$. So we conclude there is a unique $c$-decreasing maximal chain in $\overleftarrow{[w, wc]}$. Finally, Theorem 7.1 also implies every maximal chain in $\mathrm{NC}(W, c)$ belongs to at least one $\overleftarrow{[w, wc]}$. ☐

## 8. The Coxeter Flag Variety

In this section we introduce the $c$-Coxeter Richardson varieties and the $c$-Coxeter flag variety.

### 8.1. $c$-Coxeter Richardson varieties.

A Bruhat interval $[u, v]$ is said to be *toric* if the Richardson variety $X_u^v$ is a toric variety with respect to $T$. Because Richardson varieties are normal [57, Proposition 1.22], toric Richardson varieties are normal projective toric varieties, and so each such $X_u^v$ is the toric variety $X_\Sigma$ of the complete fan $\Sigma$ arising as the normal fan of the moment polytope.

**Proposition 8.1.** Each face of $P_{[u,v]}$ is a polytope of the form $P_{[u',v']}$ for a subinterval $[u', v'] \subset [u, v]$. Furthermore the $T$-invariant subvarieties of $X_u^v$ with $[u, v]$ toric are exactly the Richardson varieties $X_{u'}^{v'}$ with $[u', v'] \subset [u, v]$.

*Proof.* The first part is [64, Theorem 7.13]. The second part follows as the faces of the moment polytope are in bijection with $T$-invariant subvarieties. □

Now recall from the introduction that a $c$-Coxeter Richardson variety is one of the form $X_w^{w\cdot c}$.

**Theorem 8.2.** Every $c$-Coxeter Richardson variety $X_w^{w\cdot c}$ is toric.

*Proof.* By [64, Proposition 7.12], for any Bruhat interval $[u,v]$ one has

$$\dim P_{[u,v]} \leqslant \ell(v) - \ell(u),$$

and equality holds if and only if $[u,v]$ is toric. Setting $u = w$ and $v = wc$ implies that

$$\dim P_{[w,wc]} \leqslant \ell(wc) - \ell(w) = n$$

Thus it suffices to show that $\dim P_{[w,wc]} \geqslant n$.

Write $c = s_1 \cdots s_n$ as a reduced product of simple reflections. Consider the sequence of vertices

$$w \cdot \lambda_{\mathrm{reg}}, \quad ws_1 \cdot \lambda_{\mathrm{reg}}, \quad ws_1 s_2 \cdot \lambda_{\mathrm{reg}}, \quad \ldots, \quad ws_1 \cdots s_n \cdot \lambda_{\mathrm{reg}} = wc \cdot \lambda_{\mathrm{reg}} \tag{8.1}$$

in $P_{[w,wc]}$. Consider the vectors $v_i$ for $1 \leqslant i \leqslant n$ obtained by taking consecutive differences:

$$v_i := ws_1 \cdots s_{i-1} \cdot \lambda_{\mathrm{reg}} - ws_1 \cdots s_i \cdot \lambda_{\mathrm{reg}} = ws_1 \cdots s_{i-1} \big(\mathrm{id} - s_i\big) \cdot \lambda_{\mathrm{reg}}.$$

Since $\lambda_{\mathrm{reg}}$ is regular dominant we have $(\mathrm{id} - s_i) \cdot \lambda_{\mathrm{reg}} = (\lambda_{\mathrm{reg}}, r(s_i)^\vee)\, r(s_i)$ with $(\lambda_{\mathrm{reg}}, r(s_i)^\vee) > 0$. Thus, $v_i$ is parallel to $ws_1 \cdots s_{i-1}\big(r(s_i)\big)$. Equivalently, if we translate by $w^{-1}$, the directions are given by:

$$r(s_1), \quad s_1 r(s_2), \quad s_1 s_2 r(s_3), \ \ldots, \ s_1 \cdots s_{n-1} r(s_n).$$

We now check that these $n$ vectors are linearly independent. We have for any root $\beta$ that

$$s_j(\beta) = \beta - (\beta, r(s_j)^\vee)\, r(s_j),$$

which inductively implies that $s_1 \cdots s_{i-1}\big(r(s_i)\big)$ is a linear combination of $r(s_1), \ldots, r(s_i)$ with nonzero coefficient on $r(s_i)$ and no contribution from $r(s_j)$ for $j > i$. Hence a routine triangularity argument implies that these vectors are linearly independent. Left-translating by $w$ does not affect linear independence, and hence so are the $v_i$ vectors as well. Therefore the $n+1$ vertices in (8.1) are affinely independent, and we obtain $\dim P_{[w,wc]} \geqslant n$, as desired. □

8.2. **The toric complex $\mathrm{CFl}_c$.** Recall that *$c$-Coxeter flag variety* is the union

$$\mathrm{CFl}_c = \bigcup w^{-1} X_w^{w\cdot c} \subset G/B.$$

By Theorem 8.2, each $w^{-1} X_w^{w\cdot c}$ is a $T$-orbit closure in $G/B$ so $\mathrm{CFl}_c$ is a toric complex.

**Theorem 8.3.** The following hold.

(1) We have $\mathrm{CFl}_c^T = \mathrm{NC}(W,c)$, and the $T$-invariant curves $\mathbb{P}_{u,v}$ (Definition 5.3) contained in $\mathrm{CFl}_c$ are

$$\{\mathbb{P}_{u,v} \mid u, v \text{ adjacent in } \mathrm{NC}(W,c)\}.$$

In particular (from the fixed point statement) we have $\mathrm{CFl}_c \subset \mathrm{PV}_{\mathrm{NC}(W,c)}$.

(2) Distinct $T$-orbit closures in $\mathrm{CFl}_c$ have distinct fixed point sets $X^T \subset \mathrm{NC}(W,c)$.

(3) For $T$-orbit closures $Y$ and $Z$ in $\mathrm{CFl}_c$ we have

$$Y^T \subset Z^T \iff Y \subset Z.$$

*Proof.* (1) follows from Theorem 7.1. For (2) and (3) it suffices to show that $T$-orbit closures in $\mathrm{CFl}_c$ are rigid in the sense of Theorem 6.6. By Proposition 8.1, every $T$-orbit closure in $\mathrm{CFl}_c$ is a translated Richardson $w^{-1}X_u^v$ for some $w \in W$ with $\ell(wc) = \ell(w) + \ell(c)$, and so we conclude by Fact 6.7. □

Later in Theorem 9.3 we will improve Theorem 8.3(1), showing the equality $\mathrm{CFl}_c = \mathrm{PV}_{\mathrm{NC}(W,c)}$.

8.3. $(W,c)$**-polypositroids.** We now proceed to identify the faces of the polytopes in $\mathrm{Complex}(\mathrm{CFl}_c)$ from Section 2 as members of the class of $(W,c)$-polypositroids introduced by Lam–Postnikov [46]. The results that follow are a substantial generalization of [8, Theorem 7.6].

**Definition 8.4** ([46, Definition 13.1])**.** A Coxeter matroid polytope is called a *$(W,c)$-polypositroid* if it can be defined by inequalities $\mathbf{x}\cdot\vec{n} \leqslant a$ using vectors $\vec{n}$ with the property that $(\mathrm{id} - c)\,\vec{n}$ is parallel to a root in $\Phi$.

We now show that the moment polytopes arising in this complex are $(W,c)$-polypositroids.

**Theorem 8.5** (Theorem 2.2)**.** All faces of polytopes in $\mathrm{Complex}(\mathrm{CFl}_c)$ are $(W,c)$-polypositroids.

*Proof.* Any face of a $(W,c)$-polypositroid is a $(W,c)$-polypositroid, so it suffices to show that the top-dimensional $w^{-1}P_{[w,wc]}$ are $(W,c)$-polypositroids. By Proposition 8.1, the facets of $w^{-1}P_{[w,wc]}$ are exactly the faces of the form $w^{-1}P_{[w,wv]}$ with $w \leqslant_B wv <_B wc$ and $w^{-1}P_{[wu,wc]}$ with $w <_B wu \leqslant_B wc$, so we want to show that the normal vectors $\vec{n}$ to these facets have the property that $(\mathrm{id} - c)\vec{n}$ are parallel to roots in $\Phi$.

We first consider facets of the form $w^{-1}P_{[w,wv]}$ with $w \leqslant_B wv <_B wc$. Take any maximal chain

$$\mathcal{C} = w <_B w\tau_1 <_B w\tau_2\tau_1 <_B \ldots <_B w\tau_{n-1}\cdots\tau_1 <_B w\tau_n\cdots\tau_1 = wc \qquad \text{with } \tau_{n-1}\cdots\tau_1 = v.$$

Since $\tau_n\cdots\tau_1$ is a minimal reflection factorization for $c$, the collection of roots $\{r(\tau_1),\ldots,r(\tau_n)\}$ is linearly independent. Translating $\mathcal{C}$ by $w^{-1}$, the corresponding edges in $w^{-1}P_{[w,wc]}$ are in the linearly independent directions $r(\tau_1),\ldots,r(\tau_{n-1})$. Therefore the facet $w^{-1}P_{[w,wv]}$ is spanned by $r(\tau_1),\ldots,r(\tau_{n-1})$. The normal vector $\vec{n}$ is therefore fixed by $\tau_1,\ldots,\tau_{n-1}$, so we have

$$(\mathrm{id} - c)\,\vec{n} \;=\; (\mathrm{id} - \tau_n)\,\vec{n} \;=\; (\vec{n}, r(\tau_n)^\vee)\; r(\tau_n),$$

which is parallel to $r(\tau_n) \in \Phi$.

For facets of the form $w^{-1}P_{[wu,wc]}$ with $w <_B wu \leqslant_B wc$, we take a maximal chain

$$w <_B w\tau_1 <_B w\tau_2\tau_1 <_B \ldots <_B w\tau_{n-1}\cdots\tau_1 <_B w\tau_n\cdots\tau_1 = wc \qquad \text{with } \tau_1 = u.$$

Arguing as before we see that the facet $w^{-1}P_{[wu,wc]}$ is spanned by the linearly independent roots given by $r(\tau_2), \ldots, r(\tau_n)$, and thus the facet normal $\vec{n}$ is fixed by $\tau_2, \ldots, \tau_n$.

Now we write

$$c = \tau\tau_2\cdots\tau_n \text{ where } \tau = (\tau_n\cdots\tau_2)\ \tau_1\ (\tau_n\cdots\tau_2)^{-1} \in \mathsf{T}.$$

We then have $(\mathrm{id} - c)\ \vec{n} = (\mathrm{id} - \tau)\ \vec{n}$, which is parallel to $r(\tau) \in \Phi$. □

## 9. Plücker vanishing

In this section we characterize the Coxeter flag variety using the combinatorics of noncrossing partitions and clusters. Both Theorem A and Theorem B are established in Theorem 9.3. Recall the map $\mathrm{Clust}^+ : \mathrm{NC}(W,c) \to \mathrm{Cl}_c^+$ from Section 3.5. We show in Fact 9.7 that the set $r(\mathrm{Inv}_{\mathrm{NC}}(u)) \subseteq r(\mathrm{Inv}(u))$ is a closed set of roots, allowing us to make the following definition.

**Definition 9.1.** For $u \in \mathrm{NC}(W,c)$ we define the *Coxeter Schubert cell* to be

$$\mathring{X}^u_{\mathrm{NC}} := U_{r(\mathrm{Inv}_{\mathrm{NC}}(u))}uB/B \subset U_{r(\mathrm{Inv}(u))}uB/B = BuB/B = \mathring{X}^u$$

and the *Coxeter Schubert variety* $X^u_{\mathrm{NC}}$ to be the closure of $\mathring{X}^u_{\mathrm{NC}}$.

**Remark 9.2.** Recalling Section 5.2, there is a $T$-equivariant isomorphism $N_{r(\mathrm{Inv}_{\mathrm{NC}}(u))} \cong \mathring{X}^u_{\mathrm{NC}}$. A similar construction is given by Gelfand–Graev–Postnikov in [34], which studies the charts

$$\bigoplus_{\alpha\in\mathrm{Clust}^+(u)} \mathbb{C}_{-\alpha} \cong u^{-1}\mathring{X}^u_{\mathrm{NC}} \subset U^-B,$$

where $U^- = U_{\Phi^-}$ is the unipotent subgroup of the opposite Borel $B^-$. These charts in [34] are considered in the coordinate chart compactification of $U^- \cong \mathbb{A}^{\ell(w_\circ)} \subset \mathbb{P}^{\ell(w_\circ)}$, which makes their closures isomorphic to projective spaces. Our closures $X^u_{\mathrm{NC}}$ are not these projective spaces because our charts are compactified under the Plücker embedding into $\mathbb{P}(V^{\lambda_{reg}})$.

An *affine paving* of a closed subvariety $X$ is a sequence of closed subvarieties $X_1 \subset X_2 \subset \cdots \subset X_m = X$ such that $X_k \backslash X_{k-1}$ is isomorphic to an affine space for all $k$.

**Theorem 9.3** (Theorem A, Theorem B)**.** We have

$$\mathrm{CFl}_c = \mathrm{PV}_{\mathrm{NC}(W,c)} = \bigsqcup_{u\in\mathrm{NC}(W,c)} \mathring{X}^u_{\mathrm{NC}}.$$

Moreover, for any linear extension $u_1, u_2, \ldots, u_{\mathrm{Cat}_W}$ of the Bruhat order restricted to $\mathrm{NC}(W,c)$,

$$X_i = X^{u_i} \cap \mathrm{CFl}_c = \bigsqcup_{k\leqslant i} \mathring{X}^{u_k}_{\mathrm{NC}}$$

defines an affine paving of $\mathrm{CFl}_c$.

**Corollary 9.4.** There are exactly $\mathrm{Cat}_W^+$-many distinct $w^{-1}X_w^{w\cdot c}$ comprising $\mathrm{CFl}_c$, and if $u \in \overleftarrow{[w, w\cdot c]}$ is the Bruhat-maximum element, then $u \in \mathrm{NC}(W,c)^+$ and $w^{-1}X_w^{w\cdot c} = X_{\mathrm{NC}}^u$.

*Proof.* The irreducible components of $\mathrm{PV}_{\mathrm{NC}(W,c)}$ are the closures of the top-dimensional $X_{\mathrm{NC}}^u$, and the Bruhat-maximum element of $X_{\mathrm{NC}}^u$ is $u$. Because $\dim X_{\mathrm{NC}}^u = |\mathrm{Inv}_{\mathrm{NC}}(u)| \leqslant n$ with equality if and only if $u \in \mathrm{NC}(W,c)^+$, the result follows. ☐

In later sections we will determine the $u \in \mathrm{NC}(W,c)^+$ with $w^{-1}X_w^{w\cdot c} = X_{\mathrm{NC}}^u$.

**Remark 9.5.** As is well-known, the existence of an affine paving of a variety $X$ tells us that the $i$th Betti number $H^{2i}(X) = H_{2i}(X)$ is the number of pieces isomorphic to $\mathbb{A}^i$. In the case of $\mathrm{CFl}_c$ this is the the number of positive $c$-clusters of size $i$. These numbers are enumerated by the diagonal of the $F$-triangle of Chapoton [27, §2]. In type A [27, §4] one obtains the *ballot numbers* occurring in the Catalan triangle [56, A009766]. In type B [27, §5] one obtains [56, A059481].

The $\mathrm{B}_2$ case is instructive. Using the table in Example 3.9 we have that the Betti numbers are $1, 2, 3$. The Betti numbers of $G/B$ on the other hand are $1, 2, 2, 2, 1$. This implies that we do not have a surjection $H^\bullet(G/B)$ to $H^\bullet(\mathrm{CFl}_c)$ (see also the examples in Appendix B and Appendix C).

The proof of Theorem 9.3 is involved and occupies the next two subsections: in Section 9.1 we prove several technical results about Plücker coordinates, and in Section 9.2 we relate these results to Coxeter Schubert cells and $\mathrm{CFl}_c$.

9.1. **Coordinate subspaces of $\mathring{X}^u$ associated to strongly closed subsets.** Say that a subset $C \subset \Phi^+$ of roots is *strongly closed* if the only roots in the nonnegative cone spanned by $C$ are $C$ itself. Every strongly closed subset $C$ is closed in the sense of Section 5.2, but the converse is not true.

**Example 9.6.** In the type $\mathrm{B}_2$ root system $\Phi = \{\pm\epsilon_1 \pm \epsilon_2\} \cup \{\pm\epsilon_1, \pm\epsilon_2\}$, the set $C = \{\epsilon_1 + \epsilon_2, \epsilon_1 - \epsilon_2\}$ is closed but not strongly closed, as

$$\epsilon_1 = \frac{1}{2}(\epsilon_1 + \epsilon_2) + \frac{1}{2}(\epsilon_1 - \epsilon_2) \in \mathrm{Cone}(C) \cap \Phi.$$

This property is highly dependent on the root system: the same set is both closed and strongly closed in type $\mathrm{D}_2$ ($\Phi = \{\pm\epsilon_1 \pm \epsilon_2\}$), and is neither closed nor strongly closed in type $\mathrm{C}_2$ ($\Phi = \{\pm\epsilon_1 \pm \epsilon_2\} \cup \{\pm 2\epsilon_1, \pm 2\epsilon_2\}$).

**Fact 9.7** (*Cluster cone property*)**.** For $u \in \mathrm{NC}(W,c)$ the subset $r(\mathrm{Inv}_{\mathrm{NC}}(u)) \subset r(\mathrm{Inv}(u))$ is strongly closed.

*Proof.* As $r(\mathrm{Inv}_{\mathrm{NC}}(u)) = \{-u^{-1}\alpha \mid \alpha \in \mathrm{Clust}^+(u)\}$ and the action of $W$ preserves strong closure, it suffices to show that the positive subset of any $c$-clusters are closed. Recall that the $c$-cluster fan $\mathrm{Cl}_c$ is a complete, simplicial fan that includes every almost-positive root as a generating ray.

Thus, every positive root in each cone of $\mathrm{Cl}_c^+$ must be one of its generators, for any others would contradict the assumption that the cones form a fan. □

The aim of this section is to prove the following.

**Proposition 9.8.** For $w \in W$ and $C \subset r(\mathrm{Inv}(w))$ a strongly closed subset, the isomorphism $N_w \cong U_w wB/B = \mathring{X}^w$ from Section 5.2 restricts to an isomorphism

$$N_C \cong U_C wB/B = \mathring{X}^w \cap \bigcap_{\substack{\tau \in \mathrm{Inv}(w) \\ r(\tau) \notin C}} \{\mathrm{Pl}_{\tau w} = 0\}.$$

In particular for $u \in \mathrm{NC}(W, c)$ we have

$$\mathring{X}^u_{\mathrm{NC}} = \mathring{X}^u \cap \bigcap_{\tau \in \mathrm{Inv}(u) \setminus \mathrm{Inv}_{\mathrm{NC}}(u)} \{\mathrm{Pl}_{\tau u} = 0\}.$$

**Example 9.9.** We illustrate this proposition and its proof with an example, using the notation and matrices from type A as in Appendix A. Take $c = s_3 s_2 s_1 \in S_4$, and $u = w_\circ = 4321$. Then

$$X^{w_\circ} = \begin{bmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{bmatrix} w_\circ B/B = \begin{bmatrix} c & b & a & 1 \\ e & d & 1 & 0 \\ f & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} B/B$$

with $a, b, c, d, e, f$ corresponding to positive roots $\epsilon_i - \epsilon_j$ for the transpositions given by

$$(i, j) = (1\,2), (1\,3), (1\,4), (2\,3), (2\,4), (3\,4)$$

respectively. The Plücker coordinates associated to $(i\,j)w_\circ$ are given by

| $(i\,j)$ | $(1\,2)$ | $(1\,3)$ | $(1\,4)$ | $(2\,3)$ | $(2\,4)$ | $(3\,4)$ |
|---|---|---|---|---|---|---|
| $\mathrm{Pl}_{(i\,j)w_\circ}$ | $a$ | $b(ad-b)$ | $c(cd-be)(bf+ae-c-adf)$ | $d$ | $e(df-e)$ | $f$ |

each of which is computed as a product of determinants of submatrices using the first $\ell \in \{1, 2, 3\}$ columns and rows $(i, j)w_\circ\{1, \ldots, \ell\}$. We have $\mathrm{Inv}(w_\circ) \setminus \mathrm{Inv}_{\mathrm{NC}}(w_\circ) = \{(1\,2), (2\,4), (3\,4)\}$, so

$$\mathring{X}^{w_\circ}_{\mathrm{NC}} = \begin{bmatrix} c & b & 0 & 1 \\ 0 & d & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} B/B,$$

obtained by setting $a = e = f = 0$. On the other hand, the corresponding Plücker equations indexed by $\mathrm{Inv}(w_\circ) \setminus \mathrm{Inv}_{\mathrm{NC}}(w_\circ)$ are $a, e(df-e), f$.

As in the proof, we will order these equations by removing successive extreme ray generators to ensure—as proved in Lemma 9.11 below—that all but one variable is eliminated. Note that

- $\epsilon_1 - \epsilon_2$ is an extreme ray of $\mathbb{R}_{\geq 0}\Phi^+$,

- $\epsilon_3 - \epsilon_4$ is an extreme ray of $\mathbb{R}_{\geqslant 0}(\Phi^+ \backslash \{\epsilon_1 - \epsilon_2\})$, and
- $\epsilon_2 - \epsilon_4$ is an extreme ray of $\mathbb{R}_{\geqslant 0}(\Phi^+ \backslash \{\epsilon_1 - \epsilon_2, \epsilon_3 - \epsilon_4\})$,

so if we impose vanishing conditions in the same order, i.e. $(1\,2), (3\,4), (2\,4)$ or $a = 0$, $f = 0$, $e(df - e) = 0$, we see that $e(df-e) = 0$ becomes $-e^2 = 0$, so $a = f = e(df-e) = 0$ defines the same subspace as $a = f = e = 0$.

The proof follows the next two technical results. The first, which we suspect is known to experts in the area, describes the behavior of Plücker coordinates under the action of $W$.

**Lemma 9.10.** Let $\alpha \in \Phi^+$ and $w \in W$. There exists a nonzero constant $k \in \mathbb{C}^\times$ such that

$$\mathrm{Pl}_{s_\alpha w}(v) = kx^{-(\alpha^\vee, w\lambda_{\mathrm{reg}})}\mathrm{Pl}_w(s_\alpha(-x)v) \qquad \text{for all } x \in \mathbb{C}^\times \text{ and } v \in V^{\lambda_{\mathrm{reg}}}.$$

*Proof.* Recall the fixed extremal weight vectors $\{v_{u\lambda_{\mathrm{reg}}} \mid u \in W\}$ from (6.1), and recall that $s_\alpha(-x)$ sends each of these vectors to a scalar multiple of another; we will calculate this scalar for $v_{s_\alpha w\lambda_{\mathrm{reg}}}$. We begin by applying the Chevalley relations from Section 4.2:

$$s_\alpha(-x)v_{s_\alpha w\lambda_{\mathrm{reg}}} \overset{(4.6)}{=} h_{\alpha^\vee}(-x)s_\alpha(-1)^{-1}v_{s_\alpha w\lambda_{\mathrm{reg}}} \overset{(4.3),(4.4)}{=} h_{\alpha^\vee}(-x)s_\alpha(1)v_{s_\alpha w\lambda_{\mathrm{reg}}}.$$

Define $k \in \mathbb{C}^\times$ by the equation $ks_\alpha(1)v_{s_\alpha w\lambda_{\mathrm{reg}}} = (-1)^{(\alpha^\vee, w\lambda_{\mathrm{reg}})}v_{w\lambda_{\mathrm{reg}}}$, so that the above is equal to

$$\begin{aligned} k^{-1}(-1)^{(\alpha^\vee, w\lambda_{\mathrm{reg}})}h_{\alpha^\vee}(-x)v_{w\lambda_{\mathrm{reg}}} &= k^{-1}(-1)^{(\alpha^\vee, w\lambda_{\mathrm{reg}})}(-x)^{(\alpha^\vee, w\lambda_{\mathrm{reg}})}v_{w\lambda_{\mathrm{reg}}} \\ &= k^{-1}x^{(\alpha^\vee, w\lambda_{\mathrm{reg}})}v_{w\lambda_{\mathrm{reg}}}. \end{aligned}$$

Thus $\mathrm{Pl}_w(s_\alpha(-x)v) = k^{-1}x^{(\alpha^\vee, w\lambda_{\mathrm{reg}})}\mathrm{Pl}_{s_\alpha w}(v)$, from which the claim follows. ∎

Before stating the second lemma, we make some general observations about the computation of Plücker coordinates. For $w \in W$ and $C = \{\beta_1, \ldots, \beta_{|C|}\} \subseteq r(\mathrm{Inv}(w))$,

$$e_{\beta_1}(x_{\beta_1}) \cdots e_{\beta_{|C|}}(x_{\beta_{|C|}})v_{w\lambda_{\mathrm{reg}}} = \sum_{\vec{a} \in \mathbb{Z}_{\geqslant 0}^{|C|}} x_{\beta_1}^{a_1} \cdots x_{\beta_{|C|}}^{a_{|C|}} E_{\beta_1}^{a_1} \cdots E_{\beta_{|C|}}^{a_{|C|}} v_{w\lambda_{\mathrm{reg}}}.$$

With the caveat that all but finitely many summands above are zero, we observe that the summand for $\vec{a} \in \mathbb{Z}_{\geqslant 0}^{|C|}$ belongs to the $w\lambda_{\mathrm{reg}} + a_1\beta_1 + \cdots + a_{|C|}\beta_{|C|}$ weight space. Thus for $u \in W$,

$$\mathrm{Pl}_u\left(e_{\beta_1}(x_{\beta_1}) \cdots e_{\beta_{|C|}}(x_{\beta_{|C|}})v_{w\lambda_{\mathrm{reg}}}\right) = \mathrm{Pl}_u\left(\sum_{\text{solutions } \vec{a}} x_{\beta_1}^{a_1} \cdots x_{\beta_{|C|}}^{a_{|C|}} E_{\beta_1}^{a_1} \cdots E_{\beta_{|C|}}^{a_{|C|}} v_{w\lambda_{\mathrm{reg}}}\right), \tag{9.1}$$

where the sum in the right hand side is over all $\vec{a} \in \mathbb{Z}_{\geqslant 0}^{|C|}$ for which

$$u\lambda_{\mathrm{reg}} - w\lambda_{\mathrm{reg}} = a_1\beta_1 + \cdots + a_{|C|}\beta_{|C|}.$$

**Lemma 9.11.** Let $w \in W$ and $C \subset r(\mathrm{Inv}(w))$. For any ordering $\beta_1, \ldots, \beta_{|C|}$ of $C$ so that $\alpha = \beta_{|C|}$ is an extremal ray generator of $\mathrm{Cone}(C)$, and any collection $\{x_{\beta_i} \in \mathbb{C} \mid \beta_i \in C\}$,

$$\mathrm{Pl}_{s_\alpha w}\left(e_{\beta_1}(x_{\beta_1}) \cdots e_{\beta_{|C|}}(x_{\beta_{|C|}})v_{w\lambda_{\mathrm{reg}}}\right) = kx_\alpha^{-(\alpha^\vee, w\lambda_{\mathrm{reg}})}$$

where $k \in \mathbb{C}^\times$ is the proportionality constant in Lemma 9.10 for $w$ and $\alpha$.

Note that $w^{-1}\alpha$ is a negative root, so the exponent $-(\alpha^\vee, w\lambda_{\mathrm{reg}})$ above is in fact nonnegative.

*Proof.* We first reduce to the case wherein $C = \{\alpha\}$ by proving that

$$\mathrm{Pl}_{s_\alpha w}\left(e_{\beta_1}(x_{\beta_1})\cdots e_{\beta_{|C|}}(x_{\beta_{|C|}})v_{w\lambda_{\mathrm{reg}}}\right) = \mathrm{Pl}_{s_\alpha w}\left(e_\alpha(x_\alpha)v_{w\lambda_{\mathrm{reg}}}\right). \tag{9.2}$$

By (9.1) and the fact that $s_\alpha w\lambda_{\mathrm{reg}} = w\lambda_{\mathrm{reg}} - (\alpha^\vee, w\lambda_{\mathrm{reg}})\alpha$, all contributions to the left hand side of (9.2) must correspond to solutions $\vec{a} \in \mathbb{Z}_{\geqslant 0}^{|C|}$ to the equation

$$-\big(a_{|C|} + (\alpha^\vee, w\lambda_{\mathrm{reg}})\big)\alpha = a_1\beta_1 + \cdots + a_{|C|-1}\beta_{|C|-1}.$$

By assumption $\mathrm{Cone}(C\backslash\{\alpha\}) \cap \mathbb{Z}\alpha = \{\vec{0}\}$, so any possible solution has $a_{|C|} = -(\alpha^\vee, w\lambda_{\mathrm{reg}})$ and $\sum_{i=1}^{|C|-1} a_i\beta_i = \vec{0}$. As each $\beta_i \in \Phi^+$, the only solution to the latter equation is $a_1 = \cdots = a_{|C|-1} = 0$; this can be seen for example by taking the inner product of both sides with $\lambda_{\mathrm{reg}}$, as $(\beta_i, \lambda_{\mathrm{reg}}) > 0$. Therefore the $s_\alpha w$-Plücker coordinate comes solely from $x_\alpha^{-(\alpha^\vee, w\lambda_{\mathrm{reg}})} E_\alpha^{-(\alpha^\vee, w\lambda_{\mathrm{reg}})} v_{w\lambda_{\mathrm{reg}}}$. This is also the case for the right hand side of (9.2), proving that (9.2) holds.

Now assume $C = \{\alpha\}$. If $x_\alpha = 0$, then $e_\alpha(x_\alpha)v_{w\lambda_{\mathrm{reg}}} = v_{w\lambda_{\mathrm{reg}}}$ is entirely in the $w\lambda_{\mathrm{reg}}$ weight space and therefore has $s_\alpha w$-Plücker coordinate zero. Therefore we assume that $x_\alpha \neq 0$ and consider $s_\alpha(-x_\alpha)e_\alpha(x_\alpha)v_{w\lambda_{\mathrm{reg}}}$, with an eye toward applying Lemma 9.10.

We first claim that $s_\alpha(-x_\alpha)e_\alpha(x_\alpha)v_{w\lambda_{\mathrm{reg}}} = e_\alpha(-x_\alpha)v_{w\lambda_{\mathrm{reg}}}$. Using the Chevalley relations we have

$$s_\alpha(-x_\alpha)e_\alpha(x_\alpha) \overset{(4.3)}{=} e_\alpha(-x_\alpha)e_{-\alpha}(x_\alpha^{-1})e_\alpha(-x_\alpha)e_\alpha(x_\alpha) \overset{(4.4)}{=} e_\alpha(-x_\alpha)e_{-\alpha}(x_\alpha^{-1}).$$

To prove our claim we show that $e_{-\alpha}(x_\alpha^{-1})$ acts trivially on $v_{w\lambda_{\mathrm{reg}}}$. Let $w = s_{\alpha_1}\cdots s_{\alpha_\ell}$ be a reduced word for $w$. As $\lambda_{\mathrm{reg}}$ is regular, there exists a $h \in T$ such that $\dot{w} = s_{\alpha_1}(1)s_{\alpha_2}(1)\cdots s_{\alpha_\ell}(1)h \in N_G(T)$ maps $v_{\lambda_{\mathrm{reg}}}$ to $v_{w\lambda_{\mathrm{reg}}}$. We therefore have

$$e_{-\alpha}(x_\alpha^{-1})v_{w\lambda_{\mathrm{reg}}} = e_{-\alpha}(x_\alpha^{-1})\dot{w}v_{\lambda_{\mathrm{reg}}} \overset{(4.8),(4.9)}{=} \dot{w}e_{-w^{-1}\alpha}(y)v_{\lambda_{\mathrm{reg}}} \qquad \text{for some } y \in \mathbb{C}^\times.$$

Now $v_{\lambda_{\mathrm{reg}}}$ is a highest weight vector, and $-w^{-1}\alpha \in \Phi^+$ (as $s_\alpha \in \mathrm{Inv}(w)$), so $E_{-w^{-1}\alpha}v_{\lambda_{\mathrm{reg}}} = 0$, and therefore $e_{-w^{-1}\alpha}(y)v_{\lambda_{\mathrm{reg}}} = v_{\lambda_{\mathrm{reg}}}$. Therefore the above expression is equal to $\dot{w}v_\lambda = v_{w\lambda_{\mathrm{reg}}}$.

As a consequence,

$$\mathrm{Pl}_w\left(s_\alpha(-x_\alpha)e_\alpha(x_\alpha)v_{w\lambda_{\mathrm{reg}}}\right) = \mathrm{Pl}_w\left(e_\alpha(-x_\alpha)v_{w\lambda_{\mathrm{reg}}}\right) = 1.$$

Lemma 9.10 now gives the desired formula for $\mathrm{Pl}_{s_\alpha w}\left(e_\alpha(x_\alpha)v_{w\lambda_{\mathrm{reg}}}\right)$. ☐

*Proof of Proposition 9.8.* As $C$ is strongly closed, by greedily removing extreme rays from the cone spanned by $r(\mathrm{Inv}(w))$ we can order $r(\mathrm{Inv}(w))\backslash C$ as $\beta_{|C|+1}, \ldots, \beta_{\ell(w)}$ in such a way that, for each $|C| < i \leqslant \ell(w)$, $\beta_i$ is an extreme ray of the cone spanned by $r(\mathrm{Inv}(w))\backslash\{\beta_{i+1}, \ldots, \beta_{\ell(w)}\}$.

Now using Lemma 9.11, the $s_{\beta_{\ell(w)}}w$-Plücker function composed with $N_w \cong U_w wB$ is a power of the $\beta_{\ell(w)}$-coordinate function of $N_w$ and as $\lambda_{\text{reg}}$ is regular, this power is not zero. Thus the isomorphism descends to $\mathring{X}^w \cap \{\text{Pl}^{\lambda}_{\beta_{\ell(w)}} = 0\}$ and $N_{r(\text{Inv}(w))\setminus\{\beta_{\ell(w)}\}}$. Repeating this descent argument for $N_{r(\text{Inv}(w))\setminus D}$ with $D = \{\beta_{|C|+i}, \ldots, \beta_{\ell(w)}\}$ for each $1 \leqslant i < \ell(w) - |C|$, we arrive at the desired isomorphism. □

9.2. **Coxeter Schubert cells and** $\text{CFl}_c$ **via Plücker vanishing.** We now apply results of the previous section to Coxeter Schubert cells and eventually $\text{CFl}_c$. The proof of Theorem 9.3 is given at the end of the section.

**Proposition 9.12.** We have $\text{PV}_{\text{NC}(W,c)} \subset \bigsqcup_{u \in \text{NC}(W,c)} \mathring{X}^u_{\text{NC}}$.

*Proof.* Let $x \in \text{PV}_{\text{NC}(W,c)}$ and take $u \in W$ so that $x \in \mathring{X}^u$. Then $\overline{T \cdot x} \subset \text{PV}_{\text{NC}(W,c)}$ by Theorem 6.6. For a strictly antidominant cocharacter $\psi : \mathbb{C}^* \to T$ (as in the proof of Fact 6.7) we have $u = \lim_{t\to 0} \psi(t)x \in \overline{T \cdot x}$ and so $u \in \text{NC}(W, c)$. The fact that $x \in \mathring{X}^u_{\text{NC}}$ now follows from Fact 9.7 and Proposition 9.8. □

**Corollary 9.13.** If $u \in \overleftarrow{[w, w \cdot c]}$ is the Bruhat-maximum element then $w^{-1}X^{w\cdot c}_w = X^u_{\text{NC}}$.

*Proof.* We have $u \in w^{-1}X^{w\cdot c}_w \subset X^u$, and so $w^{-1}X^{wc}_w \cap \mathring{X}^u$ is a dense open subset of the irreducible $n$-dimensional variety $w^{-1}X^{wc}_w$. Furthermore by Theorem 8.3(1) and Proposition 9.8 we have

$$(w^{-1}X^{wc}_w) \cap \mathring{X}^u \subset \text{PV}_{\text{NC}(W,c)} \cap \mathring{X}^u \subset \mathring{X}^u_{\text{NC}},$$

and $\mathring{X}^u_{\text{NC}} \cong \mathbb{A}^n$. Therefore $w^{-1}X^{w\cdot c}_w \cap \mathring{X}^u$ is dense in $\mathring{X}^u_{\text{NC}}$, and taking closures we obtain

$$w^{-1}X^{w\cdot c}_w = X^u_{\text{NC}}.$$

□

For the next result, we make a few observations about descending to a standard parabolic subgroup $W' \subseteq W$. First, if $c' \leqslant_B c$ is a Coxeter element of $W'$, then $\text{NC}(W', c') \subseteq \text{NC}(W, c)$. Second, if $w' \in W$ has $\ell(w'c') = \ell(w') + \ell(c')$ , then $\overleftarrow{[w', w' \cdot c']} \subseteq \text{NC}(W', c')$ by Theorem 7.1.

**Corollary 9.14.** Let $W' \subseteq W$ be a standard parabolic subgroup and let $c'$ be a Coxeter element of $W'$ with $c' \leqslant_B c$. If $w' \in W$ and $u \in \overleftarrow{[w', w' \cdot c']} \subset \text{NC}(W', c')$ is the Bruhat-maximum element, then $(w')^{-1}X^{w'\cdot c'}_{w'} = X^u_{\text{NC}}$.

*Proof.* Let $L \subseteq LB \subseteq G$ be the Levi and parabolic subgroups associated to $W'$, so that $B_L := B \cap L$ is a Borel subgroup for $L$. Then $(w')^{-1}X^{w'c'}_{w'} \subseteq LB/B$, and under $L/B_L \cong LB/B$ it must map to $\text{CFl}_{c'}$. We can therefore apply the argument in the proof of Corollary 9.13 above to equate the $\ell(c')$-dimensional varieties

$$\widetilde{(w')^{-1}X^{w'c'}_{w'}} = \widetilde{X^u_{\text{NC}}},$$

where $\widetilde{\cdot}$ indicates that these varieties are viewed in $L/B_L$. Transporting back along $L/B_L \cong LB/B \subseteq G/B$ identifies $\widetilde{X^u_{\text{NC}}}$ with $X^u_{\text{NC}}$. □

**Lemma 9.15.** If in Corollary 9.14 we have $w' \in W'$, then we have $\ell(w'c) = \ell(w') + \ell(c)$.

*Proof.* If $y, z$ are in the subgroup of $W$ generated by simple reflections in $\mathsf{S}\backslash\{s_j\}$ and $\ell(y) + \ell(z) = \ell(yz)$ then we claim that $\ell(ys_jz) = \ell(yz)+1$. Indeed, $ys_jy^{-1} \notin \mathrm{Inv}(yz)$ as $\mathrm{Inv}(yz)$ is in the subgroup of $W$ generated by $\mathsf{S}\backslash\{s_j\}$, so $\ell(yz) + 1 \geqslant \ell(ys_jz) > \ell(yz)$.

Now, start with a reduced word for $w'c'$ obtained by concatenating reduced words for $w'$ and $c'$ and apply the claim iteratively, inserting the missing letters from $c$ that are not in $c'$. □

**Proposition 9.16.** Let $u \in \mathrm{NC}(W, c)$ and let $c' \leqslant_B c$ be the sub-Coxeter corresponding to a standard parabolic subgroup $W' \subset W$ such that $u \in \mathrm{NC}(W', c')^+$. Then there exists $w' \in W'$ with $\ell(w'c') = \ell(w') + \ell(c')$ with $u \in \overleftarrow{[w', w'c']} \subset \mathrm{NC}(W', c')$.

*Proof.* It suffices to prove the statement for $u \in \mathrm{NC}(W, c)^+$, as we can apply that statement directly to $W'$ and $c'$. Note that by Corollary 7.3 and Corollary 3.6 we know that there are at least $\mathrm{Cat}_W^+$-many translated $\overleftarrow{[w, wc]}$, so there are at least $\mathrm{Cat}_W^+$ many distinct $w^{-1}X_w^{w\cdot c}$ comprising the irreducible components of $\mathrm{CFl}_c$. On the other hand, by Theorem 8.3(1) and Proposition 9.12 we know that

$$\mathrm{CFl}_c \subseteq \mathrm{PV}_{\mathrm{NC}(W,c)} \subseteq \bigsqcup_{u\in\mathrm{NC}(W,c)} \mathring{X}_{\mathrm{NC}}^u \subseteq \bigcup_{u\in\mathrm{NC}(W,c)} X_{\mathrm{NC}}^u,$$

and $\dim \mathring{X}_{\mathrm{NC}}^u = |\mathrm{Inv}_{\mathrm{NC}}(u)| \leqslant n$ with equality if and only if $u \in \mathrm{NC}(W, c)^+$, so there are $\mathrm{Cat}_W^+$-many top-dimensional irreducible components of $\bigcup_{u\in\mathrm{NC}(W,c)} X_{\mathrm{NC}}^u$. We conclude that each $X_{\mathrm{NC}}^u$ must equal at least one $w^{-1}X_w^{w\cdot c}$. □

*Proof of Theorem 9.3.* By Theorem 8.3(1) and Proposition 9.12 we know that

$$\mathrm{CFl}_c \subseteq \mathrm{PV}_{\mathrm{NC}(W,c)} \subseteq \bigsqcup_{u\in\mathrm{NC}(W,c)} \mathring{X}_{\mathrm{NC}}^u,$$

so to show these containments are equalities it suffices to show that $\bigsqcup_{u\in\mathrm{NC}(W,c)} \mathring{X}_{\mathrm{NC}}^u \subseteq \mathrm{CFl}_c$.

For each $u \in \mathrm{NC}(W, c)^+$, there exists $w \in W$ with $\ell(w) + \ell(c) = \ell(wc)$ such that $u$ is the Bruhat-maximum element of $\overleftarrow{[w, wc]}$ (which exists by Proposition 9.16). Then by Corollary 9.13 we have

$$X_{\mathrm{NC}}^u = w^{-1}X_w^{wc} \subset \mathrm{CFl}_c\,.$$

For an arbitrary $u \in \mathrm{NC}(W, c)$ if $c'$ is the associated sub-Coxeter element in the minimal sub-Weyl group $(W', \mathsf{S}') \subset (W, \mathsf{S})$ containing $u$, then $u \in \mathrm{NC}(W', c')^+$ and we again take by Proposition 9.16 a $w' \in W'$ such that $u$ is the Bruhat-maximum element of $\overleftarrow{[w', w'c']} \subset W'$. By Lemma 9.15 we have $\ell(w') + \ell(c) = \ell(w'c)$ and so by Corollary 9.14 we have

$$X_{\mathrm{NC}}^u = (w')^{-1}X_{w'}^{w'c'} \subset (w')^{-1}X_{w'}^{w'c} \subset \mathrm{CFl}_c\,.$$

Finally we establish the affine paving. Recall that $u_1$ through $u_{\mathrm{Cat}_W}$ is an ordering of $\mathrm{NC}(W, c)$ compatible with Bruhat order on $W$. Because $X_{k+1}\backslash X_k = \mathring{X}_{\mathrm{NC}}^{u_{k+1}}$, which is isomorphic to an affine space, it suffices to show that $X_k$ is closed. But this follows as

$$X^{u_k} \cap \mathrm{CFl}_c = \left( \bigsqcup_{u \leqslant u_k} BuB \right) \cap \bigsqcup_{i=1}^{\mathrm{Cat}_W} \mathring{X}^{u_i}_{\mathrm{NC}} = \bigsqcup_{i=1}^{k} \mathring{X}^{u_i}_{\mathrm{NC}} = X_k. \qquad \square$$

**Remark 9.17.** The affine paving implies that $\mathrm{Complex}(\mathrm{CFl}_c)$ is contractible. Indeed, let $P_k = \mu(X^{u_k}_{\mathrm{NC}})$. Let $C^{\leqslant k} \subset \mathrm{Complex}(\mathrm{CFl}_c)$ be the part of the complex associated to $X_k = \bigcup_{i=1}^{k} \mathring{X}^{u_i}_{\mathrm{NC}} = \bigcup_{i=1}^{k} X^{u_i}_{\mathrm{NC}}$, i.e. the part of the complex associated to just the polytopes $P_1, \ldots, P_k$. Then $C^{\leqslant k+1}$ is obtained from $C^{\leqslant k}$ by gluing $P_{k+1}$ to $C^{\leqslant i}$ along the subset $\mu(X^{u_{k+1}}_{\mathrm{NC}} \backslash \mathring{X}^{u_{k+1}}_{\mathrm{NC}}) = (\partial P_k)' \subset \partial P_k$, the part of $\partial P_{k+1}$ not touching the simple vertex $\mu(u_{k+1}) \in P_{k+1}$. Hence we can successively collapse $\mathrm{Complex}(\mathrm{CFl}_c)$ by collapsing in reverse order each $P_k$ onto $(\partial P_k)'$.

## 10. Cohomology

In this section we describe the torus-equivariant cohomology of $\mathrm{CFl}_c$ using the suite of tools known as *GKM theory*. This culminates in the proof of Theorem D in Corollary 10.7. For technical reasons, we will need to use the *adjoint torus* $T_{ad} = T/Z(G)$ on $G/B$ (Section 5.1). As a consequence, the rings computed in this section depend only on the root system $\Phi$ of $G$.

10.1. **GKM Theory.** In this section we recall specific facts from GKM theory; beginning with Goresky–Kottwitz–MacPherson [36], versions of these statements have been proven by several authors under varying conditions. In a few cases we have not found statements that match our exact hypothesis, and therefore include proofs for the sake of completeness.

For us, a *GKM variety* is a (possibly reducible) algebraic variety $X$ with the action of a torus $T$ such that

(1) $X$ has finitely many $T$-fixed points,
(2) $X$ has finitely many $T$-invariant curves, and
(3) $X$ embeds $T$-equivariantly into $\mathbb{P}(V)$, the projectivization of a $T$-representation $V$.

All of our GKM varieties will come from the generalized flag variety $G/B$, equipped with the action of $T_{ad}$. As $Z(G)$ acts trivially on $G/B$, the $T_{ad}$-orbits are exactly the same as the $T$-orbits, so that the fixed points are $\{wB \mid w \in W\}$ and the $T_{ad}$-invariant curves are still $\mathbb{P}_{u,v}$ for $u = \tau v$ and $\tau \in \mathsf{T}$. For embedding, we take the generalized Plücker embedding $\mathrm{Pl} : G/B \to V^{\lambda_{\mathrm{reg}}}$ coming from our choice of regular dominant $\lambda_{\mathrm{reg}} \in \mathrm{Char}(T_{ad}) = \mathbb{Z}\Phi$.

Any $T$-invariant subvariety $Y$ of a GKM variety is again a GKM variety, so every $T$-invariant subvariety of $G/B$ is GKM. In particular any $T_{ad}$-invariant (or simply $T$-invariant) subvariety $Y \subseteq G/B$ is also GKM.

In any GKM variety $X$, each $T$-fixed point corresponds to a line in some weight space $V_\mu$, $\mu \in \mathrm{Char}(T)$, and each $T$-invariant curve $Y$ contains two fixed points coming from distinct weights, $\mu, \nu \in \mathrm{Char}(T)$. The tangent spaces to $Y$ at these points afford the characters $(\nu - \mu)$ and $-(\nu - \mu)$.

**Definition 10.1.** The *GKM graph* of a GKM variety $X$ is the edge-labeled graph $\mathrm{GKM}(X)$ with vertex set $X^T$, edges determined by the $T$-invariant curves, and edge labels "$\pm\chi$" where $\pm\chi$ are the characters of the tangent spaces to the two fixed points.

For $G/B$, identifying $wB$ with $w \in W$ realizes the GKM graph as the Cayley graph $\mathrm{Cayley}(W, \mathsf{T})$, with edges $\{w, \tau w\}$ labelled by $\pm r(\tau)$. If $Y \subseteq G/B$ is $T$-invariant, then its GKM graph will be a subgraph of $\mathrm{Cayley}(W, \mathsf{T})$ with the same edge labels; this will be particularly important if $Y$ is toric or if $Y$ is a Plücker vanishing variety. In the former case, it is well-known that the GKM graph is exactly the 1-skeleton of the moment polytope $P_Y$, and each edge label is the direction of the corresponding edge in $P_X$. The latter case is described in the following result.

**Proposition 10.2.** For $\mathcal{A} \subseteq W$, the GKM graph of the Plücker vanishing variety $\mathrm{PV}_{\mathcal{A}}$ is the induced subgraph of $\mathrm{Cayley}(W, \mathsf{T})$ on $\mathcal{A}$.

*Proof.* By Theorem 6.6, $\mathrm{PV}_{\mathcal{A}}$ consists of all torus orbit-closures whose $T$-fixed point set lies in the set $\mathcal{A}B = \{wB \mid w \in \mathcal{A}\}$. The zero-dimensional orbits in this set are $\mathcal{A}B$, and the one-dimensional orbits are the edges $\{w, \tau w\} \subseteq \mathcal{A}$. □

Under certain assumptions, the cohomology ring of a GKM variety can be computed from its GKM graph. The $T$-equivariant cohomology ring of a point is given by the polynomial ring

$$H_T^\bullet(pt) := \mathrm{Sym}^\bullet\left(\mathrm{Char}(T)\right)$$

As is standard in algebraic combinatorics, we denote by $t_\lambda \in H_T^\bullet(pt)$ for the element corresponding to $-\lambda \in \mathrm{Char}(T)$, the negative character, and grade this ring so that each nonzero $t_\lambda$ has degree 2. Let

$$H_T^+(pt) := \text{ the ideal generated by elements of degree} \geqslant 2.$$

The adjoint torus $T_{ad}$ has character lattice equal to the root lattice $\mathbb{Z}\Phi$, so that in $T_{ad}$-equivariant cohomology we have

$$H_{T_{ad}}^\bullet(pt) = \mathbb{Z}[t_\alpha \mid \alpha \in \Delta].$$

**Definition 10.3.** For a GKM graph $\mathcal{G}$, we define the *graph cohomology ring* to be the $H_T^\bullet(pt)$-algebra

$$H_T^\bullet(\mathcal{G}) := \{(f_v)_{v \in V(\mathcal{G})} \mid t_{\chi(vv')} \text{ divides } f_v - f_{v'} \text{ for all edges } vv' \text{ in } \mathcal{G}\} \subset (H_T^\bullet(pt))^{\oplus V(\mathcal{G})}.$$

We identify $H_T^\bullet(pt)$ with its diagonal embedding in $(H_T^\bullet)^{\oplus V(\mathcal{G})}$.

Under certain conditions the graph cohomology ring of the GKM graph computes the actual cohomology ring of a GKM space $X$. Say that $X$ has a *good affine paving* if it has a filtration by $T$-invariant subvarieties

$$\varnothing = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_\ell = X$$

such that the following holds for each $i \geqslant 1$:

(1) $X_i \backslash X_{i-1}$ is $T$-equivariantly isomorphic to a linear $T$-representation $V_i$, and therefore contains a unique $T$-fixed point $w_i$;
(2) the representation $V_i$ decomposes into a direct sum of one-dimensional $T$-representations
$$V_i = \bigoplus_{j \in A_i} V_{i,j} \qquad \text{for } A_i \subseteq \{1, \ldots, i-1\},$$
such that $\overline{V_{i,j}} = V_{i,j} \sqcup \{w_j\}$; and
(3) The character $f_{i,j}$ of $V_{i,j}$ is nonzero, and is moreover reduced in the sense that $\frac{1}{r} f_{i,j} \notin \mathrm{Char}(T)$ for any integral $r \neq \pm 1$.

Returning to our example of $X = G/B$, we obtain a good affine paving by fixing a linear extension of the Bruhat order on $W$, $w_1, w_2, \ldots, w_N$, and taking $X_i = X^{w_1} \cup X^{w_2} \cup \cdots \cup X^{w_i}$. Then the isomorphism $\mathring{X}^{w_i} \cong N_{r(\mathrm{Inv}(w_i))}$ in Section 5.2 satisfies condition (1) and the $V_{i,j}$ are the root subspaces $\mathfrak{g}_\alpha$, which correspond to $x_{r(\tau)}(\mathbb{C})wB \subseteq \mathring{X}^{w_i}$, whose closure is $\mathbb{P}_{w_i, w_i\tau}$. While the character of $\mathfrak{g}_\alpha$, which is the root $\alpha \in \Phi$, may not be reduced in $\mathrm{Char}(T)$, it is always reduced in $\mathrm{Char}(T_{ad}) = \mathbb{Z}\Phi$. Satisfying condition (3) is the primary reason for us to pass to the adjoint torus.

**Fact 10.4.** If $T = T_{ad}$ and all characters $f_{i,j}$ above are roots in $\Phi$, then condition (3) is automatically satisfied.

**Theorem 10.5** ([8, Theorem 11.3])**.** For a GKM variety $X$ with a good affine paving, we have
(1) $H_\bullet(X)$ has a homology basis $\{[\overline{X_i \backslash X_{i-1}}]\}_{i \in \{1, \ldots, \ell\}} \subset H_\bullet(X)$,
(2) $H_T^\bullet(X) \cong H_T^\bullet(\mathcal{G})$, and
(3) $H_T^\bullet(X)$ is a free $H_T^\bullet(pt)$-module and $H^\bullet(X) \cong H_T^\bullet(\mathcal{G})/(H_T^+(pt))$.

*Proof.* All parts were shown in [8, Theorem 11.3], except that in (3) the freeness was a hypothesis rather than a conclusion. However, the freeness follows from (1) as this implies that the Leray–Hirsch spectral sequence degenerates at the $E_2$–page. □

For $\mathrm{CFl}_c$ we will actually be able to combinatorially establish the existence of a free $H^\bullet_{T_{ad}}(pt)$-basis with additional properties, see Theorem 11.2.

10.2. **The cohomology of** $\mathrm{CFl}_c$**.** We now apply the machinery developed in the previous section to describe the equivariant cohomology of $\mathrm{CFl}_c$, and in particular prove Theorem D. We continue to use the action of the adjoint torus $T_{ad}$. For this reason, the results in this section are independent of the choice of $G$ up to the underlying root system $\Phi$.

**Theorem 10.6.** Let $\Phi$ be a root system, $W$ its Weyl group, and $c \in W$ a Coxeter element.
(i) For any linear extension $u_1, u_2, \ldots, u_{\mathrm{Cat}_W}$ of the Bruhat order on $\mathrm{NC}(W, c)$, the filtration of $\mathrm{CFl}_c$ given by
$$X_i = \mathring{X}_{\mathrm{NC}}^{u_1} \cup \cdots \cup \mathring{X}_{\mathrm{NC}}^{u_i}$$
is a good affine paving with respect to the action of $T_{ad}$.

(ii) We have

$$H^\bullet_{T_{ad}}(\mathrm{CFl}_c) = H^\bullet_{T_{ad}}(\mathrm{Cayley}(W,\mathsf{T})|_{\mathrm{NC}(W,c)})$$

and

$$H^\bullet(\mathrm{CFl}_c) = H^\bullet_{T_{ad}}(\mathrm{CFl}_c)/H^+_{T_{ad}}(pt).$$

*Proof.* Point (i) follows from Theorem 9.3 and the fact that the associated charts for the affine paving are coordinate subspaces of the good affine paving for $G/B$ by Schubert cells. By Theorem 10.5, point (i) implies point (ii). □

**Corollary 10.7** (Theorem D)**.** For Coxeter elements $c, c' \in W$, and $u \in W$ such that $c = wc'w^{-1}$, there is a ring isomorphism $\Psi_{c,w} : H^\bullet(\mathrm{CFl}_c) \cong H^\bullet(\mathrm{CFl}_{c'})$. Furthermore these isomorphisms satisfy $\Psi_{c,\mathrm{id}} = \mathrm{id}$, and if $w'c'(w')^{-1} = c'' \in W$ is another Coxeter element, then $\Psi_{c',w'}\Psi_{c,w} = \Psi_{c,w'w}$.

*Proof.* We define a ring isomorphism on $T_{ad}$-equivariant cohomology rings

$$\widetilde{\Psi_{c,w}} : H^\bullet_{T_{ad}}(\mathrm{CFl}_c) \to H^\bullet_{T_{ad}}(\mathrm{CFl}_{c'}) \qquad \text{by} \qquad (f_u)_{u\in\mathrm{NC}(W,c)} \mapsto \left(w \cdot f_{w^{-1}vw}\right)_{v\in\mathrm{NC}(W,c')},$$

which maps $H^+_{T_{ad}}(\mathrm{CFl}_c)$ to $H^+_{T_{ad}}(\mathrm{CFl}_{c'})$. This therefore descends to ring isomorphisms $\Psi_{c,w}$, and the compatibilities are trivially checked on the lifts $\widetilde{\Psi_{c,w}}$. □

**Remark 10.8.** The map used to prove Corollary 10.7 is an isomorphism of rings between the equivariant cohomology rings as well, but not an isomorphism of $H^\bullet_{T_{ad}}(pt)$-modules.

## 11. Duality bases

Let $X$ be a GKM variety with a good affine paving $X_1, \ldots, X_N$. Because $H^\bullet_T(X)$ is a free $H^\bullet_T(pt)$-module, we can look to find distinguished $H^\bullet_T(pt)$-bases.

**Definition 11.1.** A *flowup basis* for $H^\bullet_T(X)$ is a subset $\{f^{(1)}, f^{(2)}, \ldots, f^{(N)}\} \subset H^\bullet_T(X)$ such that

$$f^{(i)}_j = 0 \text{ for all } j < i \qquad \text{and} \qquad f^{(i)}_i = \pm \prod_{\text{edges } w_iw_j \text{ and j < i}} t_{\chi(w_iw_j)}. \tag{11.1}$$

A straightforward upper-triangularity argument shows that a flowup basis must be a free $H^\bullet_T(pt)$-basis, justifying the name. In general a flowup basis, if it exists, is not unique. One condition we might hope to impose is that $f^{(i)}_j = 0$ whenever $i \neq j$ and $\dim X_i = \dim X_j$ (see for example [39, Proposition 4.3]), but because of the strange way that our cells fit together, $H^\bullet_{T_{ad}}(\mathrm{CFl}_c)$ does not have a flowup basis satisfying this extra condition.

We will instead create a basis which is dual with respect to certain degree maps. We work under the assumption in this section that there is a *$T$-equivariant degree map* $\int^T_{\overline{X_i\setminus X_{i-1}}} : H^\bullet_T(\overline{X_i\setminus X_{i-1}}) \to$

$H_T^\bullet(pt)$ for each $i$ (see Section 11.1). We say that a collection of elements $f^{(1)}, f^{(2)}, \dots, f^{(N)}$ in $H_T^\bullet(X)$ is a *duality basis* for this paving if

$$\int_{\overline{X_i \backslash X_{i-1}}}^{T} f^{(j)} = \delta_{i,j} \in H_T^\bullet(pt).$$

The papers [7, 8] define a duality basis with respect to the affine paving in Theorem 10.6 in the special case that $G = \mathrm{GL}_{n+1}$ and $c = (n+1 \cdots 2\,1)$, which we called *double forest polynomials*. We now show that analogues of forest polynomials exist for any Coxeter flag variety.

**Theorem 11.2.** Let $W$ be a Weyl group, and $c \in W$ a Coxeter element. Then there exist unique elements

$$\{\widetilde{\mathfrak{S}_u^{\mathrm{NC}}} \mid u \in \mathrm{NC}(W, c)\} \subset H_{T_{ad}}^\bullet(\mathrm{CFl}_c)$$

we call *Coxeter Schubert classes* which give a (unique) duality basis with respect to any affine paving of the form described in Theorem 10.6. These elements satisfy, but are not uniquely characterized by, the conditions that

$$(\widetilde{\mathfrak{S}_u^{\mathrm{NC}}})_u = \prod_{\tau \in \mathrm{Inv}_{\mathrm{NC}}(u)} r(\tau)$$

and for $v \in \mathrm{NC}(W, c)$

$$(\widetilde{\mathfrak{S}_u^{\mathrm{NC}}})_v = 0 \text{ whenever } u \nleq v \text{ (in the Bruhat order).}$$

The remainder of the section is concerned with the proof of Theorem 11.2: Section 11.1 establishes general properties of duality bases needed for the proof and Section 11.2 applies these properties to the combinatorial structure of $\mathrm{CFl}_c$.

11.1. **Duality bases via Atiyah–Bott localization.** In this section, let $X$ be a GKM variety with a good affine paving $X_1, \dots, X_N$. If $\overline{X_i \backslash X_{i-1}}$ is smooth, then $\int_{\overline{X_i \backslash X_{i-1}}}^T f$ is the $T$-equivariant pushforward to a point. In general, the pushforward is not a priori well-defined. We say that the paving *admits $T$-equivariant degree maps* if for each $i$ we are able to find a fixed $T$-equivariant resolution

$$\pi_i : \widetilde{X_i \backslash X_{i-1}} \to \overline{X_i \backslash X_{i-1}},$$

by a smooth irreducible GKM variety $\widetilde{X_i \backslash X_{i-1}}$, which is an isomorphism on the open set $X_i \backslash X_{i-1}$. In this case we define the $T$-equivariant degree

$$\int_{\overline{X_i \backslash X_{i-1}}}^T f := \int_{\widetilde{X_i \backslash X_{i-1}}}^T \pi_i^* f \in H_T^\bullet(pt).$$

For each class $f \in H_T^\bullet(X)$ and each $T$-fixed-point $p$, let $f_p$ be the pullback of $f$ to $H_T^\bullet(p)$. Then $T$-equivariant Atiyah–Bott localization (see e.g. [6, 10]) computes the $T$-equivariant degree of $f$ on

$Y = \overline{X_i \backslash X_{i-1}}$ by constructing a family of elements $C^Y_{w_j} \in \mathrm{Frac}(H^\bullet_T(pt))$, $1 \leqslant j \leqslant i$ for which:

$$\int_Y^T f = \sum_{j=1}^{i} C^Y_{w_j} f_w \in H^\bullet_T(pt) \qquad \text{and} \qquad C^Y_{w_i} = \pm \prod_{\text{edges } w_i w_j \text{ and } j<i} t^{-1}_{\chi(w_i w_j)}.$$

We will primarily be interested in the case where $X$ is a toric complex with a good affine paving. Recalling the well-known facts that

(1) every projective toric variety admits a toric resolution by a smooth projective toric variety
(2) the associated degree maps are independent of the choice of resolution,

we see that such an $X$ in fact admits $T$-equivariant degree maps in an essentially unique way.

**Remark 11.3.** For $\mathrm{CFl}_c$, all of whose $\overline{X_i \backslash X_{i-1}}$ are normal toric varieties associated to Coxeter matroid polytopes, one can always resolve using the toric variety $X_{\mathrm{Perm}_W}$ associated to the $W$-permutahedron (Section 6.4).

**Theorem 11.4.** Suppose that $X$ is a GKM variety with a good affine paving admitting $T$-equivariant degree maps. If a duality basis $\{f^{(1)}, f^{(2)}, \ldots, f^{(N)}\}$ exists for $H^\bullet_T(X)$, then it is a flowup basis. Moreover, if $H^\bullet_T(X)$ has a flowup basis, then it has a unique duality basis.

*Proof.* Let $F$ be the $N \times N$ matrix with $i, j$ entry $f^{(j)}_{w_i}$ and $C$ be the matrix with $j, k$ entry is equal to $C^{\overline{X_k \backslash X_{k-1}}}_{w_j} \in \mathrm{Frac}(H^\bullet_T(pt))$ if $w_j \in \overline{X_k \backslash X_{k-1}}$ and 0 otherwise. Then

$$\int^T_{\overline{X_i \backslash X_{i-1}}} f^{(j)} = (CF)_{i,j},$$

so we have a duality basis if and only if $F = C^{-1}$. As $C$ is lower triangular with diagonal entries

$$C_{j,j} = \pm \prod_{\text{edges } w_i w_j \text{ with i < j}} t^{-1}_{\chi(w_i w_j)},$$

we see that $F = C^{-1}$ implies that (11.1) holds.

On the other hand, if (11.1) holds then $F$ is lower triangular with diagonal entries $F_{i,i} = 1/C_{i,i}$. In this case $CF$ is unipotent lower triangular, so $(CF)^{-1} = \sum_{k=0}^{N} (I - CF)^k$ is also unipotent lower triangular with entries in $H^\bullet_T(pt)$. Thus the columns of $C^{-1} = F(CF)^{-1}$ determine a duality basis of $H^\bullet_T(X)$ whose elements are $H^\bullet_T(pt)$-linear combinations of the $f^{(i)}$. □

For $G/B$ and any linear extension of the Bruhat order, the paving admits $T$-equivariant degree maps via a choice of Bott–Samelson $\mathcal{B}^{\boldsymbol{w}}$ resolution for each Schubert cell $X^w$ associated to a choice of reduced word $\boldsymbol{w}$ for $w$. The unique duality basis corresponds to the Schubert classes $\{\widetilde{\mathfrak{S}_w} \mid w \in W\}$ in $H^\bullet_{T_{ad}}(\mathrm{Cayley}(W, \mathsf{T})$. These are recursively defined by

$$\widetilde{\mathfrak{S}_{w_\circ}} = \big(\delta_{w,w_\circ} \prod_{\alpha \in \Phi^+} t_\alpha\big)_{w \in W},$$

where $\delta_{w,w_\circ}$ is the Kronecker delta, and for $w \in W$ and $s_i \in \mathrm{Des}(w)$ with associated simple root $\alpha_i$,

$$\widetilde{\mathfrak{S}}_{ws_i} = \partial_i \widetilde{\mathfrak{S}_w} \qquad \text{where} \qquad \partial_i f = \left(\frac{f_v - f_{vs_i}}{t_{\alpha_i}}\right)_{v \in W}.$$

We now describe how to construct duality bases under stronger hypotheses on $X$.

**Theorem 11.5.** Let $X$ be a toric complex with good affine paving. If the characters of each affine chart $X_i \backslash X_{i-1}$ can be extended to a basis of $\mathrm{Char}(T)$, then $X$ has a duality basis.

To prove this theorem, we need the following lemma, which was proved in [24, Proposition 5.3] in a slightly weaker setting over the rational numbers.

**Lemma 11.6.** Under the assumptions of Theorem 11.5, for any $h \in H_T^\bullet(X_{i-1})$ there is a class $h' \in H_T^\bullet(X_i)$ with $h_{w_j} = h'_{w_j} \in H_T^\bullet(pt)$ for all $j \leqslant i-1$.

*Proof.* Suppose that the vertex $w_i$ in $\mathrm{GKM}(X_i)$ is incident to the edges $w_{j_1}w_i, w_{j_2}w_i, \ldots, w_{j_m}w_i$, with respective edge labels $\lambda_1, \lambda_2, \ldots, \lambda_m$. Fix an extension $\lambda_{m+1}, \ldots, \lambda_n$ of this set to a basis of $\mathrm{Char}(T)$ and define $t_j = t_{\lambda_j}$, so that $H_T^\bullet(pt) \cong \mathbb{Z}[t_1, \ldots, t_n]$.

Now we must construct an $h'_{w_i} \in H_T^\bullet(pt)$ which is compatible with each $h_{w_{j_r}}$ under the divisibility condition imposed by each edge $w_{j_r}w_i$. This amounts to solving the system

$$h'_{w_i} \equiv h_{w_{j_1}} \mod (t_1), \qquad h'_{w_i} \equiv h_{w_{j_2}} \mod (t_2), \qquad \ldots \qquad , \qquad h'_{w_i} \equiv h_{w_{j_m}} \mod (t_m).$$

It turns out that we have a solution if and only if

$$h_{w_{j_k}} \equiv h_{w_{j_\ell}} \mod (t_k, t_\ell) \qquad \text{for } 1 \leqslant k < \ell \leqslant m.$$

Indeed, this condition is equivalent to requiring that each monomial $t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n}$ has the same coefficient in all $h_{w_r}$ for which $a_r = 0$. We denote this coefficient by $c_{a_1,\ldots,a_n}$ and for convenience set $c_{a_1,\ldots,a_n} = 0$ when all $a_i > 0$. We then set

$$h'_{w_i} = \sum c_{a_1,\ldots,a_n} t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n},$$

which clearly solves the given system.

We conclude by demonstrating the congruence $h_{w_{j_k}} \equiv h_{w_{j_\ell}} \mod (t_k, t_\ell)$ by considering the GKM conditions arising from the edges of the moment polytope for $\overline{X_i \backslash X_{i-1}}$. Let $F$ be the 2-face containing the vertices corresponding to the fixed points $w_i$, $w_{j_k}$, and $w_{j_\ell}$, and let $w_{j_k} = v_1, v_2, \ldots, v_r = w_{j_\ell}, w_i$ be the path in $\mathrm{GKM}(X_i)$ corresponding to the boundary of $F$. As $h$ satisfies the divisibility condition for each edge in this path except the last one, there exist $a_1, a_2, \ldots, a_{r-1} \in H_T^\bullet(pt)$ for which

$$h_{w_{j_k}} = h_{v_2} + a_1 t_{\chi(v_1v_2)} = h_{v_3} + a_1 t_{\chi(v_1v_2)} + a_2 t_{\chi(v_2v_3)} = \cdots = h_{w_{j_\ell}} + \sum_{q=1}^{r-1} a_q t_{\chi(v_qv_{q+1})}.$$

Finally, each $t_{\chi(v_q v_{q+1})}$ belongs to $(t_k, t_\ell)$: each character $\chi(v_q v_{q+1})$ is a real multiple of the corresponding edge $v_{q+1} - v_q$, which lies in the real span of $\lambda_{j_k}$ and $\lambda_{j_\ell}$. As $\lambda_{j_k}, \lambda_{j_\ell}$ are part of a basis, this means each $\chi(v_q v_{q+1})$ is in the integral span of $\{\lambda_{j_k}, \lambda_{j_\ell}\}$. □

*Proof of Theorem 11.5.* We construct a flowup basis, and then conclude by Theorem 11.4.

If $X = X_1 = pt$, then the claim is immediate. Otherwise, we assume the existence of a flowup basis $\{h^{(1)}, h^{(2)}, \ldots, h^{(i-1)}\}$ for $H_T^\bullet(X_{i-1})$ and construct a flowup basis $\{g^{(1)}, \ldots, g^{(i)}\} \subset H_T^\bullet(X_i)$. For this we define $g^{(i)}$ by setting

$$g^{(i)}_{w_j} = \delta_{i,j} \prod_{\text{edges } w_j w_i \text{ and } j<i} t_{\chi(w_j w_i)},$$

and to construct $g^{(j)}$ for $j \leqslant i-1$ we apply Lemma 11.6 to $h^{(j)}$. □

11.2. **A duality basis for $\mathrm{CFl}_c$.** We now show Theorem 11.2.

**Lemma 11.7.** For any Coxeter element $c \in W$, every $c$-cluster is a basis for the root lattice $\mathbb{Z}\Phi$. As a consequence, every positive $c$-cluster can be extended to a basis of $\mathbb{Z}\Phi$.

In order to prove the lemma, we recall the definition of a *bipartite Coxeter element*. Given a bipartition $\mathsf{S}_+ \sqcup \mathsf{S}_-$ of the Coxeter diagram of $\mathsf{S}$, we define the corresponding bipartite Coxeter element $c$ by $\prod_{s \in \mathsf{S}_+} s \prod_{s \in \mathsf{S}_-} s$.

*Proof.* If $c$ is a bipartite Coxeter element this is [31, Theorem 1.8]. Otherwise, there exists a $w \in W$ such that $c' = wcw^{-1}$ is a bipartite Coxeter element [42, Proposition 3.16], and $wuw^{-1} \in \mathrm{NC}(W, c')$ if and only if $u \in \mathrm{NC}(W, c)$. Thus for each $u \in \mathrm{NC}(W, c)$, $r(\mathrm{Inv}_{\mathrm{NC}}(wuw^{-1})) = w \cdot r(\mathrm{Inv}_{\mathrm{NC}}(u))$, so that $w$ maps $\mathrm{Cl}_c$ to $\mathrm{Cl}_{c'}$. As the reflection action of $w$ is defined over $\mathbb{Z}$ and the image of each $c$-cluster under $w$ is a basis for $\mathbb{Z}\Phi$, the proof is complete. □

*Proof of Theorem 11.2.* The characters of the affine spaces $\mathrm{Clust}^+(u)$ in the affine pavings are $W$-translates of subsets of $c$-clusters, so Lemma 11.7 verifies the necessary basis extension property to apply Theorem 11.5. The duality basis remains the same as we vary the linear extension of $\mathrm{NC}(W, c)$ determining the order the pieces of the affine paving appear in. This implies the duality basis is a flowup basis for every linear extension of $\mathrm{NC}(W, c)$, verifying the remaining vanishing conditions. □

## 12. Cohomology in type $A$ and permuted quasisymmetry

Fix $G = \mathrm{GL}_{n+1}$ so that $W = S_{n+1}$, and $G/B = \mathrm{Fl}_{n+1}$ is the complete flag variety parametrizing complete flags of subspaces

$$\mathrm{Fl}_{n+1} = \{0 \subsetneq \mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \cdots \subsetneq \mathcal{F}_n \subsetneq \mathbb{C}^{n+1} \mid \dim \mathcal{F}_i = i\}.$$

We summarize the work of [7, 8] on the *quasisymmetric flag variety* $\mathrm{QFl}_{n+1} = \mathrm{CFl}_{s_n \cdots s_1} \subset \mathrm{Fl}_{n+1}$, and then prove Theorem E in Theorem 12.2. In these references the diagonal torus $T$ of $\mathrm{GL}_{n+1}$ was used instead of the adjoint torus $T_{ad} = T/\mathbb{C}^* \subset \mathrm{PGL}_{n+1}$. In type $A$ this is inessential to all cohomology results as the sequence $1 \to \mathbb{C}^* \to T \to T_{ad} \to 1$ splits, and hence if $X$ is a variety with a $T_{ad}$ action we have canonical identifications

$$H^\bullet_T(X) = H^\bullet_{T_{ad}}(X) \otimes_{H^\bullet_{T_{ad}}(pt)} H^\bullet_T(pt).$$

Moving forward in this section, we work with $T$ rather than $T_{ad}$.

The noncrossing partitions in $\mathrm{NC}_{n+1} := \mathrm{NC}(S_{n+1}, s_n s_{n-1} \cdots s_1)$ are combinatorially determined by set partitions $\mathcal{P}$ of $\{1, \ldots, n+1\}$ with an additional *noncrossing* property: if $A, B \in \mathcal{P}$ are distinct blocks and $i < j < k < \ell$, then we do not have $i, k \in A$ and $j, \ell \in B$. To such a set partition, the associated element of $\mathrm{NC}_{n+1}$ is given by multiplying the backwards cycles on each of the blocks. The finest partition corresponds to the identity, the coarsest partition corresponds to $(n+1, n, \ldots, 1) = s_n s_{n-1} \cdots s_1$, and the Kreweras order is identified with the refinement order on set partitions restricted to the noncrossing set partitions.

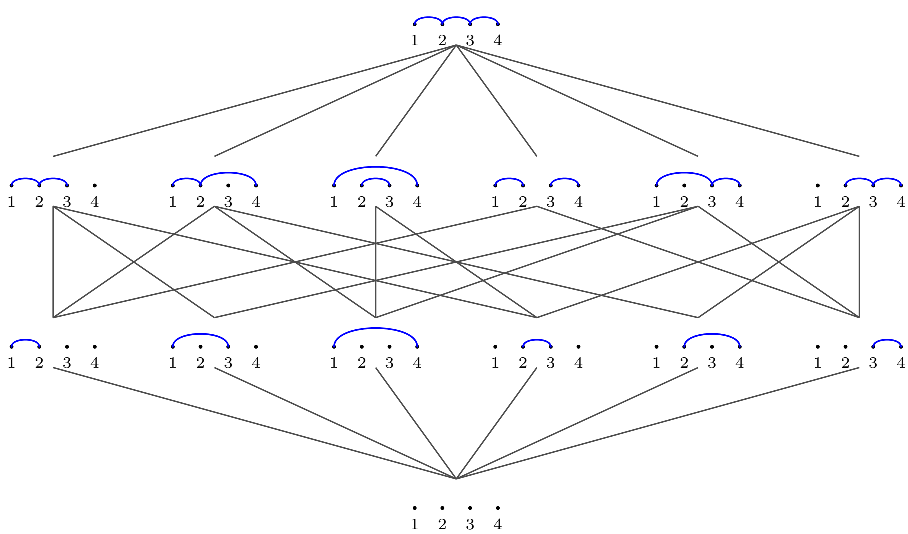

FIGURE 7. $\mathrm{NC}_4$ drawn as a lattice on noncrossing set partitions

The combinatorics of $\mathrm{QFl}_{n+1}$ centers around the notion of *equivariant quasisymmetry* defined in [7] for polynomials $f(\mathbf{x}; \mathbf{t}) \in \mathbb{Z}[x_1, \ldots, x_{n+1}][t_1, \ldots, t_{n+1}]$. In particular $\mathrm{QFl}_{n+1}$ has combinatorially defined Coxeter and equivariant Coxeter Schubert classes given by the forest polynomials and double forest polynomials respectively. We refer the reader to [7, 8] for details about the forest and double forest polynomials.

**Definition 12.1.** Define the *equivariant Bergeron–Sottile maps* $\mathsf{R}_i^-, \mathsf{R}_i^+$ by

$$\mathsf{R}_i^- f = f(x_1, \ldots, x_{i-1}, t_i, x_i, \ldots, x_n; \mathbf{t})$$
$$\mathsf{R}_i^+ f = f(x_1, \ldots, x_{i-1}, x_i, t_i, \ldots, x_n; \mathbf{t})$$

A polynomial $f(\mathbf{x}; \mathbf{t})$ is *equivariantly quasisymmetric* if for $1 \leqslant i \leqslant n$ we have $\mathsf{R}_i^- f = \mathsf{R}_i^+ f$, and we denote $\mathrm{EQSym}_{n+1} \subset \mathbb{Z}[x_1, \ldots, x_{n+1}][t_1, \ldots, t_{n+1}]$ for the subring of equivariantly quasisymmetric polynomials.

Setting $t_i = 0$, we have $\mathsf{R}_i^-$ and $\mathsf{R}_{i-1}^+$ become identified with the Bergeron–Sottile maps $\mathsf{R}_i$ [9, 50] on the polynomial ring $\mathbb{Z}[x_1, \ldots, x_{n+1}]$ defined by

$$\mathsf{R}_i f = f(x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_n).$$

As shown in [7] this specialization induces a surjection from $\mathrm{EQSym}_{n+1}$ to the ring of *quasisymmetric polynomials* $\mathrm{QSym}_{n+1} \subset \mathbb{Z}[x_1, \ldots, x_{n+1}]$ of Gessel [35] and Stanley [59], defined by either of the equivalent conditions that $\mathsf{R}_1 f = \cdots = \mathsf{R}_{n+1} f$, or that for any sequence $(a_1, \ldots, a_k)$ of positive integers we have equality of the coefficients $[x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}]f = [x_{j_1}^{a_1} \cdots x_{j_k}^{a_k}]f$ for all increasing sequences $1 \leqslant i_1 < \cdots < i_k \leqslant n+1$ and $1 \leqslant j_1 < \cdots < j_k \leqslant n+1$.

Define the ideal

$$\mathrm{EQSym}_{n+1}^+ = \langle f(\mathbf{x}; \mathbf{t}) - f(\mathbf{t}; \mathbf{t}) \mid f \in \mathrm{EQSym}_{n+1} \rangle \subset \mathbb{Z}[x_1, \ldots, x_{n+1}][t_1, \ldots, t_{n+1}].$$

It was shown in [8] that the natural map $H_T^\bullet(\mathrm{Fl}_{n+1}) \to H_T^\bullet(\mathrm{QFl}_{n+1})$ is surjective, giving a Borel-type presentation

$$\begin{aligned} \mathbb{Z}[x_1, \ldots, x_{n+1}][t_1, \ldots, t_{n+1}]/\mathrm{EQSym}_{n+1}^+ &\xrightarrow{\cong} H_T^\bullet(\mathrm{CFl}_{s_n \cdots s_1}) \\ f(x_1, \ldots, x_{n+1}; t_1, \ldots, t_{n+1}) &\mapsto \big(f(t_{w(1)}, \ldots, t_{w(n+1)}; t_1, \ldots, t_{n+1})\big)_{w \in \mathrm{NC}_{n+1}}. \end{aligned}$$

We now prove Theorem E, relating the quasisymmetric coinvariants to the cohomology rings of $\mathrm{CFl}_c$ for other choices of $c$.

**Theorem 12.2.** For $G/B = \mathrm{Fl}_{n+1}$ and any Coxeter element $c \in S_{n+1}$, the restriction map $H^\bullet(\mathrm{Fl}_{n+1}) \to H^\bullet(\mathrm{CFl}_c)$ is surjective. Moreover, if $c$ is the cycle $(w(n+1)\, w(n)\, \cdots\, w(1))$ for some $w \in S_{n+1}$ then under this restriction map we can realize $H^\bullet(\mathrm{CFl}_c)$ as a quotient of the presentation in (1.1):

$$H^\bullet(\mathrm{CFl}_c) \cong \mathbb{Z}[x_1, \ldots, x_{n+1}] \Big/ \Big\langle f(x_{w(1)}, \ldots, x_{w(n+1)}) - f(0, \ldots, 0) \,\Big|\, f \in \mathrm{QSym}_{n+1} \Big\rangle,$$

the ring of *permuted quasisymmetric coinvariants*.

*Proof.* Because $c = w s_n \cdots s_1 w^{-1}$, by Corollary 10.7, we have a map

$$\widetilde{\Psi_{c,w}} : H_T^\bullet(\mathrm{CFl}_{s_n \cdots s_1}) \cong H_T^\bullet(\mathrm{CFl}_c) \qquad \text{given by} \qquad (f_v)_{v \in \mathrm{NC}_{n+1}} \mapsto (w \cdot f_{w^{-1} v w})_{v \in \mathrm{NC}_{n+1}}.$$

This map descends to an isomorphism $\Psi_{c,w} : H^\bullet(\mathrm{CFl}_{s_n\cdots s_1}) \cong H^\bullet(\mathrm{CFl}_c)$ after quotienting by the ideal $H_T^+(pt) = (t_1, \ldots, t_{n+1})$. Let $\mathrm{EQSym}^+_{n+1,w}$ be the image of $\mathrm{EQSym}^+_{n+1}$ under the automorphism of $\mathbb{Z}[x_1, \ldots, x_{n+1}][t_1, \ldots, t_{n+1}]$ given by

$$f(x_1, \ldots, x_{n+1}; t_1, \ldots, t_{n+1}) \mapsto f(x_{w(1)}, \ldots, x_{w(n+1)}; t_{w(1)}, \ldots, t_{w(n+1)}). \tag{12.1}$$

Then there is a commutative square

$$\begin{array}{ccc} \mathbb{Z}[x_1, \ldots, x_{n+1}][t_1, \ldots, t_{n+1}]/\mathrm{EQSym}^+_{n+1} & \xrightarrow{\sim} & \mathbb{Z}[x_1, \ldots, x_{n+1}][t_1, \ldots, t_{n+1}]/\mathrm{EQSym}^+_{n+1,w} \\ \downarrow\wr & & \downarrow\wr \\ H_T^\bullet(\mathrm{CFl}_{s_n\cdots s_1}) & \xrightarrow{\sim} & H_T^\bullet(\mathrm{CFl}_c), \end{array}$$

where the top map is induced by the automorphism in (12.1). The result then follows by quotienting out the ideal $H_T^+(pt) = (t_1, \ldots, t_{n+1})$. ☐

**Remark 12.3.** As in the introduction we remark that the map $\Psi_{c,w}$ does not preserve flowup bases. While the image of the flowup basis for $H_T^\bullet(\mathrm{CFl}_c)$ certainly maps to a basis, conjugation by $w$ only preserves the absolute order and *not* the Bruhat order used for our flowup condition.

## 13. Preliminaries of $c$-sortability

In this section we define and recall key properties of $c$-sortability. Throughout, we use the right weak order $\leqslant_R$ defined in Section 3.2. Additionally we adopt the convention that if $\boldsymbol{w}$ is a word in simple reflections, then $w \in W$ denotes the group element it represents.

Fix a reduced word $\boldsymbol{c}$ for $c$ and let $\boldsymbol{c}^\infty$ denote the string obtained by repeating this reduced word infinitely. We will depict each copy of $\boldsymbol{c}$ in $\boldsymbol{c}^\infty$ as separated by vertical bars, so for example if $W = S_5$ and $\boldsymbol{c} = s_2s_1s_3s_4$ we have

$$\boldsymbol{c}^\infty = s_2\ s_1\ s_3\ s_4 \mid s_2\ s_1\ s_3\ s_4 \mid s_2\ s_1\ s_3\ s_4 \mid \cdots$$

For any subword $\boldsymbol{x}$ of $\boldsymbol{c}^\infty$, we refer to the part of $\boldsymbol{x}$ in the $i$th copy of $\boldsymbol{c}$ in $\boldsymbol{c}^\infty$ as the *$i$th syllable of $\boldsymbol{x}$*. The *$c$-sorting word* $\boldsymbol{x}_{\boldsymbol{c}^\infty}$ of an element $x \in W$ is the lexicographically first reduced word for $x$ in $\boldsymbol{c}^\infty$.

**Definition 13.1.** A *$c$-sortable element* $x \in W$ is an element such that each syllable of the $c$-sorting word $\boldsymbol{x}_{\boldsymbol{c}^\infty}$ contains every letter in the following syllable. We denote the set of $c$-sortable elements in $W$ by $\mathrm{Sort}(W, c)$.

We illustrate these notions by an example.

**Example 13.2.** Let $W = S_5$ and $\boldsymbol{c} = s_2s_1s_3s_4 \in S_5$. We display $\boldsymbol{c}^\infty$ with the letters of the sorting word unstruck (and unused letters struck out in gray): the permutations $x = 35421$ and $y = 53421$ in one-line notation have $c$-sorting words

$$\boldsymbol{x}_{\boldsymbol{c}^\infty} = s_2\ s_1\ s_3\ s_4 \mid s_2\ \cancel{s_1}\ s_3\ s_4 \mid s_2\ \cancel{s_1}\ \cancel{s_3}\ \cancel{s_4} \mid \cdots, \qquad \text{and}$$

$$\boldsymbol{y}_{\boldsymbol{c}^\infty} = s_2\ s_1\ s_3\ s_4 \mid s_2\ \cancel{s_1}\ s_3\ s_4 \mid s_2\ s_1\ \cancel{s_3}\ \cancel{s_4} \mid \cdots .$$

Then $x$ is $c$-sortable, but $y$ is not, as $\{s_2, s_3, s_4\} \not\supseteq \{s_1, s_2\}$.

The following lemma shows that the syllable containment property of a reduced word in $\boldsymbol{c}^\infty$ implies it is already the $c$-sorting word of a $c$-sortable permutation.

**Lemma 13.3.** Let $\boldsymbol{x}$ be a reduced subword of $\boldsymbol{c}^\infty$ representing $x \in W$. If, syllable by syllable, the set of letters used in $\boldsymbol{x}$ is weakly decreasing under inclusion (i.e. each syllable contains every letter appearing in the following syllable), then $\boldsymbol{x}$ is the $c$-sorting word of $x$. In particular, $x \in \mathrm{Sort}(W, c)$.

*Proof.* Let $\boldsymbol{x}_{c^\infty}$ be the $c$-sorting word of $x$, and assume $\boldsymbol{x} \neq \boldsymbol{x}_{c^\infty}$. Let $p$ be the leftmost position in $\boldsymbol{c}^\infty$ where the chosen subwords differ, so $\boldsymbol{x}_{c^\infty}$ uses $s := \boldsymbol{c}^\infty_p$ at position $p$ while $\boldsymbol{x}$ skips it. Writing

$$\boldsymbol{x}_{c^\infty} = \boldsymbol{y}\, s\, \boldsymbol{z} \qquad \text{and} \qquad \boldsymbol{x} = \boldsymbol{y}\, \boldsymbol{w}$$

with $\boldsymbol{y}$ the common chosen prefix before $p$, we claim that $\boldsymbol{w}$ contains no letter $s$. Indeed, since each simple reflection occurs at most once per syllable of $\boldsymbol{c}^\infty$, skipping $s$ at $p$ means the syllable of $\boldsymbol{x}$ containing $p$ has no $s$, and by the syllable-containment hypothesis no later syllable can contain $s$.

Let $y, w, z \in W$ be the elements represented by $\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{z}$. Then $y^{-1}x = w \in W_{\mathsf{S}\setminus\{s\}}$, while also $y^{-1}x = sz$ has a reduced word beginning with $s$, a contradiction. Hence $\boldsymbol{x} = \boldsymbol{x}_{c^\infty}$, and $x$ is $c$-sortable by definition. ☐

We now record some fundamental results due to Reading [53] that revolve around the study of $c$-sortability that we shall appeal to in the sequel.

**Fact 13.4.**

(1) By [53, Theorem 1.1], there is a unique maximum $c$-sortable element $\pi_\downarrow(x) \in \mathrm{Sort}(W, c)$ below $x$ in the right weak order $\leqslant_R$ called the *downward projection*.

(2) The downward projection determines an equivalence relation on $W$ called *Cambrian equivalence* saying $x \equiv_c y$ if $\pi_\downarrow(x) = \pi_\downarrow(y)$. The proof of [53, Theorem 1.1] shows that the Cambrian equivalence class of $x \in W$ is a $\leqslant_R$-interval $[\pi_\downarrow(x), \pi_\uparrow(x)]$, whose maximum $\pi_\uparrow(x)$ is called the *upward projection*.

(3) $\pi_\uparrow(x)$ can be computed (see [53, §3]) using the downward projection $\pi_\downarrow^{(c^{-1})}$ for $c^{-1}$ and the antiautomorphism $x \to xw_\circ$ of the right weak order:

$$\pi_\uparrow(x) = \pi_\downarrow^{(c^{-1})}(xw_\circ)\, w_\circ. \tag{13.1}$$

Figure 8 depicts the Hasse diagram of the weak order on $S_4$ and highlights the intervals arising from Cambrian equivalence for $c = s_1s_3s_2$. So, for instance, $\{1324, 3124, 1342, 3142, 3412\}$ constitutes a single Cambrian class with 1324 and 3412 being the bottom and top elements respectively.

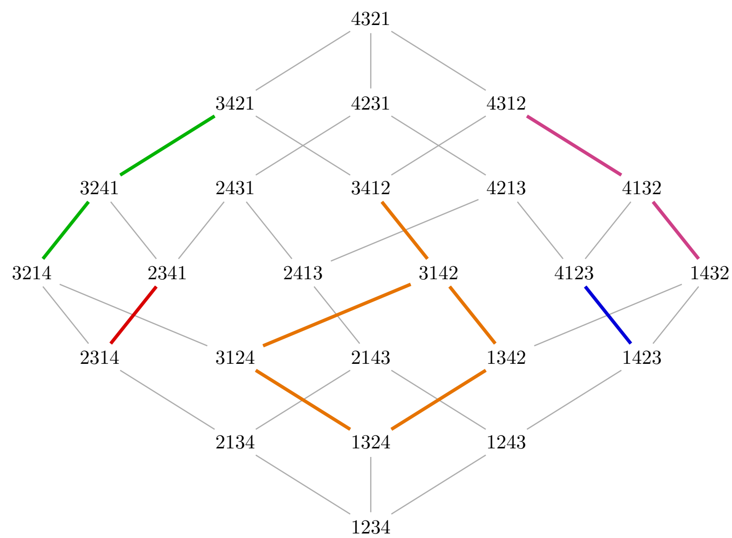


FIGURE 8. The right weak order on $S_4$ with non-singleton $c$-Cambrian classes emphasized using bolded edges, for $c = s_1 s_3 s_2$

The longest element $w_\circ \in W$ is $c$-sortable [52, Corollary 4.4], and (recalling the notation of (3.1)) the inversion factorization for its $c$-sorting word $\boldsymbol{w}_\circ$ determines the *c-reflection order* $\leqslant_c$ [4] by

$$\tau_1^{\boldsymbol{w}_\circ} <_c \tau_2^{\boldsymbol{w}_\circ} <_c \cdots <_c \tau_{\ell(w_\circ)}^{\boldsymbol{w}_\circ}$$

on the set of reflections $\mathsf{T}$. We now describe the connection between $c$-sortability and $c$-reflection orders.

**Fact 13.5.** Let $x \in \mathrm{Sort}(W, c)$ with $c$-sorting word $\boldsymbol{x}$. Then the sequence $\tau_1^{\boldsymbol{x}}, \ldots, \tau_{\ell(x)}^{\boldsymbol{x}}$ can be reordered into a $\leqslant_c$-increasing sequence by repeatedly swapping adjacent pairs of reflections that commute.

*Proof.* By employing a certain bilinear pairing on the roots, Reading and Speyer show [54, Proposition 3.11] that the relative order in which two *noncommuting* reflections appear in the reflection sequence of a $c$-sorting word is forced (equivalently, any ambiguity in the sequence comes only from commuting adjacent factors). In particular, whenever $\tau_p^{\boldsymbol{x}}$ and $\tau_q^{\boldsymbol{x}}$ do not commute, their order for $p < q$ already agrees with $\leqslant_c$. Thus the only way the list $\tau_1^{\boldsymbol{x}}, \ldots, \tau_{\ell(x)}^{\boldsymbol{x}}$ can fail to be $\leqslant_c$-increasing is via commuting reflections appearing in the "wrong" $\leqslant_c$-order. The claim now follows. ☐

13.1. **Cover reflections and skip reflections.** For the remainder of this section we fix

$$x \in \mathrm{Sort}(W, c) \text{ with } c\text{-sorting word } \boldsymbol{x} = \boldsymbol{x}_{\boldsymbol{c}^\infty} \subset \boldsymbol{c}^\infty.$$

We now proceed to describe how the $c$-sortable elements are related to noncrossing partitions. To this end we need the notion of cover reflections.

**Definition 13.6.** The set of *cover reflections* for $x$ is given by

$$\mathrm{Covers}(x) = \{xsx^{-1} \mid s \in \mathrm{Des}(x)\} \subset \mathrm{Inv}(x).$$

Recalling the inversion factorization notation of (3.1) we define

$$\mathrm{nc}_c(x) = \tau^{\boldsymbol{x}}_{i_1} \cdots \tau^{\boldsymbol{x}}_{i_k} \text{ where } \mathrm{Covers}(x) = \{\tau^{\boldsymbol{x}}_{i_1}, \ldots, \tau^{\boldsymbol{x}}_{i_k}\} \text{ and } i_1 < \cdots < i_k.$$

By Fact 13.5 the order in which the cover reflections appear in the factorization of $\mathrm{nc}_c(x)$ is the $c$-reflection order up to swapping adjacent commuting reflections.

**Fact 13.7** ([52, Theorem 6.1])**.** The map $x \mapsto \mathrm{nc}_c(x)$ is a bijection

$$\mathrm{Sort}(W, c) \to \mathrm{NC}(W, c).$$

Furthermore, this restricts to a bijection

$$\{x \in \mathrm{Sort}(W, c) \mid c \leqslant_R x\} \to \mathrm{NC}(W, c)^+.$$

In the introduction, we write $\mathrm{nc}_c(x) = \mathrm{nc}_c(\pi_\downarrow(x))$ for any $x \in W$. The cover reflections can be recovered from $\mathrm{nc}_c(x)$ in the following manner.

**Fact 13.8** ([13, Proposition 3.3])**.** Let $W'$ be the smallest reflection subgroup containing $\mathrm{nc}_c(x)$. With respect to the induced positive system $\Phi^+_{W'} = \Phi_{W'} \cap \Phi^+$ and simple system $\Delta_{W'}$, one has $\mathrm{Covers}(x) = \{s_\beta : \beta \in \Delta_{W'}\}$, and $\mathrm{nc}_c(x)$ is a Coxeter element of $W'$.

We therefore have that the product $\tau^{\boldsymbol{x}}_{i_1} \cdots \tau^{\boldsymbol{x}}_{i_k} = \mathrm{nc}_c(x)$ is a minimal length reflection factorization, and by [13, Proposition 3.4] any two minimal length reflection factorizations $t_1 \cdots t_k = t'_1 \cdots t'_k = \mathrm{nc}_c(x)$ into the cover reflections in some order are related to each other by commuting adjacent transpositions – we call any such factorization a *cover reflection factorization* of $\mathrm{nc}_c(x)$.

**Example 13.9.** Continuing Example 13.2 with $W = S_5$ and $c = s_2s_1s_3s_4 \in S_5$, and $x = 35421 \in \mathrm{Sort}(S_5, c)$, the cover reflections of $x$ are $(4\,5)$, $(2\,4)$, and $(1\,2)$. Moreover,

$$\mathrm{nc}_c(x) = (2\,4)(1\,2)(4\,5) = (2\,4)(4\,5)(1\,2) = 41352.$$

The minimal reflection subgroup containing $\mathrm{nc}_c(x)$ is the subgroup $S_{\{1,2,4,5\}} \subseteq S_5$ comprising all permutations which fix 3, which has simple generating set $(1\,2)$, $(2\,4)$, and $(4\,5)$.

**Definition 13.10.** Say that $p \in \mathbb{Z}_{>0}$ is a *skip position* for $x$ if $\boldsymbol{x}$ does not include the $p$th letter $s = \boldsymbol{c}^\infty_p$, and moreover $\boldsymbol{x}$ contains all copies of $s$ in $\boldsymbol{c}^\infty$ before position $p$. We enumerate the skip positions for $x$ in increasing order (i.e. from left to right) as

$$p_1 < p_2 < \cdots < p_n.$$

See Example 13.15 where the circled positions denote skip positions. Each skip position $p_j$ determines a special reflection as follows.

**Definition 13.11.** For $1 \leqslant j \leqslant n$, let $a_1 a_2 \cdots a_{r_j}$ be the prefix of $\boldsymbol{x}$ consisting of all letters to the left of the skip position $p_j$, and let $s_j = \boldsymbol{c}^{\infty}_{p_j}$. We define the $j$th *skip reflection* (or just "skip") by

$$\phi_j = a_1 \cdots a_{r_j} s_j a_{r_j} \cdots a_1,$$

and set $\mathrm{Skips}(x) \coloneqq \{\phi_1, \dots, \phi_n\}$

We will use the following fact repeatedly.

**Proposition 13.12.** For any $1 \leqslant i_1 < \cdots < i_k \leqslant n$, the product $\phi_{i_1}\phi_{i_2}\cdots\phi_{i_k}x$ has a (possibly non-reduced) word obtained from $\boldsymbol{x} \subset \boldsymbol{c}^{\infty}$ by filling in the skips associated to $\phi_{i_1}, \dots, \phi_{i_k}$.

*Proof.* Write $\boldsymbol{x} = a_1 a_2 \cdots a_{\ell(x)}$ and take $r_{i_1}, r_{i_2}, \dots, r_{i_k}$ and $s_{i_1}, \dots, s_{i_k}$ as in Definition 13.11. Then

$$\begin{aligned}\phi_{i_1}\phi_{i_2}\cdots\phi_{i_k}x &= (a_1 \cdots a_{r_{i_1}})s_{i_1}(a_{r_{i_1}} \cdots a_1)\cdots(a_1 \cdots a_{r_{i_k}})s_{i_k}(a_{r_{i_k}} \cdots a_1)a_1 a_2 \cdots a_{\ell(x)} \\ &= a_1 \cdots a_{r_{i_1}}\ s_{i_1}\ a_{r_{i_1}+1} \cdots a_{r_{i_2}}\ s_{i_2}\ a_{r_{i_2}+1} \cdots a_{r_{i_k}}\ s_{i_k}\ a_{r_{i_k}+1} \cdots a_{\ell(x)}\end{aligned}$$

as desired. □

As a consequence we obtain a minimal reflection factorization for $c$ from the skip reflections.

**Corollary 13.13.** We have $\phi_1 \cdots \phi_n = c$.

*Proof.* Let $\boldsymbol{x}'$ be the (possibly nonreduced) word for $(\phi_1 \cdots \phi_n)x$ given by Proposition 13.12, obtained from $\boldsymbol{x} \subset \boldsymbol{c}^{\infty}$ by filling all skip positions.

Fix a simple reflection $s \in \mathsf{S}$, and let $m_s$ be the number of occurrences of $s$ used by $\boldsymbol{x}$ in $\boldsymbol{c}^{\infty}$. Since the syllables of $\boldsymbol{x}$ are weakly decreasing under inclusion, $\boldsymbol{x}$ uses precisely the first $m_s$ occurrences of $s$ in $\boldsymbol{c}^{\infty}$. Hence the skip position for $s$ is its $(m_s + 1)$st occurrence, and filling all skips forces $\boldsymbol{x}'$ to use the first $m_s + 1$ occurrences of $s$.

In particular, every $s \in \mathsf{S}$ occurs in the first syllable of $\boldsymbol{x}'$, so the first syllable of $\boldsymbol{x}'$ is exactly $\boldsymbol{c}$, and the $(i+1)$st syllable of $\boldsymbol{x}'$ coincides with the $i$th syllable of $\boldsymbol{x}$ for all $i \geqslant 1$. Therefore $\boldsymbol{x}' = \boldsymbol{c}\,\boldsymbol{x}$ as subwords of $\boldsymbol{c}^{\infty}$, and hence $(\phi_1 \cdots \phi_n)x = cx$. The claim follows. □

13.2. **Forced and unforced skip reflections.** Following Reading's Cambrian theory [52], the $c$-sorting word of a $c$-sortable element carries combinatorial data via cover reflections that determines its associated noncrossing partition. Reading–Speyer [54] show that this data can be read directly from the pattern of omitted letters in $\boldsymbol{c}^{\infty}$: the skips that are *forced* by reducedness are precisely the cover reflections, and hence encode $\mathrm{nc}_c(x)$, while the remaining skips are optional insertions, motivating the distinction between forced and unforced skips [54, Proposition 5.2].

Like before, we write $x \in \mathrm{Sort}(W, c)$ as $\boldsymbol{x} = a_1 a_2 \cdots a_{\ell(x)}$.

**Definition 13.14.** Say that a skip $\phi_j$ is *unforced* if $a_1 \cdots a_{r_j} s_j$ is a reduced expression in $W$, and *forced* otherwise. Equivalently, letting $y_j = a_1 \cdots a_{r_j}$, the skip $\phi_j$ is unforced iff $\ell(y_j s_j) = \ell(y_j) + 1$, and forced iff $\ell(y_j s_j) = \ell(y_j) - 1$.

If we let $\mathrm{USkips}(x)$ and $\mathrm{FSkips}(x)$ denote the respective sets of unforced and forced skips, then we have a decomposition

$$\mathrm{Skips}(x) = \mathrm{USkips}(x) \sqcup \mathrm{FSkips}(x).$$

We will enumerate elements of these sets according to their skip positions in $\boldsymbol{c}^\infty$ as

$$\begin{aligned} \mathrm{FSkips}(x) &= \{\mathrm{fs}_1, \ldots, \mathrm{fs}_k\}, && \text{where } k \coloneqq |\,\mathrm{FSkips}(x)|, \text{ and} \\ \mathrm{USkips}(x) &= \{\mathrm{ufs}_1, \ldots, \mathrm{ufs}_{n-k}\}. \end{aligned}$$

This is the same order used for $\mathrm{Skips}(x)$, so $\phi_1, \ldots, \phi_n$ is a shuffle of the lists $\mathrm{ufs}_1, \ldots, \mathrm{ufs}_{n-k}$ and $\mathrm{fs}_1, \ldots, \mathrm{fs}_k$.

**Example 13.15.** As in Example 13.2 take $c = s_2 s_1 s_3 s_4 \in S_5$ and $x = 35421 \in \mathrm{Sort}(W, c)$. The skip positions are the circled letters in the $c$-sorting word

$$\boldsymbol{x} = s_2\ s_1\ s_3\ s_4 \mid s_2\ \enclose{circle}{\cancel{s_1}}\ s_3\ s_4 \mid s_2\ \cancel{s_1}\ \enclose{circle}{\cancel{s_3}}\ \enclose{circle}{\cancel{s_4}} \mid \enclose{circle}{\cancel{s_2}}\ \cancel{s_1}\ \cancel{s_3}\ \cancel{s_4} \mid \cdots$$

The corresponding skip reflections are, in order, $(3\,4) = \mathrm{ufs}_1$, $(2\,4) = \mathrm{fs}_1$, $(1\,2) = \mathrm{fs}_2$, and $(4\,5) = \mathrm{fs}_3$.

The agreement of forced skips and cover reflections in Example 13.15 is not accidental.

**Fact 13.16.** $\mathrm{Covers}(x) = \mathrm{FSkips}(x)$, and $\mathrm{fs}_1 \cdots \mathrm{fs}_k = \mathrm{nc}_c(x)$ is a cover reflection factorization.

*Proof.* By [54, Proposition 5.2], the forced skips are precisely the cover reflections, so $\mathrm{Covers}(x) = \mathrm{FSkips}(x)$ as sets. Let $u \coloneqq \mathrm{fs}_1 \cdots \mathrm{fs}_k$, i.e. the product in skip-position order.

By definition, $\mathrm{nc}_c(x)$ is the product of the cover reflections in the order they appear in the reflection sequence of the $c$-sorting word (equivalently, in $c$-reflection order up to commuting adjacent commuting reflections). Moreover, [13, Remark 3.2(2)] implies that for any two noncommuting reflections in $\mathrm{Covers}(x)$, their relative order agrees in any cover reflection factorization. Hence the words $u$ and $\mathrm{nc}_c(x)$ differ only by swapping adjacent commuting reflections, and in particular $u = \mathrm{nc}_c(x)$.

Finally, by [13, Proposition 3.4] any minimal reflection factorization of $\mathrm{nc}_c(x)$ into the cover reflections is unique up to swapping adjacent commuting factors, so $\mathrm{fs}_1 \cdots \mathrm{fs}_k$ is a cover reflection factorization. $\square$

**Proposition 13.17.** The following hold.

(1) We have $x <_B \mathrm{ufs}_i\, x$ for all $1 \leqslant i \leqslant n - k$.
(2) $\mathrm{ufs}_i\, x$ is $c$-sortable, and $\mathrm{ufs}_1, \ldots, \mathrm{ufs}_{i-1}$ are unforced skips of $\mathrm{ufs}_i\, x$.

*Proof.* Fix $i$. By Proposition 13.12, the product $\mathrm{ufs}_i\, x$ admits an expression $\boldsymbol{x}'$ obtained from the $c$-sorting word $\boldsymbol{x} \subset \boldsymbol{c}^\infty$ by filling the skip corresponding to $\mathrm{ufs}_i$ (i.e. inserting the skipped letter at its skip position). In particular, $\boldsymbol{x}'$ has length $\ell(x) + 1$, so $\ell(\mathrm{ufs}_i\, x) \leqslant \ell(x) + 1$. On the other hand, [54, Lemma 5.7] gives $\ell(\mathrm{ufs}_i\, x) > \ell(x)$, hence $\ell(\mathrm{ufs}_i\, x) = \ell(x) + 1$. Therefore $\boldsymbol{x}'$ is reduced and $x <_B \mathrm{ufs}_i\, x$, proving (1).

Since $\boldsymbol{x}'$ is a reduced subword of $\boldsymbol{c}^\infty$ and is obtained from $\boldsymbol{x}$ by adding a single letter in one syllable, it still satisfies the syllable-containment condition. Thus Lemma 13.3 implies that $\boldsymbol{x}'$ is the $c$-sorting word of $\mathrm{ufs}_i\, x$, so $\mathrm{ufs}_i\, x$ is $c$-sortable.

Finally, for any $j < i$, the skip position of $\mathrm{ufs}_j$ occurs before that of $\mathrm{ufs}_i$, so the prefix of $\boldsymbol{x}'$ up to that position agrees with the corresponding prefix of $\boldsymbol{x}$. Hence the reduced/nonreduced status of the word $a_1 \cdots a_{r_j} s_j$ is unchanged, and $\mathrm{ufs}_j$ remains an unforced skip for $\mathrm{ufs}_i\, x$. This proves (2). ☐

**Proposition 13.18.** Let $\phi_i, \phi_j \in \mathrm{Skips}(x)$ with $i < j$. If $\phi_i$ is forced and $\phi_j$ is unforced, then $\phi_i\phi_j = \phi_j\phi_i$.

*Proof.* We may truncate $\boldsymbol{x}$ immediately after the skip position of $\phi_j$; this does not change $\phi_i$ or $\phi_j$ since both are defined using prefixes before that position. In particular, we may assume the skip position of $\phi_j$ occurs after all letters of $\boldsymbol{x}$, so $\phi_j = x s_j x^{-1}$ and hence $\phi_j x = x s_j$.

Since $\phi_i$ is forced, it is a cover reflection by Fact 13.16, so $\phi_i = x s x^{-1}$ for some $s \in \mathrm{Des}(x)$. Now consider the element $x s_j = \phi_j x$, which is $c$-sortable by Proposition 13.17. Appending $s_j$ does not change the prefix data for the earlier skip position $i$, so $\phi_i$ is still a forced skip for $x s_j$, hence again a cover reflection of $x s_j$. Thus $\phi_i = (x s_j)\, s'\, (x s_j)^{-1}$ for some $s' \in \mathrm{Des}(x s_j)$.

Equating the two expressions for $\phi_i$ gives

$$x s x^{-1} = x s_j s' s_j x^{-1}, \quad \text{so} \quad s = s_j s' s_j.$$

Since $s$ and $s'$ are simple reflections, the conjugate $s_j s' s_j$ can be simple only if $s_j$ commutes with $s'$, in which case $s_j s' s_j = s'$. Therefore $s = s'$ and $s$ commutes with $s_j$. Conjugating by $x$ shows $\phi_i$ commutes with $\phi_j$. ☐

Applying these commutation relations to the identity in Corollary 13.13 gives us the following alternate factorizations of $c$ into skip reflections.

**Corollary 13.19.** $\mathrm{ufs}_1 \cdots \mathrm{ufs}_{n-k}\, \mathrm{fs}_1 \cdots \mathrm{fs}_k = c$.

## 14. Characterizing equivalent translated intervals

We now determine when $\overleftarrow{[w', w' \cdot c]} = \overleftarrow{[w'', w'' \cdot c]}$ using Cambrian equivalence by identifying certain distinguished decreasing chains in the $c$-reflection order. The main theorem of this section was also independently shown by Defant–Sherman-Bennett–Williams [28, Theorem 1.9 and Corollary 1.10]. To transition between the condition $\ell(wc) = \ell(w) + \ell(c)$ and $c$-sortable combinatorics, we need the following transformation.

**Definition 14.1.** For each $w \in W$, define $w_{op} := w^{-1} w_\circ$.

Note that $w \mapsto w_{op}$ is a bijection, with inverse $x \mapsto w_\circ x^{-1}$.

**Proposition 14.2.**

(1) $\ell(wc) = \ell(w) + \ell(c)$ is equivalent to $c \leqslant_R w_{op}$.
(2) $\{x \mid c \leqslant_R x\}$ is the union of $\mathrm{Cat}_W^+$-many Cambrian equivalence classes.

*Proof.* (1) follows because $c \leqslant_R w_{op}$ is equivalent to $\ell(c) + \ell(c^{-1}w^{-1}w_\circ) = \ell(w^{-1}w_\circ)$, and we can apply the identity $\ell(y^{-1}w_\circ) = \ell(w_\circ) - \ell(y)$ to both sides for $y = wc$ and $y = w$ respectively. For (2), by Fact 13.7 and the fact that $c \leqslant_R x$ implies $c \leqslant_R \pi_\downarrow(x)$ by definition of $\pi_\downarrow(x)$, we have that

$$c \leqslant_R x \iff \mathrm{nc}_c(\pi_\downarrow(x)) \in \mathrm{NC}(W, c)^+.$$

Because $|\mathrm{NC}(W, c)^+| = \mathrm{Cat}_W^+$ we conclude. □

**Theorem 14.3.** For $w', w'' \in \{w \mid \ell(wc) = \ell(w) + \ell(c)\}$, or equivalently with $(w')_{op}, (w'')_{op} \in \{x \mid c \leqslant_R x\}$, the following are equivalent.

(1) $(w')_{op} \equiv_c (w'')_{op}$
(2) $\overleftarrow{[w', w'c]} = \overleftarrow{[w'', w''c]}$
(3) $\overleftarrow{[w', w'c]}$ and $\overleftarrow{[w'', w''c]}$ have the same $c$-decreasing chain.

In particular, there are $\mathrm{Cat}_W^+$-many distinct translated Bruhat intervals $\overleftarrow{[w, wc]}$ in $\mathrm{NC}(W, c)$.

*Proof.* By Corollary 6.9 and Corollary 9.4 there are exactly $\mathrm{Cat}_W^+$-many translates $\overleftarrow{[w, wc]}$, and by Corollary 7.3 each of these contains a unique $c$-decreasing chain. By Proposition 14.2 there are exactly $\mathrm{Cat}_W^+$-many $c$-decreasing chains in $\mathrm{NC}(W, c)$, which establishes the equivalence between (2) and (3). By Corollary 3.6 the number of $c$-Cambrian equivalence classes in $\{x \mid c \leqslant_R x\}$ is also equal to $\mathrm{Cat}_W^+$, so to show (1) and (3) are equivalent it therefore suffices to show the following.

**Goal.** For $w$ with $w_{op} \in \{x \mid c \leqslant_R x\}$, the $c$-decreasing (reflection-labeled) maximal chain in $\overleftarrow{[w, wc]}$ depends only on the $c$-Cambrian class of $w_{op}$.

Fix such a $w$, and set

$$x := \pi_\downarrow(w_{op}).$$

Then $x$ depends only on the $c$-Cambrian class of $w_{op}$. Moreover $c \leqslant_R x$ and, by Lemma 13.3, deleting the first syllable $c$ from the $c$-sorting word $\boldsymbol{x}$ produces the $c$-sorting word $\boldsymbol{c^{-1}x}$ for $c^{-1}x$, so in particular $c^{-1}x \in \mathrm{Sort}(W, c)$.

Let

$$\mathrm{Skips}(c^{-1}x) = \{\phi_1, \ldots, \phi_n\}.$$

Define reflections $\psi_1, \ldots, \psi_n$ by

$$\psi_i := \phi_1 \cdots \phi_{i-1}\, \phi_i\, \phi_{i-1} \cdots \phi_1 \qquad (1 \leqslant i \leqslant n).$$

A $c$-decreasing maximal chain is determined by its set of labels, so it is enough to prove that the unique $c$-decreasing maximal chain in $\overleftarrow{[w, wc]}$ is labeled by $\psi_1, \dots, \psi_n$ (in $c$-reflection order). The remainder of the proof is organized in three steps.

**Step 1: a distinguished Bruhat chain in $[c^{-1}x, x]$.** Set

$$x_i := \psi_i \psi_{i+1} \cdots \psi_n \, x \qquad (1 \leqslant i \leqslant n), \qquad \text{and} \qquad x_{n+1} := x.$$

Then $x_1 = c^{-1}x$ and $x_{i+1} = \psi_i x_i$ for $1 \leqslant i \leqslant n$. We claim that

$$x_1 \lessdot_B x_2 \lessdot_B \cdots \lessdot_B x_{n+1}$$

is a maximal Bruhat chain in the interval $[c^{-1}x, x]$.

To verify the covering relations, write the $c$-sorting word $\boldsymbol{c^{-1}x}$ (as a subword of $\boldsymbol{c}^\infty$) and fix $1 \leqslant i \leqslant n$. Let $\boldsymbol{a_i b_i}$ denote the portions of $\boldsymbol{c^{-1}x}$ occurring before and after the $i$th skip position, and let the letter at that skip position be $\boldsymbol{s_i}$ so that, on the level of group elements,

$$\phi_i \, c^{-1}x = a_i s_i b_i.$$

Let $\boldsymbol{a'_i}$ be the word obtained from $\boldsymbol{a_i}$ by filling in the first $i-1$ skips of $\boldsymbol{c^{-1}x}$. Then

$$\phi_1 \cdots \phi_{i-1} \, c^{-1}x \;=\; a'_i b_i, \qquad \text{hence} \qquad x_i = \psi_i \cdots \psi_n x = \phi_1 \cdots \phi_{i-1} c^{-1}x = a'_i b_i.$$

We show by induction on $i$ that $\boldsymbol{a'_i b_i}$ is a reduced $c$-sorting word for a $c$-sortable element (namely $x_i$). For $i = 1$ this is exactly $\boldsymbol{c^{-1}x}$. Assuming the claim for $\boldsymbol{a'_i b_i}$, the $i$th skip position of $c^{-1}x$ is also a skip position for $\boldsymbol{a'_i b_i}$, and inserting the letter $\boldsymbol{s_i}$ fills this position:

$$\boldsymbol{a'_{i+1} b_i} = \boldsymbol{a'_i s_i b_i}.$$

This insertion is unforced, because $\boldsymbol{a'_i s_i}$ is a prefix of $\boldsymbol{x}$ and hence reduced. Therefore Proposition 13.17 implies that $\boldsymbol{a'_{i+1} b_i}$ remains reduced. Finally, Lemma 13.3 implies that $\boldsymbol{a'_{i+1} b_{i+1}}$ is again a $c$-sorting word of a $c$-sortable element. This completes the induction and establishes that each step $x_i \lessdot_B x_{i+1}$ is a Bruhat cover.

**Step 2: lifting the chain to $[c^{-1}w_{op}, w_{op}]$.** Since $x \leqslant_R w_{op}$, let $z := x^{-1}w_{op}$ and choose a reduced word $\boldsymbol{z}$ for $z$ so that $\boldsymbol{xz}$ is a reduced word for $w_{op}$. For each $i$ set

$$y_i := \psi_i \psi_{i+1} \cdots \psi_n \, w_{op} \qquad (1 \leqslant i \leqslant n), \qquad \text{and} \qquad y_{n+1} := w_{op}.$$

Then $y_1 = c^{-1}w_{op}$ and $y_{i+1} = \psi_i y_i$.

We claim that

$$y_1 \lessdot_B y_2 \lessdot_B \cdots \lessdot_B y_{n+1}$$

is a maximal Bruhat chain in $[c^{-1}w_{op}, w_{op}]$. Using the notation from Step 1, we have

$$y_i = \psi_i \cdots \psi_n w_{op} = (a'_i b_i) \, z,$$

so it suffices to prove that $\boldsymbol{a'_i b_i z}$ is reduced for each $i$. Proceed by induction on $i$, with the base case $i = 1$ given by the reduced word $\boldsymbol{c^{-1}xz}$ for $c^{-1}w_{op}$.

Assume $\boldsymbol{a'_i b_i z}$ is reduced. Since $y_{i+1} = \psi_i y_i$ and $\psi_i a'_i = a'_i s_i$, the word

$$\boldsymbol{a'_{i+1} b_i z} = \boldsymbol{a'_i s_i b_i z}$$

represents $y_{i+1}$ and has one more letter than $\boldsymbol{a'_i b_i z}$. Thus it is enough to show that $\psi_i \notin \mathrm{Inv}(a'_i b_i z)$ (equivalently, that left-multiplication by $\psi_i$ increases length by 1). Using

$$\mathrm{Inv}(a'_i b_i z) = \mathrm{Inv}(a'_i) \cup (a'_i)\,\mathrm{Inv}(b_i z)(a'_i)^{-1},$$

and the fact that $\psi_i \notin \mathrm{Inv}(a'_i)$ (since $\psi_i a'_i = a'_i s_i$ and $\boldsymbol{a'_i s_i}$ is reduced from Step 1), it remains to show that $\psi_i \notin (a'_i)\,\mathrm{Inv}(b_i z)(a'_i)^{-1}$, i.e. that $s_i \notin \mathrm{Inv}(b_i z)$. As before, by Lemma 13.3, we have $\pi_\downarrow(c^{-1}w_{op}) = c^{-1}\pi_\downarrow(w_{op}) = c^{-1}x$. If $s_i \in \mathrm{Inv}(b_i z)$, then one can delete the letter of $\boldsymbol{b_i z}$ corresponding to this inversion and prepend $\boldsymbol{s_i}$, contradicting the $\leqslant_R$-maximality of the $c$-sorting word $\boldsymbol{c^{-1}x} = \boldsymbol{a_i b_i}$ for $c^{-1}x = \pi_\downarrow(c^{-1}w_{op})$. Hence $s_i \notin \mathrm{Inv}(b_i z)$, completing the induction and proving the Bruhat covering claims.

**Step 3: transporting to the interval $\overleftarrow{[w, wc]}$.** Applying the map $v \mapsto w_\circ v^{-1}$ to the chain in Step 2 yields a maximal chain

$$w \lessdot_B w\psi_n \lessdot_B w\psi_n\psi_{n-1} \lessdot_B \cdots \lessdot_B w\psi_n\psi_{n-1}\cdots\psi_1,$$

whose reflection labels are, in order, $\psi_n, \ldots, \psi_1$.

These same reflections appear (in reverse order) in the reflection sequence associated to the $c$-sorting word of $w_{op}$, corresponding to the final occurrence of each simple reflection. Therefore, by Fact 13.5, we can repeatedly swap adjacent commuting reflections to obtain a reordering

$$\psi'_1, \psi'_2, \ldots, \psi'_n$$

that is in reverse $c$-reflection order. Since only commuting reflections are swapped, the chain

$$w \lessdot_B w\psi'_n \lessdot_B w\psi'_n\psi'_{n-1} \lessdot_B \cdots \lessdot_B w\psi'_n\psi'_{n-1}\cdots\psi'_1$$

remains inside the same Bruhat interval $[w, wc]$, hence is the unique $c$-decreasing maximal chain in $\overleftarrow{[w, wc]}$ by Corollary 7.3. In particular, its label set is $\{\psi_1, \ldots, \psi_n\}$, which depends only on $x = \pi_\downarrow(w_{op})$, and hence only on the $c$-Cambrian class of $w_{op}$.

This proves the Goal. Combining this with the counting discussion at the start of the proof yields the equivalences (1)–(3), and the final assertion follows. □

**Example 14.4.** Work in type $B_4$ in one-line notation $w(1)\,w(2)\,w(3)\,w(4)$, writing $\bar{i} := -i$. Let $s_0, s_1, s_2, s_3$ be the standard simple reflections, where $s_0$ negates the first entry. Fix $\boldsymbol{c} = s_0 s_1 s_2 s_3$.

Say we take $w_{op} = \bar{3}\,\bar{4}\,2\,\bar{1}$, so that $c \leqslant_R w_{op}$. Its $c$-sorting word $\boldsymbol{w_{op}}$ is $s_0\, s_1\, s_2\, s_3 \;\; s_1\, s_2 \;\; s_0\, s_1 \;\; s_0\, s_1$. The downward Cambrian projection is $x = \pi_\downarrow(w_{op}) = 4\;3\;2\;\bar{1}$, whose $c$-sorting word $\boldsymbol{x}$ is given by $\boldsymbol{x} = s_0\, s_1\, s_2\, s_3 \;\; s_1\, s_2\, s_1$. Deleting the initial syllable $\boldsymbol{c}$ from $\boldsymbol{x}$ gives $c^{-1}x = 3\;2\;1\;4$ with $c$-sorting word $\boldsymbol{c^{-1}x} = s_1s_2s_1$. The skip positions of $c^{-1}x$ are $\{1,4,7,10\}$ with simple reflections at these positions being $s_0$, $s_3$, $s_2$, and $s_1$ respectively. We thus get

$$\mathrm{Skips}(c^{-1}x) = \{\phi_1,\ldots,\phi_4\} = \{s_0, s_1s_2s_3s_2s_1, s_1, s_2\} = \{\overline{1}\,2\,3\,4, 4\,2\,3\,1, 2\,1\,3\,4, 1\,3\,2\,4\}.$$

The conjugated reflections $\psi_i = \phi_1\cdots\phi_{i-1}\,\phi_i\,\phi_{i-1}\cdots\phi_1$ are then computed to equal

$$\psi_1 = \overline{1}\,2\,3\,4, \qquad \psi_2 = \overline{4}\,2\,3\,\overline{1}, \qquad \psi_3 = 1\,4\,3\,2, \qquad \psi_4 = 1\,2\,4\,3.$$

The Bruhat chain in $[c^{-1}x, x]$ constructed in Step 1 in the proof is given by

$$3\,2\,1\,4 \xrightarrow{\psi_1} 3\,2\,\overline{1}\,4 \xrightarrow{\psi_2} 3\,2\,4\,\overline{1} \xrightarrow{\psi_3} 3\,4\,2\,\overline{1} \xrightarrow{\psi_4} 4\,3\,2\,\overline{1}.$$

These reflections then give the following Bruhat chain in $[c^{-1}w_{op}, w_{op}]$ as per Step 2 in the proof.

$$\overline{2}\,\overline{3}\,1\,4 \xrightarrow{\psi_1} \overline{2}\,\overline{3}\,\overline{1}\,4 \xrightarrow{\psi_2} \overline{2}\,\overline{3}\,4\,\overline{1} \xrightarrow{\psi_3} \overline{4}\,\overline{3}\,2\,\overline{1} \xrightarrow{\psi_4} \overline{3}\,\overline{4}\,2\,\overline{1}.$$

We now transport to the translated interval $\overleftarrow{[w, wc]}$ as per Step 3 in the proof. We have $w_\circ = \overline{1}\,\overline{2}\,\overline{3}\,\overline{4}$ and so $w := w_\circ w_{op}^{-1} = 4\,\overline{3}\,1\,2$. Then $wc = \overline{3}\,1\,2\,\overline{4}$. and applying the inverse of the *op*-map to the previous chain gives the maximal chain in $[w, wc]$:

$$w = 4\,\overline{3}\,1\,2 \xrightarrow{\psi_4} 4\,\overline{3}\,2\,1 \xrightarrow{\psi_3} 4\,1\,2\,\overline{3} \xrightarrow{\psi_2} 3\,1\,2\,\overline{4} \xrightarrow{\psi_1} \overline{3}\,1\,2\,\overline{4} = wc.$$

Thus the labels in the translated interval $\overleftarrow{[w, wc]}$ are exactly $\psi_1, \psi_2, \psi_3, \psi_4$ (appearing along the chain as $\psi_4, \psi_3, \psi_2, \psi_1$), as in the proof. We reconcile this with the $c$-reflection order for our choice of $c$. The $c$-sorting word for $w_\circ$ is $\boldsymbol{c}^4$ and in the resulting reflection order, the $\psi_1$ through $\psi_4$ appear in positions 1, 4, 15 and 16 respectively. Thus the chain constructed last is the $c$-decreasing maximal chain in $[w, wc]$.

**Corollary 14.5.** Let $W' \subseteq W$ be a standard parabolic subgroup and let $c' \leqslant_B c$ be the corresponding Coxeter element. If $w \in W$ has $\pi_\downarrow(w_{op}) \in W'$, then denoting $x \in W$ for the element with $x_{op} = \pi_\downarrow(w_{op})$, we have $\overleftarrow{[w, wc']} = \overleftarrow{[x, xc']}$.

*Proof.* We first recall the well-known fact [41, §1.10] that $W'$ has a complete set of right coset representatives $M$ such that for every $m \in M$ and every $v \in W'm$, we have $v = v'\cdot m$ for $v' = vm^{-1} \in W'$. If $u \in W'$ has $u \leqslant_R v$, then applying the above to $u^{-1}v$ shows that $u \leqslant_R v'$. Furthermore, if $v \lessdot_B \tau v$ is a Bruhat cover in $W'$, then $vm \lessdot_B \tau vm$ is a Bruhat cover in $W$.

Now factor $w_{op} = w'_{op}\cdot m$. As $c'$-sortablility and $c$-sortability coincide in $W'$, the $\leqslant_R$-preserving property implies that $x_{op}$ is the maximal $c'$-sortable below $w'$. Thus Theorem 14.3 implies that $[(c')^{-1}x_{op}, x_{op}]$ and $[(c')^{-1}w'_{op}, w'_{op}]$ are isomorphic as edge-labeled posets with each $v \lessdot_B \tau v$ labeled by $\tau$. By the cover-preserving property, $[(c')^{-1}w'_{op}, w'_{op}]$ is isomorphic in the same way to $[(c')^{-1}w_{op}, w_{op}]$. Transporting intervals as in Step 3 of the proof above, we have $\overleftarrow{[x, xc']} = \overleftarrow{[w, wc']}$. ☐

## 15. Clusters, noncrossing inversions, and Bruhat maximal elements

We now give a novel characterization of Reading's map from $c$-sortable elements to noncrossing partitions; see Fact 13.7. The following result implies Theorem C, as we have established in Corollary 9.4 that $w^{-1}X_w^{w\cdot c} = X_{\mathrm{NC}}^u$ for $u \in w^{-1}[w, w\cdot c]$ the Bruhat-maximum element, and in Corollaries 9.14 and 14.5 that all cases are equivalent to considering noncrossing partitions which are fully supported in an appropriate standard parabolic subgroup.

**Theorem 15.1.** The Bruhat-maximal element of $\overleftarrow{[w, w\cdot c]}$ is $u = \mathrm{nc}_c(\pi_\downarrow(w_{op}))$, so the Bruhat maximal elements of $w^{-1}[w, w\cdot c]$ and $(w')^{-1}[w', w'\cdot c]$ are equal if and only if $w_{op} \equiv_c w'_{op}$.

In particular, the $n$ elements adjacent to $u$ in $\overleftarrow{[w, w\cdot c]} \subset \mathrm{NC}(W, c)$ are $\{\tau u \mid \tau \in \mathrm{Inv}_{\mathrm{NC}}(u)\}$.

We illustrate Theorem 15.1 before laying the groundwork for the proof.

**Example 15.2.** We take $W = S_5$ and $c = s_2s_1s_3s_4$ as in Example 13.15. For $w = 12534$ it is seen that $\ell(wc) = \ell(w) + \ell(c)$. One checks that $w_{op} = 35421$, which is $c$-sortable and thus satisfies $\pi_\downarrow(w_{op}) = w_{op}$. Using the forced skips computed in Example 13.15 together with Fact 13.16 we see that $u = \mathrm{nc}_c(\pi_\downarrow(w_{op})) = 41352$. A direct computation tells us that

$$\overleftarrow{[w, wc]} = \left\{ \begin{array}{llll} 12345, & 12354, & 13245, & 13254, \\ 13425, & 13452, & 14325, & 14352, \\ 21345, & 21354, & 31245, & 31254, \\ 31425, & 31452, & 41325, & 41352 \end{array} \right\},$$

and that $u = 41352$ is indeed the Bruhat-maximal element. Finally, Theorem 15.1 says that the elements of $\overleftarrow{[w, wc]}$ adjacent to $u$ are precisely the four permutations $\tau u$ with $\tau \in \mathrm{Inv}_{\mathrm{NC}}(u)$, namely

$$(2\,4)u = 21354, \qquad (1\,4)u = 14352, \qquad (2\,5)u = 41325, \qquad (3\,4)u = 31452.$$

We remark that in the Bruhat interval $[w, wc]$, the element $wu$ equals 31542, and among the four $w\tau u$ for $\tau \in \mathrm{Inv}_{\mathrm{NC}}(u)$, one covers $wu$ while the remaining are covered by $wu$.

Biane and Josuat-Vergès describe the right noncrossing inversion set $\mathrm{Inv}^R_{\mathrm{NC}}(u)$ used to define $\mathrm{Clust}^+$ explicitly in terms of cover reflection factorizations. For $u \in \mathrm{NC}(W, c)$, note that $u^{-1}c \in \mathrm{NC}(W, c)$ as well. Indeed, if $c = \tau_1 \cdots \tau_n$ with $u = \tau_1 \cdots \tau_k$, then $u^{-1}c = \tau_{k+1} \cdots \tau_n$, and we have a minimal factorization $c = \tau_{k+1} \cdots \tau_n \sigma_1 \cdots \sigma_k$ where $\sigma_i = (u^{-1}c)^{-1}\tau_i(u^{-1}c)$.

**Fact 15.3** ([13])**.** Let $u \in \mathrm{NC}(W, c)^+$. Fix cover reflection factorizations

$$u = t_1 \cdots t_k \qquad \text{and} \qquad u^{-1}c = t_{k+1} \cdots t_n.$$

The set of right noncrossing inversions $\mathrm{Inv}^R_{\mathrm{NC}}(u) = \{u^{-1}\tau u \mid \tau \in \mathrm{Inv}_{\mathrm{NC}}(u)\}$ of $u$ are given by

(1) the reflections $p_1, \ldots, p_k$ such that

$$t_1 \cdots t_k p_i = t_1 \cdots \widehat{t_i} \cdots t_k,$$

(2) the reflections $q_1, \ldots, q_{n-k}$ such that

$$q_i t_{k+1} \cdots t_n = t_{k+1} \cdots \widehat{t_{k+i}} \cdots t_n.$$

Of all cover reflection factorizations of $u$ and $u^{-1}c$ from Fact 15.3, there are two which are most useful for us. They will be determined by the forced and unforced skips of the $c$-sortable $x \in \mathrm{Sort}(W, c)$ with $c \leqslant_R x$ (Fact 13.7) for which $\mathrm{nc}_c(x) = u$.

**Lemma 15.4.** For $w \in W$, the element $\pi_\uparrow(w_{op})w_\circ$ is $c^{-1}$-sortable, and

$$\mathrm{Covers}\left(\pi_\uparrow(w_{op})w_\circ\right) = \mathrm{USkips}(\pi_\downarrow(w_{op})),$$

i.e. its set of cover reflections is the unforced skip set of $\pi_\downarrow(w_{op})$.

*Proof.* By (13.1) we have

$$\pi_\uparrow^{(c)}(w^{-1}) = \pi_\downarrow^{(c^{-1})}(w^{-1}w_\circ)\, w_\circ,$$

so in particular $\pi_\uparrow(w_{op})w_\circ \in \mathrm{Sort}(W, c^{-1})$.

Let $x := \pi_\downarrow(w_{op})$, and let $\mathcal{I}$ be the $c$-Cambrian class of $w_{op}$ (equivalently of $x$). Then $x$ is the unique $c$-sortable element in $\mathcal{I}$. Write $\{\mathrm{fs}_1, \ldots, \mathrm{fs}_k\}$ and $\{\mathrm{ufs}_1, \ldots, \mathrm{ufs}_{n-k}\}$ for the forced and unforced skips of $x$. Let $D = \mathbb{R}_{\geqslant 0}\Lambda^+$ be the dominant chamber. By [54, Theorem 6.3] we have

$$\bigcup_{z \in \mathcal{I}} zD = \left(\bigcap_{i=1}^{k} \{\langle v, r(\mathrm{fs}_i)\rangle \leqslant 0\}\right) \cap \left(\bigcap_{i=1}^{n-k} \{\langle v, r(\mathrm{ufs}_i)\rangle \geqslant 0\}\right).$$

Thus the data of forced versus unforced skips for $x$ are encoded by which side of each of these $n$ facet hyperplanes the cone lies on.

Now consider the image class $\mathcal{I}w_\circ$ under the anti-automorphism $z \mapsto zw_\circ$. Since $w_\circ D = -D$, we have

$$\bigcup_{z \in \mathcal{I}w_\circ} zD = \bigcup_{z \in \mathcal{I}} zw_\circ D = -\bigcup_{z \in \mathcal{I}} zD.$$

Therefore the defining inequalities for the cone $\bigcup_{z \in \mathcal{I}w_\circ} zD$ are obtained from those for $\bigcup_{z \in \mathcal{I}} zD$ by reversing all inequality signs. Equivalently, in the class $\mathcal{I}w_\circ$ the roles of "forced" and "unforced" skips are interchanged. Thus for the unique $c^{-1}$-sortable element in the class $\mathcal{I}w_\circ$, namely $\pi_\uparrow(w_{op})w_\circ$, we have

$$\mathrm{Covers}\left(\pi_\uparrow(w_{op})w_\circ\right) = \{\mathrm{ufs}_1, \ldots, \mathrm{ufs}_{n-k}\},$$

as claimed. □

**Proposition 15.5.** If $u \in \mathrm{NC}(W, c)^+$ and $c \leqslant_R x \in \mathrm{Sort}(W, c)$ is such that $\mathrm{nc}_c(x) = u$ (Fact 13.7), then we have

(1) a cover reflection factorization $u = t_1 \cdots t_k$ is given by $t_i = \mathrm{fs}_i$,
(2) a cover reflection factorization $u^{-1}c = t_{k+1} \cdots t_n$ is given by $t_{k+i} = c^{-1}\, \mathrm{ufs}_i\, c$.

Here $\mathrm{fs}_1, \ldots, \mathrm{fs}_k$ and $\mathrm{ufs}_1, \ldots, \mathrm{ufs}_{n-k}$ are the forced and unforced skips of $x$ respectively.

*Proof.* Part (1) has been shown in Fact 13.16. For (2), note that we have $\mathrm{ufs}_1 \cdots \mathrm{ufs}_{n-k}\, \mathrm{fs}_1 \cdots \mathrm{fs}_k = c$ by Corollary 13.19. Rewriting this as

$$\mathrm{fs}_1 \cdots \mathrm{fs}_k\, (c^{-1}\, \mathrm{ufs}_1\, c) \cdots (c^{-1}\, \mathrm{ufs}_{n-k}\, c) = c$$

and using $u = \mathrm{fs}_1 \cdots \mathrm{fs}_k$ yields

$$u^{-1}c = (c^{-1}\, \mathrm{ufs}_1\, c) \cdots (c^{-1}\, \mathrm{ufs}_{n-k}\, c).$$

Thus it remains to show that the reflections $c^{-1}\, \mathrm{ufs}_i\, c$ are the set of cover reflections associated to the noncrossing element $u^{-1}c$.

By Lemma 15.4 applied for $w_{op}$ equal to $x$, the reflections $\mathrm{ufs}_1, \ldots, \mathrm{ufs}_{n-k}$ are the cover reflections of

$$y := \pi_{\uparrow}(x) w_\circ \in \mathrm{Sort}(W, c^{-1}).$$

We know that $c \leqslant_R x \leqslant_R \pi_{\uparrow}(x)$ so $\ell(c^{-1}) + \ell(y) = \ell(c^{-1}y)$, and so $c^{-1}y \in \mathrm{Sort}(W, c^{-1})$, with $c^{-1}$-sorting word obtained by appending $c^{-1}$ onto the front of the $c^{-1}$-sorting word for $y$. This shows that $c^{-1}\, \mathrm{ufs}_1\, c, \ldots, c^{-1}\, \mathrm{ufs}_{n-k}\, c$ are still skip reflections of $c^{-1}y$, and we claim that they are a subset of the cover reflections. Indeed,

$$(c^{-1}\, \mathrm{ufs}_i\, c)(c^{-1}y) = c^{-1}\, \mathrm{ufs}_i\, y,$$

which has smaller length than $\ell(c^{-1}y) = n + \ell(y)$. This is because $\ell(\mathrm{ufs}_i\, y) < \ell(y)$ due to $\mathrm{ufs}_i$ being a cover reflection of $y$.

Finally, since the cover reflections of a noncrossing element are the simple generators relative to the induced simple system $\Delta_{W'}$ for the minimal reflection subgroup $W'$ containing that noncrossing element (see Fact 13.8), we see that $c^{-1}y$ is the Coxeter element for the reflection subgroup $W'$ generated by $\mathrm{Covers}(y)$, and so $u^{-1}c$ is the Coxeter element associated to the standard parabolic subgroup $W''$ of $W'$ with simple generators the reflections $c^{-1}\, \mathrm{ufs}_1\, c, \ldots, c^{-1}\, \mathrm{ufs}_{n-k}\, c$. Therefore $W'' \subset W$ is the minimal reflection subgroup containing the $c$-noncrossing element $u^{-1}c$, establishing that the $c^{-1}\, \mathrm{ufs}_i\, c$ are exactly the cover reflections of $u^{-1}c$. □

**Example 15.6.** Let $W = S_8$ with $c = s_2 s_5 s_1 s_3 s_6 s_7 s_4$. Set $x := c^2$. We have that $c \leqslant_R x$ and that $x$ is $c$–sortable. Let $\leqslant_c$ denote the reflection order coming from the $c$–sorting word of $w_\circ$. The cover reflections, ordered by $\leqslant_c$, are the transpositions $(5\,8),\ (3\,4),\ (2\,5),\ (1\,7)$. Therefore

$$u \;=\; \mathrm{nc}_c(x) \;=\; (5\,8)(3\,4)(2\,5)(1\,7) \;=\; 78432615.$$

In particular $u \in \mathrm{NC}(W, c)^+$.

The Kreweras complement of $u$ is $u^{-1}c = 47365128$, and the unique $c$-sortable $x' \in \mathrm{Sort}(W, c)$ such that $\mathrm{nc}_c(x') = u^{-1}c$ is $x' = 36417258$. Its cover reflections are $\mathrm{Covers}(x') = \{(1\,4), (2\,7), (4\,6)\}$. Ordered by the same reflection order $\leqslant_c$, we have $(1\,4) \leqslant_c (2\,7) \leqslant_c (4\,6)$, and therefore

$$u^{-1}c = (1\,4)(2\,7)(4\,6),$$

which one can check agrees with the conjugated unforced skips of $x$.

15.1. **The Bruhat maximal element.** The proof of Theorem 15.1 follows the next two lemmas.

**Lemma 15.7.** Fix $w \in W$ and let $d = s_1 \cdots s_k$ be the product of $k$ distinct simple reflections.

(1) If $\ell(wd) = \ell(w) + k$ and $w[\mathrm{id}, d] \subset [w, wd]$ then $w[\mathrm{id}, d] = [w, wd]$.
(2) If $\ell(wd) = \ell(w) - k$ and $w[\mathrm{id}, d] \subset [wd, w]$ then $w[\mathrm{id}, d] = [wd, w]$.

*Proof.* (2) is the special case of (1) with $w$ replaced with $ws_k \cdots s_1$. The hypothesis $w[\mathrm{id}, d] \subseteq [w, wd]$ in (1) is equivalent to $[\mathrm{id}, d] \subseteq w^{-1}[w, wd]$. Interpreting these as $T$-fixed point sets of Richardson varieties and applying the rigidity statement (Fact 6.7) gives an inclusion

$$X_{\mathrm{id}}^d \subseteq w^{-1} X_w^{wd}.$$

Both Richardson varieties are irreducible of dimension $\ell(d) = k$, hence they are equal. Taking $T$-fixed points yields the claim. □

**Lemma 15.8.** For $w \in W$ and distinct $s_1, \ldots, s_k \in \mathrm{Des}(w)$, multiplication by $w$ gives an anti-isomorphism from the interval $[\mathrm{id}, s_1 \cdots s_k]$ to $[ws_1 \cdots s_k, w]$.

*Proof.* By [16, Corollary 2.8(i)] we have that if $s_i$ and $s_j$ are distinct simple reflections so that $ws_i <_B w$ and $ws_j <_B w$, then $ws_is_j <_B ws_i$ and $ws_is_j <_B ws_j$. Applying this repeatedly implies that for all $\{i_1 < i_2 < \cdots < i_j\} \subseteq [k]$, we have $ws_1s_2 \cdots s_k <_B ws_{i_1} \cdots s_{i_j} <_B w$. Since by the subword property the interval $[\mathrm{id}, s_1 \cdots s_k]$ contains all elements of the form $s_{i_1} \cdots s_{i_j}$ where $1 \leqslant i_1 < \cdots < i_j \leqslant k$, the claim now follows from Lemma 15.7. □

*Proof of Theorem 15.1.* By Theorem 14.3 we may assume that $w_{op} = w^{-1}w_\circ$ is $c$-sortable, so that $w_{op} = \pi_\downarrow(w_{op})$. Take $w'$ so that $(w')_{op} = \pi_\uparrow(w_{op})$. Then by Theorem 14.3 we have

$$\overleftarrow{[w, wc]} = \overleftarrow{[w', w'c]}.$$

We will prove the following three claims. Let $p_1, \ldots, p_k, q_1, \ldots, q_{n-k}$ be as in Fact 15.3, associated to the specific cover reflection factorizations from Proposition 15.5. Then

(1) $u \in \overleftarrow{[w, wc]} = \overleftarrow{[w', w'c]}$
(2) The elements covered by $wu$ in $[w, wu]$ are $wup_1, \ldots, wup_k$.
(3) The covers of $w'u$ in $[w'u, w'c]$ are the elements $w'uq_1, \ldots, w'uq_{n-k}$.

Suppose that (1)-(3) are established. Then by Theorem 14.3, the adjacent elements of $\overleftarrow{[w, wc]}$ which are adjacent to $u \in \overleftarrow{[w, wc]}$ are

$$\{up_1, \ldots, up_k, uq_1, \ldots, uq_{n-k}\} = \{\tau u \mid \tau \in \mathrm{Inv}_{\mathrm{NC}}(u)\}.$$

In particular, each element of $\overleftarrow{[w, wc]}$ which is adjacent to $u$ precedes it in the Bruhat order. This characterizes the unique Bruhat-maximum element in any Coxeter matroid, and $\overleftarrow{[w, wc]}$ is a Coxeter matroid, so $u$ must be this maximum, completing the proof.

We now show (1). Let $s_1, \dots, s_k \in \mathrm{Des}(w_{op})$ be the descents associated to the forced skips $\mathrm{fs}_1, \dots, \mathrm{fs}_k$ of $w_{op}$, i.e. with $\mathrm{fs}_i\, w_{op} = w_{op}s_i$. Then as $u = \mathrm{fs}_1 \cdots \mathrm{fs}_k$, by Lemma 15.8 applied to $w_{op}$ and the descents $s_k, \dots, s_1$ we have

$$w_{op}[\mathrm{id}, s_k \cdots s_1] = [w_{op}s_k \cdots s_1, w_{op}] = [u^{-1}w_{op}, w_{op}], \tag{15.1}$$

and hence $u^{-1}w_{op} \leqslant w_{op}$. As $c \leqslant_R w_{op}$, we have that $c^{-1}w_{op}$ is $c$-sortable and the conjugates $c^{-1}\,\mathrm{ufs}_1\, c, \dots, c^{-1}\,\mathrm{ufs}_{n-k}\, c$ are an ordered subset of the unforced skips of $c^{-1}w_{op}$. We can therefore apply Proposition 13.17 repeatedly followed by Corollary 13.19 to get

$$c^{-1}w_{op} \lessdot_B (c^{-1}\,\mathrm{ufs}_{n-k}\, c)(c^{-1}w_{op}) \lessdot_B \cdots \lessdot_B \prod_{i=1}^{n-k}(c^{-1}\,\mathrm{ufs}_i\, c)(c^{-1}w_{op}) = u^{-1}w_{op}.$$

This shows that $c^{-1}w_{op} \leqslant u^{-1}w_{op} \leqslant w_{op}$, which after inverting and multiplying by $w_\circ$ on the left shows $w \leqslant wu \leqslant wc$ as desired.

We record for future use that, because we have now established (1), we have the sequence

$$[c^{-1}w'_{op}, u^{-1}w'_{op}] \xrightarrow{op} [w'u, w'c] \xrightarrow{\text{shape}} [wu, wc] \xrightarrow{op} [c^{-1}w_{op}, u^{-1}w_{op}] \tag{15.2}$$

between length $n-k$ Bruhat intervals, where the first and last are anti-equivalences and the second is an equivalence.

Now we show (2). By Proposition 13.17, the elements covering $w_{op}s_k \cdots s_1$ in (15.1) are given by $w_{op}s_k \cdots \hat{s}_i \cdots s_1$, where $\hat{s}_i$ denotes omission, so the elements covering $u^{-1}w_{op}$ in (15.1) are

$$w_{op}s_k \cdots \widehat{s_i} \cdots s_1 = \mathrm{fs}_k \cdots \widehat{\mathrm{fs}_i} \cdots \mathrm{fs}_1\, w_{op} = p_i u^{-1}w_{op}.$$

Inverting and multiplying by $w_\circ$ on the left then shows that the elements covered by $wu$ in this interval are exactly the $wup_i$ for $i = 1, \dots, k$.

Finally, we show (3). The strategy is similar to (2), but with added complications to account for the reversal in order. Lemma 15.4 states that the cover reflections of $w'_{op}w_\circ$ are $\mathrm{ufs}_1, \dots, \mathrm{ufs}_{n-k}$, the unforced skips of $w_{op}$. Let $s'_1, \dots, s'_{n-k}$ be the associated descents of $w'_{op}w_\circ$, so that $(w'_{op}w_\circ)s'_i = \mathrm{ufs}_i(w'_{op}w_\circ)$. Noting that $\mathrm{ufs}_1 \cdots \mathrm{ufs}_{n-k} = cu^{-1}$ by Fact 13.16 and Corollary 13.19, we thus have by Lemma 15.8 that

$$w'_{op}w_\circ[\mathrm{id}, s'_1 \cdots s'_{n-k}] = [w'_{op}w_\circ s'_1 \cdots s'_{n-k}, w'_{op}w_\circ] = [(cu^{-1})w'_{op}w_\circ, w'_{op}w_\circ].$$

Writing $s''_i = w_\circ s'_i w_\circ$, we have $s''_1, \dots, s''_i$ are the ascents of $w'_{op}$ and $w'_{op}s''_i = \mathrm{ufs}_i\, w'_{op}$. Therefore

$$w'_{op}[\mathrm{id}, s''_1 \cdots s''_{n-k}] = [w'_{op}, w'_{op}s''_1 \cdots s''_{n-k}] = [w'_{op}, (cu^{-1})w'_{op}],$$

providing an analogue of (15.1) which we will use to prove (3).

We claim that $c^{-1}w'_{op}[\mathrm{id}, s''_1 \cdots s''_{n-k}] = [c^{-1}w'_{op}, u^{-1}w'_{op}]$ (note the right hand interval is the left-most length $n-k$ Bruhat interval from (15.2)). This will follow from Lemma 15.7 if we can show that $c^{-1}w'_{op}[\mathrm{id}, s''_1 \cdots s''_{n-k}] \subset [c^{-1}w'_{op}, u^{-1}w'_{op}]$, or equivalently

$$\left\{\prod_{j=1}^{\ell}(c^{-1}\,\mathrm{ufs}_{i_j}\,c)(c^{-1}w'_{op}) \mid 1 \leqslant i_1 < i_2 < \cdots < i_\ell \leqslant n-k\right\} \subset [c^{-1}w'_{op}, u^{-1}w'_{op}]$$

Applying the sequence of equivalences and anti-equivalence in (15.2) to this condition, it suffices to show that each $z = \prod_{j=1}^{\ell}(c^{-1}\,\mathrm{ufs}_{i_j}\,c)(c^{-1}w_{op})$ is contained in $[c^{-1}w_{op}, u^{-1}w_{op}]$. To see this, note that by applying Proposition 13.17 repeatedly, a reduced word for $z$ is obtained by filling in the unforced skips of $c^{-1}w_{op}$ corresponding to each $c^{-1}\,\mathrm{ufs}_{i_j}\,c$, so we see that $c^{-1}w_{op}$ is naturally a subword of $z$, and $u^{-1}w_{op} = \prod_{i=1}^{n-k}(c^{-1}\,\mathrm{ufs}_i\,c)(c^{-1}w_{op})$ has a reduced word which contains the reduced word for $z$.

Finally, by Lemma 15.8 the covers of $u^{-1}w'_{op}$ in $[c^{-1}w'_{op}, u^{-1}w'_{op}]$ are the elements

$$\begin{aligned} c^{-1}w'_{op}s''_1 \cdots \widehat{s''_i} \cdots s''_{n-k} &= (c^{-1}\,\mathrm{ufs}_1\,c)\cdots(\widehat{c^{-1}\,\mathrm{ufs}_i}\,c)\cdots(c^{-1}\,\mathrm{ufs}_{n-k}\,c)(c^{-1}w_{op}) \\ &= (q_i u^{-1}c)(c^{-1}w_{op}) \\ &= q_i u^{-1}w'_{op}. \end{aligned}$$

Inverting and multiplying by $w_\circ$ on the left shows that the covers of $w'u$ in $[w'u, w'c]$ are given by the elements $w'uq_1, \ldots, w'uq_{n-k}$ as desired. □

## Appendix A. Type $A$ Examples

A.1. **The type** A **flag variety.** Our choice of $G, B, B^-, T$ for type A that we use in this section is $G = \mathrm{GL}_{n+1}$, $B, B^-$ are upper and lower triangular matrices respectively, and $T = B \cap B^-$ are the diagonal matrices. The complete flag variety $\mathrm{GL}_{n+1}/B$ parametrizes complete flags of subspaces $\{0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n \subsetneq \mathbb{C}^{n+1} \mid \dim V_i = i\}$. A coset $MB$ is determined by $M$ up to invertible forward column operations. To recover the flag from $M$ we take $V_i$ to be the span of the first $i$ columns.

We denote $\epsilon_i \in \mathrm{Char}(T)$ for the $i$th standard character written additively so that $\mathrm{Char}(T) = \bigoplus_{i=1}^{n+1}\mathbb{Z}\epsilon_i$. The Weyl group is the group $S_{n+1} = \langle s_1, \ldots, s_n\rangle$ of permutations on $\{1, \ldots, n+1\}$, where $s_i = (i, i+1)$. The root system is $\Phi = \{\epsilon_i - \epsilon_j \mid i \neq j\}$, and the positive roots are $\Phi^+ = \{\epsilon_i - \epsilon_j \mid i < j\}$. The weight lattice is $\mathbb{Z}^{n+1}$ and the root lattice is $\{(x_1, \ldots, x_{n+1}) \mid \sum x_i = 0\} \subset \mathbb{Z}^{n+1}$.

The fundamental dominant weights are $\omega_i = \epsilon_1 + \cdots + \epsilon_i$, with associated Plücker functions

$$\mathrm{Pl}_w^{\omega_i}(M) = \det M_{w(1),\ldots,w(i)},$$

where $M_A$ is the submatrix of $M$ with rows chosen from $A$ and columns chosen from $1, \ldots, |A|$. Since each $\omega_i$ lies in the character lattice $\mathrm{Char}(T)$, we can take the regular dominant weight $\lambda_{\mathrm{reg}} = \sum \omega_i = (n, n-1, \ldots, 0)$. The associated Plücker functions for $\lambda_{\mathrm{reg}}$ are given by

$$\mathrm{Pl}_w(M) = \prod_{i=1}^{n}\det M_{w(1),\ldots,w(i)}.$$

A.2. **Classical presentation of Schubert cells.** Each permutation in $w \in S_{n+1}$ corresponds to the permutation matrix in $\mathrm{GL}_{n+1}$ with 1's in the entries $(w(i), i)$ and 0 elsewhere. We can represent $BwB$ as a matrix with entries in $\{0, 1, *\}$, where $* \in \mathbb{C}$ is a free entry (an omitted entry is by default zero). To do so, we take the permutation matrix associated to $w$, and replace each 0 which has no 1 either to the left or above it with a $*$.

The 6 charts $\mathring{X}^u = BwB = U_w wB$ for $w \in S_3$ are given by

$$\underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}}_{\mathring{X}^{123}}, \underbrace{\begin{bmatrix} * & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}}_{\mathring{X}^{213}}, \underbrace{\begin{bmatrix} 1 & & \\ & * & 1 \\ & 1 & \end{bmatrix}}_{\mathring{X}^{132}}, \underbrace{\begin{bmatrix} * & * & 1 \\ 1 & & \\ & 1 & \end{bmatrix}}_{\mathring{X}^{231}}, \underbrace{\begin{bmatrix} * & 1 & \\ * & & 1 \\ 1 & & \end{bmatrix}}_{\mathring{X}^{312}}, \underbrace{\begin{bmatrix} * & * & 1 \\ * & 1 & \\ 1 & & \end{bmatrix}}_{\mathring{X}^{321}}.$$

As an example, if we write $\mathring{X}^{321}$ as $\begin{bmatrix} c & a & 1 \\ b & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, then the Plücker functions in this chart are

$$(\mathrm{Pl}_{123}, \mathrm{Pl}_{213}, \mathrm{Pl}_{132}, \mathrm{Pl}_{231}, \mathrm{Pl}_{312}, \mathrm{Pl}_{321}) = (c(c - ab), b(c - ab), -ac, -b, -a, -1).$$

We see that these do not induce an injection on $\mathring{X}^{321}$ – for example if $c = \pm 1, a = 0, b = 0$.

A.3. **Examples of type $A$ Coxeter flag varieties.** For $c = s_2 s_1 = 132$ we have

$$\underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}}_{\mathring{X}^{123}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}}_{\mathring{X}^{213}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} 1 & & \\ & * & 1 \\ & 1 & \end{bmatrix}}_{\mathring{X}^{132}_{\mathrm{NC}}}, \qquad\qquad , \underbrace{\begin{bmatrix} * & 1 & \\ * & & 1 \\ 1 & & \end{bmatrix}}_{\mathring{X}^{312}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & * & 1 \\ 0 & 1 & \\ 1 & & \end{bmatrix}}_{\mathring{X}^{321}_{\mathrm{NC}}}.$$

For $c = s_1 s_2 = 231$ we have

$$\underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}}_{\mathring{X}^{123}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}}_{\mathring{X}^{213}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} 1 & & \\ & * & 1 \\ & 1 & \end{bmatrix}}_{\mathring{X}^{132}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & * & 1 \\ 1 & & \\ & 1 & \end{bmatrix}}_{\mathring{X}^{231}_{\mathrm{NC}}}, \qquad\qquad , \underbrace{\begin{bmatrix} * & 0 & 1 \\ * & 1 & \\ 1 & & \end{bmatrix}}_{\mathring{X}^{321}_{\mathrm{NC}}}.$$

For $c = s_3 s_2 s_1 = 4123$ we have

$$\underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}}_{\mathring{X}^{1234}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & 1 & & \\ 1 & & & \\ & & 1 & \\ & & & 1 \end{bmatrix}}_{\mathring{X}^{2134}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} 1 & & & \\ & * & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}}_{\mathring{X}^{1324}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & * & 1 \\ & & 1 & \end{bmatrix}}_{\mathring{X}^{1243}_{\mathrm{NC}}}$$

$$\underbrace{\begin{bmatrix} * & 1 & & \\ * & & 1 & \\ 1 & & & \\ & & & 1 \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{3124}}, \underbrace{\begin{bmatrix} 1 & & & \\ & * & 1 & \\ & * & & 1 \\ & 1 & & \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{1423}}, \underbrace{\begin{bmatrix} * & 1 & & \\ 1 & & & \\ & & * & 1 \\ & & 1 & \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{2143}}, \underbrace{\begin{bmatrix} * & * & 1 & \\ 0 & 1 & & \\ 1 & & & \\ & & & 1 \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{3214}}, \underbrace{\begin{bmatrix} 1 & & & \\ & * & * & 1 \\ & 0 & 1 & \\ & 1 & & \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{1432}}$$

$$\underbrace{\begin{bmatrix} * & 1 & & \\ * & & 1 & \\ * & & & 1 \\ 1 & & & \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{4123}}, \underbrace{\begin{bmatrix} * & * & 1 & \\ 0 & 1 & & \\ * & & & 1 \\ 1 & & & \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{4213}}, \underbrace{\begin{bmatrix} * & 1 & & \\ * & & * & 1 \\ 0 & & 1 & \\ 1 & & & \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{4132}}, \underbrace{\begin{bmatrix} * & * & * & 1 \\ 0 & 1 & & \\ 0 & & 1 & \\ 1 & & & \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{4231}}, \underbrace{\begin{bmatrix} * & * & 0 & 1 \\ 0 & * & 1 & \\ 0 & 1 & & \\ 1 & & & \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{4321}}$$

For $c = s_2s_1s_3 = 3142$ we have

$$\underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{1234}}, \underbrace{\begin{bmatrix} * & 1 & & \\ 1 & & & \\ & & 1 & \\ & & & 1 \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{2134}}, \underbrace{\begin{bmatrix} 1 & & & \\ & * & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{1324}}, \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & * & 1 \\ & & 1 & \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{1243}}$$

$$\underbrace{\begin{bmatrix} * & 1 & & \\ * & & 1 & \\ 1 & & & \\ & & & 1 \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{3124}}, \underbrace{\begin{bmatrix} 1 & & & \\ & * & 0 & 1 \\ & * & 1 & \\ & 1 & & \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{1432}}, \underbrace{\begin{bmatrix} * & 1 & & \\ 1 & & & \\ & & * & 1 \\ & & 1 & \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{2143}}, \underbrace{\begin{bmatrix} * & * & 1 & \\ 0 & 1 & & \\ 1 & & & \\ & & & 1 \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{3214}}, \underbrace{\begin{bmatrix} 1 & & & \\ & * & * & 1 \\ & 1 & & \\ & & 1 & \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{1342}}$$

$$\underbrace{\begin{bmatrix} * & 1 & & \\ * & & 0 & 1 \\ * & & 1 & \\ 1 & & & \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{4132}}, \underbrace{\begin{bmatrix} * & * & 0 & 1 \\ 0 & 1 & & \\ * & & 1 & \\ 1 & & & \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{4231}}, \underbrace{\begin{bmatrix} * & 1 & & \\ * & & * & 1 \\ 1 & & & \\ & & 1 & \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{3142}}, \underbrace{\begin{bmatrix} * & * & * & 1 \\ 0 & 1 & & \\ 1 & & & \\ & & 1 & \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{3241}}, \underbrace{\begin{bmatrix} * & * & 1 & \\ 0 & * & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{3412}}$$

## APPENDIX B. TYPE $B$ EXAMPLES

B.1. **Type** B **flag variety.** We write the coordinates on $\mathbb{C}^{2n+1}$ in order as $x_{\bar{n}}, \ldots, x_0, \ldots, x_n$, and fix the symmetric form $x_0^2 + \sum x_i x_{\bar{i}}$. In type B we take $G = SO(2n+1)$, $B, B^- \subset G$ the subsets of upper and lower triangular matrices, and $T = B \cap B^- \subset G$ are the diagonal matrices. Recall that a subspace $V \subset \mathbb{C}^{2n+1}$ is isotropic if $\langle v, w \rangle = 0$ for all $v, w \in V$. For Type B the flag variety is

$$SO(2n+1)/B = \{0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n \subsetneq \mathbb{C}^{2n+1} \mid V_i \text{ is isotropic and } \dim V_i = i\}.$$

For $i \in \{-n, \dots, n\}$ we denote $\epsilon_i$ for the *negative* of the standard character associated to $i$, and $\mathrm{Char}(T) = \bigoplus_{i=-n}^{n} \mathbb{Z}\epsilon_i/\langle \epsilon_{-i} + \epsilon_i \rangle$. The root system is $\Phi = \{\pm\epsilon_j \pm \epsilon_i \mid 1 \leqslant i < j \leqslant n\} \cup \{\pm\epsilon_i \mid 1 \leqslant i \leqslant n\}$, and the simple roots are $\alpha_0 = \epsilon_1$ and $\alpha_i = -\epsilon_i + \epsilon_{i+1}$ for $i \in \{1, \dots, n-1\}$. The associated Weyl group is $H_n$, the permutations of $\{\pm 1, \dots, \pm n\}$ with $w(-i) = -w(i)$. These are called the *signed permutations*, and $W$ is called the *hyperoctahedral group*. As is standard we write $\bar{i}$ for $-i$, and we write the generators $s_0 = (1, \bar{1})$ and $s_i = (i, i+1)(\bar{i}, \overline{i+1})$. Every permutation can be written in one-line notation as $w = w(1)w(2)\cdots w(n)$.

The rows and columns of the matrices in $SO(2n+1)$ are labeled in order by $\overline{n}, \dots, 0, \dots, n$. An element of $SO(2n+1)/B$ is determined by its restriction to the first $n$ columns (the ones labeled $\bar{n}, \dots, \bar{1}$), and a $(2n+1) \times n$ matrix represents an element of $SO(2n+1)$ if it has rank $n$, and for column vectors $[a_{\bar{n}}, \dots, a_n]^T$ and $[b_{\bar{n}}, \dots, b_n]^T$ we have $a_0 b_0 + \sum_{i=1}^{n}(a_i b_{\bar{i}} + a_{\bar{i}} b_i) = 0$. Two matrices are equivalent if one can be obtained from the other by forward column operations, and such a matrix is determined by the flag of isotropic subspaces where $V_i$ is the span of the first $i$ columns.

The fundamental dominant weights are

$$\omega_0 = \frac{1}{2}\sum_{i=1}^{n} \epsilon_i \qquad \text{and, for } 1 \leqslant i \leqslant n-1, \qquad \omega_i = \epsilon_{i+1} + \cdots + \epsilon_n.$$

Each of $2\omega_0, \omega_1, \dots, \omega_{n-1}$ lie in the character lattice $\mathrm{Char}(T)$, with associated Plücker coordinates

$$\det M_{\overline{w(n)}, \dots, \overline{w(i+1)}}, \tag{B.1}$$

where $M_A$ is the submatrix consisting of the first $|A|$ columns (indexed by $\overline{n}, \dots, \overline{n - |A| + 1}$) and the rows indexed by $A$. Therefore if we take the regular dominant weight $\lambda_{reg} = 2\omega_0 + \omega_1 + \cdots + \omega_{n-1}$, the associated Plücker function is

$$\mathrm{Pl}_w = \prod_{i=1}^{n} \det M_{\overline{w(n)}, \dots, \overline{w(i+1)}}. \tag{B.2}$$

B.2. **Classical presentation of Schubert cells.** We learned the following presentation from [1]. The chart $BwB$ can be written as an $(2n+1) \times n$ matrix containing $0, 1, *, \otimes$, where $0$ is often omitted from the notation, $* \in \mathbb{C}$ is a free variable, and $\otimes$ are variables which are the unique polynomials in the $*$ that ensure isotropy between the column they are in and previous columns. Concretely, we put $1$ in the entry $(w(i), i)$ for $i \in \{-n, \dots, -1\}$, and then we put $*$ in all entries which are not below and not to the right of a 1, and then finally convert all $*$ to $\otimes$ which are *weakly* to the right of $(\overline{w(i)}, i)$ for any $i \in \{-n, \dots, -1\}$. For $n = 2$ the $8$ charts $M(w) = BwB$ for $w \in H_2$, with rows labeled by $\bar{2}, \bar{1}, 1, 2$ and columns labeled by $\bar{2}, \bar{1}$ are given by

$$\underbrace{\begin{bmatrix} 1 & \\ & 1 \\ & \\ & \\ & \end{bmatrix}}_{\mathring{X}^{12}}, \underbrace{\begin{bmatrix} 1 & \\ & \otimes \\ & * \\ & 1 \\ & \end{bmatrix}}_{\mathring{X}^{\bar{1}2}}, \underbrace{\begin{bmatrix} * & 1 \\ 1 & \\ & \\ & \\ & \end{bmatrix}}_{\mathring{X}^{21}}, \underbrace{\begin{bmatrix} * & 1 \\ \otimes & \\ * & \\ 1 & \\ & \end{bmatrix}}_{\mathring{X}^{2\bar{1}}}, \underbrace{\begin{bmatrix} * & \otimes \\ 1 & \\ & * \\ & \otimes \\ & 1 \end{bmatrix}}_{\mathring{X}^{\bar{2}1}}, \underbrace{\begin{bmatrix} \otimes & \otimes \\ * & 1 \\ * & \\ * & \\ 1 & \end{bmatrix}}_{\mathring{X}^{1\bar{2}}}, \underbrace{\begin{bmatrix} * & \otimes \\ \otimes & \otimes \\ * & * \\ 1 & \\ & 1 \end{bmatrix}}_{\mathring{X}^{\bar{2}\bar{1}}}, \underbrace{\begin{bmatrix} \otimes & \otimes \\ * & \otimes \\ * & * \\ * & 1 \\ 1 & \end{bmatrix}}_{\mathring{X}^{\bar{1}\bar{2}}}.$$

B.3. **Examples of type $B$ Coxeter flag varieties.** For the Coxeter $s_1 s_0 = \bar{2}1$ we have the charts

$$\underbrace{\begin{bmatrix} 1 & \\ & 1 \\ & \\ & \\ & \end{bmatrix}}_{\mathring{X}^{12}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} 1 & \\ & \otimes \\ & * \\ & 1 \\ & \end{bmatrix}}_{\mathring{X}^{\bar{1}2}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & 1 \\ 1 & \\ & \\ & \\ & \end{bmatrix}}_{\mathring{X}^{21}_{\mathrm{NC}}}, \qquad , \underbrace{\begin{bmatrix} * & \otimes \\ 1 & \\ & * \\ & \otimes \\ & 1 \end{bmatrix}}_{\mathring{X}^{\bar{2}1}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} \otimes & \otimes \\ * & 1 \\ * & \\ 0 & \\ 1 & \end{bmatrix}}_{\mathring{X}^{1\bar{2}}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & \otimes \\ \otimes & \otimes \\ * & 0 \\ 1 & \\ & 1 \end{bmatrix}}_{\mathring{X}^{\bar{2}\bar{1}}_{\mathrm{NC}}}, \qquad .$$

For the Coxeter $s_2 s_1 s_0 = \bar{3}12$ we have the $20$ charts

$$\underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \\ & & \\ & & \\ & & \\ & & \end{bmatrix}}_{\mathring{X}^{123}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} 1 & & \\ & * & 1 \\ & 1 & \\ & & \\ & & \\ & & \\ & & \end{bmatrix}}_{\mathring{X}^{213}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & 1 & \\ 1 & & \\ & & 1 \\ & & \\ & & \\ & & \\ & & \end{bmatrix}}_{\mathring{X}^{132}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & 0 & 1 \\ * & 1 & \\ 1 & & \\ & & \\ & & \\ & & \\ & & \end{bmatrix}}_{\mathring{X}^{321}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & & \otimes \\ & & * \\ & & 1 \\ & & \\ & & \end{bmatrix}}_{\mathring{X}^{\bar{1}23}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} 1 & & \\ & \otimes & \otimes \\ & * & 1 \\ & * & \\ & 0 & \\ & 1 & \\ & & \end{bmatrix}}_{\mathring{X}^{1\bar{2}3}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} \otimes & & \\ * & 1 & \\ * & & 1 \\ * & & \\ 0 & & \\ 0 & & \\ 1 & & \end{bmatrix}}_{\mathring{X}^{12\bar{3}}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} 1 & & \\ & * & \otimes \\ & \otimes & \otimes \\ & * & 0 \\ & 1 & \\ & & 1 \\ & & \end{bmatrix}}_{\mathring{X}^{\bar{2}\bar{1}3}_{\mathrm{NC}}}$$

$$\underbrace{\begin{bmatrix} * & \otimes & \otimes \\ \otimes & \otimes & \otimes \\ 0 & * & 1 \\ 0 & 0 & \\ 0 & 0 & \\ 1 & & \\ & 1 & \end{bmatrix}}_{\mathring{X}^{1\bar{3}\bar{2}}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & 0 & \otimes \\ * & 1 & \\ \otimes & & \otimes \\ * & & 0 \\ 1 & & \\ & & \otimes \\ & & 1 \end{bmatrix}}_{\mathring{X}^{\bar{3}2\bar{1}}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & 0 & \otimes \\ \otimes & \otimes & \otimes \\ 0 & 1 & \\ * & & 0 \\ * & & \otimes \\ 1 & & \\ & & 1 \end{bmatrix}}_{\mathring{X}^{\bar{3}1\bar{2}}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & * & 1 \\ 1 & & \\ & 1 & \\ & & \\ & & \\ & & \\ & & \end{bmatrix}}_{\mathring{X}^{312}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & * & \otimes \\ 1 & & \\ & \otimes & \otimes \\ & * & 0 \\ & 1 & \\ & & \otimes \\ & & 1 \end{bmatrix}}_{\mathring{X}^{\bar{3}\bar{1}2}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & 1 & \\ 1 & & \\ & & \otimes \\ & & * \\ & & 1 \\ & & \\ & & \end{bmatrix}}_{\mathring{X}^{\bar{1}32}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & 0 & \otimes \\ * & \otimes & \otimes \\ \otimes & \otimes & \otimes \\ 0 & * & 0 \\ 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}}_{\mathring{X}^{\bar{3}\bar{2}\bar{1}}_{\mathrm{NC}}},$$

$$\underbrace{\begin{bmatrix} \otimes & \otimes & \otimes \\ * & * & 1 \\ * & 1 & \\ * & & \\ 0 & & \\ 0 & & \\ 1 & & \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{21\bar{3}}}, \underbrace{\begin{bmatrix} * & \otimes & \otimes \\ 1 & & \\ & * & 1 \\ & * & \\ & 0 & \\ & \otimes & \\ & 1 & \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{1\bar{3}2}}, \underbrace{\begin{bmatrix} * & 0 & \otimes \\ * & 1 & \\ 1 & & \\ & & * \\ & & \otimes \\ & & \otimes \\ & & 1 \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{\bar{3}21}}, \underbrace{\begin{bmatrix} 1 & & \\ & * & \otimes \\ & 1 & \\ & & * \\ & & \otimes \\ & & 1 \\ & & \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{\bar{2}13}}, \underbrace{\begin{bmatrix} * & * & \otimes \\ 1 & & \\ & 1 & \\ & & * \\ & & \otimes \\ & & \otimes \\ & & 1 \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{\bar{3}12}}.$$

## APPENDIX C. TYPE $C$ EXAMPLES

C.1. **The type** C **flag variety.** We write the coordinates on $\mathbb{C}^{2n}$ in order as $x_{\bar{n}}, \dots, x_{\bar{1}}, x_1, \dots, x_n$, and fix the symplectic form $\sum dx_i \wedge dx_{\bar{i}}$. In type C we take $G = Sp(2n)$, $B, B^- \subset G$ the subsets of upper and lower triangular matrices, and $T = B \cap B^- \subset G$ are the diagonal matrices.

Recall that a subspace $V \subset \mathbb{C}^{2n}$ is isotropic if $\langle v, w \rangle = 0$ for all $v, w \in V$. For type C the flag variety is

$$Sp(2n)/B = \{0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n \subsetneq \mathbb{C}^{2n} \mid V_i \text{ is isotropic and } \dim V_i = i\}.$$

For $i \in \{-n, \dots, n\}$ we denote $\epsilon_i$ for the *negative* of the standard character associated to $i$, and $\mathrm{Char}(T) = \bigoplus_{i=-n}^{n} \mathbb{Z}\epsilon_i/(\epsilon_{-i} + \epsilon_i)$. The root system is $\Phi = \{\pm\epsilon_i - \pm\epsilon_j \mid 1 \leqslant i < j \leqslant n\} \cup \{\pm 2\epsilon_i \mid 1 \leqslant i \leqslant n\}$, and the simple roots are $\alpha_0 = 2\epsilon_0$ and $\alpha_i = -\epsilon_i + \epsilon_{i+1}$ for $i \in \{2, \dots, n-1\}$. The Weyl group is $H_n$ like for type B.

The rows and columns of the matrices in $Sp_{2n}$ are labeled in order $\bar{n}, \dots, \bar{1}, 1, \dots, n$. An element of $Sp_{2n}/B$ is determined by its restriction to the first $n$ columns, and a $2n \times n$ matrix represents an element of $Sp_{2n}$ if it is rank $n$, and for column vectors $[a_{\bar{n}}, \dots, a_{\bar{1}}, a_1, \dots, a_n]^T$ and $[b_{\bar{n}}, \dots, b_{\bar{1}}, b_1, \dots, b_n]^T$ we have $\sum_{j=1}^{n} a_j b_{\bar{j}} = \sum_{j=1}^{n} a_{\bar{j}} b_j$. Two matrices are equivalent if one can be obtained from the other by forward column operations, and such a matrix is determined by the flag of isotropic subspaces where $V_i$ is the span of the first $i$ columns. The fundamental weights are given by $\omega_i = \epsilon_{i+1} + \cdots + \epsilon_n$ for each $0 \leqslant i \leqslant n-1$, so the fundamental Plücker coordinates are given by the determinantal expressions in (B.1) for all $i$, and with $\lambda_{\mathrm{reg}} = \sum_{i=0}^{n-1} \omega_i$, the $\lambda_{\mathrm{reg}}$-Plücker coordinate is given by (B.2).

C.2. **Classical presentation of Schubert cells.** We learned the following presentation from [1]. The chart $BwB$ can be written as an $(2n) \times n$ matrix containing $0, 1, *, \otimes$, where $0$ is often omitted from the notation, $* \in \mathbb{C}$ is a free variable, and $\otimes$ are variables which are the unique polynomials in the $*$ that ensure isotropy between the columns. Concretely, we put $1$ in the entry $(w(i), i)$ for $i \in \{-n, \dots, -1\}$, and then we put $*$ in all entries which are not below and not to the right of a $1$, and then finally convert all $*$ to $\otimes$ which are *strictly* to the right of $(\overline{w(i)}, i)$ for any $i \in \{-n, \dots, -1\}$.

For $n = 2$ the 8 charts $M(w) = BwB$ for $w \in H_2$, with rows labeled by $\bar{2}, \bar{1}, 1, 2$ and columns labeled by $\bar{2}, \bar{1}$ are given by

$$\underbrace{\begin{bmatrix} 1 & \\ & 1 \\ & \\ & \end{bmatrix}}_{\mathring{X}^{12}}, \underbrace{\begin{bmatrix} 1 & \\ & * \\ & 1 \\ & \end{bmatrix}}_{\mathring{X}^{\bar{1}2}}, \underbrace{\begin{bmatrix} * & 1 \\ 1 & \\ & \\ & \end{bmatrix}}_{\mathring{X}^{21}}, \underbrace{\begin{bmatrix} * & 1 \\ * & \\ 1 & \\ & \end{bmatrix}}_{\mathring{X}^{2\bar{1}}}, \underbrace{\begin{bmatrix} * & * \\ 1 & \\ & \otimes \\ & 1 \end{bmatrix}}_{\mathring{X}^{\bar{2}1}}, \underbrace{\begin{bmatrix} * & \otimes \\ * & 1 \\ * & \\ 1 & \end{bmatrix}}_{\mathring{X}^{1\bar{2}}}, \underbrace{\begin{bmatrix} * & * \\ * & \otimes \\ 1 & \\ & 1 \end{bmatrix}}_{\mathring{X}^{\bar{2}\bar{1}}}, \underbrace{\begin{bmatrix} * & * \\ * & \otimes \\ * & 1 \\ 1 & \end{bmatrix}}_{\mathring{X}^{\bar{1}\bar{2}}}.$$

For the Coxeter $s_1 s_0 = \bar{2}1$ we have the charts

$$\underbrace{\begin{bmatrix} 1 & \\ & 1 \\ & \\ & \end{bmatrix}}_{\mathring{X}^{12}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} 1 & \\ & * \\ & 1 \\ & \end{bmatrix}}_{\mathring{X}^{\bar{1}2}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & 1 \\ 1 & \\ & \\ & \end{bmatrix}}_{\mathring{X}^{21}_{\mathrm{NC}}}, \qquad , \underbrace{\begin{bmatrix} * & * \\ 1 & \\ & \otimes \\ & 1 \end{bmatrix}}_{\mathring{X}^{\bar{2}1}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & \otimes \\ * & 1 \\ 0 & \\ 1 & \end{bmatrix}}_{\mathring{X}^{1\bar{2}}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & \\ * & \otimes \\ 1 & \\ & 1 \end{bmatrix}}_{\mathring{X}^{\bar{2}\bar{1}}_{\mathrm{NC}}}, \qquad .$$

For the Coxeter $s_2 s_1 s_0$ we have the 20 charts

$$\underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \\ & & \\ & & \\ & & \end{bmatrix}}_{\mathring{X}^{123}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} 1 & & \\ & * & 1 \\ & 1 & \\ & & \\ & & \\ & & \end{bmatrix}}_{\mathring{X}^{213}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & 1 & \\ 1 & & \\ & & 1 \\ & & \\ & & \\ & & \end{bmatrix}}_{\mathring{X}^{132}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & 0 & 1 \\ * & 1 & \\ 1 & & \\ & & \\ & & \\ & & \end{bmatrix}}_{\mathring{X}^{321}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & & * \\ & & 1 \\ & & \\ & & \end{bmatrix}}_{\mathring{X}^{\bar{1}23}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} 1 & & \\ & * & \otimes \\ & * & 1 \\ & 0 & \\ & 1 & \\ & & \end{bmatrix}}_{\mathring{X}^{1\bar{2}3}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & & \\ * & 1 & \\ * & & 1 \\ 0 & & \\ 0 & & \\ 1 & & \end{bmatrix}}_{\mathring{X}^{12\bar{3}}_{\mathrm{NC}}},$$

$$\underbrace{\begin{bmatrix} 1 & & \\ & * & 0 \\ & * & \otimes \\ & 1 & \\ & & 1 \\ & & \end{bmatrix}}_{\mathring{X}^{\bar{2}\bar{1}3}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & 0 & \otimes \\ 0 & \otimes & \otimes \\ 0 & * & 1 \\ 0 & 0 & \\ 1 & & \\ & 1 & \end{bmatrix}}_{\mathring{X}^{1\bar{3}\bar{2}}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & 0 & 0 \\ * & 1 & \\ * & & \otimes \\ 1 & & \\ & & \otimes \\ & & 1 \end{bmatrix}}_{\mathring{X}^{\bar{3}\bar{2}\bar{1}}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & 0 & 0 \\ * & \otimes & \otimes \\ 0 & 1 & \\ * & & \otimes \\ 1 & & \\ & & 1 \end{bmatrix}}_{\mathring{X}^{\bar{3}1\bar{2}}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & * & 1 \\ 1 & & \\ & 1 & \\ & & \\ & & \\ & & \end{bmatrix}}_{\mathring{X}^{312}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & * & 0 \\ 1 & & \\ & * & \otimes \\ & 1 & \\ & & \otimes \\ & & 1 \end{bmatrix}}_{\mathring{X}^{\bar{3}12}_{\mathrm{NC}}},$$

$$\underbrace{\begin{bmatrix} * & 1 & \\ 1 & & \\ & & * \\ & & 1 \\ & & \\ & & \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{\bar{1}32}}, \underbrace{\begin{bmatrix} * & 0 & 0 \\ * & * & \otimes \\ 0 & \otimes & \otimes \\ 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{\bar{3}\bar{2}\bar{1}}}, \underbrace{\begin{bmatrix} * & \otimes & \otimes \\ * & * & 1 \\ * & 1 & \\ 0 & & \\ 0 & & \\ 1 & & \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{21\bar{3}}}, \underbrace{\begin{bmatrix} * & * & \otimes \\ 1 & & \\ & * & 1 \\ & 0 & \\ & \otimes & \\ & 1 & \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{1\bar{3}2}}, \underbrace{\begin{bmatrix} * & 0 & * \\ * & 1 & \\ 1 & & \\ & & \otimes \\ & & \otimes \\ & & 1 \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{\bar{3}21}}, \underbrace{\begin{bmatrix} 1 & & \\ & * & * \\ & 1 & \\ & & \otimes \\ & & 1 \\ & & \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{\bar{2}13}}, \underbrace{\begin{bmatrix} * & * & * \\ 1 & & \\ & 1 & \\ & & \otimes \\ & & \otimes \\ & & 1 \end{bmatrix}}_{\mathring{X}_{\mathrm{NC}}^{\bar{3}12}}.$$

## Appendix D. Type $D$ Examples

D.1. **The type** D **flag variety.** We write the coordinates on $\mathbb{C}^{2n}$ in order as $x_{\bar{n}}, \ldots, x_{\bar{1}}, x_1, \ldots, x_n$, and fix the symmetric form $\sum x_i x_{\bar{i}}$. In type D we take $G = SO(2n)$, $B, B^-$ the subsets of upper and lower triangular matrices, and $T = B \cap B^- \subset G$ are the diagonal matrices.

Recall that a subspace $V \subset \mathbb{C}^{2n}$ is isotropic if $\langle v, w\rangle = 0$ for all $v, w \in V$. For type D the flag variety is

$$SO(2n)/B = \{0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n \subsetneq \mathbb{C}^{2n} \mid V_i \text{ is isotropic and } \dim V_i = i\}.$$

For $i \in \{-n, \ldots, n\}$ we denote $\epsilon_i$ for the *negative* of the standard character associated to $i$, and $\mathrm{Char}(T)$ is the index 2 sublattice in the type $\mathrm{B}_n$ consisting of elements with even coordinate sum lattice. The root system is $\Phi = \{\pm\epsilon_i - \pm\epsilon_j \mid 1 \leqslant i < j \leqslant n\}$, and the simple roots are $\alpha_{\hat{1}} = \epsilon_1 + \epsilon_2$ and $\alpha_i = -\epsilon_i + \epsilon_{i+1}$ for $i \in \{1, \ldots, n-1\}$. The associated Weyl group is $H_n^{even} = \langle s_{\hat{1}}, s_1, \ldots, s_{n-1}\rangle$, where $s_{\hat{1}} = s_0 s_1 s_0$. This is the set of permutations $w \in H_n$ with $\#\{w(1), \ldots, w(n)\} \cap \{1, \ldots, n\}$ even.

The rows and columns of the matrices in $SO(2n)$ are labeled in order by $\overline{n}, \ldots, \overline{1}, 1, \ldots, n$. An element of $SO(2n)/B$ is determined by its restriction to the first $n$ columns (the ones labeled $\bar{n}, \ldots, \bar{1}$), and a $2n \times n$ matrix represents an element of $SO(2n)$ if it has rank $n$, and for column vectors $[a_{\bar{n}}, \ldots, a_n]^T$ and $[b_{\bar{n}}, \ldots, b_n]^T$ we have $\sum_{i=1}^n (a_i b_{\bar{i}} + a_{\bar{i}} b_i) = 0$. Two matrices are equivalent if one can be obtained from the other by forward column operations, and such a matrix is determined by the flag of isotropic subspaces where $V_i$ is the span of the first $i$ columns.

The fundamental dominant weights are

$$\begin{aligned} \omega_{\hat{1}} &= \tfrac{1}{2}(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n), \\ \omega_1 &= \tfrac{1}{2}(-\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n), \end{aligned} \qquad \text{and, for } 2 \leqslant i \leqslant n-1, \qquad \omega_i = \epsilon_{i+1} + \cdots + \epsilon_n$$

Each of $\omega_1 + \omega_{\hat{1}}, \omega_2, \ldots, \omega_{n-1}$ lie in the character lattice $\mathrm{Char}(T)$, with associated Plücker coordinates given by the determinantal expressions in (B.1) for $i = 1, 2, \ldots, n-1$, and with $\lambda_{\mathrm{reg}} = (\omega_1 + \omega_{\hat{1}}) + \sum_{i=2}^{n-1} \omega_i$, the $\lambda_{\mathrm{reg}}$-Plücker coordinate is given by

$$\mathrm{Pl}_w = \prod_{i=1}^{n-1} \det M_{\overline{w(n)}, \ldots, \overline{w(i+1)}}.$$

D.2. **Classical presentation of Schubert cells.** We adapt the presentations in types $B$ and $C$ we learned from [1] to type $D$. The chart $BwB$ can be written as an $2n \times n$ matrix containing $0, 1, *, \otimes$, where $0$ is often omitted from the notation, $* \in \mathbb{C}$ is a free variable, and $\otimes$ are variables which are the unique polynomials in the $*$ that ensure isotropy between the column they are in and previous columns. Concretely, we put $1$ in the entry $(w(i), i)$ for $i \in \{-n, \dots, -1\}$, and then we put $*$ in all entries which are not below and not to the right of a $1$, and then finally convert all $*$ to $\otimes$ which are *weakly* to the right of $(\overline{w(i)}, i)$ for any $i \in \{-n, \dots, -1\}$. For $n = 2$ the $4$ charts $M(w) = BwB$ for $w \in H_2^{even}$, with rows labeled by $\bar{2}, \bar{1}, 0, 1, 2$ and columns labeled by $\bar{2}, \bar{1}$ are given by

$$\underbrace{\begin{bmatrix} 1 & \\ & 1 \\ & \\ & \end{bmatrix}}_{\mathring{X}^{12}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & 1 \\ 1 & \\ & \\ & \end{bmatrix}}_{\mathring{X}^{21}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & \otimes \\ \otimes & \otimes \\ 1 & \\ & 1 \end{bmatrix}}_{\mathring{X}^{\bar{2}\bar{1}}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} \otimes & \otimes \\ * & \otimes \\ * & 1 \\ 1 & \end{bmatrix}}_{\mathring{X}^{\bar{1}\bar{2}}_{\mathrm{NC}}}.$$

For $c = s_2 s_1 s_{\hat{1}} = \bar{1}\bar{3}2$ we have

$$\underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \\ & & \\ & & \\ & & \end{bmatrix}}_{\mathring{X}^{123}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} 1 & & \\ & * & 1 \\ & 1 & \\ & & \\ & & \\ & & \end{bmatrix}}_{\mathring{X}^{213}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & 1 & \\ 1 & & \\ & & 1 \\ & & \\ & & \\ & & \end{bmatrix}}_{\mathring{X}^{132}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & 0 & 1 \\ * & 1 & \\ 1 & & \\ & & \\ & & \\ & & \end{bmatrix}}_{\mathring{X}^{321}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} 1 & & \\ & * & \otimes \\ & \otimes & \otimes \\ & 1 & \\ & & 1 \\ & & \end{bmatrix}}_{\mathring{X}^{\bar{2}\bar{1}3}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & \otimes & \otimes \\ \otimes & \otimes & \otimes \\ * & 0 & 1 \\ * & 0 & \\ 1 & & \\ & 1 & \end{bmatrix}}_{\mathring{X}^{1\bar{3}\bar{2}}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & 0 & \otimes \\ * & 1 & \\ \otimes & & \otimes \\ 1 & & \\ & & * \\ & & 1 \end{bmatrix}}_{\mathring{X}^{\bar{3}2\bar{1}}_{\mathrm{NC}}}$$

$$\underbrace{\begin{bmatrix} * & * & \otimes \\ 1 & & \\ & \otimes & \otimes \\ & 1 & \\ & & \otimes \\ & & 1 \end{bmatrix}}_{\mathring{X}^{\bar{3}\bar{1}2}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} \otimes & \otimes & \otimes \\ * & 1 & \\ * & & \otimes \\ * & & 1 \\ 0 & & \\ 1 & & \end{bmatrix}}_{\mathring{X}^{\bar{1}2\bar{3}}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & \otimes & \otimes \\ * & * & 1 \\ \otimes & \otimes & \\ 1 & & \\ & 0 & \\ & 1 & \end{bmatrix}}_{\mathring{X}^{\bar{2}\bar{3}\bar{1}}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & * & 1 \\ 1 & & \\ & 1 & \\ & & \\ & & \\ & & \end{bmatrix}}_{\mathring{X}^{312}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} 1 & & \\ & \otimes & \otimes \\ & * & \otimes \\ & * & 1 \\ & 1 & \\ & & \end{bmatrix}}_{\mathring{X}^{\bar{1}\bar{2}3}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & \otimes & \otimes \\ * & * & \otimes \\ 1 & & \\ & \otimes & \otimes \\ & 0 & 1 \\ & 1 & \end{bmatrix}}_{\mathring{X}^{\bar{2}31}_{\mathrm{NC}}}, \underbrace{\begin{bmatrix} * & \otimes & \otimes \\ 1 & & \\ & * & \otimes \\ & * & 1 \\ & \otimes & \\ & 1 & \end{bmatrix}}_{\mathring{X}^{\bar{1}32}_{\mathrm{NC}}}$$

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