

THE QUASISYMMETRIC GRASSMANNIAN

NANTEL BERGERON, LUCAS GAGNON, HUNTER SPINK, AND VASU TEWARI

ABSTRACT. We construct a complex of toric varieties we call the quasisymmetric Grassmannian inside the Grassmannian of r -planes in $\mathbb{C}^n$. Each irreducible component is a positroid variety and an S_n translate of a toric Richardson variety of ribbon shape. We describe it as the vanishing locus of equations $\Delta_A \Delta_{A'} = 0$ in Plücker coordinates determined by a new noncrossing combinatorial object we call the quasisymmetric Johnson graph. We give an affine paving, and show that its cohomology ring is a quasisymmetric modification of the Borel presentation of the Grassmannian's cohomology, with fundamental quasisymmetric polynomials playing the role of Schur polynomials.

1. INTRODUCTION

There is a rich connection between symmetric polynomials and the Grassmanian manifold $\text{Gr}(r; n)$ of r -planes in $\mathbb{C}^n$. Let $\text{Part}_{r,n}$ be the set of integer partitions contained in an $r \times (n - r)$ rectangle. Every $\text{Gr}(r; n)$ is stratified by Schubert cells $\{\check{X}^\lambda \mid \lambda \in \text{Part}_{r,n}\}$, and the Kronecker dual of the homology basis of Schubert cycles $X^\lambda := \overline{\check{X}^\lambda}$ is the cohomology basis of Schur symmetric polynomials $s_\lambda(x_1, \dots, x_r) \in H^\bullet(\text{Gr}(r; n))$ in the negative Chern roots $x_1, \dots, x_r$ of the tautological subbundle. This fact is intimately tied to two equivalent and classical cohomology presentations:

$$H^\bullet(\text{Gr}(r; n)) \cong \frac{\text{Sym}_r}{\langle s_\lambda(x_1, \dots, x_r) \mid \lambda \notin \text{Part}_{r,n} \rangle} \cong \frac{\text{Sym}_r \otimes \text{Sym}_{n-r}}{\langle f - f(0) \mid f \in \text{Sym}_n \rangle}$$

where Sym_k is the ring of symmetric polynomials in k variables. In fact, the ideal in the first presentation has a free $\mathbb{Z}$ -basis $\{s_\lambda(x_1, \dots, x_r) \mid \lambda \notin \text{Part}_{r,n}\}$.

Our aim is to give a parallel construction adapted to the quasisymmetric polynomials, the *quasisymmetric Grassmannian*. This will be an equidimensional algebraic variety $\text{QGr}(r; n) \subset \text{Gr}(r; n)$ composed of $\binom{n-2}{r-1}$ many $(n - 1)$ -dimensional torus-orbit closures.

For an integer composition $\alpha = (\alpha_1, \dots, \alpha_\ell)$ of m , the *fundamental quasisymmetric polynomial* is

$$F_\alpha(x_1, \dots, x_r) = \sum_{\substack{1 \leq i_1 \leq \dots \leq i_m \leq r \\ i_k < i_{k+1} \text{ if } k \in \text{Des}(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_m} \quad \text{where } \text{Des}(\alpha) = \{\sum_{i=1}^k \alpha_i \mid 1 \leq k < \ell\}.$$

Stanley [32] identified a natural decomposition of Schur polynomials into fundamental quasisymmetric polynomials, and Gessel [14] observed that the $\{F_\alpha(x_1, \dots, x_r) \mid \ell(\alpha) \leq r\}$ span a ring

NB was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC) and York Research Chair in Applied Algebra. HS and VT acknowledge the support of the NSERC, respectively [RGPIN-2024-04181] and [RGPIN-2024-05433].

containing Sym_r , the *ring of quasisymmetric polynomials* QSym_r . In order to state the main result, let $\text{Comp}_{r,n}$ be the set of compositions whose ribbon diagram is contained in an $r \times (n-r)$ rectangle.

Theorem A (Theorems 8.2 and 8.7). The variety $\text{QGr}(r; n) \subseteq \text{Gr}(r; n)$ has

$$H^\bullet(\text{QGr}(r; n)) \cong \frac{\text{QSym}_r}{\langle F_\alpha(x_1, \dots, x_r) \mid \alpha \notin \text{Comp}_{r,n} \rangle} \cong \frac{\text{QSym}_r \otimes \text{QSym}_{n-r}}{\langle f - f(0) \mid f \in \text{QSym}_n \rangle}.$$

In fact, the ideal in the first presentation has a free $\mathbb{Z}$ -basis $\{F_\alpha(x_1, \dots, x_r) \mid \alpha \notin \text{Comp}_{r,n}\}$.

The definition of $\text{QGr}(r; n)$ follows naturally from the authors' work with P. Nadeau on the *quasisymmetric flag variety* QFl_n [6], an equidimensional toric complex inside the complete flag variety Fl_n . The cohomology compares with that of the flag variety as follows.

Borel's theorem [10] states: $H^\bullet(\text{Fl}_n) \cong \mathbb{Z}[x_1, \dots, x_n] / \langle f - f(0) \mid f \in \text{Sym}_n \rangle$, and

[6, Theorem A] states: $H^\bullet(\text{QFl}_n) \cong \mathbb{Z}[x_1, \dots, x_n] / \langle f - f(0) \mid f \in \text{QSym}_n \rangle$.

This geometrically realizes the “quasisymmetric coinvariants” first studied in [3]. In contrast, the quasisymmetric Grassmannian gives a direct cohomological interpretation of QSym_r .

Definition 1.1. The Quasisymmetric Grassmannian $\text{QGr}(r; n)$ is the projection $\pi(\text{QFl}_n) \subseteq \text{Gr}(r; n)$ under the natural projection map $\pi : \text{Fl}_n \rightarrow \text{Gr}(r; n)$.

Morally, one can view elements of $H^\bullet(\text{QGr}(r; n))$ as *quasisymmetric polynomials* in the Chern roots of the tautological subbundle on $\text{QGr}(r; n)$. As the containment $\text{Sym}_r \subset \text{QSym}_r$ is proper for $r \geq 2$, these classes often do not come from the pullback $H^\bullet(\text{Gr}(r; n)) \rightarrow H^\bullet(\text{QGr}(r; n))$.

Quasisymmetry is a specialization of a stronger condition called *equivariant quasisymmetry* on polynomials in two sets of variables $f(x_1, \dots, x_r; t_1, \dots, t_n)$ introduced by us and P. Nadeau [5]. The incidence structure of T -invariant curves in $\text{QGr}(r; n)$ imposes constraints on its T -equivariant cohomology via GKM theory [15], which corresponds to equivariant quasisymmetry.

Our construction substantially differs from the infinite-dimensional James space $\Omega\Sigma\mathbb{P}^\infty$ of Baker–Richter [4], Oesinghaus [26], and Pechenik–Satriano [27, 28], which realizes its cohomology ring as quasisymmetric power series. The primary difference is that the James space construction has a natural cohomological basis corresponding to the *monomial quasisymmetric functions* $M_\alpha = \sum_{i_1 < \dots < i_\ell} x_{i_1}^{\alpha_1} \cdots x_{i_\ell}^{\alpha_\ell}$. Our construction of $\text{QGr}(r; n) \subset \text{Gr}(r; n)$ allows us to leverage the relationship between Schur and fundamental quasisymmetric polynomials; see Section 1.2.

Question 1.2. Is there a geometric relationship between the constructions $\text{QGr}(r; n)$ and $\Omega\Sigma\mathbb{P}^\infty$?

1.1. Structure and characterizations. Beyond Definition 1.1, we give two intrinsic characterizations of $\text{QGr}(r; n)$. Each exploits the fact that the incidences of T -invariant curves in $\text{Gr}(r; n)$ are encoded by the *Johnson graph* $J_{r,n}$ on r -element subsets of $\{1, \dots, n\}$, with edges given by single-element swaps. In Section 3 we define a spanning subgraph, the *quasisymmetric Johnson graph*

$QJ_{r,n}$ encoding the incidences of curves in $\text{QGr}(r; n)$ by removing all edges that violate a certain non-crossing condition; see Figure 4 or the 1-skeleton of Figure 1 for an example.

Our first characterization of $\text{QGr}(r; n)$ makes use of the $\binom{n}{r}$ -many Plücker coordinates on $\text{Gr}(r; n)$, which we write as Δ_A for each vertex A of the Johnson graph.

Theorem B (Theorem 5.1). As a subvariety of $\text{Gr}(r; n)$, $\text{QGr}(r; n)$ is the common vanishing locus of the Plücker coordinate products $\{\Delta_A \Delta_B \mid AB \in E(J_{r,n}) \setminus E(QJ_{r,n})\}$.

For the second characterization, we write $\lambda_A \in \text{Part}_{r,n}$ for the partition canonically associated to $A \in V(J_{r,n})$, as recalled in Section 2. If we orient $J_{r,n}$ according to inclusion of partitions, then the incoming edges to A correspond precisely to the boxes in λ_A , which index the coordinates of $\dot{X}^{\lambda_A} \cong \mathbb{A}^{|\lambda_A|}$. We then define the *quasisymmetric Grassmannian Schubert cell* $\dot{X}_{\text{NC}}^\lambda$ to be the coordinate subspace of $\dot{X}^\lambda$ corresponding to the incoming edges from $QJ_{r,n}$. The dimension of $\dot{X}_{\text{NC}}^\lambda$ is the *outer hook length* $\text{OHL}(\lambda) = \lambda_1 + \ell(\lambda) - 1$, the length of the longest hook.

Theorem C (Theorem 5.1). There is an affine paving of $\text{QGr}(r; n)$ by quasisymmetric Grassmannian Schubert cells,

$$\text{QGr}(r; n) = \bigsqcup_{\lambda \in \text{Part}_{r,n}} \dot{X}_{\text{NC}}^\lambda \quad \text{with} \quad \dot{X}_{\text{NC}}^\lambda \cong \mathbb{A}^{\text{OHL}(\lambda)}.$$

Consequently, the odd Betti numbers of $\text{QGr}(r; n)$ vanish and the even Betti numbers are

$$\dim H_{2k}(\text{QGr}(r; n)) = \sum_{\substack{a \leq r, b \leq n-1, \\ \text{and } a+b = k+1}} \binom{k-2}{a-1}.$$

From this result we see that $\text{QGr}(r; n)$ is the union of the cell closures $X_{\text{NC}}^\lambda := \overline{\dot{X}_{\text{NC}}^\lambda}$ which we call *quasisymmetric Grassmannian Schubert cycles*. We find that they are *positroid varieties* in the sense of Knutson–Lam–Speyer [18], the complex closures of Postnikov’s positroid cells that stratify the totally nonnegative Grassmannian [29], and use this to show that the complex of moment polytopes faithfully captures the combinatorics of $\text{QGr}(r; n)$ as a complex of toric varieties.

1.2. A geometric meaning for F_α . We conclude with a re-interpretation of the transition from Schur polynomials to fundamental quasisymmetric polynomials. By Theorem C, the cell closures X_{NC}^λ give a homology basis for $\text{QGr}(r; n)$. While $\text{QGr}(r; n)$ does not enjoy Poincaré duality, we find a *Kronecker dual* of this basis under the natural pairing $H^k(\text{QGr}(r; n)) \otimes H_k(\text{QGr}(r; n)) \rightarrow \mathbb{Z}$, exactly paralleling the Kronecker duality between the bases of $s_\lambda(x_1, \dots, x_r)$ and X^λ for $\text{Gr}(r; n)$.

In Section 3 we introduce an apparently novel bijection $\alpha \mapsto \lambda_\alpha$ from compositions to partitions.

Theorem D (Theorem 8.8). The cohomological basis $\{F_\alpha(x_1, \dots, x_r) \mid \alpha \in \text{Comp}_{r,n}\} \subset H^\bullet(\text{QGr}(r; n))$ is Kronecker dual to the homology basis $\{[X_{\text{NC}}^\lambda] \mid \lambda \in \text{Part}_{r,n}\} \subset H_\bullet(\text{QGr}(r; n))$, or equivalently

$$\int_{X_{\text{NC}}^{\lambda_\beta}} F_\alpha(x_1, \dots, x_r) = \langle [X_{\text{NC}}^{\lambda_\beta}], F_\alpha(x_1, \dots, x_r) \rangle_{\text{QGr}(r; n)} = \delta_{\alpha, \beta}.$$

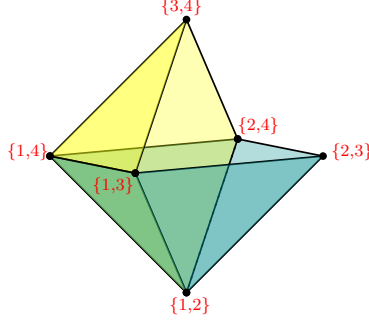


FIGURE 1. $\text{QGr}(2; 4)$ is the union of two isomorphic 3-dimensional singular projective toric varieties whose moment polytopes are square pyramids. The intersection corresponds to their common faces, two triangles which share an edge, which is geometrically two $\mathbb{P}^2$ s sharing a $\mathbb{P}^1$. The 1-skeleton of this complex is the quasisymmetric Johnson graph $Q_{J_{2,4}}$, and $\text{QGr}(2; 4) = \{\Delta_{34}\Delta_{23} = 0\} \subset \text{Gr}(2; 4)$.

This statement allows us to generalize a classical result of Gessel [14]. If we interpret a polynomial $f(x_1, \dots, x_r) \in \text{QSym}_r$ as an element of $H^\bullet(\text{QGr}(r; n))$, then the Kronecker duality implies

$$\int_{X_{\text{NC}}^{\lambda_\alpha}} f = [F_\alpha](f) \quad \text{where} \quad [F_\alpha]f := \text{the coefficient of } F_\alpha \text{ in } f.$$

For $f \in \text{Sym}_r$, Gessel [14] shows (in combinatorial language) that the same equation holds if we replace X_{NC}^λ by a Richardson variety $X_\nu^\eta \subset \text{Gr}(r; n)$ with η/ν of ribbon shape α , so

$$[X_{\text{NC}}^\lambda] = \sum_{\mu} ([F_\alpha]s_\mu)[X^\mu] = [X_\nu^\eta] \in H_\bullet(\text{Gr}(r; n)).$$

This coincidence is satisfactorily explained by the fact that X_{NC}^λ is in fact an S_n -translate of X_ν^η , a computation that was essentially carried out in [24] which we shall recall in Section 6.

1.3. Outline. Section 2 recalls the classic combinatorics of $\text{Gr}(r; n)$ and Section 3 introduces new “noncrossing” generalizations thereof. Sections 4 and 5 give alternative characterizations of $\text{QGr}(r; n)$ by affine paving and Plücker equations. Section 6 relates the paving strata to positroid and translated Richardson varieties before establishing a rigidity theorem for $\text{QGr}(r; n)$. Section 7 recalls properties of quasisymmetric polynomials for Section 8, which proves all cohomological claims. Section 9 establishes the equidimensionality of $\text{QGr}(r; n)$ and describes its moment polytopes.

2. COMBINATORICS OF THE GRASSMANNIAN

This section recalls the essential combinatorics of Grassmannian Schubert calculus, including r -subsets, partitions, and Grassmannian permutations [2, 9, 23, 33, 34]. We let n be a nonnegative integer, $[n] := \{1, \dots, n\}$, and S_n be the symmetric group on n letters, generated by the simple transpositions $s_i = (i, i+1)$ for $1 \leq i \leq n-1$.

Our primary combinatorial objects will be r -element subsets $A \subseteq [n]$, and we denote by $\binom{[n]}{r}$ the collection of all such subsets. We sometimes represent $A \in \binom{[n]}{r}$ using the sets

$$L := [r] \setminus A = \{a_k < \cdots < a_1\} \quad \text{and} \quad R := A \setminus [r] = \{b_1 < \cdots < b_k\},$$

which necessarily have the same size, and whose union is the symmetric difference of A and $[r]$. We will use the notation $A = (L \mid_r R)$, and from this representation we recover $A = ([r] \setminus L) \sqcup R$.

The *Johnson graph* $J_{r,n}$ is the graph on vertex set $\binom{[n]}{r}$ with edges AB whenever $B = (A \setminus j) \cup i$ for $i \neq j$. See Figure 2 for an example. Section 4.1 explains how $J_{r,n}$ encodes the geometry of the Grassmannian $\text{Gr}(r; n)$ by identifying vertices and edges with torus-invariant points and curves.

The *inversion set* of $A \in \binom{[n]}{r}$ is

$$\text{Inv}(A) := \{(i, j) \mid i < j, j \in A \text{ and } i \notin A\}.$$

The *Gale order* $\leq$ is the partial order on $\binom{[n]}{r}$ generated by $A < B$ if $B = (A \setminus j) \cup i$ for $(i, j) \in \text{Inv}(A)$. In this ordering the minimum is $[r]$, the maximum is $\{n - r + 1, \dots, n\}$, and

$$A \leq B \quad \text{if} \quad |A \cap [k]| \geq |B \cap [k]| \quad \text{for all } 1 \leq k \leq n.$$

Lemma 2.1. Let $A = (a_k, \dots, a_1 \mid_r b_1, \dots, b_k)$ and set $b_0 := r$. Then $\text{Inv}(A)$ consists of:

- (1) (i, b_p) where $i \notin A$ and $i < b_p$, for $1 \leq p \leq k$;
- (2) (a_p, j) where $j \in A \cap [r]$ and $a_p < j$, for $1 \leq p \leq k$.

Proof. Cases (1) and (2) describe those $(i, j) \in \text{Inv}(A)$ with $r < j$ and $r \geq j$ respectively. □

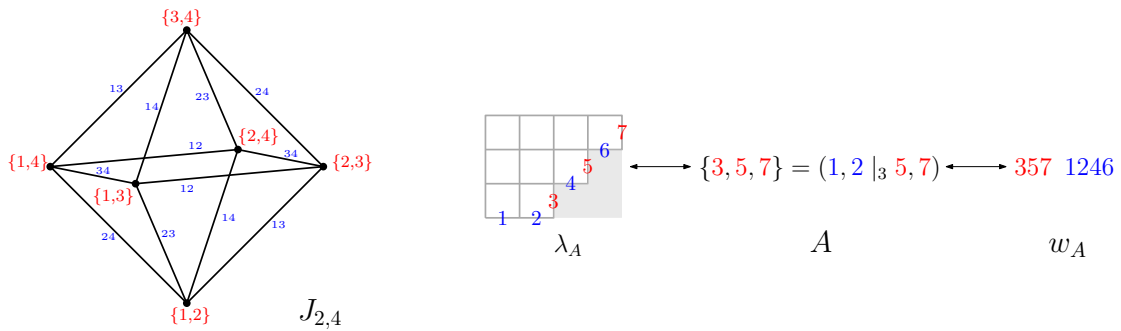


FIGURE 2. The Johnson graph $J_{2,4}$ (left) and an example of the bijections $\text{Part}_{r,n} \leftrightarrow \binom{[n]}{r} \leftrightarrow \text{Grass}_{r,n}$ (right) for $r = 3, n = 7$.

In addition to r -subsets of $[n]$, partitions and Grassmannian permutations are also used in the study of the Grassmannian. We now recall the bijections between these equivalent combinatorial structures; see Figure 2 for an example. As in the introduction, let $\text{Part}_{r,n}$ be the set of partitions

$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 0)$ with $\lambda_1 \leq n - r$. There is a bijection

$$(2.1) \quad \begin{aligned} \binom{[n]}{r} &\rightarrow \text{Part}_{r,n} \\ \{i_1 < i_2 < \dots < i_r\} = A &\mapsto \lambda_A := (i_r - r, \dots, i_2 - 2, i_1 - 1). \end{aligned}$$

Under the bijection $A \leftrightarrow \lambda_A$, Gale order corresponds to inclusion order, and each box of λ_A corresponds to an inversion of A . Reading the boundary of λ from bottom left to top right as shown in Figure 2, A indexes the north steps, and each $(i, j) \in \text{Inv}(A)$ determines a box of λ_A in the row of north step j and column of east step i .

We can also characterize edges in the Johnson graph using *rim hooks* [34, Chapter 7.17], which are connected skew shapes containing no 2×2 square. For $(i, j) \in \text{Inv}(A)$, the partition $\lambda_{(A \setminus j) \cup i}$ is obtained from λ_A by removing the rim hook of boundary boxes between steps i and j .

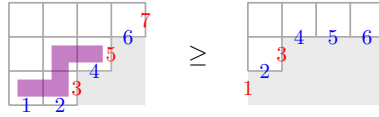


FIGURE 3. Rim hook removal corresponding to inversion $(1, 5)$ from λ_A where $A = \{3, 5, 7\}$

The *Frobenius symbol* [23, Chapter I.1] of λ records arm and leg sizes of each box b on the main diagonal of λ (not including b itself in the sizes). If $A = (L|_r R)$, the Frobenius symbol of λ_A is

$$(2.2) \quad \begin{pmatrix} \text{Arm}_1 & \dots & \text{Arm}_k \\ \text{Leg}_1 & \dots & \text{Leg}_k \end{pmatrix} = \begin{pmatrix} b_k - r - 1 & \dots & b_1 - r - 1 \\ r - a_k & \dots & r - a_1 \end{pmatrix}.$$

We now describe the combinatorial equivalence between $\binom{[n]}{r}$ and Grassmannian permutations. For $w \in S_n$, the *(left) inversion set* and *length* of w are

$$\text{Inv}(w) := \{(i, j) \mid i < j, w^{-1}(i) > w^{-1}(j)\} \quad \text{and} \quad \ell(w) := |\text{Inv}(w)|.$$

The *Bruhat order* $u \leq v$ on S_n is generated by $u < (i, j)u$ for $(i, j) \in \text{Inv}(u)$. The *descent set* of w is $\text{Des}(w) := \{i \mid w(i) > w(i+1)\}$. If $k \in \text{Des}(w)$ then $ws_k < w$ and $ws_k w^{-1} \in \text{Inv}(w)$. Let $\text{Grass}_{r,n}$ be the set of *r-Grassmannian permutations* $\{w \in S_n \mid \text{Des}(w) \subseteq \{r\}\}$. There is a bijection

$$(2.3) \quad \begin{aligned} \binom{[n]}{r} &\rightarrow \text{Grass}_{r,n} \\ A &\mapsto w_A \end{aligned} \quad \text{defined by} \quad \begin{aligned} A &= \{w_A(1) < \dots < w_A(r)\}, \text{ and} \\ [n] \setminus A &= \{w_A(r+1) < \dots < w_A(n)\} \end{aligned}$$

There is a natural action of S_n on $\binom{[n]}{r}$. By construction w_A is the unique minimal length permutation with $w_A \cdot [r] = A$ and each inversion $(i, j) \in \text{Inv}(w_A)$ satisfies $w_A^{-1}(i) \leq r < w_A^{-1}(j)$. By construction $\text{Inv}(w_A) = \text{Inv}(A)$ and the inversions of w_A correspond to boxes of λ_A as above.

3. COMBINATORICS OF THE QUASISYMMETRIC GRASSMANNIAN

This section introduces the combinatorial objects essential for the quasisymmetric Grassmannian including noncrossing partitions, the quasisymmetric Johnson graph, quasigrassmannian permutations, and compositions, analogous to the constructions recalled in Section 2.

3.1. Noncrossing partitions and the quasisymmetric Johnson graph. A *set partition* of $[n]$ is a collection of non-empty disjoint subsets $B_1, \dots, B_k$, called *blocks*, whose union is $[n]$. A *crossing* is a 4-tuple $a < b < c < d$ in $[n]$ such that a, c belong to one block and b, d belong to another. A set partition is *noncrossing* if it has no crossing.

Coxeter-Catalan combinatorics, when specialized to the product of simple transpositions for the standard Coxeter element $s_{n-1} \cdots s_1 = (n \ n-1 \ \dots \ 1)$, transforms the noncrossing set partitions into a collection $\text{NC}_n \subseteq S_n$ as follows. From a noncrossing set partition of $[n]$, each block $B = \{b_1 < \dots < b_s\}$ determines a backwards cycle $(b_s, b_{s-1}, \dots, b_1)$. The product of these (disjoint) cycles over all blocks is an *(algebraic) noncrossing partition* and NC_n is the set of all algebraic noncrossing partitions. Given this bijection we shall use the terms blocks and cycles interchangeably. For example, the noncrossing set partition with blocks $\{\{1, 5, 6\}, \{2, 3\}, \{4\}\}$ of $[6]$ gives the algebraic noncrossing partition $(6 \ 5 \ 1)(3 \ 2) = 632415 \in \text{NC}_6$.

The *Kreweras order* [21] is the partial order on NC_n induced by refinement of set partitions, with id at the bottom and $(n \ n-1 \ \dots \ 1)$ at the top. We are primarily concerned with the Hasse diagram of this order, so we describe its covering relations: $u, w \in \text{NC}_n$ have a covering relation if there is a transposition $\tau = (i \ j)$ such that $\tau u = w$. See Figure 4 for an example of the Kreweras order on NC_4 .

Definition 3.1. The *quasisymmetric Johnson graph* $QJ_{r,n}$ is the spanning subgraph of $J_{r,n}$ with edges from vertices A to B if and only if under the natural action of S_n on $\binom{[n]}{r}$ we have

$$A = w \cdot [r] \quad \text{and} \quad B = u \cdot [r] \quad \text{for some } w, u \in \text{NC}_n \text{ Kreweras-adjacent.}$$

See Figure 4 for an example of $QJ_{4,2}$.

We define the *quasisymmetric inversion set* of $A \in \binom{[n]}{r}$ to be

$$\text{Inv}^Q(A) := \{(i, j) \mid (i, j) \in \text{Inv}(A) \text{ and } AB \in QJ_{r,n} \text{ for } B = (A \setminus j) \cup i\}.$$

The structure of $J_{r,n}$ and each $\text{Inv}^Q(A)$ is controlled by the fibers

$$\text{NC}_n^A := \{w \in \text{NC}_n \mid w \cdot [r] = A\},$$

each of which is nonempty: if $A = (a_k, \dots, a_1 \mid_r b_1, \dots, b_k) \in \binom{[n]}{r}$ then the permutation $w = \prod_{1 \leq i \leq k} (a_i, b_i)$ is noncrossing and also has $w \cdot [r] = A$, so $w \in \text{NC}_n^A$.

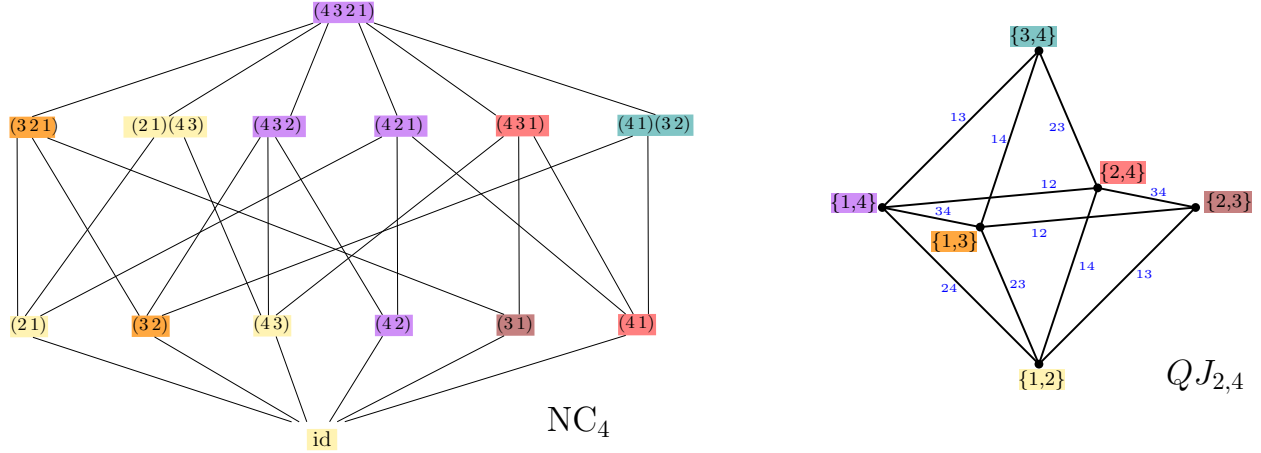


FIGURE 4. The Kreweras lattice NC_4 (left) written using cycle notation and the Quasisymmetric Johnson graph $QJ_{2,4}$ (right), with color denoting the fibers NC_4^A .

3.2. Quasigrassmannian permutations, $\text{Comp}_{r,n}$ and $\text{Part}_{r,n}$. For $w \in NC_n$, the *noncrossing descent set* of w is defined to be

$$\text{Des}_{NC}(w) := \{k \in \text{Des}(w) \mid ws_k \in NC_n\}.$$

Fact 3.2 ([5, Lemma 7.4]). For $w \in NC_n$ and $i \in \text{Des}(w)$, we have $i \in \text{Des}_{NC}(w) \leftrightarrow i \in \{w(i), w(i+1)\}$.

We now introduce the noncrossing analogue of Grassmannian permutations.

Definition 3.3. A permutation $w \in NC_n$ is *r -Quasigrassmannian* if $\text{Des}_{NC}(w) \subset \{r\}$. Denote the set of Quasigrassmannian permutations by $\text{QGrass}_{r,n} \subset NC_n$.

Despite the similar definitions, we emphasize here that Quasigrassmannian permutations are *not* in general Grassmannian; for example $w = 4321 = (41)(32) \in \text{QGrass}_{2,4}$ is not Grassmannian.

Lemma 3.4. With $b_0 := r$, the map

$$(3.1) \quad \begin{aligned} \binom{[n]}{r} &\rightarrow \text{QGrass}_{r,n} \\ A = (a_k, \dots, a_1 \mid_r b_1, \dots, b_k) &\mapsto z_A := \prod_{i=1}^k (b_i (b_i - 1) \dots (b_{i-1} + 1) a_i), \end{aligned}$$

is a bijection. We will henceforth call z_A the *quasigrassmannian permutation associated to A* .

Proof. Either $z_A = id$ or $\text{Des}(z_A) = \{a_1, \dots, a_k, r, b_{k-1}, \dots, b_1\}$. Of these descents, only r meets the criterion of Fact 3.2, so $z_A \in \text{QGrass}_{r,n}$. On the other hand, $z_A \cdot [r] = A$, giving an inverse. $\square$

For example, if $n = 10$, $r = 4$, and $A = \{1, 3, 7, 9\} = (2, 4 \mid_4 7, 9)$ then $z_A = (982)(7654)$.

Lemma 3.5. For $A \in \binom{[n]}{r}$, z_A belongs to the fiber NC_n^A , and every noncrossing partition in NC_n^A can be reduced to z_A by a sequence of transformations $u \mapsto us_k$ with $k \in \text{Des}_{NC}(u)$. In particular, z_A is the unique Bruhat-minimal element of NC_n^A .

Proof. The first claim is immediate. The second follows from induction on length, as every element of $\text{NC}_n \setminus \text{QGrass}_{r,n}$ has a noncrossing descent in $[n] \setminus \{r\}$. $\square$

Remark 3.6. It is shown in [5, Corollary 7.9] that NC_n is the equivalence class of the identity permutation under the equivalence relation on S_n where $w \sim ws_i$ whenever $i \in \{w(i), w(i+1)\}$. Lemma 3.5 and Fact 3.2 then say that if we restrict this relation to $i \neq r$ then the equivalence classes are the fibers NC_n^A .

As in the introduction, let $\text{Comp}_{r,n}$ denote the set of compositions $\alpha = (\alpha_1, \dots, \alpha_k)$ whose ribbon diagram fit in an $r \times (n-r)$ box, i.e. with $k \leq r$ and $\sum \alpha_i - k + 1 \leq n-r$.

Lemma 3.7. For $z_A \in \text{QGrass}_{r,n}$, let B_i be the block with $\min(B_i) = i$ for $1 \leq i \leq r$. The map

$$(3.2) \quad \begin{aligned} \text{QGrass}_{r,n} &\rightarrow \text{Comp}_{r,n} \\ z_A &\mapsto \alpha_A := (|B_{a_k}| - 1, |B_{(a_k)+1}|, \dots, |B_{r-1}|, |B_r|) \end{aligned}$$

is a bijection.

Proof. This map is the composition of the bijections in [5, Def. 7.10] and [25, Theorem 8.3]. $\square$

As $\text{Comp}_{r,n}$ and $\text{Part}_{r,n}$ are both in bijection with $\binom{[n]}{r}$, we obtain a direct bijection which appears to be new; see Figure 5. Define the *outer hook length* $\text{OHL}(\lambda) = \lambda_1 + \ell(\lambda) - 1$ (and $\text{OHL}(\emptyset) = 0$).

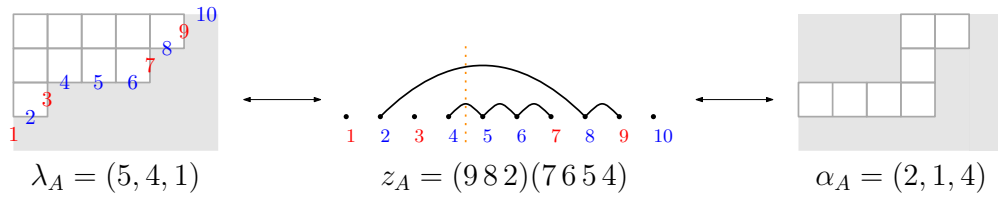


FIGURE 5. Bijections $\text{Part}_{4,10} \leftrightarrow \text{QGrass}_{4,10} \leftrightarrow \text{Comp}_{4,10}$ for $A = \{1, 3, 7, 9\} \in [10]$.

Theorem 3.8. The composite bijection $\text{Comp}_{r,n} \rightarrow \text{Part}_{r,n}$, $\alpha \mapsto \lambda_\alpha$ are described as follows.

- (1) Each $\alpha \in \text{Comp}_{r,n}$ can be written as $\alpha = d_1 1^{e_1-1} (d_2 + 1) 1^{e_2-1} \dots (d_k + 1) 1^{e_k-1}$ for numbers $d_1, \dots, d_k, e_1, \dots, e_k \geq 1$. (If α starts with 1^a then $d_1 = 1$ and $e_1 = a$.) Using Frobenius notation for partitions, the bijection takes

$$\alpha = d_1 1^{e_1-1} \dots (d_k + 1) 1^{e_k-1} \mapsto \lambda_\alpha := \begin{pmatrix} (\sum_{i=1}^k d_i) - 1 & (\sum_{i=2}^k d_i) - 1 & \dots & d_k - 1 \\ (\sum_{i=1}^k e_i) - 1 & (\sum_{i=2}^k e_i) - 1 & \dots & e_k - 1 \end{pmatrix}.$$

- (2) The inverse map $\text{Part}_{r,n} \rightarrow \text{Comp}_{r,n}$ takes a partition $\lambda = \begin{pmatrix} A_1 & \dots & A_k \\ L_1 & \dots & L_k \end{pmatrix}$ to

$$(A_1 - A_2) 1^{L_1 - L_2 - 1} (A_2 - A_3 + 1) \dots 1^{L_{k-2} - L_{k-1} - 1} (A_{k-1} - A_k + 1) 1^{L_{k-1} - L_k - 1} (A_k + 2) 1^{L_k}$$

if $k \geq 2$ and $(A_1 + 1)1^{L_1}$ if $k = 1$.

Under this bijection, if α corresponds to λ then $|\alpha| = \text{OHL}(\lambda)$, the outer hook length of λ .

Proof. We omit the proof of the bijections as it is a matter of carefully composing previous bijections. The union of blocks of z_A for $A = (a_k, \dots, a_1 | b_1, \dots, b_k)$ is the closed interval $[a_k, b_k]$, and the outer hook length of λ_A is given by $b_k - a_k$ by (2.2), so we conclude $|\alpha_A| = \text{OHL}(\lambda)$ by (3.2). $\square$

3.3. Quasisymmetric inversions and noncrossing inversions. We now relate the quasisymmetric inversions of $A \in \binom{[n]}{r}$ with a distinguished subset of ordinary inversions defined for NC_n , following [5, Section 8.3].

Definition 3.9. For $w \in \text{NC}_n$, define the set of *noncrossing inversions* by

$$\text{Inv}_{\text{NC}}(w) := \{(i, j) \in \text{Inv}(w) \mid (i, j)w \in \text{NC}_n\}.$$

Explicitly we have $(i, j) \in \text{Inv}_{\text{NC}}(w)$ if and only if j is the largest element in its cycle and either

- (1) i belongs to the same cycle as j , or
- (2) $i < j < w^{-1}(i)$ and i is maximal with respect to this property.

Remark 3.10. Note that [5, Section 8.3] defines noncrossing inversions as right inversions, which in the present notation would be the set of $(w^{-1}(i), w^{-1}(j)) = w^{-1}(i, j)w$ for $(i, j) \in \text{Inv}_{\text{NC}}(w)$. The content of the second half of Definition 3.9 is then [5, Proposition 8.16].

Lemma 3.11. Let $A = (a_k, \dots, a_1 \mid b_1, \dots, b_k)$ and set $a_0 := r + 1$, $b_0 := r$. The noncrossing inversions of z_A are:

- (1) (*Split*) for $1 \leq p \leq k$, (a_p, b_p) and (q, b_p) where $b_{p-1} < q < b_p$;
- (2) (*Merge*) (a_{p+1}, b_p) for $1 \leq p \leq k-1$, and (a_{p+1}, q) where $a_{p+1} < q < a_p$ for $0 \leq p \leq k-1$.

In particular $|\text{Inv}_{\text{NC}}(z_A)| = \text{OHL}(\lambda_A)$.

Proof. The combinatorial criterion in Definition 3.9 implies the description of the noncrossing inversions. As for the cardinality of $\text{Inv}_{\text{NC}}(z_A)$, observe that we have $b_k - r$ noncrossing inversions in item (1) and $r - a_k$ noncrossing inversions in item (2). Their sum $b_k - a_k$ equals $\text{OHL}(\lambda_A)$. $\square$

Remark 3.12. The labels refer to the effect of left-multiplying z_A by the corresponding transposition: the inversions in (1) split a cycle into two, while those in (2) merge two cycles into one.

The description in Lemma 3.11 implies that $\text{Inv}_{\text{NC}}(z_A) \subseteq \text{Inv}(w_A) = \text{Inv}(A)$. Therefore the set $\text{Inv}_{\text{NC}}(z_A)$ identifies a distinguished collection of boxes of λ_A under the usual correspondence between boxes and $\text{Inv}(w_A) = \text{Inv}(A)$. Let k be the number of boxes along the main diagonal of λ_A . Then Lemma 3.11 translates to the following: noncrossing inversions in (1) (resp. (2)) correspond to the boxes along the main diagonal (resp. subdiagonal) of λ_A as well as boxes in the first k rows (resp. columns) that do not have a box below them (resp. to their right) in λ_A .

Figure 6 lists the noncrossing inversions for the z_A in Figure 5 and identifies them with a subset of boxes in λ_A . The dots (resp. crosses) in Figure 6 represent inversions in (1) (resp. (2)).

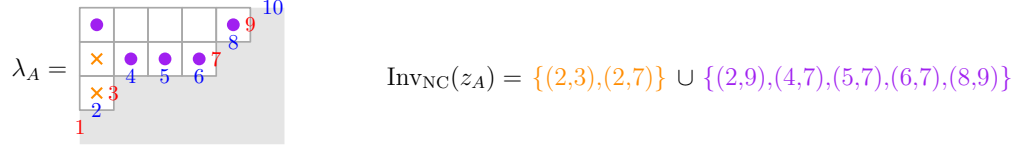


FIGURE 6. Noncrossing inversions of z_A and the corresponding boxes in λ_A .

The next proposition cements the role of r -Quasigrassmannian permutations as the *correct* noncrossing analogue of Grassmannian permutations.

Theorem 3.13. For $A \in \binom{[n]}{r}$, we have $|\text{Inv}^Q(A)| = \text{OHL}(\lambda_A)$ and

$$\text{Inv}^Q(A) = \text{Inv}_{\text{NC}}(z_A).$$

We shall prove this theorem after a preparatory lemma. Say that a block B of a noncrossing set partition *extends over* r if $\min B \leq r < \max B$.

Lemma 3.14. For $w \in \text{NC}_n$ with associated noncrossing set partition $\mathcal{P}$, order the blocks extending over r as $B_1, \dots, B_k$ so that $a_i := \max(B_i \cap [r])$ satisfies $a_1 > \dots > a_k$, and set $b_i := \max B_i$. Then $b_1 < \dots < b_k$ and

$$w \cdot [r] = (a_k, \dots, a_1 \mid_r b_1, \dots, b_k).$$

Proof. First note if $j > i$ and $b_j > b_i$, then $a_i < a_j < b_i < b_j$ is a crossing between B_i and B_j , a contradiction. Therefore $b_1 < \dots < b_k$.

The backwards cycle of a block B is supported on B , so blocks not extending over r fix $[r]$ setwise. For a block B_i extending over r with $B_i \cap [r] = \{c_1 < \dots < c_p = a_i\}$ and $B_i \setminus [r] = \{d_1 < \dots < d_q = b_i\}$, the backwards cycle $(d_q, \dots, d_1, c_p, \dots, c_1)$ sends $c_j \mapsto c_{j-1}$ for $j \geq 2$ and $c_1 \mapsto d_q$, so the image of $B_i \cap [r]$ is $((B_i \cap [r]) \setminus a_i) \cup b_i$.

Since distinct blocks act on disjoint sets, these replacements are independent, giving $w \cdot [r] = ([r] \setminus \{a_1, \dots, a_k\}) \cup \{b_1, \dots, b_k\} = (a_k, \dots, a_1 \mid_r b_1, \dots, b_k)$ as desired. $\square$

Call a block B in a noncrossing set partition *nested* if there exists another block B' containing i, j so that for all $k \in B$ we have $i < k < j$. If this happens we say B is *nested below* B' .

Proof of Theorem 3.13. The equality $|\text{Inv}^Q(A)| = \text{OHL}(\lambda_A)$ follows from Lemma 3.11 once we show $\text{Inv}^Q(A) = \text{OHL}(\lambda)$. As the noncrossing inversions described in Lemma 3.11 are a subset of the inversions described in Lemma 2.1, we know $\text{Inv}_{\text{NC}}(z_A) \subset \text{Inv}(A)$. The inclusion $\text{Inv}_{\text{NC}}(z_A) \subseteq \text{Inv}^Q(A)$ then follows as $z_A \in \text{NC}_n^A$ and

$$\text{Inv}^Q(A) = \bigcup_{u \in \text{NC}_n^A} (\text{Inv}_{\text{NC}}(u) \cap \text{Inv}(A)).$$

We prove the reverse inclusion. Let $(i, j) \in \text{Inv}^Q(A)$, so there exists $w \in \text{NC}_n^A$ with $(i, j) \in \text{Inv}_{\text{NC}}(w) \cap \text{Inv}(A)$. Fix such a w . In what follows we let B_s be the block of w extending over r with

$$(\max(B_s \cap [r]), \max B_s) = (a_s, b_s)$$

from Proposition 3.14. By Lemma 3.11, the element $j \in A$ is the maximal element of its cycle B . There are two cases in this lemma and we address both.

Suppose first that $i \notin A$ has $i \in B$. Then B extends over r as it is neither contained in A , nor is disjoint from A . This allows us to deduce that $B = B_p$ for some $1 \leq p \leq k$, and that $j = b_p$. Furthermore, as $i \notin A$ we know that $i \geq a_p$, so either $i = a_p$, in which case $(i, j) \in \text{Inv}_{\text{NC}}(z_A)$ by the first part of Lemma 3.11(1), or $r < i < b_p$ (as $a_p = \max B_p \cap [r]$). But in this latter case because B_{p-1} is nested inside B_p we know that $B_p \cap [a_{p-1}, b_{p-1}] = \emptyset$ so in fact $b_{p-1} < i < b_p$, establishing $(i, j) \in \text{Inv}_{\text{NC}}(z_A)$ by the second part of Lemma 3.11(1).

Suppose now that $i \notin A$ lies in a different cycle B' to B , and is the maximal element such that $i < j < w^{-1}(i)$. For $j \in A$ we must have $\min B \leq r$. The condition $i < j < w^{-1}(i)$ forces B to be nested below B' , so we must have $i \leq \min B$ and hence $i \leq r$. As $i \notin A$, we can only have $i \leq r$ if $B' = B_{p+1}$ extends over r for some $0 \leq p \leq k-1$, and $i = a_{p+1}$. On the one hand if B extends over r then $B = B_{p'}$ for some $p' \leq p$. But we cannot have $p' < p$ as then a_p is a larger element than i with $a_p < i < w^{-1}(a_p)$. We deduce $p' = p$ and so $s = b_p$, implying $(i, j) \in \text{Inv}_{\text{NC}}(z_A)$ by the first part of Lemma 3.11(2). If on the other hand B does not extend over r then we claim that $a_{p+1} < j < a_p$. Indeed, if not then $a_p < j \leq r$ and we arrive at a contradiction that $i < a_p$ has $a_p < j < w^{-1}(a_p)$. We conclude from $a_{p+1} < j < a_p$ that $(i, j) \in \text{Inv}_{\text{NC}}(z_A)$ by the second part of Lemma 3.11(2). $\square$

Theorem 3.15. For $A, B \in \binom{[n]}{r}$, we have $z_A \geq z_B$ in the Bruhat order if and only if $A \geq B$.

Proof. For the order statement, recall that $x \geq y$ in Bruhat order if and only if $x \cdot [i]$ is elementwise at least as large as $y \cdot [i]$ for all $1 \leq i \leq n$. Since $z_A \cdot [r] = A$ and $z_{A'} \cdot [r] = A'$, the condition at $i = r$ gives $z_A \geq z_{A'}$ whenever $A \geq A'$.

For the converse, it suffices to treat the case $B < A$, so that $AB \in QJ_{r,n}$ by Lemma 3.11 (see also the discussion after Remark 3.12 where we can see clearly that each removable corner belongs to $\text{Inv}^Q(A)$). By Theorem 3.13 we conclude there exists $(i, j) \in \text{Inv}_{\text{NC}}(z_A)$ with $B = (A \setminus j) \cup i$. We then have $(i, j)z_A \in \text{NC}_n^A$, so by minimality of $z_A \in \text{NC}_n^A$ we conclude that $z_A > (i, j)z_A \geq z_A$. $\square$

4. PLÜCKER VANISHING AND $QJ_{r,n}$

In this section we define two subsets

$$\text{PV}_G \subset \bigsqcup_{A \in \binom{[n]}{r}} \mathring{X}_{\text{NC}}^A \subset \text{Gr}(r; n)$$

of the Grassmannian that we will show in the next section are both equal to $Q\text{Gr}(r; n)$.

4.1. Recollections on the Grassmannian. The Grassmannian $\text{Gr}(r; n)$ is the space of all r -planes in $\mathbb{C}^n$, which we identify with its image under the *Plücker embedding*

$$\begin{aligned} \text{Gr}(r; n) &\hookrightarrow \mathbb{P}(\bigwedge^r \mathbb{C}^n) \\ V &\mapsto \langle v_1 \wedge v_2 \wedge \cdots \wedge v_r \rangle \end{aligned} \quad \text{for } \{v_1, v_2, \dots, v_r\} \text{ a basis of } V.$$

This map is equivariant with respect to the standard actions of $T = (\mathbb{C}^*)^n$ on $\text{Gr}(r; n)$ and $\mathbb{P}(\bigwedge^r \mathbb{C}^n)$. For $A = \{i_1 < \cdots < i_r\}$, let

$$e_A = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_r} \in \bigwedge^r \mathbb{C}^n.$$

The T -fixed points of $\text{Gr}(r; n)$ are represented by the coordinate r -planes $\langle e_A \rangle$, and the T -invariant curves are $\langle e_A, e_{A'} \rangle$ for $AA' \in E(J_{r,n})$. We denote by $\{\Delta_A \mid A \in \binom{[n]}{r}\}$ the coordinate functions of the e_A , which are defined up to simultaneous scaling.

Each point $\langle v_1 \wedge v_2 \wedge \cdots \wedge v_r \rangle \in \text{Gr}(r; n)$ can be represented by the full-rank $n \times r$ matrix with columns $v_1, v_2, \dots, v_r \in \mathbb{C}^n$, which is unique up to right multiplication by $\text{GL}_r(\mathbb{C})$. Denoting $\text{Mat}_{n \times r}^\circ \subset \text{Mat}_{n \times r}$ for the subset of full column rank $n \times r$ matrices, we may thus identify $\text{Gr}(r; n) = \text{Mat}_{n \times r}^\circ / \text{GL}_r(\mathbb{C})$. We will **transpose all $n \times r$ matrix representatives** for the sake of space, so the entry (i, j) in the transposed matrix represents the entry in the i th column and j th row.

The Grassmannian $\text{Gr}(r; n)$ has a stratification into Schubert cells, which we write as either $\mathring{X}^A = \mathring{X}^\lambda$ depending on whether we index with $A \in \binom{[n]}{r}$ or $\lambda \in \text{Part}_{r,n}$. For each $A = \{a_1 < \cdots < a_r\} \in \binom{[n]}{r}$, the Grassmannian Schubert cell $\mathring{X}^A$ is the subset represented by matrices with 1's in the positions (a_i, i) , * (arbitrary entries) in the positions (i, j) for $(i, a_j) \in \text{Inv}(A)$, and 0's elsewhere. The * in $\mathring{X}^{\lambda_A} := \mathring{X}^A$ bijectively correspond to the boxes of the Young diagram for λ_A .

Example 4.1. For $n = 10$ and $r = 4$, let $A = \{1, 3, 7, 9\}$ and $\lambda = (5, 4, 1)$. Then

$$\mathring{X}^{\{1,3,7,9\}} = \begin{matrix} 1 \\ 3 \\ 7 \\ 9 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & * & * & * & 1 & 0 & 0 & 0 \\ 0 & * & 0 & * & * & * & 0 & * & 1 & 0 \end{bmatrix}^\top = \mathring{X}^{\begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square \\ \hline \end{array}} \cong \mathbb{A}^{10}.$$

In terms of a matrix representative M of a point in $\text{Gr}(r; n)$, the Plücker coordinates Δ_A can be simultaneously computed as $\Delta_A = \det(M_{i,j})_{i \in A, j \in [r]}$. This gives the following result.

Fact 4.2. At any point of $\mathring{X}^A$ we have $\Delta_{A'} = 0$ if $A' \not\leq A$, $\Delta_A \neq 0$, and if $(i, a_j) \in \text{Inv}(A)$ and we set $A' = (A \setminus a_j) \cup i$ then $\Delta_{A'}/\Delta_A = \pm M_{i,j}$ for M the canonical matrix representative of our point.

Recall that an *affine paving* of a variety X is a sequence $\emptyset = X_0 \subset X_1 \subset \cdots \subset X_k = X$ of closed subvarieties such that each $X_i \setminus X_{i-1}$ is isomorphic to an affine space. We have $\mathbb{A}^{|\text{Inv}(A)|} \cong \mathring{X}^A = \mathring{X}^{\lambda_A} \cong \mathbb{A}^{|\lambda_A|}$, and the Schubert cells in listed in any linear extension of the Gale order induce an affine paving

$$\text{Gr}(r; n) = \bigsqcup_{A \in \binom{[n]}{r}} \mathring{X}^A = \bigsqcup_{\lambda \in \text{Part}_{r,n}} \mathring{X}^\lambda.$$

4.2. Plücker vanishing and Quasisymmetric Schubert cells. For any subgraph $G \subseteq J_{r,n}$, we define an associated *Plücker vanishing variety* by

$$\text{PV}_G := \left(\bigcap_{A \in V(J_{r,n}) \setminus V(G)} \{\Delta_A = 0\} \right) \cap \left(\bigcap_{AB \in E(J_{r,n}) \setminus E(G)} \{\Delta_A \Delta_B = 0\} \right) \subset \text{Gr}(r; n).$$

Write $\text{Inv}^G(A)$ for the set of inversions (i, j) of A such that for $B = (A \setminus j) \cup i$ we have $AB \in E(G)$. From its definition there is a natural union of affine subspaces of the Schubert cells which always contains PV_G . For $A = \{a_1 < \dots < a_r\} \in V(G)$, let

$$\mathring{X}_G^A := \{M \in \mathring{X}^A \mid M_{i,j} = 0 \text{ for } (i, a_j) \in \text{Inv}(A) \setminus \text{Inv}^G(A)\} \cong \mathbb{A}^{|\text{Inv}^G(A)|}.$$

See Example 4.5 for the application of this construction to $G = QJ_{r,n}$.

Proposition 4.3. We have $\mathring{X}_G^A = \mathring{X}^A \cap \bigcap_{(i,j) \in \text{Inv}(A) \setminus \text{Inv}^G(A)} \{\Delta_{(A \setminus j) \cup i} = 0\}$ and

$$\text{PV}_G \subset \bigsqcup_{A \in V(G)} \mathring{X}_G^A \subset \text{Gr}(r; n)$$

Proof. The equality follows directly from Fact 4.2. We now show the containment of PV_G . Since $\Delta_A \neq 0$ on $\mathring{X}^A$, we have $\text{PV}_G \subset \bigcup_{A \in V(G)} \mathring{X}^A$, so it suffices to show for $A \in V(G)$ that $\text{PV}_G \cap \mathring{X}^A \subset \mathring{X}_G^A$. By Fact 4.2 we have $\Delta_A \neq 0$ on $\mathring{X}^A$, so for $(i, j) \in \text{Inv}(A) \setminus \text{Inv}^G(A)$ we have $\Delta_{(A \setminus j) \cup i} = 0$ is equivalent to $\Delta_A \Delta_{(A \setminus j) \cup i} = 0$ on $\mathring{X}^A$, and this is a defining relation for PV_G . $\square$

Now consider $G = QJ_{r,n}$. Then $\text{Inv}^G(A) = \text{Inv}^Q(A)$ has size $\text{OHL}(\lambda)$ (Theorem 3.13) and

$$\text{PV}_{QJ_{r,n}} = \bigcap_{AA' \in E(J_{r,n}) \setminus E(QJ_{r,n})} \{\Delta_A \Delta_{A'} = 0\}.$$

Definition 4.4. Define the *quasisymmetric Schubert cell* $\mathring{X}_{\text{NC}}^A \cong \mathbb{A}^{|\text{OHL}(\lambda_A)|}$ to be the subspace of $\mathring{X}^A$ obtained by setting to zero all matrix entries $M_{j,i}$ for $(a_i, j) \in \text{Inv}(A) \setminus \text{Inv}^Q(A)$.

Equivalently $\mathring{X}_{\text{NC}}^A = \mathring{X}_G^A$ for $G = QJ_{r,n}$. Under the correspondence between boxes of λ and $*$'s in $\mathring{X}^\lambda$, these entries are characterized by the results of Section 3.2.

Example 4.5. Continuing Example 4.1, we have $\text{Inv}(A) \setminus \text{Inv}^Q(A) = \{(4, 9), (5, 9), (6, 9)\}$ corresponding to the unfilled cells in Figure 6, and additionally

$$\mathring{X}_{\text{NC}}^A = \begin{matrix} 1 \\ 3 \\ 7 \\ 9 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & * & * & * & 1 & 0 & 0 & 0 \\ 0 & * & 0 & \boxed{0} & \boxed{0} & \boxed{0} & 0 & * & 1 & 0 \end{bmatrix}^T \cong \mathbb{A}^7.$$

Corollary 4.6. We have

$$\mathrm{PV}_{QJ_{r,n}} \subset \bigsqcup_{A \in \binom{[n]}{r}} \mathring{X}_{\mathrm{NC}}^A \subset \mathrm{Gr}(r; n).$$

What is special about $QJ_{r,n}$ is that this first containment is an equality, and we will show in the next section that these are equal to the quasisymmetric Grassmannian $\mathrm{QGr}(r; n)$.

5. THE QUASISYMMETRIC GRASSMANNIAN

The complete flag variety Fl_n parametrizes flags of subspaces $\{0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq \mathbb{C}^n\}$. The *quasisymmetric flag variety* of [5] is a complex of projective toric varieties $\mathrm{QFl}_n \subset \mathrm{Fl}_n$. The natural projection map $\pi : \mathrm{Fl}_n \rightarrow \mathrm{Gr}(r; n)$ sends the flag $V_\bullet$ to the subspace V_r . Recalling Definition 1.1, the *quasisymmetric Grassmannian* is defined to be the projection

$$\mathrm{QGr}(r; n) := \pi(\mathrm{QFl}_n).$$

Since π is proper and T -equivariant, $\mathrm{QGr}(r; n) \subset \mathrm{Gr}(r; n)$ is also a complex of projective toric varieties. Recall the sets $\mathrm{PV}_{QJ_{r,n}}$ and $\mathring{X}_{\mathrm{NC}}^\lambda$ from Section 4.2.

Theorem 5.1. We have the equalities

$$\mathrm{QGr}(r; n) = \mathrm{PV}_{QJ_{r,n}} = \bigsqcup \mathring{X}_{\mathrm{NC}}^\lambda = \bigcup X_{\mathrm{NC}}^\lambda.$$

Furthermore the quasisymmetric Schubert cells $\mathring{X}_{\mathrm{NC}}^\lambda$ give an affine paving of $\mathrm{QGr}(r; n)$ under any linear extension of the Gale order $\leq$.

The proof of Theorem 5.1 is given at the end of the section; we first recall an affine paving of QFl_n in Section 5.1 and relate this paving to the cells $\mathring{X}_{\mathrm{NC}}^\lambda$ in Section 5.2.

5.1. Noncrossing Schubert cells in QFl_n . We identify Fl_n with GL_n/B , where $B \subset \mathrm{GL}_n$ is the subgroup of upper triangular matrices. The flag associated to a coset $MB \in \mathrm{GL}_n/B$ is obtained by taking V_i to be the span of the first i columns of M . The matrix representative of the projection $\pi(MB) \in \mathrm{Gr}(r; n)$ is the principle $n \times r$ submatrix $(M_{i,j})_{\substack{1 \leq i \leq n, \\ 1 \leq j \leq r}}$, or any right GL_r -multiple thereof.

For $w \in S_n$, the *Schubert cell* $\mathring{X}^w$ is BwB/B . The Bruhat decomposition states

$$\mathrm{GL}_n/B = \bigsqcup_{w \in S_n} \mathring{X}^w \quad \text{with} \quad \mathring{X}^w \cong \mathbb{A}^{\ell(w)},$$

and this gives an affine paving of Fl_n with under any linear extension of the Bruhat order. Furthermore every Schubert variety $X^w = \overline{\mathring{X}^w}$ is a union of Schubert cells, though this is not required by the definition of an affine paving.

Each $\mathring{X}^w$ has a canonical affine chart that can be represented as a $\{0, 1, *\}$ matrix in the manner of Section 4.1. Beginning with the permutation matrix for w , which has 1's in positions $(w(j), j)$

and 0's elsewhere, fill each position which is not below or right of a 1 with *, representing a free coordinate. This amounts to placing an * in position $(i, w^{-1}(j))$ for each inversion $(i, j) \in \text{Inv}(w)$.

Following [6, Definition 9.6], define the *noncrossing Schubert cell*¹ $\mathring{X}_{\text{NC}}^w$ for $w \in \text{NC}_n$ to be the subspace of $\mathring{X}^w$ obtained by zeroing out the * in positions $(i, w^{-1}(j))$ for $(i, j) \notin \text{Inv}_{\text{NC}}(w)$. Also define the *noncrossing Schubert cycle* $X_{\text{NC}}^w = \overline{\mathring{X}_{\text{NC}}^w}$.

Example 5.2. Let $w = 53214$, so that $\text{Inv}_{\text{NC}}(w) = \{(13), (15), (23), (45)\}$ and $\text{Inv}(w) \setminus \text{Inv}_{\text{NC}}(w) = \{(12), (25), (35)\}$. Then we have

$$\mathring{X}^{53214} = \begin{bmatrix} * & * & * & 1 & 0 \\ * & * & 1 & 0 & 0 \\ * & 1 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \cong \mathbb{A}^7 \quad \text{and} \quad \mathring{X}_{\text{NC}}^{53214} = \begin{bmatrix} * & * & \boxed{0} & 1 & 0 \\ \boxed{0} & * & 1 & 0 & 0 \\ \boxed{0} & 1 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \cong \mathbb{A}^4.$$

In [6] these cells were shown to induce an affine paving of QFl_n with respect to any linear extension of the Bruhat order restricted to NC_n :

$$\text{QFl}_n = \bigsqcup_{w \in \text{NC}_n} \mathring{X}_{\text{NC}}^w \quad \text{with} \quad \mathring{X}_{\text{NC}}^w \cong \mathbb{A}^{|\text{Inv}_{\text{NC}}(w)|}.$$

Unlike the paving of Fl_n by Schubert cells, some of the cell closures X_{NC}^w are strictly contained in (and are not equal to) a union of noncrossing Schubert cells.

5.2. Projection of noncrossing Schubert cells in Fl_n . For $w \in S_n$, if $w \cdot [r] = A = \{a_1 < \dots < a_r\}$ then the projection π maps $\mathring{X}^w$ onto $\mathring{X}^A$. More specifically, $\text{Inv}(A) \subseteq \text{Inv}(w)$ and $\pi|_{\mathring{X}^w}$ is the coordinate projection sending the $(i, w^{-1}(j))$ -coordinate subspace of $\mathring{X}^w$ to zero unless $w^{-1}(j) \leq r < w^{-1}(i)$, in which case $(i, j) = (i, a_k) \in \text{Inv}(A)$ and the $(i, w^{-1}(j))$ -coordinate subspace maps to the (i, k) -coordinate subspace of $\mathring{X}^A$.

Theorem 5.3. If $u \in \text{NC}_n^A$, then $\pi(\mathring{X}_{\text{NC}}^u) \subseteq \mathring{X}^A$. Moreover:

- (1) if $u = z_A$, then $\pi|_{\mathring{X}_{\text{NC}}^u}$ is an isomorphism onto $\mathring{X}_{\text{NC}}^A$; and
- (2) if $u \neq z_A$ then $\pi|_{\mathring{X}_{\text{NC}}^u}$ has positive-dimensional fibers.

Proof. By definition $u \in \text{NC}_n^A$ has $u \cdot [r] = A$, so the remarks above imply that $\pi(\mathring{X}_{\text{NC}}^u) \subseteq \mathring{X}^A$.

If $u = z_A$, Theorem 3.13 tells us that $\text{Inv}_{\text{NC}}(u) = \text{Inv}^Q(A)$, so $\mathring{X}_{\text{NC}}^u$ contains none of the linear subspaces which generate $\ker(\pi|_{\mathring{X}^u})$ and maps onto $\mathring{X}_{\text{NC}}^A$.

If $u \neq z_A$, we show that $\mathring{X}_{\text{NC}}^u$ intersect the kernel $\ker(\pi|_{\mathring{X}^u})$ nontrivially. Indeed, by Lemma 3.5 there is some $i \in \text{Des}_{\text{NC}}(u) \setminus \{r\}$, and $(u(i+1), u(i)) \in \text{Inv}_{\text{NC}}(u)$ with $u(i+1), u(i) \leq r$ or $r < u(i+1), u(i)$. Then the corresponding coordinate subspace in $\mathring{X}_{\text{NC}}^u$ is in the kernel of $\pi|_{\mathring{X}^u}$. $\square$

¹Noncrossing Schubert cells are called noncrossing Bruhat cells in [6].

Example 5.4. Let $\lambda = (3, 3, 2)$. Then $A = \{3, 5, 6\}$, $z_A = 653241$, and the noncrossing inversions are $\text{Inv}_{\text{NC}}(z_A) = \{(1, 6), (1, 5), (4, 5), (2, 5), (2, 3)\}$. We then compute

$$\{(i, z_A^{-1}(j)) \mid (i, j) \in \text{Inv}_{\text{NC}}(z_A)\} = \{(1, 1), (1, 2), (4, 2), (2, 2), (2, 3)\},$$

so the associated noncrossing Schubert cell is given by

$$\dot{X}_{\text{NC}}^{z_A} = \begin{bmatrix} * & * & \boxed{0} & \boxed{0} & \boxed{0} & 1 \\ \boxed{0} & * & * & 1 & 0 & 0 \\ \boxed{0} & \boxed{0} & 1 & 0 & 0 & 0 \\ \boxed{0} & * & 0 & 0 & 1 & 0 \\ \boxed{0} & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cong \mathbb{A}^5.$$

The column span of the first 3 columns then gives the isomorphic projection to $\dot{X}_{\text{NC}}^\lambda = \text{Gr}(3; 6)$. This agrees with the earlier construction: $\text{Inv}(A) \setminus \text{Inv}^Q(A) = \{(2, 6), (4, 6), (1, 3)\}$ and

$$\begin{array}{c} 1 \quad 2 \quad 4 \\ 6 \quad \bullet \quad \bullet \quad \bullet \\ 5 \quad \bullet \quad \bullet \quad \bullet \\ 3 \quad \bullet \quad \bullet \quad \bullet \end{array} \rightarrow \dot{X}_{\text{NC}}^A = \begin{array}{c} 3 \\ 5 \\ 6 \end{array} \begin{bmatrix} \boxed{0} & * & 1 & 0 & 0 & 0 \\ * & * & 0 & * & 1 & 0 \\ * & \boxed{0} & 0 & \boxed{0} & 0 & 1 \end{bmatrix}^\top \cong \mathbb{A}^5.$$

We now consider the (noncrossing) Schubert cells under the action of the subgroup $T \subseteq \text{GL}_n$ of diagonal matrices. We will denote by χ_i the character of T that maps $\text{Diag}(h_1, \dots, h_n) \in T$ to h_i , and write $\mathbb{C}_\chi$ for the corresponding one-dimensional representation.

The left action of T on GL_n/B makes $\dot{X}^w$ into a linear T -representation, and this action scales the $*$ in entry $(i, w^{-1}(j))$ by the character χ_i/χ_j . We therefore have

$$\dot{X}^w \cong \bigoplus_{(i,j) \in \text{Inv}(w)} \mathbb{C}_{\chi_i/\chi_j} \quad \text{and} \quad \dot{X}_{\text{NC}}^u \cong \bigoplus_{(i,j) \in \text{Inv}_{\text{NC}}(u)} \mathbb{C}_{\chi_i/\chi_j}.$$

The T action on Grassmannian Schubert cells is similar: for $A \in \binom{[n]}{r}$, T acts on $\dot{X}^A$ by scaling the $*$ corresponding to $(i, j) \in \text{Inv}(A)$ by χ_i/χ_j , so that

$$\dot{X}^A \cong \bigoplus_{(i,j) \in \text{Inv}(A)} \mathbb{C}_{\chi_i/\chi_j} \quad \text{and} \quad \dot{X}_{\text{NC}}^A \cong \bigoplus_{(i,j) \in \text{Inv}^Q(A)} \mathbb{C}_{\chi_i/\chi_j}.$$

The projection map $\pi : \text{Fl}_n \rightarrow \text{Gr}(r; n)$ induces maps $\dot{X}^w \rightarrow \dot{X}^{w \cdot [r]}$ and $\dot{X}_{\text{NC}}^u \rightarrow \dot{X}_{\text{NC}}^A$ by sending $\mathbb{C}_{\chi_i/\chi_j}$ to either $\mathbb{C}_{\chi_i/\chi_j}$ or 0 according to whether $(i, j) \in \text{Inv}(A)$.

Proof of Theorem 5.1. By Proposition 4.3 and Theorem 5.3 we know that

$$\text{PV}_{QJ_{r,n}} \subseteq \bigsqcup \dot{X}_{\text{NC}}^\lambda \subseteq \bigcup X_{\text{NC}}^\lambda \subseteq \text{QGr}(r; n),$$

so what remains is to show that $\text{QGr}(r; n) \subseteq \text{PV}_{QJ_{r,n}}$.

We start by showing that any 1-dimensional T -orbit closure in $\text{QGr}(r; n)$ is of the form $\langle e_A, e_B \rangle$ with $AB \in E(QJ_{r,n})$. By definition $\text{QGr}(r; n) = \pi(\text{QFl}_n) = \bigcup_{u \in \text{NC}_n} \pi(\mathring{X}_{\text{NC}}^u)$. Since no two characters of $\mathring{X}^A$ are linearly dependent, any 1-dimensional T -orbit in $\pi(\mathring{X}_{\text{NC}}^u)$ is the T -orbit of an interior point x on a coordinate line $\langle e_A, e_B \rangle$. Moreover $\pi|_{\mathring{X}_{\text{NC}}^u}$ is a coordinate projection onto a coordinate subspace of $\mathring{X}_{\text{NC}}^A$ for $A = u \cdot [r]$. Using the coordinate projection property we can find a point $z \in \mathring{X}_{\text{NC}}^u$ such that $\pi(z) = x$ and z lies in the interior of a coordinate line connecting u to some $(i, j)u$ with $(i, j) \in \text{Inv}_{\text{NC}}(u)$. We therefore have

$$\overline{T \cdot x} = \langle e_A, e_B \rangle, \quad A = u \cdot [r], \quad B = (i, j)u \cdot [r],$$

and $(i, j) \in \text{Inv}^Q(A)$ by the definition of $\mathring{X}_{\text{NC}}^A$.

To show $\text{QGr}(r; n) \subset \text{PV}_{QJ_{r,n}}$, by the definition of $\text{PV}_{QJ_{r,n}}$ it suffices to show that for any $x \in \text{QGr}(r; n)$ and any edge $AB \in E(J_{r,n}) \setminus E(QJ_{r,n})$, we have $(\Delta_A \Delta_B)(x) = 0$. Suppose on the contrary that $(\Delta_A \Delta_B)(x) \neq 0$. Writing $B = (A \setminus j) \cup i$, consider the cocharacter $\rho : \mathbb{C}^* \rightarrow T$ defined by

$$\rho(t) = (t^{-b_1}, \dots, t^{-b_n}) \quad \text{where} \quad b_k = \delta_{k \in A} + \delta_{k \in B}.$$

Then $x' = \lim_{t \rightarrow 0} \rho(t)x \in \text{QGr}(r; n)$ because $\text{QGr}(r; n) = \pi(\text{QFl}_n)$ is closed and T -invariant (as π is proper and T -equivariant). This x' lies in the interior of $\langle e_A, e_{A'} \rangle$, as in $\mathbb{P}(\Lambda^r \mathbb{C}^n)$ the cocharacter $\rho(t)$ scales each e_C by t^{-N_C} with

$$N_C = 2|C \cap (A \cap B)| + |C \cap \{i, j\}|,$$

a quantity that is maximized simultaneously at $C \in \{A, B\}$. But then $\langle e_A, e_B \rangle = \overline{T \cdot x'} \subset \text{QGr}(r; n)$, contradicting the fact that every 1-dimensional T -orbit closure in $\text{QGr}(r; n)$ satisfies $AB \in E(QJ_{r,n})$.

For the affine paving statement, since $\mathring{X}_{\text{NC}}^\lambda \cong \mathbb{A}^{\text{OHL}(\lambda)}$ it suffices to show that under the total ordering $\varnothing = \lambda_1, \lambda_2, \dots$ we have $X_k := \bigsqcup_{i \leq k} \mathring{X}_{\text{NC}}^{\lambda_i}$ is closed. This follows as

$$X_k = \text{QGr}(r; n) \cap \bigsqcup_{i=1}^k \mathring{X}^{\lambda_i} = \text{QGr}(r; n) \cap \bigcup_{i=1}^k X^{\lambda_i},$$

where the last equality uses the fact that $X^\lambda = \bigsqcup_{\lambda' \subset \lambda} \mathring{X}^{\lambda'}$. □

See Figure 7 which shows the affine paving

$$\text{QGr}(2; 4) = \underbrace{\mathring{X}^{12}}_{\mathbb{A}^0} \sqcup \underbrace{\mathring{X}^{13}}_{\mathbb{A}^1} \sqcup \underbrace{\mathring{X}^{14}}_{\mathbb{A}^2} \sqcup \underbrace{\mathring{X}^{23}}_{\mathbb{A}^2} \sqcup \underbrace{\mathring{X}^{24}}_{\mathbb{A}^3} \sqcup \underbrace{\begin{bmatrix} * & * & 1 & 0 \\ * & \boxed{0} & 0 & 1 \end{bmatrix}^\top}_{\mathring{X}_{\text{NC}}^{34} \cong \mathbb{A}^3}$$

in terms of moment polytopes: the closures of the top-dimensional cells $X_{\text{NC}}^{24} = X^{24}$ and X_{NC}^{34} correspond respectively to the blue and yellow pyramids.

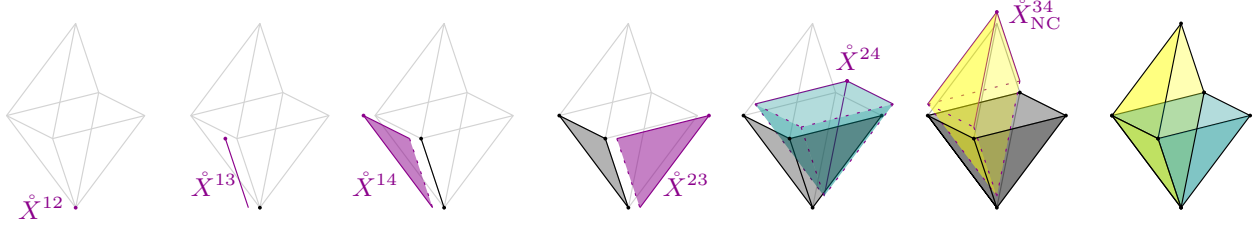


FIGURE 7. Paving $\text{QGr}(2; 4)$ with quasisymmetric Schubert cells. The T -orbits added by the affine cell $\hat{X}_{\text{NC}}^{ij}$ correspond to faces of the moment polytope for $\hat{X}_{\text{NC}}^{ij}$ that touch vertex ij . The union of the new faces is homeomorphic to a half-open cube.

6. POSITROIDS, RICHARDSONS, AND COMBINATORIAL RIGIDITY

In this section we connect the quasisymmetric Schubert cycles to positroid varieties and Richardson varieties. Using these connections we show that the complex of moment polytopes determines the inclusion relations between the T -orbit closures in $\text{QGr}(r; n)$.

6.1. Positroids. We now recall both geometric and combinatorial notions pertaining to total positivity as first described by Postnikov [29] and further developed by Knutson–Lam–Speyer [18]; see also [31, 35] for a survey of the various mathematical fields this touches.

Definition 6.1. A matrix $M \in \text{Mat}_{n \times r}^{\circ}$ with real entries is *totally nonnegative* if all its maximal minors satisfy $\Delta_A(M) \geq 0$. Its *positroid* is

$$\mathcal{A}(M) := \{ A \in \binom{[n]}{r} \mid \Delta_A(M) > 0 \}.$$

A subset $\mathcal{A} \subseteq \binom{[n]}{r}$ is a *positroid* if $\mathcal{A} = \mathcal{A}(M)$ for some totally nonnegative M . The *totally nonnegative Grassmannian* $\text{Gr}(r; n)_{\geq 0}$ is the set of real linear spaces determined by the column spans of the totally nonnegative matrices. A *positroid variety* is the closure in the complex Grassmannian $\text{Gr}(r; n)$ of the points in $\text{Gr}(r; n)_{\geq 0}$ with the same positroid.

Positroids admit a combinatorial description in terms of Le diagrams [29, Definition 6.1]. We isolate a subset of such diagrams that is relevant to us.

Definition 6.2. A subset $L \subseteq \lambda$ is called a *noncrossing Le diagram* if it satisfies the following two conditions:

- (1) (*Le condition*) If a square in λ lies strictly below a square of L in the same column and strictly to the right of a square of L in the same row, then it also belongs to L .
- (2) (*noncrossing condition*) For every box $b \in L$, either there is no box of L strictly above b in its column, or there is no box of L strictly to the left of b in its row.

Remark 6.3. Every noncrossing Le diagram determines a unique *noncrossing alternating forest* in the sense of [1]. Suppose that $L \subset \lambda_A$ for $\lambda_A \in \text{Part}_{r,n}$ and consider the subset of $\text{Inv}(A)$ determined by the boxes of L . If we represent each inversion (i, j) by an arc from i to j , as shown in Figure 8, the result is a noncrossing alternating forest. If L is maximal under inclusion, then the corresponding forest will be a *noncrossing alternating tree* [13, 30] with $\text{OHL}(\lambda)$ -many arcs.

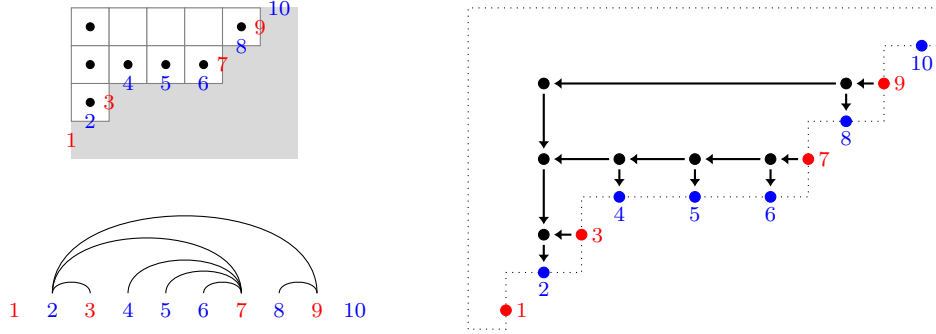


FIGURE 8. Clockwise from the top left: a noncrossing Le diagram $L \subseteq \lambda = (5, 4, 1)$ marked by $\bullet$, the Γ -network for L , and the noncrossing alternating forest for L .

Proposition 6.4. For a noncrossing Le diagram L , let $\hat{X}_L^\lambda \subseteq \hat{X}^\lambda$ be the subspace with all coordinates outside L equal to zero. Then $X_L^\lambda = \overline{\hat{X}_L^\lambda}$ is the positroid variety with Le diagram L .

Proof. Each Le diagram defines an acyclic directed planar “ Γ -network” on the vertices $[n] \cup L$; see Figure 8. Recall by Fact 4.2 that in the coordinatization of $\hat{X}^\lambda$, the $*$ corresponding to the inversion $(i, j) \in \text{Inv}(A_\lambda)$ is given by $\pm \Delta_{(A_\lambda \setminus j) \cup i} / \Delta_{A_\lambda}$. The positroid variety associated to L is the closure of a subset of $\hat{X}^\lambda$ parametrized by $\vec{c} \in (\mathbb{C}^*)^L$, whose coordinates are uniquely determined by

$$\Delta_{(A_\lambda \setminus j) \cup i} / \Delta_{A_\lambda} = \sum_{P: j \rightarrow i} \text{wt}_P(\vec{c}),$$

where the sum above is over directed walks P from j to i in the network and $\text{wt}_P(\vec{c})$ is the product of c_b for each $b \in L$ entered on a horizontal edge in P . If L is noncrossing, there is a walk $P: j \rightarrow i$ only if $(i, j) \in \text{Inv}^Q(A)$, in which case P is unique and $\text{wt}_P(\vec{c})$ is the product of c_b weakly right of the box corresponding to (i, j) . This parametrizes a dense subset of $\hat{X}_L^\lambda$ (where the $*$ are nonzero), so the closure X_L^λ is the positroid variety associated to L . $\square$

Every T -orbit in $\text{QGr}(r; n)$ is obtained by taking the $0, 1, *$ chart for some X_{NC}^λ and replacing each $*$ with either a $\times$ to represent a nonzero indeterminate $\mathbb{C}^*$ or a 0 . Furthermore, it is immediate to check that the inversions in $\text{QGr}(r; n)$ satisfy the noncrossing Le condition. We therefore conclude the following.

Corollary 6.5. The Le diagram of the associated positroid varieties in $\text{QGr}(r; n)$ are precisely the noncrossing Le diagrams associated to the subset of $\hat{X}_{\text{NC}}^\lambda$ occupied by $\times$ as described above.

6.2. Projected and Translated Richardsons. Let $w_o \in S_n$ be the longest permutation. For $u \leq v$ in Bruhat order, the *Richardson variety* $X_u^v \subset \text{Fl}_n$ is defined by

$$X_u^v = X^v \cap w_o X^u,$$

the intersection of a Schubert and an opposite Schubert variety. In the Grassmannian, one defines analogously the Richardson varieties $X_\eta^\nu = X^\nu \cap w_o X^\eta$ for $\eta \subset \nu$.

We describe two ways in which X_{NC}^λ is related to Richardson varieties. First, by [18, 19], each positroid stratum $X_{\text{NC}}^\lambda \subset \text{Gr}(r; n)$ is a *projected Richardson variety* of the form

$$X_{\text{NC}}^\lambda = \pi(X_u^{w_\lambda}),$$

where $\pi|_{X_u^{w_\lambda}}$ is birational onto its image. Here w_λ is the r -Grassmannian permutation w_A for $A \in \binom{[n]}{r}$ corresponding to λ , introduced in Section 2. The permutation u is read from the Le diagram as a pipe dream [29, Section 19]: replace each box of L with an elbow tile and each box of $\lambda \setminus L$ with a cross tile, then follow the strands from the south-east boundary to the north-west to obtain u in one-line notation. See Figure 9, where this procedure yields $u = 123485679\ 10$.



FIGURE 9. The Le diagram determining the permutation u via a pipe dream construction

Corollary 6.6. X_{NC}^λ is a projected Richardson variety $\pi(X_u^{w_\lambda})$ for u as described above.

Now we give an alternate description: X_{NC}^λ is in fact a *translated Richardson variety*. This follows from an analysis carried out in [24, Section 10] that applies to the noncrossing Schubert cycle $X_{\text{NC}}^{z_A}$ where $z_A \in \text{QGrass}_{r,n}$ corresponds to λ . We record the details of the construction in the terminology of the present paper.

Let $\alpha \in \text{Comp}_{r,n}$ be the composition associated to λ under the bijection of Theorem 3.8. Suppose η/ν is the skew shape associated to the ribbon α (i.e. η/ν is a rimhook with $\eta_i - \nu_i = \alpha_i$). We choose η and ν so that η/ν has its extreme boxes in the first column and first row. Set $w = 12 \cdots (r - \ell(\alpha))r(r - 1) \cdots (r - \ell(\alpha) + 1)(r + 1)(r + 2) \cdots$, and suppose further that w_η and w_ν are the r -Grassmannian permutations associated with η and ν respectively. Then for $(x, y) = (w_\nu w, w_\eta w)$ we have that

$$X_{\text{NC}}^{z_A} = x^{-1} X_x^y.$$

As discussed in [24, Proposition 10.5], since the same $w \in S_r \times S_{n-r}$ is applied on the right to both w_μ and w_η to obtain x, y , we have

$$\pi(X_x^y) = \pi(X_{w_\nu}^{w_\eta}) = X_\nu^\eta.$$

We thus obtain the following.

Theorem 6.7. With notation as before, we have the equality $X_{\text{NC}}^\lambda = x^{-1}X_\nu^\eta$.

Corollary 6.8. Every T -orbit closure in $\text{QGr}(r; n)$ is a translated Richardson variety.

Proof. This follows from Theorem 6.7 and the fact that every T -orbit closure contained in a toric Richardson variety is a toric Richardson variety [7, Proposition 8.1]. $\square$

Figure 10 shows the relevant information for $\lambda = (4, 3, 1) \in \text{Part}_{4,9}$ corresponding to $\alpha = (2, 1, 3) \in \text{Comp}_{4,9}$ under the bijection from Theorem 3.8.

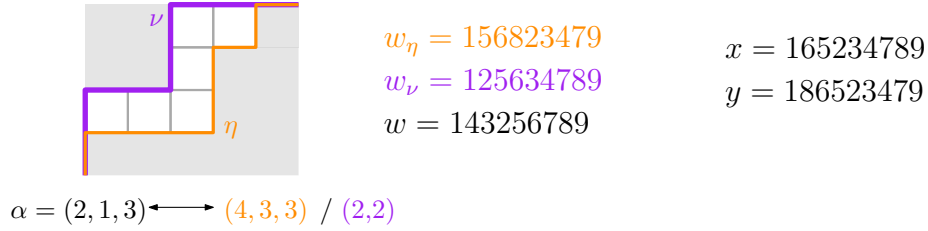


FIGURE 10. The data of η , ν , and x determining the translated Richardson for $X_{\text{NC}}^{(4,3,1)}$

6.3. Rigidity. Using equivariant rigidity theorems we now show the following.

Theorem 6.9. Each T -orbit closure in $\text{QGr}(r; n)$ is

- (1) uniquely determined as a T -invariant subvariety of $\text{Gr}(r; n)$ by its T -fixed point set, and
- (2) containment of T -orbit closures in $\text{QGr}(r; n)$ corresponds to containment of T -fixed points.

Proof. To show (1) and (2) it suffices to show that if X is a T -orbit closure in $\text{QGr}(r; n)$ and $Y \subset \text{Gr}(r; n)$ is a T -invariant subvariety with $Y^T \subset X$ then $Y \subset X$. This follows either by the fact that these T -orbit closures are positroid varieties (Corollary 6.5) by [11, Corollary 6.4], or the fact that they are translates of Richardson varieties (Corollary 6.8) by [11, Theorem 6.3]. $\square$

These facts imply that if a face is common to two moment polytopes then the corresponding T -orbit closures share the same sub- T -orbit closure, and so the “moment complex” of $\text{QGr}(r; n)$ completely determines the combinatorics of how it is assembled as a toric complex.

7. EQUIVARIANTLY QUASISYMMETRIC POLYNOMIALS

This section summarizes the essential facts about fundamental and double fundamental polynomials from [5, 6]. In what follows we let $\mathbf{x} = (x_1, x_2, \dots)$ and $\mathbf{t} = (t_1, t_2, \dots)$ be infinite variable sets, and let $\mathbf{x}_k = (x_1, \dots, x_k)$ and $\mathbf{t}_k = (t_1, \dots, t_k)$ be their finite truncations, obtained by setting $x_{k+1} = \dots = 0$ and $t_{k+1} = \dots = 0$. The adjective *non-equivariant* will mean that only $\mathbf{x}$ variables are used, and equivariant will mean that $\mathbf{x}$ and $\mathbf{t}$ variables are used. To pass from equivariant to non-equivariant we will set $\mathbf{t} = 0$.

The *quasisymmetric polynomials* $\text{QSym}_n \subset \mathbb{Z}[\mathbf{x}_n]$ are those polynomials which satisfy the coefficient equality of $x_1^{a_1} \dots x_k^{a_k}$ and $x_{i_1}^{a_1} \dots x_{i_k}^{a_k}$ for every increasing sequence $i_1 < \dots < i_k$. There is an equivalent condition defining quasisymmetric polynomials that we now recall. Define the *Bergeron–Sottile map* [8, 25] to be the map

$$R_i f(\mathbf{x}) = f(x_1, \dots, x_{i-1}, 0, x_i, x_{i+1}, \dots).$$

In particular because of the reindexing we have $R_i f(\mathbf{x}_n) \in \mathbb{Z}[\mathbf{x}_{n-1}]$ for $1 \leq i \leq n$. Then $f(\mathbf{x}_n) \in \text{QSym}_n$ if and only if $R_i f(\mathbf{x}_n) = R_{i+1} f(\mathbf{x}_n)$ for $i = 1, \dots, n-1$ [25, Theorem 2.1].

In [5] we introduced with P. Nadeau a deformation of the above equivalent definition of quasisymmetry. Define the *equivariant Bergeron–Sottile operations* are the maps

$$\begin{aligned} R_i^- f(\mathbf{x}; \mathbf{t}) &= f(x_1, \dots, x_{i-1}, t_i, x_i, x_{i+1}, \dots; \mathbf{t}) \\ R_i^+ f(\mathbf{x}; \mathbf{t}) &= f(x_1, \dots, x_{i-1}, x_i, t_i, x_{i+1}, \dots; \mathbf{t}). \end{aligned}$$

We say that $f(x_1, \dots, x_r; \mathbf{t})$ is *equivariantly quasisymmetric* if $R_i^- f = R_i^+ f$ for $i = 1, \dots, r-1$.

Definition 7.1. Write EQSym_r for the set of equivariantly quasisymmetric polynomials $f(\mathbf{x}_r; \mathbf{t}_r)$, and write $\text{EQSym}_r[t_{r+1}, \dots, t_n] := \text{EQSym}_r \otimes \mathbb{Z}[t_{r+1}, \dots, t_n]$.

In analogy to the fundamental basis of QSym_r , [5] constructs a family of *double fundamental quasisymmetric polynomials* $F_\alpha(\mathbf{x}_r; \mathbf{t})$ indexed by compositions, which specialize to the fundamental polynomials recalled in the introduction when $\mathbf{t} = 0$. Here we consider only their truncations at $t_{n+1} = t_{n+2} = \dots = 0$, which we denote by $F_\alpha(\mathbf{x}_r; \mathbf{t}_n)$ (when $\alpha \in \text{Comp}_{r,n}$ it turns out that F_α had no dependence on the variables outside of $\mathbf{x}_r$ and $\mathbf{t}_n$ before truncation).

Fact 7.2 ([5, Theorem 4.14]). The ring $\text{EQSym}_r[t_{r+1}, \dots, t_n]$ has a free $\mathbb{Z}[\mathbf{t}_n]$ -basis of double fundamental polynomials $\{F_\alpha(\mathbf{x}_r; \mathbf{t}_n) \mid \ell(\alpha) \leq r\}$.

The second result is one of the main theorems of [5]; we restate the special case for quasisymmetric permutations only. Identifying S_n as a subset of S_{n+1} in the usual way makes the r -quasisymmetric permutations into a tower

$$\text{QGrass}_{r,r+1} \subseteq \text{QGrass}_{r,r+2} \subseteq \dots$$

Let $\text{QGrass}_{r,\infty}$ to be the union of all $\text{QGrass}_{r,n}$ under this identification. We then have the following consequence of the “AJS–Billey”-type formula for double forest polynomials.

Fact 7.3 ([5, Theorem 8.3]). Let $u \in \text{QGrass}_{r,\infty}$, and let α be the composition corresponding to u under the bijection of (3.2), which is unique up to trailing zeros. Then

$$F_\alpha(t_{u(1)}, t_{u(2)}, \dots, t_{u(n)}; \mathbf{t}_n) = \prod_{(i,j) \in \text{Inv}_{\text{NC}}(u)} (t_j - t_i),$$

and if $w \in \text{QGrass}_{r,\infty}$ with $w \not\geq u$ in Bruhat order, then $F_\alpha(t_{w(1)}, t_{w(2)}, \dots, t_{w(n)}; \mathbf{t}_n) = 0$.

8. COHOMOLOGY

This section proves Theorems A and D, about the cohomology of $\text{QGr}(r; n)$ and quasisymmetric polynomials. In order to do so, we first prove several results of independent interest about the equivariant cohomology ring $H_T^\bullet(\text{QGr}(r; n))$ with respect to the standard torus $T = (\mathbb{C}^*)^n$. We identify the T -equivariant cohomology of a point with the polynomial ring

$$H_T^\bullet(pt) = \text{Sym}^\bullet \text{Char}(T) = \mathbb{Z}[\mathbf{t}_n],$$

where we take the standard convention used in double Schubert calculus that t_i is the generator corresponding to χ_i^{-1} , the inverse of the i 'th standard character of T . This convention ensures for example that double Schur polynomials $s_\lambda(\mathbf{x}_r; \mathbf{t}_n) \in H_T^\bullet(\text{Gr}(r; n))$ are T -equivariantly Kronecker dual to the T -equivariant homology classes of Grassmannian Schubert cycles X^λ in $H_\bullet^T(\text{Gr}(r; n))$.

8.1. Graph Cohomology Rings. Let G be a finite simple graph with a labelling χ that assigns a linear form $\chi(uv) \in \mathbb{Z}[\mathbf{t}_n]$ to each edge $uv \in E(G)$. The *equivariant graph cohomology ring* associated to G and χ is the $\mathbb{Z}[\mathbf{t}_n]$ -algebra

$$H_T^\bullet(G) := \left\{ (f_v)_{v \in V(G)} \in \mathbb{Z}[\mathbf{t}_n]^{V(G)} \mid \chi(vw) \text{ divides } f_v - f_w \text{ for all } vw \in E(G) \right\}.$$

with pointwise addition, multiplication, and scaling. We define the *singular graph cohomology ring*

$$H^\bullet(G) := H_T^\bullet(G) / (t_1, t_2, \dots, t_n),$$

where t_i denotes the constant assignment in $H_T^\bullet(G)$ with value t_i .

Goresky–Kottwitz–MacPherson [15] show that graph cohomology rings compute the T -equivariant and singular cohomology rings of suitably nice T -spaces. For example, taking G to be the Johnson graph $J_{r,n}$ with edge labelling

$$(8.1) \quad \chi(AB) = t_i - t_j \quad \text{if } B = (A \setminus i) \cup j \text{ for } (i, j) \in \text{Inv}(A),$$

produces T -equivariant and ordinary cohomology rings of the Grassmannian $\text{Gr}(r; n)$, i.e.

$$H_T^\bullet(\text{Gr}(r; n)) \cong H_T^\bullet(J_{r,n}) \quad \text{and} \quad H^\bullet(\text{Gr}(r; n)) \cong H^\bullet(J_{r,n}).$$

Recall from Section 4.1 that $V(J_{r,n})$ and $E(J_{r,n})$ can be identified with the T -fixed points and T -invariant curves in $\text{Gr}(r; n)$. For $X \subseteq \text{Gr}(r; n)$, the *moment graph of X* is the subgraph of $J_{r,n}$ with vertex set X^T and edges the T -invariant curves in X . By the proof of Theorem 5.1, the moment graph of $\text{QGr}(r; n)$ is the quasisymmetric Johnson graph $QJ_{r,n}$.

Theorem 8.1. With the edge labellings from Equation (8.1),

$$H_T^\bullet(\text{QGr}(r; n)) \cong H_T^\bullet(QJ_{r,n}) \quad \text{and} \quad H^\bullet(\text{QGr}(r; n)) \cong H^\bullet(QJ_{r,n}).$$

Proof. A standard application of GKM theory gives the above isomorphisms for any variety $X \subseteq \text{Gr}(r; n)$ and its moment graph provided certain conditions hold. A simple statement of these conditions appears under the name “good affine paving” in [5, §11.1 items (1)–(3)], adapted from [17].

The affine paving of $\text{QGr}(r; n)$ in Theorem 5.1 is “good” in this sense, as each affine cell $\mathring{X}_{\text{NC}}^A$ decomposes into T -representations with characters χ_i/χ_j for $(i, j) \in \text{Inv}^Q(A)$, no two characters in the same chart divide one another, and each standard character χ_i appears in each character with an exponent of ± 1 or 0. $\square$

8.2. A map from quasisymmetric polynomials. We now state an equivariant version of the first presentation in Theorem A. Recall the ring EQSym_n and the polynomials $F_\alpha(\mathbf{x}_n; \mathbf{t}_n)$ from Section 7.

Theorem 8.2. The variety $\text{QGr}(r; n) \subseteq \text{Gr}(r; n)$ has

$$H_T^\bullet(\text{QGr}(r; n)) \cong \frac{\text{EQSym}_r[t_{r+1}, \dots, t_n]}{\langle F_\alpha(\mathbf{x}_r; \mathbf{t}_n) \mid \alpha \notin \text{Comp}_{r,n} \rangle},$$

and the ideal has a free $\mathbb{Z}[\mathbf{t}_n]$ -basis $\{F_\alpha(\mathbf{x}_r; \mathbf{t}_n) \mid \alpha \notin \text{Comp}_{r,n}\}$. As a consequence, the first presentation of Theorem A holds.

Theorem 8.2 is proved at the end of the subsection. We construct an isomorphism from the right hand side of the theorem to the graph cohomology ring $H_T^\bullet(QJ_{r,n})$. For $f(\mathbf{x}_n; \mathbf{t}_n) \in \mathbb{Z}[\mathbf{t}_n][\mathbf{x}_n]$ and $w \in S_n$, let

$$f_w = f(t_{w(1)}, t_{w(2)}, \dots, t_{w(n)}; \mathbf{t}_n) \in \mathbb{Z}[\mathbf{t}_n].$$

Now recall the element $z_A \in \text{QGrass}_{r,n} \subseteq S_n$ from Section 3.2 associated to $A \in \binom{[n]}{r}$. In what follows when we write $\text{EQSym}_r \otimes \text{EQSym}_{n-r}$, the elements from the factor EQSym_{n-r} are to be considered as polynomials $f(x_{r+1}, \dots, x_n; t_{r+1}, \dots, t_n)$.

Lemma 8.3. If $f(\mathbf{x}_n; \mathbf{t}_n) \in \text{EQSym}_r \otimes \text{EQSym}_{n-r}$ and $w \in \text{NC}_n^A$ then $f_w = f_{z_A}$.

Proof. By Lemma 3.5 there is a sequence $w_1 = w, \dots, w_\ell = z_A$ in NC_n such that for each $1 \leq k < \ell$, $w_{k+1} = w_k s_{i_k}$ for $i_k \in \text{Des}_{\text{NC}}(w_k) \setminus \{r\}$. By Fact 3.2, each $i_k \in \{w_k(i_k), w_k(i_k + 1)\}$, so by the definition of equivariant quasisymmetry $f_{w_k} = f_{w_{k+1}}$. $\square$

Now define a map

$$\begin{aligned} \Psi : \text{EQSym}_r \otimes \text{EQSym}_{n-r} &\rightarrow \mathbb{Z}[\mathbf{t}_n]^{\oplus \binom{[n]}{r}} \\ f(\mathbf{x}_n; \mathbf{t}_n) &\mapsto (f_{z_A})_{A \in \binom{[n]}{r}}. \end{aligned}$$

Proposition 8.4. The image of Ψ is contained in the equivariant graph cohomology ring $H_T^\bullet(QJ_{r,n})$.

Proof. We show that each $f \in \text{EQSym}_r \otimes \text{EQSym}_{n-r}$ satisfies the divisibility condition for each edge AB with $B = (A \setminus j) \cup i$ and $(i, j) \in \text{Inv}^Q(A)$. Let $w = (ij)z_A$, so that $w \cdot [r] = B$ and Lemma 8.3 implies $f_w = f_{z_B}$. Then $f_{z_A} - f_{z_B} = f_{z_A} - f_w$, which must be divisible by $t_j - t_i$. $\square$

We now show that Ψ surjects EQSym_r onto $H_T^\bullet(QJ_{r,n})$. For an edge labeled graph G as in the previous section, a collection of elements $\{f^{(v)} \in H_T^\bullet(G) \mid v \in V(G)\}$ is a *flowup basis* with respect to an ordering $<$ on $V(G)$ [5, Section 11.2] if:

$$f_u^{(v)} = 0 \text{ whenever } v \not\prec u \quad \text{and} \quad f_v^{(v)} = \prod_{\substack{uv \in E(G) \\ u < v}} \chi(uv).$$

If a flowup basis exists, then it is a free $\mathbb{Z}[\mathbf{t}_n]$ -basis for $H_T^\bullet(G)$ [5, Proposition 11.9].

Proposition 8.5. The map Ψ sends the set $\{F_\alpha(\mathbf{x}_r; \mathbf{t}_n) \mid \alpha \in \text{Comp}_{r,n}\}$ to a flowup basis of $H_T^\bullet(QJ_{r,n})$ with respect to the Gale order. Further, if $\alpha \notin \text{Comp}_{r,n}$ then $\Psi(F_\alpha(\mathbf{x}_r; \mathbf{t}_n)) = 0$.

Proof. As $A \leq B$ if and only if $z_A \leq z_B$ by Theorem 3.15 and $\text{Inv}^Q(A) = \text{Inv}_{\text{NC}}(z_A)$ by Theorem 3.13, the first claim follows from Fact 7.3. The second claim also follows from Fact 7.3: if $\alpha \notin \text{Comp}_{r,n}$ then the associated noncrossing partition Z does not belong to NC_n , and therefore cannot precede any element of $\text{QGrass}_{r,n}$ in the Bruhat order. $\square$

Proof of Theorem 8.2. As $\text{EQSym}_r[t_{r+1}, \dots, t_n]$ has a free $\mathbb{Z}[\mathbf{t}_n]$ -basis of double fundamental quasisymmetric polynomials, Theorem 8.1 and Proposition 8.5 immediately imply that

$$H_T^\bullet(\text{QGr}(r; n)) \cong \frac{\text{EQSym}_r[t_{r+1}, \dots, t_n]}{\langle F_\alpha(\mathbf{x}_r; \mathbf{t}_n) \mid \alpha \notin \text{Comp}_{r,n} \rangle},$$

and moreover that the ideal has a free $\mathbb{Z}$ -basis $\{F_\alpha(\mathbf{x}_r; \mathbf{t}_n) \mid \alpha \notin \text{Comp}_{r,n}\}$. Quotienting by $(t_1, \dots, t_n)$ then gives the first isomorphism in Theorem A. $\square$

Example 8.6. Figure 11 depicts show the GKM graph for $\text{QGr}(2; 4)$ as well as examples of flowup basis elements obtained by evaluating double fundamental quasisymmetric polynomials. The combinatorial vine model introduced in [5, Section 5] implies the following expansions

$$F_{12}(\mathbf{x}_2; \mathbf{t}_4) = (x_1 - t_3)(x_1 - t_1)(x_2 - t_1)$$

$$F_2(\mathbf{x}_2; \mathbf{t}_4) = x_2^2 + x_1x_2 - x_2t_3 - x_2t_2 + x_1^2 - x_1t_3 - x_1t_2 + t_2t_3 - x_2t_1 - x_1t_1 + t_1t_3 + t_1t_2$$

using which the reader can check that evaluations at elements in $\text{QGrass}_{2,4}$ produces the polynomials highlighted in red.

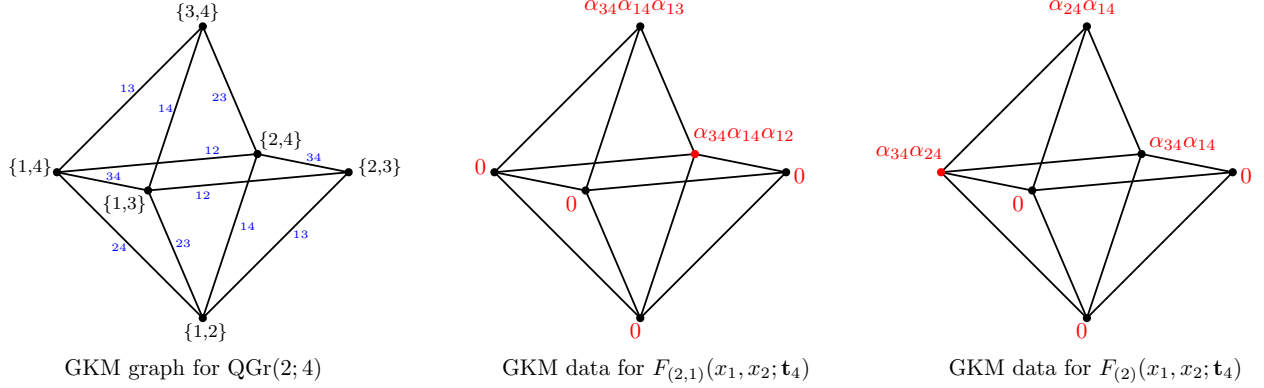


FIGURE 11. Examples of flowup basis elements from evaluations of $F_\alpha(\mathbf{x}_r; \mathbf{t}_n)$, where $\alpha_{ij} := t_j - t_i$.

8.3. A quasisymmetric Borel presentation. We now give the second presentation in Theorem A. As in the previous section, we state an equivariant version which implies the ordinary version.

Theorem 8.7. The variety $\text{QGr}(r; n) \subseteq \text{Gr}(r; n)$ has

$$H_T^\bullet(\text{QGr}(r; n)) \cong \frac{\text{EQSym}_r \otimes \text{EQSym}_{n-r}}{\langle f(\mathbf{x}_n; \mathbf{t}_n) - f(\mathbf{t}_n; \mathbf{t}_n) \mid f \in \text{EQSym}_n \rangle}.$$

As a consequence, the second presentation of Theorem A holds.

Proof. By Proposition 8.5, composing the map Ψ with the isomorphism of Theorem 8.1 gives a surjection from $\text{EQSym}_r \otimes \text{EQSym}_{n-r}$ onto $H_T^\bullet(\text{QGr}(r; n))$. We therefore only need to show that every element of $\text{EQSym}_r \otimes \text{EQSym}_{n-r}$ can be written modulo the ideal as a $\mathbb{Z}[\mathbf{t}_n]$ -linear combination of $\{F_\alpha(\mathbf{x}_r; \mathbf{t}_n) \mid \alpha \in \text{Comp}_{r,n}\}$.

By Fact 7.2, the top $\mathbf{x}_n$ -degree part of any element of $\text{EQSym}_r \otimes \text{EQSym}_{n-r}$ is an ordinary quasisymmetric polynomial. Thus by induction on $\mathbf{x}_n$ -degree, it is sufficient to show that every element of $\text{QSym}_r \otimes \text{QSym}_{n-r}$ can be written as a linear combination of the ordinary fundamentals $\{F_\alpha(\mathbf{x}_r) \mid \alpha \in \text{Comp}_{r,n}\}$ modulo the ideal $\langle f - f(0) \mid f \in \text{QSym}_n \rangle$. The coproduct formula for fundamental quasisymmetric polynomials $F_\beta(\mathbf{x}_n)$ induces an upper triangular change of basis

$$\{F_\alpha(\mathbf{x}_r)F_\beta(x_{r+1}, \dots, x_n) \mid \ell(\alpha) \leq r, \ell(\beta) \leq n-r\} \rightarrow \{F_\alpha(\mathbf{x}_r)F_\beta(\mathbf{x}_n) \mid \ell(\alpha) \leq r, \ell(\beta) \leq n-r\}$$

with respect to $|\beta|$. The elements $F_\alpha(\mathbf{x}_r)F_\beta(\mathbf{x}_n)$ belong to our ideal if $|\beta| > 0$, so we need only reduce the elements $F_\alpha(\mathbf{x}_r)$ modulo the ideal.

To do so, we appeal to the algorithm constructed in [25, §9.2]; while this algorithm was originally phrased in the language of forest polynomials, we summarize the relevant case using compositions. Every $\alpha \notin \text{Comp}_{r,n}$ determines two compositions $\beta = (\beta_1, \dots, \beta_k) \in \text{Comp}_{r,n}$ and $\gamma = (\gamma_1, \dots, \gamma_\ell) \neq \emptyset$ so that $\alpha = (\gamma_1, \dots, \gamma_\ell + \beta_1, \dots, \beta_k)$ and $|\beta|$ is as large as possible. Then the

proofs of [25, Theorem 9.7, Lemma 9.8] show that we can write

$$F_\alpha(\mathbf{x}_r) = F_\beta(\mathbf{x}_r)F_\gamma(\mathbf{x}_n) + \sum_{\substack{\theta \in \text{Comp}_{r,n} \\ |\theta| < |\beta|}} F_\theta(\mathbf{x}_r)f_\theta(\mathbf{x}_n) \quad \text{with } f_\theta(\mathbf{x}_n) \in \text{QSym}_n \text{ of degree } |\alpha| - |\theta|.$$

As $|\alpha| - |\theta| > |\gamma| > 0$, each term on the right is contained in $\langle f - f(0) \mid f \in \text{QSym}_n \rangle$. $\square$

8.4. Kronecker duality and positivity. We conclude the section with a proof of Theorem D. If X is a projective (not necessarily irreducible) variety with a T -action, then there are *Kronecker pairings*

$$\langle -, - \rangle_X^T : H_\bullet^T(X) \otimes_{H^\bullet(pt)} H_T^\bullet(X) \rightarrow H_T^\bullet(pt) = \mathbb{Z}[\mathbf{t}_n]$$

and

$$\langle -, - \rangle_X : H_\bullet(X) \otimes H^\bullet(X) \rightarrow H^\bullet(pt) = \mathbb{Z},$$

where $H_\bullet^T(X)$ and $H_T^\bullet(X)$ are T -equivariant cohomology and T -equivariant Borel–Moore homology respectively. The pairing is defined to be zero if the cohomological degree and homological dimension differ. If $Y \subset X$ is a projective toric variety in X , then for cohomology classes $\tilde{f} \in H_T^\bullet(X)$ and $f \in H^\bullet(X)$ we can express the Kronecker pairings

$$\langle [Y], \tilde{f} \rangle_X^T = \int_Y^T \tilde{f} \in H_T^\bullet(pt) = \mathbb{Z}[\mathbf{t}_n] \quad \text{and} \quad \langle [Y], \tilde{f} \rangle_X = \int_Y f \in H^\bullet(pt) = \mathbb{Z},$$

where $\tilde{f}$ and f are considered in $H_T^\bullet(Y)$ and $H^\bullet(Y)$ via pullback from the inclusion $Y \hookrightarrow X$, and the *degree maps* $\int_Y^T$ and $\int_Y$ are computed directly on Y by $\kappa_* \rho^*$, where $\rho : \tilde{Y} \rightarrow Y$ is any toric resolution and $\kappa : \tilde{Y} \rightarrow pt$ is the projection.

Recall the affine paving $\{\check{X}_{\text{NC}}^A \mid A \in \binom{[n]}{r}\}$ into T -representations from Theorem 5.1. A standard property of affine pavings implies the cell closures $X_{\text{NC}}^A = \overline{\check{X}_{\text{NC}}^A}$ determine a free $\mathbb{Z}$ -basis for $H_\bullet(\text{QGr}(r; n))$ [12, Ex. 19.1.11(b)] and a free $\mathbb{Z}[\mathbf{t}_n]$ basis for $H_\bullet^T(\text{QGr}(r; n))$ [16, Proposition 2.1].

Theorem 8.8. The cohomological basis $\{F_\alpha(\mathbf{x}_r; \mathbf{t}_n) \mid \alpha \in \text{Comp}_{r,n}\}$ is Kronecker dual to the homology basis $\{[X_{\text{NC}}^A] \mid A \in \binom{[n]}{r}\}$, or equivalently

$$\langle X_{\text{NC}}^A, F_\alpha(\mathbf{x}_r; \mathbf{t}_n) \rangle_{\text{QGr}(r; n)}^T = \int_{X_{\text{NC}}^A}^T F_\alpha(\mathbf{x}_r; \mathbf{t}_n) = \begin{cases} 1 & \text{if } A = A_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Passing to singular cohomology, this implies Theorem D.

Proof. In [6, Theorem 11.11] we show that the map $\text{ev}_{\text{NC}} : H_T^\bullet(\text{QFl}_n) \cong \mathbb{Z}[\mathbf{t}_n][\mathbf{x}_n] / \langle f - f(0) \mid f \in \text{EQSym}_n \rangle \rightarrow \mathbb{Z}[\mathbf{t}_n]^{\oplus \text{NC}_n}$ given by $f(\mathbf{x}_n; \mathbf{t}_n) \mapsto (f_w)_{w \in \text{NC}_n}$ is an injection. Then by Theorem 8.1 we have a commutative diagram

$$\begin{array}{ccccc}
H_T^\bullet(X_{\text{NC}}^{z_A}) & \longleftarrow & H_T^\bullet(\text{QFl}_n) & \xrightarrow{\text{ev}_{\text{NC}}} & \mathbb{Z}[\mathbf{t}_n]^{\oplus \text{NC}_n} \\
\pi^* \uparrow & & \pi^* \uparrow & & \uparrow \\
H_T^\bullet(X_{\text{NC}}^A) & \longleftarrow & H_T^\bullet(\text{QGr}(r; n)) & \xrightarrow{\Psi} & \mathbb{Z}[\mathbf{t}_n]^{\oplus \binom{[n]}{r}}
\end{array}$$

where the unlabeled horizontal arrows are pullbacks of inclusions and the unlabeled vertical arrow sends $(f_A)_{A \in \binom{[n]}{r}}$ to $(f_{w \cdot [r]})_{w \in \text{NC}_n}$.

Now we claim that the middle vertical arrow sends $f(\mathbf{x}_r; \mathbf{t}_n) \in \text{EQSym}_r$ to $f(\mathbf{x}_r; \mathbf{t}_n) \in H_T^\bullet(\text{QFl}_n)$. By Lemma 8.3, $f_w = f_{z_A}$ for $A = w \cdot [r]$, so $\text{ev}_{\text{NC}}(f)$ is the image of $\Psi(f)$ under the right vertical arrow. The right three maps are injective so this determines the middle arrow by commutativity.

By Theorem 5.3 the map $\pi : X_{\text{NC}}^{z_A} \rightarrow X_{\text{NC}}^A$ is T -equivariant and birational. Thus by the push-pull formula and the commutativity of the left part of the diagram, we have

$$\int_{X_{\text{NC}}^A}^T F_\alpha(\mathbf{x}_r; \mathbf{t}_n) = \int_{X_{\text{NC}}^{z_A}}^T F_\alpha(\mathbf{x}_r; \mathbf{t}_n),$$

where the $F_\alpha(\mathbf{x}_r; \mathbf{t}_n)$ on the right is pulled back from $H_T^\bullet(\text{QFl}_n)$. By [6, Theorem 12.12] this is the indicator function for whether z_A corresponds to α under the bijection $\text{QGrass}_{r,n} \rightarrow \text{Comp}_{r,n}$. $\square$

We conclude by considering the degree operations of other cohomology classes. Recall that $f \in \mathbb{Z}[t_1, \dots, t_n]$ is *Graham-positive* if $f \in \mathbb{N}[t_2 - t_1, \dots, t_n - t_{n-1}]$.² The T -equivariant homology class of every T -invariant subvariety $Z \subseteq \text{Gr}(r; n)$ can be written as

$$[Z] = \sum_{\eta \in \text{Part}_{r,n}} a_\eta(\mathbf{t}_n) [X^\eta] \quad \text{with} \quad a_\eta(\mathbf{t}_n) = \langle [Z], s_\eta(\mathbf{x}_r; \mathbf{t}_n) \rangle_{\text{Gr}(r;n)}^T,$$

where $s_\eta(\mathbf{x}_r; \mathbf{t}_n)$ is the double Schur polynomial. Graham showed [16] that for any Z , the $a_\eta(\mathbf{t}_n)$ above are Graham-positive. For $Z = X_\mu^\lambda$, the coefficients are the Littlewood–Richardson coefficients $c_{\mu,\eta}^\lambda(\mathbf{t}_n)$ for double Schur polynomial multiplication [20], which recover the usual Littlewood–Richardson after passing to singular cohomology.

Theorem 8.9. For any T -orbit closure $Z \subset \text{QGr}(r; n)$, the Graham-positive quantity

$$\int_Z^T s_\lambda(\mathbf{x}_r; \mathbf{t}_n) = \langle [Z], s_\lambda(\mathbf{x}_r; \mathbf{t}_n) \rangle_{\text{Gr}(r;n)}^T \in \mathbb{N}[t_2 - t_1, \dots, t_n - t_{n-1}]$$

can be computed with a manifestly positive combinatorial rule.

Proof. First, we claim that $Z = \overline{T \cdot x} \subset \text{QGr}(r; n)$ is the birational projection of a T -orbit closure $Z' = \overline{T \cdot y} \subset \text{QFl}_n$. Indeed, $\text{QGr}(r; n) = \bigsqcup \dot{X}_{\text{NC}}^A$ (Theorem 5.1) and the projection π restricts to an isomorphism $\dot{X}_{\text{NC}}^{z_A} \cong \dot{X}_{\text{NC}}^A$ (Theorem 5.3), so if $x \in \dot{X}_{\text{NC}}^A$ we can take the preimage $y \in \dot{X}_{\text{NC}}^{z_A}$. The push-pull formula shows $\int_Z^T s_\lambda(\mathbf{x}_r; \mathbf{t}_n) = \int_{Z'}^T s_\lambda(\mathbf{x}_r; \mathbf{t}_n)$, and combinatorially positive rules for degrees of $s_\lambda(\mathbf{x}_r; \mathbf{t}_n)$ on T -orbit closures in QFl_n were computed in [6, Corollary 12.14]. $\square$

²Each $t_{i+1} - t_i$ corresponds to a positive root χ_i/χ_{i+1} under our convention that t_i corresponds to χ_i^{-1} .

Applied to each X_{NC}^λ , this gives that

$$\int_{X_{\text{NC}}^\lambda}^T s_\eta(\mathbf{x}_r; \mathbf{t}_n) = \langle [X_{\text{NC}}^\lambda], s_\lambda(\mathbf{x}_r; \mathbf{t}_n) \rangle_{\text{Gr}(r;n)}^T = [F_\alpha(\mathbf{x}_r; \mathbf{t}_n)] s_\eta(\mathbf{x}_r; \mathbf{t}_n)$$

is Graham-positive, a special case of the double Schubert into double forest Graham-positivity [5].

9. FORESTS, MOMENT POLYTOPES, AND EQUIDIMENSIONALITY

We now describe the moment polytope of each cell X_{NC}^λ and prove that the moment complex of $\text{QGr}(r; n)$ is equidimensional. We do so by drawing on similar results for QFl_n from [5, 6].

9.1. Forests and noncrossing partitions. We first recall some essential combinatorics about indexed forests. A *plane binary tree* is a tree T in which each node is either a leaf or an internal node $v \in \text{IN}(T)$ with one left child v_L and one right child v_R . The size of T is $|T| := |\text{IN}(T)|$. There is a unique tree $*$ of size 0, consisting of a single node.

A *plane binary indexed forest* is a sequence $F = (T_1, T_2, \dots)$ of plane binary trees such that all but finitely many T_i are equal to $*$. Let Forest denote the set of all such forests, and let Forest_n denote the subset consisting of those indexed forests whose nontrivial components have leaf labels in $[n]$. There is a bijection

$$\text{ForToNC} : \text{Forest}_n \longrightarrow \text{NC}_n$$

constructed as follows: for each internal node $v \in \text{IN}(F)$ that is a right child, delete the edge joining v to its left child v_L . Consider the resulting connected components. To each one assign a cycle written in decreasing order on its set of leaf labels.

For $z \in \text{QGrass}_{r,n}$, the associated tree $T \in \text{Zigzag}_{r,n}$ is called a *zigzag tree*. These trees are in bijection with compositions in a manner compatible with the bijection $z_A \mapsto \alpha_A$, see Figure 12. We say that $T \in \text{Zigzag}_{r,n}$ is *maximal* if $|T| = n - 1$.

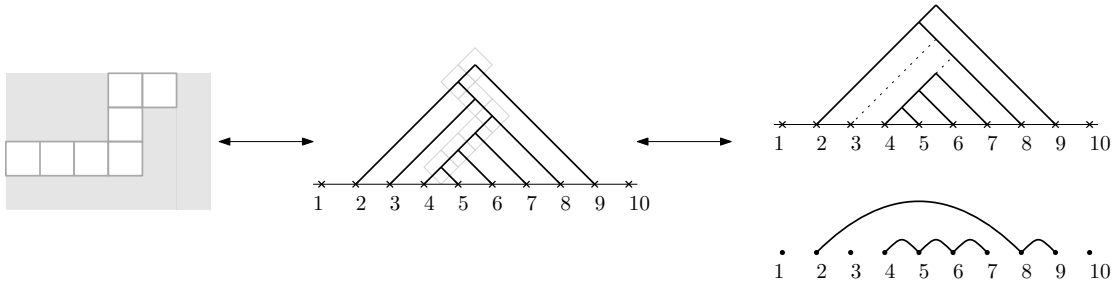


FIGURE 12. Mapping $(2, 1, 4) \in \text{Comp}_{4,10}$ to a zigzag tree in $\text{Zigzag}_{4,10}$ and then to the associated permutation in $\text{QGrass}_{4,10}$.

9.2. Moment Polytopes. We describe the moment polytopes of the X_{NC}^λ in terms of hyperplanes.

Theorem 9.1. Let $A = (a_k, \dots, a_1 \mid_r b_1, \dots, b_k) \in \binom{[n]}{r}$. The moment polytope of X_{NC}^A is the subset of $[0, 1]^n \cap \{z_1 + \dots + z_n = r\}$ cut out by:

- (i) the equalities $z_i = 1$ for $1 \leq i < a_k$ and $z_i = 0$ for $b_k < i \leq n$;
- (ii) $z_{a_i} + z_{a_i+1} + \dots + z_{b_i} \leq r + 1 - a_i$ for $1 \leq i \leq k$;
- (iii) $z_q + z_{q+1} + \dots + z_{b_{i-1}} \geq r - q$ for $q \in [r]$ with $a_i \leq q < a_{i-1}$,

where we set $a_0 = r + 1$ and $b_0 = r$ by convention.

Proof. This follows by specializing [6, Theorem 7.6] which gives an explicit description of the moment polytope of $X_{\text{NC}}^u \subset \text{Fl}_n$ for $u \in \text{NC}_n$ as a polypositroid [22]. More precisely we set the regular dominant weight in the description in [6, Theorem 7.6] to the vector $(1^r, 0^{n-r})$. $\square$

We can also give an explicit vertex description for the moment polytopes once we set up some more notation. It will be convenient for us to use compositions as our indexing objects. Let $\alpha \in \text{Comp}_{r,n}$. As in Theorem 3.8, parse α uniquely as $d_1 1^{e_1-1} (d_2 + 1) 1^{e_2-1} \dots (d_k + 1) 1^{e_k-1}$ subject to the constraints $d_i, e_i \geq 1$ for $1 \leq i \leq k$. This datum determines a decomposition of the interval $\{r - \ell(\alpha) + 1, \dots, r - \ell(\alpha) + 1 + |\alpha|\}$ into disjoint intervals from left to right:

$$(9.1) \quad \{r - \ell(\alpha) + 1, \dots, r - \ell(\alpha) + 1 + |\alpha|\} = (E_1 \sqcup \dots \sqcup E_k) \sqcup (D_k \sqcup \dots \sqcup D_1),$$

where $|E_i| = e_i, |D_i| = d_i$ for $1 \leq i \leq k$, so that

$$\{r - \ell(\alpha) + 1, \dots, r\} = E_1 \sqcup \dots \sqcup E_k \text{ and } \{r + 1, \dots, r + 1 - \ell(\alpha) + |\alpha|\} = D_k \sqcup \dots \sqcup D_1.$$

Definition 9.2. Fix $\alpha \in \text{Comp}_{r,n}$. Let $A = (a_m, \dots, a_1 \mid_r b_1, \dots, b_m) \in \binom{[n]}{r}$ where $r - \ell(\alpha) + 1 \leq a_m$ and $b_m \leq r - \ell(\alpha) + 1 + |\alpha|$. Then the data $E_1, \dots, E_k, D_k, \dots, D_1$ as in (9.1) determines unique indices $(i_1, j_1), (i_2, j_2), \dots, (i_m, j_m)$ such that $a_p \in E_{i_p}$ and $b_p \in D_{j_p}$ for all $1 \leq p \leq m$. We say that A is α -admissible if:

$$j_1 \leq i_1 < j_2 \leq i_2 < \dots < j_m \leq i_m.$$

Figure 13 gives an example of an α -admissible set A and an α -inadmissible set B for $\alpha = (2, 1, 4) \in \text{Comp}_{4,10}$. For $B = (3, 4 \mid_4 5, 6)$ we have, in terms of the notation in Definition 9.2, that $(i_1, j_1) = (1, 2)$ and $(i_2, j_2) = (2, 2)$. Since $j_1 > i_1$ we have B is α -inadmissible.

Theorem 9.3. Let $\alpha \in \text{Comp}_{r,n}$. The T -fixed points of $X_{\text{NC}}^{\lambda_\alpha}$ are

$$(X_{\text{NC}}^{\lambda_\alpha})^T = \left\{ A \in \binom{[n]}{r} \mid A \text{ is } \alpha\text{-admissible} \right\}.$$

Proof. We outline the argument, referring the reader to [5] for any errant terminology about indexed forests. Let $Z \in \text{Zigzag}_{r,n}$ correspond to α . For $v \in \text{IN}(Z)$, let τ_v be the transposition (i, j)

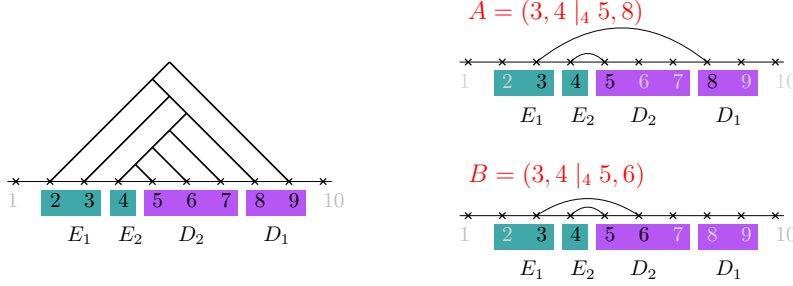


FIGURE 13. For $\alpha = (2, 1, 4) \in \text{Comp}_{4,10}$ we have α -admissible A and α -inadmissible B . The arcs connect i/j -indices determined as in Definition 9.2.

where i is the rightmost leaf descended from v_L and j is the rightmost leaf descended from v . Order the internal nodes $v_1, v_2, \dots \in \text{IN}(F)$ in the unique order from root to terminal node. Then [6, Theorem 6.3] implies that the set of T -fixed points of $X_{\text{NC}}^{\lambda_\alpha}$ is given by the following subset of $\binom{[n]}{r}$:

$$(9.2) \quad \{\tau_{v_{i_1}} \tau_{v_{i_2}} \cdots \tau_{v_{i_k}} \cdot [r] \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n-1\}.$$

Concretely, the products $\tau_{v_{i_1}} \tau_{v_{i_2}} \cdots \tau_{v_{i_k}}$ correspond to deleting all left edges emanating from internal nodes in $\text{IN}(Z) \setminus \{v_{i_1}, \dots, v_{i_k}\}$, and then multiplying the backward cycles on the leaves for each connected component. This perspective, in conjunction with Lemma 3.14, implies the claim. $\square$

For $\alpha = (2, 1) \in \text{Comp}_{2,4}$ we have five T -fixed points corresponding to the elements in $\binom{[4]}{2}$ except $\{3, 4\}$ which is α -inadmissible. These in turn give the vertices of the blue pyramid in Figure 1. Figure 14 show how these admissible sets arise following the left edge deletion procedure described in the preceding proof.

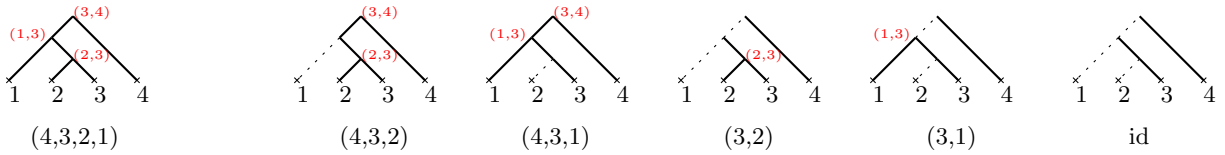


FIGURE 14. $Z \in \text{Zigzag}_{2,4}$ for $\alpha = (2, 1)$ and the subproducts giving distinct T -fixed points

Corollary 9.4. $\text{QGr}(r; n)$ is an equidimensional complex of $(n-1)$ -dimensional toric varieties.

Proof. Let T_A and T_B be zigzag trees associated to sets $A, B \in \binom{[n]}{r}$. By (9.2), if T_B is obtained by iteratively deleting the root vertex of T_A (i.e. it is a lower ideal of the tree T_A), then $(X_{\text{NC}}^B)^T \subset (X_{\text{NC}}^A)^T$, which then implies $X_{\text{NC}}^B \subset X_{\text{NC}}^A$ by Theorem 6.9. As $\dim X_{\text{NC}}^A = |T_A|$, equidimensionality follows because we can extend any zigzag tree to a maximal one, possibly in many ways. If B corresponds to the composition $\alpha = (\alpha_1, \dots, \alpha_k)$ then one uniform choice is to have A correspond to $(a, 1^b, \alpha_1, \dots, \alpha_k)$ with $b = r - k - 1$ and $a = n - 1 - b - |\alpha|$. $\square$

REFERENCES

- [1] M. Albenque and P. Nadeau. Growth function for a class of monoids. In *21st International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2009)*, Discrete Math. Theor. Comput. Sci. Proc., AK, pages 25–38. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2009. [20](#)
- [2] D. Anderson and W. Fulton. *Equivariant cohomology in algebraic geometry*, volume 210 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2024. [4](#)
- [3] J.-C. Aval, F. Bergeron, and N. Bergeron. Ideals of quasi-symmetric functions and super-covariant polynomials for S_n . *Adv. Math.*, 181(2):353–367, 2004. [2](#)
- [4] A. Baker and B. Richter. Quasisymmetric functions from a topological point of view. *Math. Scand.*, 103(2):208–242, 2008. [2](#)
- [5] N. Bergeron, L. Gagnon, P. Nadeau, H. Spink, and V. Tewari. Equivariant quasisymmetry and noncrossing partitions, 2025, 2504.15234. [2](#), [8](#), [9](#), [10](#), [15](#), [23](#), [24](#), [25](#), [26](#), [30](#), [31](#)
- [6] N. Bergeron, L. Gagnon, P. Nadeau, H. Spink, and V. Tewari. The quasisymmetric flag variety: a toric complex on noncrossing partitions, 2025, 2508.12171. [2](#), [16](#), [23](#), [28](#), [29](#), [30](#), [31](#), [32](#)
- [7] N. Bergeron, L. Gagnon, H. Spink, and V. Tewari. The Coxeter Flag Variety, 2026, 2601.23111. [22](#)
- [8] N. Bergeron and F. Sottile. Schubert polynomials, the Bruhat order, and the geometry of flag manifolds. *Duke Math. J.*, 95(2):373–423, 1998. [23](#)
- [9] A. Björner and F. Brenti. *Combinatorics of Coxeter groups*, volume 231 of *Graduate Texts in Mathematics*. Springer, New York, 2005. [4](#)
- [10] A. Borel. Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts. *Ann. of Math. (2)*, 57:115–207, 1953. [2](#)
- [11] A. S. Buch, P.-E. Chaput, and N. Perrin. Equivariant rigidity of Richardson varieties. *Pacific J. Math.*, 338(2):209–229, 2025. [22](#)
- [12] W. Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998. [28](#)
- [13] I.M. Gelfand, M.I. Graev, and A. Postnikov. Combinatorics of hypergeometric functions associated with positive roots. In *The Arnold-Gelfand mathematical seminars*, pages 205–221. Birkhäuser Boston, Boston, MA, 1997. [20](#)
- [14] I. M. Gessel. Multipartite P -partitions and inner products of skew Schur functions. In *Combinatorics and algebra (Boulder, Colo., 1983)*, volume 34 of *Contemp. Math.*, pages 289–317. Amer. Math. Soc., Providence, RI, 1984. [1](#), [4](#)
- [15] M. Goresky, R. Kottwitz, and R. MacPherson. Equivariant cohomology, Koszul duality, and the localization theorem. *Invent. Math.*, 131(1):25–83, 1998. [2](#), [24](#)
- [16] W. Graham. Positivity in equivariant Schubert calculus. *Duke Math. J.*, 109(3):599–614, 2001. [28](#), [29](#)
- [17] M. Harada, A. Henriques, and T. S. Holm. Computation of generalized equivariant cohomologies of Kac-Moody flag varieties. *Adv. Math.*, 197(1):198–221, 2005. [25](#)
- [18] A. Knutson, T. Lam, and D. E. Speyer. Positroid varieties: juggling and geometry. *Compos. Math.*, 149(10):1710–1752, 2013. [3](#), [19](#), [21](#)
- [19] A. Knutson, T. Lam, and D. E. Speyer. Projections of Richardson varieties. *J. Reine Angew. Math.*, 687:133–157, 2014. [21](#)
- [20] A. Knutson and T. Tao. Puzzles and (equivariant) cohomology of Grassmannians. *Duke Math. J.*, 119(2):221–260, 2003. [29](#)
- [21] G. Kreweras. Sur les partitions non croisées d’un cycle. *Discrete Math.*, 1(4):333–350, 1972. [7](#)
- [22] T. Lam and A. Postnikov. Polypositroids. *Forum Math. Sigma*, 12:Paper No. e42, 67 pp, 2024. [31](#)

- [23] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications. [4](#), [6](#)
- [24] P. Nadeau, H. Spink, and V. Tewari. The geometry of quasisymmetric coinvariants, 2024, 2410.12643. [4](#), [21](#), [22](#)
- [25] P. Nadeau, H. Spink, and V. Tewari. Quasisymmetric divided differences, 2024, 2406.01510. [9](#), [23](#), [27](#), [28](#)
- [26] J. Oesinghaus. Quasisymmetric functions and the Chow ring of the stack of expanded pairs. *Res. Math. Sci.*, 6(1):Paper No. 5, 18 pp, 2019. [2](#)
- [27] O. Pechenik and M. Satriano. Quasisymmetric Schubert calculus, 2023, 2205.12415. [2](#)
- [28] O. Pechenik and M. Satriano. James reduced product schemes and double quasisymmetric functions. *Adv. Math.*, 449:Paper No. 109737, 28, 2024. [2](#)
- [29] A. Postnikov. Total positivity, grassmannians, and networks, 2006, math/0609764. [3](#), [19](#), [21](#)
- [30] A. Postnikov. Permutohedra, associahedra, and beyond. *Int. Math. Res. Not. IMRN*, (6):1026–1106, 2009. [20](#)
- [31] A. Postnikov. Positive Grassmannian and polyhedral subdivisions. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. IV. Invited lectures*, pages 3181–3211. World Sci. Publ., Hackensack, NJ, 2018. [19](#)
- [32] R. P. Stanley. *Ordered structures and partitions*. Memoirs of the American Mathematical Society, No. 119. American Mathematical Society, Providence, R.I., 1972. [1](#)
- [33] R. P. Stanley. *Enumerative combinatorics. Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2012. [4](#)
- [34] R. P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 208 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, [2024] ©2024. Second edition [of 1676282], With an appendix by Sergey Fomin. [4](#), [6](#)
- [35] L. K. Williams. The positive Grassmannian, the amplituhedron, and cluster algebras. In *ICM—International Congress of Mathematicians. Vol. 6. Sections 12–14*, pages 4710–4737. EMS Press, Berlin, [2023] ©2023. [19](#)

DEPT. OF MATH. AND STAT., YORK UNIVERSITY, TORONTO, ON M3J 1P3, CANADA

Email address: bergeron@yorku.ca

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, CA 90089, USA

Email address: lgagnon@usc.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, ON M5S 2E4, CANADA

Email address: hunter.spink@utoronto.ca

DEPARTMENT OF MATHEMATICAL AND COMPUTATIONAL SCIENCES, UNIVERSITY OF TORONTO MISSISSAUGA, MISSISSAUGA, ON L5L 1C6, CANADA

Email address: vasu.tewari@utoronto.ca