TOPOLOGICAL OBSTRUCTIONS TO NONNEGATIVE CURVATURE

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Abstract. We find new obstructions to the existence of complete Riemannian metric of nonnegative sectional curvature on manifolds with infinite fundamental groups. In particular, we construct many examples of vector bundles whose total spaces admit no nonnegatively curved metric.

1. Introduction

According to the soul theorem of J. Cheeger and D. Gromoll a complete open manifold of nonnegative sectional curvature is diffeomorphic to the total space of the normal bundle of a compact totally geodesic submanifold which is called the *soul*. One of the harder questions in the subject of is what kind of normal bundles can occur.

Cheeger and Gromoll also proved that a finite cover of any closed nonnegatively curved manifold (throughout the paper by a nonnegatively curved manifold we mean a complete Riemannian manifold of nonnegative *sectional* curvature) is diffeomorphic to a product of a torus and a simply-connected closed nonnegatively curved manifold. It turns out that a similar statement holds for open complete nonnegatively curved manifolds (see [Wil98] and section 2 where a Ricci version of the statement is proved).

We use this fact to find new obstructions to nonnegative curvature and build many examples of vector bundles whose total spaces admit no complete metric of nonnegative curvature. Basic obstructions are provided by the following proposition (of which there is a more general version incorporating the Euler class).

Proposition 1.1. *Let N be an open complete nonnegatively curved manifold such that $Q(TN) \neq 0$ for some polynomial Q in rational Pontrjagin classes. Then $Q(T\tilde{N}) \neq 0$ for the universal (and hence any) cover $\pi\colon \tilde{N} \to N$.*

Note that 1.1 is true for *finite* covers without any curvature assumptions (because finite covers induce injective maps on rational cohomology). In general, the results of this paper are only interesting for manifold with infinite fundamental groups.

Previously, obstructions to the existence of nonnegatively curved metrics on vector bundles were only known for a flat soul [ÖW94]. No obstructions are known when the soul is simply-connected. Examples of nonnegatively curved metrics on vector bundles can be found in [Che73, Rig78, Yan95, GZ99, GZ].

Corollary 1.2. *Let η be a vector bundle over a closed smooth manifold C and let ξ be a vector bundle over a closed flat manifold F such that the total space of $\eta\times\xi$ admits a complete nonnegatively curved metric. Then ξ becomes stably trivial after passing to a finite cover. Furthermore, if either rank$(\eta)=0$, or η is orientable and has nonzero rational Euler class, then ξ becomes trivial in a finite cover.*

Note that a vector bundle over a flat manifold F becomes trivial in a finite cover iff its rational Euler and Pontrjagin classes vanish. Similarly, a bundle over F is stably trivial in a finite cover iff its rational Pontrjagin classes vanish (see 4.4).

In case C is a point 1.2 says that any vector bundle F with nonnegatively curved total space becomes trivial in a finite cover. Also since the Euler and Pontrjagin classes determine a vector bundle up to finite ambiguity (see e.g. [Bel98]), in every rank there are only finitely many vector bundles over F with nonnegatively curved total spaces. Thus, 1.2 is a generalization of the main result of [ÖW94].

To see how 1.2 works, note that if T is a torus of dimension ≥ 4, then there are infinitely many vector bundles over T of every rank ≥ 2 with (pairwise) different first Pontrjagin classes. Also there are infinitely many rank 2 vector bundles over T^2 and T^3 with different Euler classes. We now deduce the following.

Corollary 1.3. *Let B be a closed nonnegatively curved manifold. If $\pi_1(B)$ contains a free abelian subgroup of rank four (two, respectively), then for each $k\geq 2$ (for $k=2$, respectively) there exists a finite cover of B over which there exist infinitely many rank k vector bundles whose total spaces admit no nonnegatively curved metrics.*

By contrast, any vector bundle over $S^2\times S^1$ admits a nonnegatively curved metric as we observe in 7.3. Thus 1.3 cannot be generalized to the case when $\pi_1(B)$ is virtually-$\mathbb{Z}$.

Passing to finite covers in 1.2 and 1.3 seems necessary, in general, in order to obtain bundles without nonnegatively curved metrics. For example, one can easily construct flat $SO(n)$ vector bundles over a torus with nonzero Stiefel-Whitney classes, and obviously their total spaces are complete flat manifolds. Here is an example when we get a complete picture without passing to a finite cover.

Corollary 1.4. *Let ξ be a vector bundle over $S^3 \times S^1$ whose total space has a nonnegatively curved metric. Then either ξ is the trivial bundle or ξ is the product of a trivial bundle over S^3 and the Möbius band bundle over S^1.*

We emphasize that our method does not apply when B is simply-connected, or more generally if after passing to a finite cover $C \times T \to B$ the bundle ξ becomes a pullback of a bundle over C via the projection $C \times T \to C$. (Here, and until the end of the section C is a simply-connected manifold and T is a torus.) For instance, if B is a closed flat manifold which is an odd-dimensional rational homology sphere [Szc83], then any vector bundle over B becomes trivial in a finite cover and it is unclear whether there are bundles over B which are not nonnegatively curved.

A reasonable goal is to find an example of a rank k vector bundle over $C \times T$ with no nonnegatively curved metric, whenever there is a rank k vector bundle over $C \times T$ that does not become the pullback of a bundle over C in a finite cover. This is achieved in 1.3 when $\dim(T) \geq 4$. Otherwise, the answer may depends on the topology of $C \times T$. For example, any bundle of rank ≥ 3 over 2-torus becomes trivial, and hence nonnegatively curved, in a finite cover.

While we do not quite settle the case $\dim(T) < 4$, we get various partial results. For instance, given a closed orientable $2n$-manifold B and an integer $d \neq 0$, there always exists a map $f\colon B \to S^{2n}$ of degree d. Then, if $\pi_1(B)$ is infinite, we show that the total space of the pullback bundle $f^{\#}TS^{2n}$ admits no complete metric of nonnegative curvature. To state further results we need to review some basic bundle theory.

By a simple obstruction-theoretic argument $H^{even}(C \times T, C) = 0$ implies that any vector bundle over $C \times T$ becomes the pullback of a bundle over C after passing to a finite cover. This is the case, for example, for bundles over $CP^n \times S^1$. However, once $H^{even}(C \times T, C) \neq 0$ we immediately get a bundle with no nonnegatively curved metric.

Corollary 1.5. *If $H^{2i}(C \times T, C) \neq 0$ for some $i > 0$, then there exist infinitely many rank $2i$ vector bundles over $C \times T$ with different Euler classes whose total spaces are not nonnegatively curved.*

The Euler class is unstable and, in fact, the bundles constructed in the proof of 1.5 become pullbacks of bundles over C after taking Whitney sum with a trivial line bundle and passing to a finite cover.

To get examples that survive stabilization one has to deal with Pontrjagin classes which live in $H^{4*}(C \times T)$. Generally, if $H^{4*}(C \times T, C) = 0$, then after adding a trivial line bundle, any vector bundle over $C \times T$ becomes the pullback of a bundle over C in a finite cover. If $H^{4*}(C \times T, C) \neq 0$, one hopes to find a vector bundle without nonnegatively curved metric that survives stabilization and passing to finite covers. We do this in several cases, the simplest being when the rank of the bundle is $\geq \dim(C)$ (see section 5 for other results involving various assumptions on Pontrjagin classes of TC).

Corollary 1.6. *If $H^{4i}(C\times T, C) \neq 0$ for some $i > 0$, then for each $k \geq \dim(C)$ there exist infinitely many rank k vector bundles over $C \times T$ with different Pontrjagin classes whose total spaces admit no metric of nonnegative curvature.*

The main geometric ingredient of this paper is that a finite cover of any complete nonnegatively curved manifold N is diffeomorphic to a product of a torus T and a simply connected manifold M and this diffeomorphism can be chosen to take a soul S to the product of T and a simply-connected submanifold of M. There is also a Ricci version of this statement described in section 2. For example, the above conclusion holds if N has nonnegative Ricci curvature, S is an isometrically embedded compact submanifold of N such that the inclusion $S \hookrightarrow N$ induces an isomorphism of fundamental groups, and either S is totally convex, or there exists a distance nonincreasing retraction $N \to S$. In particular, all the theorems stated above hold in these cases.

Our methods also yield obstructions to existence of metrics of nonnegative Ricci curvature on *closed* manifolds (after all, 1.1 can be applied to closed manifolds). Here is an example. It was shown in [GW00] that the total space of the sphere bundle associated with the normal bundle to the soul has a nonnegatively curved metric. Thus, potentially, sphere bundles provide a good source of closed nonnegatively curved manifolds. Among other things, we prove the following.

Corollary 1.7. *Let ξ be a bundle over a flat manifold F with associated sphere bundle $S(\xi)$ and let C be a closed smooth simply-connected manifold. If $C \times S(\xi)$ admits a metric of nonnegative Ricci curvature , then ξ becomes trivial in a finite cover.*

Finally note that obstructions to to the existence of nonnegatively curved metrics on total spaces of vector bundles give rise to obstructions to the existence of G-invariant nonnegatively curved metrics on the associated G-principal bundles. Indeed, any vector bundle ξ with a structure group G can be written as $(P \times \mathbb{R}^k)/G$ where P is a principal G-bundle and G acts on $\mathbb{R}^k$ via a representation $G \to SO(k)$. By the O'Neill curvature submersion formula, if P has a G-invariant nonnegatively curved metric, then so does the total space of ξ.

The structure of the paper is as follows. Section 2 contains the above mentioned splitting theorem for nonnegatively curved manifolds. Section 3 summarizes the obstructions to nonnegative curvature coming from the splitting theorem. In section 4 we develop general existence and uniqueness results for bundles over $C \times T$. Section 5 contains concrete examples of vector bundles with no nonnegatively curved metrics. Various obstructions to the existence of metrics of nonnegative Ricci curvature on sphere bundles are described in section 6. Theorem 1.4 is proved in the section 7.

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2. Splitting in a finite cover

Cheeger and Gromoll proved in [CG72] that a finite cover of a closed nonnegatively curved manifold is diffeomorphic to a product of a torus and a simply connected manifold. The main geometric tool we employ in this paper is the following generalization of this result to open manifolds.

Lemma 2.1. *Let (N, g) be a complete nonnegatively curved manifold. Then there exists a finite cover N' of N diffeomorphic to a product $M \times T^k$ where M is a complete open simply connected nonnegatively curved manifold. Moreover, if S' is a soul of N', then this diffeomorphism can be chosen in such a way that it takes S' onto $C \times T^k$ where C is a soul of M.*

After obtaining this result we have learned that it follows from a more general theorem which was proved earlier by B. Wilking [Wil98]. We then realized that our proof of 2.1 in fact gives the following stronger statement.

Proposition 2.2. *Let (N, g) be a complete manifold of nonnegative Ricci curvature. Let $q\colon \tilde{N} \to N$ be the universal cover of N and let $\rho\colon \pi \to Iso(\tilde{N})$ be the deck transformation representation of $\pi = \pi_1(N)$.*

Suppose that there exists a closed manifold $S \subset N$ isometrically embedded into N such that the inclusion $S \hookrightarrow N$ induces an isomorphism of the fundamental groups, and any line in $\tilde{S} = q^{-1}(S)$ with respect to the induced metric from $\tilde{N}$ is also a line in $\tilde{N}$.

Then π is virtually abelian and, if π has no torsion, then there exists a smooth path $\rho(t)\colon [0,1] \to \mathrm{Hom}(\pi, Iso(\tilde{N}))$ such that

- *(i)* $\rho(0) = \rho$;
- *(ii) for each t the action of π on $\tilde{N}$ is free and properly discontinuous;*
- *(iii) A finite cover of $N_1 = \tilde{N}/\rho(1)(\pi)$ splits isometrically as $M \times T^k$ where $k = rank(\pi)$;*
- *(iv) There is a family of closed submanifolds $S_t \subseteq N_t = \tilde{N}/\rho(t)(\pi)$ such that*
  - *(a)* $S_0 = S$

(b) *Under the splitting from (iii) the cover of* S_1 *corresponds to the Riemannian product* $C \times T^k \subset M \times T^k$ *where* C *is a closed isometrically embedded submanifold of* M.
(c) *for each* t *there exists a diffeomorphism* $\phi_t\colon (N_t, S_t) \to (N_0, S_0)$

The assumption that any line in $\tilde{S} = q^{-1}(S)$ is also a line in $\tilde{N}$ is satisfied if S is totally convex in N or if there is a distance nonincreasing retraction $N \to S$. Both of these conditions are true if N is an open manifold of nonnegative sectional curvature and $S \subseteq N$ is its soul. In this case one can also describe the souls of the deformed manifolds N_t. Namely we have the following

Proposition 2.3. *Let* (N, g) *be a complete nonnegatively curved manifold with a free abelian fundamental group* π. *Let* $\rho\colon \pi \to Iso(\tilde{N})$ *be the deck transformation representation of* π. *Then there exists a smooth path* $\rho(t)\colon [0,1] \to \mathrm{Hom}(\pi, Iso(\tilde{N}))$ *such that in addition to (i)-(iv) of* 2.2 *the following holds.*

(v) If S *is a soul of* N, *then there exists an isometric splitting* $\tilde{N} = M \times \mathbb{R}^k$ *where* $k = rank(\pi)$ *and a soul* C *of* M *such that, for every* $t \in [0,1]$, *the projection* S_t *of* $C \times \mathbb{R}^k$ *to* $N_t = \tilde{N}/\rho(t)(\pi)$ *is a soul of* N_t. *Also for each* t, *there exists a diffeomorphism* $\phi_t\colon (N_t, S_t) \to (N_0, S_0)$.

Remark 2.4. The above mentioned result of Wilking [Wil98] implies the existence of the deformation $\rho(t)$ as in 2.2 for an arbitrary virtually abelian group. He also gives an upper bound on the order of the covering in question in terms of π and the number of connected components of $Iso(M)$. Nevertheless, we will present our proof of 2.2 for it is considerably easier than the one in [Wil98]. (In fact, our proof is very similar to the original argument of Cheeger and Gromoll in the closed manifold case.) Besides, the statements of 2.2 and 2.3 are tailored to our applications, for example the parts $(iv) - (v)$ are not discussed in [Wil98].

Proof of 2.2. Let $q\colon \tilde{N} \to N$ be the universal cover of N and let $\tilde{S} = q^{-1}(S)$. Then since inclusion $S \hookrightarrow N$ induces an isomorphism of the fundamental groups $q|_{\tilde{S}}\colon \tilde{S} \to S$ is the universal cover of S.

Let $\tilde{S} = C \times \mathbb{R}^k$ be the de Rham decomposition of $\tilde{S}$ so that C does not split off a Euclidean factor. We claim that C is compact. (Indeed, suppose C is not compact. Then C contains a ray γ. Since S is compact, there exists a point $p \in S$ such that $q(\gamma(i)) \to p$ as $i \to \infty$. Let $\tilde{p}$ be a point in $q^{-1}(p)$. By above there exists a sequence $g_i \in \pi$ such that $g_i(\gamma(i)) \to \tilde{p}$. Passing to a subsequence we can assume that $g_i(\gamma'(i)) \to v \in T_{\tilde{p}}C$. Just as in [CG72] we readily conclude that $\sigma(t) = \exp(tv)\colon \mathbb{R} \to \tilde{S}$ is a line in $C \subset \tilde{S}$. By assumption $\sigma(t)$ is also a line in $\tilde{N}$. Therefore, the splitting theorem [CG72] implies $\tilde{N}$ splits off $\sigma(t)$

isometrically. So v is invariant under the $Hol(\tilde{N})$ and hence under $Hol(\tilde{S})$ which contradicts the fact that C does not split off a Euclidean factor.)

Since any isometry of $\tilde{S}$ takes lines to lines, the isometry group $Iso(\tilde{S})$ splits as a direct product $Iso(\tilde{S}) \cong Iso(C) \times Iso(\mathbb{R}^k)$. Therefore, the natural deck transformation action of π on $\tilde{S}$ gives a monomorphism $\rho = (\rho_1, \rho_2)\colon \pi \to Iso(C) \times Iso(\mathbb{R}^k)$.

Since $Iso(C)$ is compact and π is discrete, the group $\rho_2(\pi)$ is a discrete subgroup of $Iso(\mathbb{R}^k)$. Also $\ker(\rho_2)$ is compact and hence it is finite. Thus, π is an extension of a finite group by a crystallographic one. It is well-known (see e.g. the proof of [Wil98, Thm 2.1]) that any such a group is virtually abelian.

Now suppose that π is free abelian. Then $\rho_2(\pi)$ is a discrete torsion-free subgroup of $Iso(\mathbb{R}^k)$, in particular, it acts on $\mathbb{R}^k$ by translations and $\mathbb{R}^k/\rho_2(\pi)$ is isometric to a flat torus T^k.

By above the splitting $\tilde{S} = C \times \mathbb{R}^k$ is just a part of a bigger isometric splitting $\tilde{N} = \tilde{M} \times \mathbb{R}^k$ where $\tilde{M}$ is a complete open simply connected manifold containing C as an isometrically embedded submanifold.

Since the action of π on $\tilde{N}$ leaves $\tilde{S}$ invariant, it sends lines parallel to $\mathbb{R}^k$ into lines parallel to $\mathbb{R}^k$. Hence the map ρ is a restriction of a natural monomorphism $\pi \to Iso(\tilde{M}) \times Iso(\mathbb{R}^k)$ which with a slight abuse of notations we will still denote by $\rho = (\rho_1, \rho_2)$. In fact, the image of ρ_1 lies in the subgroup $G \trianglelefteq Iso(\tilde{M})$ of isometries leaving C invariant. Since C is compact it follows that G is a *compact* subgroup of $Iso(\tilde{M})$.

Next consider the homomorphism $\rho_1\colon \pi \to G$. Let H be the closure of $\rho_1(\pi)$ in G. Then H is a compact abelian subgroup of G. Let H_0 be the identity component of H. Consider the short exact sequence $1 \to H_0 \to H \to \Gamma \to 1$ where $\Gamma = H/H_0$ is a finite abelian group. We claim that this sequence splits and hence $H \cong H_0 \times \Gamma$.

Indeed, the group Γ is a product of finite cyclic subgroups and, since H is abelian, it is enough to define the splitting on generators of these subgroups. Let $g \in \Gamma$ be a generator of order m and let $\bar{g} \in H$ be a preimage of g. The endomorphism of H sending x to x^m, takes $\bar{g}$ to H_0, and maps H_0 *onto* itself. Hence, there is $h \in H_0$ such that $h^m = \bar{g}^m$, and we can define a splitting by mapping g to $\bar{g} \cdot h^{-1}$.

Thus, $\rho_1\colon \pi \to H \cong H_0 \times \Gamma$ can be written as a product of two representations $\rho'\colon \pi \to H_0$ and $\rho''\colon \pi \to \Gamma$. Since $H_0 \cong T^l$, the representation variety $\mathrm{Hom}(\pi, H_0)$ is diffeomorphic to a torus T^{kl}. Hence, we can find a smooth deformation $\rho_1'(t) \in \mathrm{Hom}(\pi, H_0)$ such that $\rho_1'(0) = \rho'$ and $\rho_1'(1) = 1$, the trivial representation.

Crossing $\rho_1'(t)$ with ρ'' and ρ_2, we obtain a smooth path $\rho(t) \in \mathrm{Hom}(\pi, Iso(\tilde{N}))$ such that $\rho(0) = \rho$ and $\rho(1) = 1 \times \rho'' \times \rho_2$. For every t the action of π on $\tilde{N}$ via ρ_t is free and properly discontinuous because so is the action of π on $\mathbb{R}^k$. Therefore, we get a smooth family of manifolds of nonnegative Ricci curvature $N_t = \tilde{N}/\rho(t)(\pi)$ with $N_0 = N$. We also get the family $S_t = \tilde{S}/\rho(t)(\pi) \subseteq N_t$ of closed isometrically embedded submanifolds with $S_0 = S$.

The finite cover of N_1 corresponding to the kernel of ρ'' splits isometrically as $M \times T^k$. Under the splitting the cover of S_1 corresponds to the Riemannian product $C \times T^k \subset M \times T^k$.

By the (relative) covering homotopy theorem the family (N_t, S_t), considered as a bundle over $[0,1]$ is smoothly isomorphic to the trivial bundle $[0,1] \times (N, S)$. In particular, all S_t's are mutually diffeomorphic and, moreover, have isomorphic normal bundles. □

Proof of 2.3. Let S be a soul of N and let $p\colon \tilde{N} \to N$ be the universal cover of N. By the Cheeger-Gromoll soul theorem [CG72] S is totally convex and the inclusion $S \hookrightarrow N$ is a homotopy equivalence. Thus, 2.2 applies and it only remains to deduce (v).

Let $h\colon N \to \mathbb{R}$ be the Cheeger-Gromoll exhaustion function generating S and let $\tilde{h} = h \circ q\colon \tilde{N} \to \mathbb{R}$ be its lift to the universal cover $\tilde{N}$. Clearly, $\tilde{h}$ is convex. Moreover, since every line in $\tilde{N}$ parallel to $\mathbb{R}^k$ projects to an infinite geodesic lying in a compact set, $\tilde{h}$ is constant along any such line. Hence $\tilde{h}$ is given by the formula $\tilde{h}(m,t) = \bar{h}(m)$ for some convex function $\bar{h}\colon M \to \mathbb{R}$. It is easy to see that $\bar{h}$ is an exhaustion function. Let $C \subseteq M$ is the soul generated by $\bar{h}$.

By construction $\bar{h}$ is invariant under the action of $\rho_1(\pi)$ and, hence, under the action of H. In particular, $\bar{h}$ is invariant under the action of $\rho_1(t)(\pi)$ for any t. Therefore, $\bar{h}$ descends to a well defined convex exhaustion function $h_t\colon N_t \to \mathbb{R}$ generating the soul $S_t = (C \times \mathbb{R}^k)/\rho(t)(\pi)$. □

Remark 2.5. Actually, it follows from the proof of 2.2 that some versions of 2.2 and 2.1 hold without any curvature assumptions. For example, let N be a complete Riemannian manifold whose universal cover is isometric to $M \times \mathbb{R}^n$ where $Iso(M)$ is compact. Then a finite cover of N is diffeomorphic to the product $M \times T^k \times \mathbb{R}^{n-k}$. See [Wil98] for a stronger result.

3. Basic obstructions

In this section we obtain simple topological obstructions to nonnegative curvature coming from the results of the section 2.

In this section and throughout the rest of this paper we use the notation e for the Euler class, p_i for the ith Pontrjagin class, and $p = \sum_{i\geq 0} p_i$ for total

Pontrjagin class. Unless stated otherwise, all the characteristic classes live in cohomology with rational coefficients. (However, it is useful to keep in mind that e and p_i are in fact integral classes, that is they lie in the image of $H^*(B,\mathbb{Z}) \to H^*(B,\mathbb{Q})$.)

Let S be a closed manifold smoothly embedded into an open manifold N such that the inclusion $S \hookrightarrow N$ induces an isomorphism of fundamental groups. Let $q\colon \tilde{N} \to N$ be the universal cover of N; then $q\colon \tilde{S} = q^{-1}(S) \to S$ is the universal cover of S. Assume that after passing to a finite cover N becomes diffeomorphic to $M \times T$ where $\pi_1(M) = 1$ and T is a torus of positive dimension, Further, suppose that this diffeomorphism takes (a finite cover of) S onto $C \times T$ where C is a submanifold of M. Denote the normal bundles of S in N by ν_S.

Lemma 3.1. *Suppose there is a polynomial Q with rational coefficients such that $Q(e(\nu_S), p_1(TN|_S), p_2(TN|_S), ...) \neq 0$ where ν_S is assumed to be oriented if Q depends on e. Then $Q(e(q^{\#}\nu_S), p_1(T\tilde{N}|_{\tilde{S}}), p_2(T\tilde{N}|_{\tilde{S}}), ...) \neq 0$.*

Proof. Note that $Q_S = Q(e(\nu_S), p_1(TN|_S), p_2(TN|_S), ...) \neq 0$ remains true after passing to any finite cover because finite covers induce injective maps on rational cohomology. Thus, we can assume without loss of generality that N is diffeomorphic to $M \times T$ as above and this diffeomorphism identifies S with $C \times T$.

Then the normal bundle ν_C^M of C in M is the pullback of ν_S via the inclusion $i_C\colon C \to S$ and, also ν_S is the pullback of ν_C^M via the projection $\pi_C\colon S \to C$. Similarly, since T is parallelizable, $TN|_C$ is stably isomorphic to $i_C^{\#}TN|_S$ and $TN|_S$ is stably isomorphic to $\pi_C^{\#}TN|_C$. In particular,

$$Q_C = Q(e(\nu_C^M), p_1(TN|_C), p_2(TN|_C), ...) = i_C^* Q_S$$

and $\pi_C^* Q_C = Q_S$. The latter implies that $Q_C \neq 0$.

Since C is simply-connected, i_C factors through the universal covering $q\colon \tilde{S} \to S$. In particular, $q^* Q_S \neq 0$ as desired. □

Remark 3.2. Clearly, 3.1 remains true for any (not necessarily universal) cover q' in place of q because $q'^* Q_S = 0$ implies $q^* Q_S = 0$.

Remark 3.3. A particular case of 3.1 remains true even without mentioning S. Namely, assume only that N is an open manifold whose finite cover is diffeomorphic to $M \times T$. Let Q be a polynomial in rational Pontrjagin classes such that $Q(TN) \neq 0$. Then the same proof implies $Q(T\tilde{N}) \neq 0$. This applies to the geometric situation discussed in 2.5.

Base versus soul. Now we specialize to the case when N is the total space of a smooth vector bundle over a closed manifold B. We identify B with the zero section. Then the universal cover of B is $q\colon \tilde{B} = q^{-1}(B) \to B$. Assume also that the inclusion $S \hookrightarrow N$ is a homotopy equivalence.

First, we need to see how the characteristic classes of the normal bundles to the S and B are related. The homotopy equivalence $h\colon B \to S$ (defined as the composition of the inclusion $B \hookrightarrow N$ and a homotopy inverse of $S \hookrightarrow N$) clearly has the property that $h^* p_i(TN|_S) = p_i(TN|_B)$ for any i.

Furthermore, if the manifolds N, B, S are oriented, then for the rational Euler class we have $h^* e(\nu_S) = \deg(h) e(\nu_B)$. (Indeed, suppressing the inclusions we have $\langle h^* e(\nu_B), \alpha\rangle = \langle e(\nu_B), h_*\alpha\rangle = \langle [B], h_*\alpha\rangle = \langle \deg(h)[S], \alpha\rangle = \deg(h)\langle e(\nu_S), \alpha\rangle$.)

By possibly changing orientation on S we can arrange that $\deg(h) = 1$ so that $h^* e(\nu_B) = e(\nu_S)$. Thus, since h^* is an algebra homomorphism, we get

$$h^* Q(e(\nu_B), p_1(TN|_B), p_2(TN|_B), ...) = Q(e(\nu_S), p_1(TN|_S), p_2(TN|_S), ...).$$

Proposition 3.4. *Let ξ be a vector bundle over a closed smooth manifold B whose total space N admits a complete Riemannian metric of nonnegative sectional curvature. Suppose there is a polynomial Q with rational coefficients such that $Q(e(\xi), p_1(TN|_B), p_2(TN|_B), ...) \neq 0$ where ξ is assumed to be oriented if Q depends on e. Then $Q(e(q^\# \xi), p_1(T\tilde{N}|_{\tilde{B}}), p_2(T\tilde{N}|_{\tilde{B}}), ...) \neq 0$.*

Proof. Let S be a soul of N. By above the homotopy equivalence h takes $Q(e(\xi), p_1(TN|_B), p_2(TN|_B), ...)$ to $Q(e(\nu_S), p_1(TN|_S), p_2(TN|_S), ...)$ hence the latter is nonzero.

By 3.1 we have $Q(e(q^\# \nu_S), p_1(T\tilde{N}|_{\tilde{S}}), p_2(T\tilde{N}|_{\tilde{S}}), ...) \neq 0$. Let $\tilde{h}$ be the lift of h to the universal covers; note that $\tilde{h}$ is a homotopy equivalence. Then by commutativity $\tilde{h}^* e(q^\# \nu_B) = e(q^\# \nu_S)$ and $\tilde{h}^* p_i(T\tilde{N}|_{\tilde{B}}) = p_i(T\tilde{N}|_{\tilde{S}})$. So the homotopy inverse of $\tilde{h}$ takes $Q(e(q^\# \nu_S), p_1(T\tilde{N}|_{\tilde{S}}), p_2(T\tilde{N}|_{\tilde{S}}), ...)$ to the corresponding polynomial for $\tilde{B}$ which is therefore nonzero as claimed. □

Remark 3.5. The statement of 3.4 becomes especially simple if $p(TB) = 1$. Indeed, it implies that $p(TN|_B) = p(\xi \oplus TB) = p(\xi)p(TB) = p(\xi)$. We also get $p(T\tilde{B}) = 1$ which implies $p(T\tilde{N}|_{\tilde{B}}) = p(q^\# \xi)$.

We shall often use the following variation of 3.4.

Proposition 3.6. *Let ξ be an vector bundle over $B = C \times T$ where C is a closed connected smooth manifold and T is a torus. Assume the total space N of ξ admits a complete Riemannian metric of nonnegative sectional curvature. Suppose there is a polynomial Q with rational coefficients* *such that* $Q(e(\xi), p_1(TN|_B), p_2(TN|_B), ...) \neq 0$ *where* ξ *is assumed to be oriented if* Q *depends on* e. *Then* $Q(e(i_C^{\#}\xi), p_1(TN|_C), p_2(TN|_C), ...) \neq 0$ *where* $i_C\colon C \to C \times T$ *is the inclusion.*

Proof. The universal cover $q\colon \tilde{B} = \tilde{C} \times \mathbb{R}^k \to C \times T = B$ clearly factors through the inclusion $i_C\colon C \to C \times T$. By 3.4, $Q(e(q^{\#}\xi), p_1(T\tilde{N}|_{\tilde{B}}), p_2(T\tilde{N}|_{\tilde{B}}), ...) \neq 0$, therefore, $Q(e(i_C^{\#}\xi), p_1(TN|_C), p_2(TN|_C), ...)$ must be nonzero. □

4. Producing vector bundles

In this section we discuss some methods of building vector bundles. We start from several general methods and then concentrate on the case when the base is $C \times T$ where C is a finite connected CW-complex and T is a torus of positive dimension.

Example 4.1. Let B be a closed orientable $2n$-manifold and let ξ be a bundle over S^{2n}. Since there always exists a degree one map $f\colon B \to S^{2n}$, we get a pullback bundle $f^{\#}\xi$. Now if ξ has a nonzero rational characteristic class (that necessarily lives in $H^{2n}(S^{2n}, \mathbb{Q})$), so does $f^{\#}\xi$ because f induces an isomorphism on the $2n$-dimensional cohomology. In particular, every even integer $2d$ can be realized as the Euler number of a rank $2n$ bundle over B (by taking ξ to be the pullback of TS^{2n} via a self-map of S^{2n} of degree d).

Example 4.2. Any element of $H^2(B, \mathbb{Z})$ can be realized as the Euler class of an oriented rank two bundle over B (where B is any paracompact space) [Hir66, I.4.3.1].

Example 4.3. If B is a finite CW-complex of dimension d, then it is well-known that a multiple of any element of $\oplus_{i>0} H^{4i}(B, \mathbb{Q})$ can be realized as the Pontrjagin character of a vector bundle over B of rank d (and hence of any rank $\geq d$).

In particular, a multiple of any element $x \in H^{4k}(B, \mathbb{Q})$ can be realized as the kth Pontrjagin class of a bundle of rank d. (Indeed, let X be the image of x under the inclusion $H^{4k}(B, \mathbb{Q}) \to \oplus_{i>0} H^{4i}(B, \mathbb{Q})$. Realize a multiple of X as the Pontrjagin character of a bundle. Then this bundle has zero Pontrjagin classes p_i for $0 < i < k$ and the kth Pontrjagin class is a multiple of x.)

We now prove a uniqueness and existence theorem for vector bundles over $C \times T$.

Theorem 4.4. *Let* C *be a finite connected CW-complex and let* ξ *and* η *be oriented rank* n *vector bundles over* $C \times T$ *such that*

(1) $$\xi|_{C \times *} \cong \eta|_{C \times *}$$

and

(2) ξ *and* η *have the same rational characteristic classes.*

Then there exists a finite cover $\pi\colon C\times T\to C\times T$ *such that* $\pi^{\#}\xi\cong\pi^{\#}\eta$.

Proof. First, note that after passing to a finite cover we can assume that ξ and η have the same integral cohomology classes. (Indeed, look for example at the integral i-th Pontrjagin class p_i. By assumption $p_i(\xi)-p_i(\eta)$ is a torsion element of $H^{4i}(C\times T,\mathbb{Z})$. By the Künneth formula we can write $p_i(\xi)-p_i(\eta)=\sum_{j=0}^{4i}c^{4i-j}\otimes t^j$ where $c^s\in H^s(C,\mathbb{Z})$ and $t^j\in H^j(T^k,\mathbb{Z})$. Condition (1) implies $t^0\otimes c^{4i}=p_i(\xi|_{C\times *})-p_i(\eta|_{C\times *})=0$. Since any torsion element of the form $\sum_{j=1}^{4i}c^{4i-j}\otimes t^j$ becomes zero when mapped to an appropriate finite cover $C\times T\to C\times T$ along T, the integral Pontrjagin classes $p_i(\xi)$, $p_i(\eta)$ become equal in such a finite cover.)

Let $f,g\colon C\times T\to BSO(n)$ be the classifying maps for ξ and η respectively. Let γ^n be the universal bundle over $BSO(n)$. For each $i\le[n/2]$, we view the classes $p_i(\gamma^n)\in H^{4i}(BSO(n))$ as maps $p_i\colon BSO(n)\to K(\mathbb{Z},4i)$, and similarly if n is even, $e(\gamma^n)\in H^n(BSO(n))$ is thought of as a map $e\colon BSO(n)\to K(\mathbb{Z},n)$.

Consider the combined map c from $BSO(n)$ to the product of Eilenberg-MacLane spaces given by the formula

$$c=(p_1,p_2...p_{(n-1)/2})\colon BSO(n)\to X=\times_{s=1}^{(n-1)/2}K(\mathbb{Z},4s) \text{ if } n \text{ is odd}$$

and,

$$c=(e,p_1,p_2...,p_{n/2-1})\colon BSO(n)\to X=K(\mathbb{Z},n)\times(\times_{s=1}^{n/2-1}K(\mathbb{Z},4s)).$$

if n is even.

It is well known that c is a rational homotopy equivalence (i.e. the homotopy fiber F of c has finite homotopy groups) and the spaces $F,BSO(n),X$ are simply-connected (see e.g. [Bel98]).

By the condition (2) $p\circ f$ is homotopic to $p\circ g$. We shall now try to lift this homotopy to the homotopy of f and g. We view the pair $(C\times T,C)$ (where we identify $C\times *$ with C) as a relative CW-complex and we try to construct the homotopy between f and g inductively on the dimension of the skeletons. By (1) we can assume that the homotopy is already constructed on the zero skeleton $(C\times T,C)_0$.

Suppose that we have already constructed the homotopy on $(C\times T,C)_{i-1}$ for $i>0$. We want to show that after possibly passing to a finite cover we can extend it over $(C\times T,C)_i$. The relative obstruction O_i to the extension over the i-th skeleton lives in the cohomology $H^i((C\times T,C),\pi_i(F))$. Let

$m = |\pi_i(F)|$ and $k = \dim(T)$; by assumption $k > 0$. Consider the m^k cover $\Pi\colon C\times T \to C\times T$ given by the formula $(c, z_1, ..., z_k) \mapsto (c, z_1^m, ..., z_k^m)$. Notice that by the Künneth formula for the pair $(C\times T, C) = (C, \emptyset)\times(T, *)$ we have

$$H^i((C\times T, C), \pi_i(F)) = \sum_{j=0}^{i} H^{i-j}((C,\emptyset),\pi_i(F))\otimes H^j((T,*),\pi_i(F)) =$$

$$= \sum_{j=1}^{i} H^{i-j}((C,\emptyset),\pi_i(F))\otimes H^j((T,*),\pi_i(F))$$

where the last equality is due to the fact that $H^0((T,*),\pi_i(F)) = 0$. Therefore, for any $\delta \in H^i((C\times T, C),\pi_i(F))$, its pullback $\Pi^*(\delta)$ is an m-th multiple of some class, and thus is equal to zero. In particular, $\Pi^*(O_i) = 0$. On the other hand, by the naturality of obstructions $\Pi^*(O_i)$ is the obstruction to extending the homotopy between $f\circ\Pi$ and $g\circ\Pi$ over the relative i-skeleton $(C\times T, C)_i$. □

Theorem 4.5. *Let ξ be an oriented rank n vector bundle over a finite connected CW-complex C and let $i\colon C\to C\times T$ be the canonical inclusion onto $C\times *$. Let $e' \in H^n(C\times T), p_1' \in H^4(C\times T), \dots, p_{[n/2]}' \in H^{4[n/2]}(C\times T)$ be a collection of integral cohomology classes such that their restrictions onto $C\times *$ give corresponding integral characteristic classes of ξ and, furthermore, $e' = 0$ if n is odd and $p_{[n/2]}' = e'\cup e'$ if n even. Then there exists a finite cover $\Pi\colon C\times T\to C\times T$ and a rank n vector bundle η over $C\times T$ such that the integral characteristic classes of η satisfy $e(\eta) = \Pi^*(e')$ and $p_i(\eta) = \Pi^*(p_i')$ for $i = 1,\dots,[n/2]$.*

Proof. Again, consider the universal fibration $F\to BSO(n)\xrightarrow{c} X$ where X is the product of appropriate Eilenberg-MacLane spaces.

The collection of characteristic classes $(e', p_1', p_2', \dots, p_{[n/2]}')$ defines a natural map $c'\colon C\times T\to X$ (where we exclude e' for odd n and $p_{[n/2]}'$ for even n). It suffices to show that after passing to a finite cover there exists a lift of this map to the map $f\colon C\times T\to BSO(n)$.

By assumptions, we can construct the lift f over $C\times *$ by letting $f|_{C\times *}$ to be equal to a classifying map of ξ.

Now we are faced with the relative lifting problem of extending the lift from $C\times *$ to $C\times T$. As in the proof of 4.4 we will proceed by induction on the dimension of the relative skeleton $(C\times T, C\times *)_i$.

Suppose f is already defined on $(C\times T, C\times *)_{i-1}$ for some $i > 0$. As before the primary obstruction O_i to extending the lift over $(C\times T, C\times *)_i$ lives in the group $H^i((C\times T, C), \pi_{i-1}(F))$. Arguing exactly as in the proof of 4.4, we see that the cover $\Pi\colon C \times T \to C \times T$ given by the formula $(c, z_1, ..., z_k) \mapsto (c, z_1^m, ..., z_k^m)$ where $m = |\pi_{i-1}(F)|$ has the property that $\Pi^*(O_i) = 0$. Therefore, the lift of $c' \circ \Pi$ given by $f \circ \Pi$ can be extended over the relative i-th skeleton of $(C \times T, C \times *)$. This completes the proof of the induction step and hence the proof of the theorem. □

Remark 4.6. Note that by construction the cover Π depends only on n and $\dim(C \times T)$.

Remark 4.7. The proof of 4.5 shows how to compute the characteristic classes of a bundle with the classifying map f. For example, represent e' as $\sum_{j=0}^{k} e'_j$ where $e'_j \in H^{n-j}(C, \mathbb{Z}) \otimes H^j((T, *), \mathbb{Z})$. Then the Euler class of f is given by $\sum_{j=0}^{k} m^j e'_j$ where $\dim(T) = k$. The same result is of course true for any Pontrjagin class of f.

In particular, if $e' = e'_j$ for some $j > 0$, then an integer multiple of e' is realized as the Euler class of some bundle over $C \times T$.

Example 4.8. We shall often use 4.5 in the following situation. Assume $H^{4i}(C \times T, C, \mathbb{Z})$ is infinite and let $p'_i \in H^{4i}((C \times T, C), \mathbb{Z})$ be a nontorsion class and j be any nonzero integer. Let ξ be the trivial bundle of some rank $> 2i$. Then by 4.5 there exists a bundle η_j over $C \times T$ and a finite cover $\Pi\colon C \times T \to C \times T$ such that the restriction of η_j to $C \times *$ is isomorphic to ξ and $\Pi^*(jp'_i) = p_i(\eta_j)$. Clearly, the bundles η_j are pairwise nonisomorphic.

5. Vector bundles with no nonnegatively curved metrics

In this section we obtain concrete examples of bundles without nonnegatively curved metrics. Throughout this section T is a torus of positive dimension and C is a closed connected smooth manifold. (Note that C is not assumed to be simply-connected so the results of this section are slightly more general than the ones stated in the introduction.) We shall often use that the tangent bundle to $C \times T$ is stably isomorphic to the pullback of TC via the projection $C \times T \to C$. All (co)homology groups and characteristic classes in this section have rational coefficients.

Corollary 5.1. *Let η be a vector bundle over C and let ξ be a vector bundle over T such that the total space of $\eta \times \xi$ admits a complete nonnegatively curved metric. Then ξ becomes stably trivial in a finite cover. Furthermore, if either rank$(\eta) = 0$, or η is orientable with $e(\eta) \neq 0$, then ξ becomes trivial in a finite cover.*

Proof. Denote $\eta \oplus TC$ by η' so that the tangent bundle to the total space of $\eta \times \xi$ restricted to the zero section is stably isomorphic to $\eta' \times \xi$. Let i be the largest nonnegative integer such that $p_i(\eta') \neq 0$. Arguing by contradiction assume that $p_k(\xi) \neq 0$ for some $k > 0$. Using the product formula $p(\eta' \times \xi) = p(\eta') \times p(\xi)$, we conclude that the component of $p_{i+k}(\eta' \times \xi)$ in the group $H^{4i}(C) \otimes H^{4k}(T)$ is equal to $p_i(\eta') \times p_k(\xi)$. Since the cross product of nonzero classes is nonzero, $p_{i+k}(\eta' \times \xi)$ is nonzero. On the other hand, the component of $p_{i+k}(\eta' \times \xi)$ in $H^{4i+4k}(C) \otimes H^0(T)$ is $p_{i+k}(\eta') \times 1 = 0 \times 1 = 0$. We now apply 3.6 for $Q = p_{i+k}$ to get a contradiction. By 4.4, ξ becomes stably trivial in a finite cover.

Now assume that either $rank(\eta) = 0$, or $e(\eta) \neq 0$. By 4.4 it suffices to show that $e(\xi) = 0$. Of course, we can assume that $rank(\xi) > 0$. The pullback of $\eta \times \xi$ to C has zero Euler class because the pullback is the Whitney sum of η and a trivial bundle of the same rank as ξ. Hence according to 3.6, $e(\eta \times \xi) = 0$. Thus if $rank(\eta) = 0$, we get $e(\xi) = 0$. Otherwise, note that $e(\eta \times \xi) = e(\eta) \times e(\xi)$ and since $e(\eta) \neq 0$ it implies $e(\xi) = 0$ as wanted. □

Remark 5.2. The assumption that $e(\eta) \neq 0$ is certainly necessary, in general. For example, let η be the trivial line bundle over C and let ξ be the bundle over T^{2n} which is the pullback of TS^{2n} via a degree one map $T^{2n} \to S^{2n}$. Then the bundle $\eta \times \xi$ is trivial so its total space is nonnegatively curved whenever $sec(C) \geq 0$.

Theorem 5.3. *Let $H^{4i}(C \times T, C) \neq 0$ for some $i > 0$ and let Q' be a polynomial in rational Pontrjagin classes p_j where $0 < j < i$ such that the projection of $p_i(TC) + Q'(TC)$ to $H^{4i}(C)$ is zero. Then, in each rank $> 2i$, there exist infinitely many vector bundles over $B = C \times T$ whose total spaces are not nonnegatively curved.*

Proof. Set $Q = p_i + Q'$. Since $H^{4i}(C \times T, C) \neq 0$, we can use 4.5 to find a vector bundle ξ over $C \times T$ of rank $2i + 1$ such that $p_j(\xi) = 0$ for $0 < j < i$ and $p_i(\xi)$ is a nonzero class whose projection to C is zero. Using the Whitney sum formula for Pontrjagin classes we get $p_i(TB \oplus \xi) = p_i(TB) + p_i(\xi)$ and $p_j(TB \oplus \xi) = p_j(TB)$ for $0 < j < i$. Thus, looking at the projection to $H^{4i}(C \times T)$, we get

$$Q(TB \oplus \xi) = p_i(TB \oplus \xi) + Q'(TB \oplus \xi) = p_i(TB) + p_i(\xi) + Q'(TB) = p_i(\xi)$$

where the last equality is true because

$$p_i(TB) + Q'(TB) = (p_i(TC) + Q'(TC)) \times 1 = 0 \times 1 = 0.$$

Thus, we can apply 3.6. □

Corollary 5.4. *Let $H^{4i}(C \times T, C) \neq 0$ for some $i > 0$ and let $ph_i(TC) = 0$, where ph_i is the component of the Pontrjagin character that lives in the $4i$ th cohomology. Then, in each rank $> 2i$, there exist infinitely many vector bundles over $B = C \times T$ whose total spaces are not nonnegatively curved.*

Proof. Take $Q' = ph_i - p_i$. □

Corollary 5.5. *Let $H^{4i}(C \times T, C) \neq 0$ and $p_i(TC) = 0$ for some $i > 0$, then there exists a vector bundle ξ over $C \times T$ of any rank $> 2i$ such that $E(\xi)$ has no metric of nonnegative curvature.*

Proof. Take $Q = 0$. □

Remark 5.6. In particular, 5.5 shows that if $H^4(C \times T, C) \neq 0$ and $p_1(TC) = 0$, then there exist infinitely many bundles of every rank > 2 whose total spaces are not nonnegatively curved.

Corollary 5.7. *If $H^{2i}(C \times T, C) \neq 0$ for some $i > 0$, then there exist infinitely many rank $2i$ vector bundles over $C \times T$ with different Euler classes whose total spaces are not nonnegatively curved.*

Proof. It follows from 4.5 that there exists a bundle ξ (and, in fact, infinitely many such bundles) of rank $2i$ over $C \times T$ such that $e(\xi)$ is a nonzero class whose $H^{2i}(C)$ component is zero. By 3.6 $E(\xi)$ is not nonnegatively curved. □

Corollary 5.8. *Let* $\dim(C) = 4m + 2$ *and* $p_m(TC) \neq 0$, *and let* $\dim(T) \geq 2$. *Then there exist infinitely many bundles of each rank ≥ 2 over $C \times T$ whose total spaces are not nonnegatively curved.*

Proof. By Poincaré duality find $y \in H^2(C)$ such that $p_m(TC)y \neq 0 \in H^{4m+2}(C)$. Take any $t \in H^2(T)$ and realize $y \otimes 1 + 1 \otimes t$ as the Euler class of an oriented rank 2 bundle ξ over $C \times T = B$. Then $p_1(\xi) = (y \otimes 1 + 1 \otimes t)^2 = y^2 \otimes 1 + 2y \otimes t + 1 \otimes t^2$. Note that $p_m(TB)p_1(\xi) \neq 0$. (Indeed, it suffices to show that the projection of $p_m(TB)p_1(\xi)$ to $H^{4m+2}(C) \otimes H^2(T)$ is nonzero. which is true because the projection is equal to $2p_m(TC)y \otimes t$.) Also the Whitney sum formula implies that $p_{m+1}(TB \oplus \xi) = p_m(TB)p_1(\xi) \neq 0$ and the projection of $p_{m+1}(TB \oplus \xi)$ to C vanishes because $H^{4m+4}(C) = 0$. Hence, we are done by 3.6. By adding trivial bundles to ξ one can make its rank arbitrary large. □

Corollary 5.9. *If $H^{4i}(C \times T, C) \neq 0$ for some $i > 0$, then there exist infinitely many vector bundles over $B = C \times T$ of any rank $\geq \dim(C)$ whose total spaces are not nonnegatively curved.*

Proof. Let $\nu(C)$ be a rank $\dim(C)$ bundle over C which is stably isomorphic to stable normal bundle of C. By 4.5 we can find a bundle ξ (and, in fact, infinitely many such bundles) of rank $\dim(C)$ over $C \times T$ such that the pullback of ξ to C is isomorphic to $\nu(C)$ and $p_i(\xi)$ has a nonzero projection to $H^{4i}(C \times T, C)$.

Look at the bundle $TE(\xi|_B) \cong \xi \oplus TB$. Note that the pullback of $\xi \oplus TB$ to C is isomorphic to $\nu(C) \oplus TC$ which is stably trivial, hence $p_i(TE(\xi|_C)) = 0$. On the other hand the projection of $p(\xi \oplus TB) = p(\xi)p(TB)$ to $H^{4i}(C \times T, C)$

is equal to $p_i(\xi)$, in particular the projection of $p_i(\xi \oplus TB)$ to $H^{4i}(C \times T, C)$ is nontrivial. By 3.6 $E(\xi)$ is not nonnegatively curved. □

Remark 5.10. The method of 5.9 can be used with some other bundles in place of $\nu(C)$. To illustrate the idea we discuss the case when C is the total space $S(\eta)$ of the sphere bundle associated with a vector bundle η over S^4.

First, let us handle the easier case when $e(\eta) \neq 0$. Then $rank(\eta)$ is necessarily 4 and it follows from the Gysin sequence that $S(\eta)$ is a rational homology 7-sphere. In particular, all the Pontrjagin classes of $S(\eta)$ vanish. Now 4.5 implies that there are infinitely many rank 4 bundles over $S(\eta) \times T$ with nonzero p_2. By 5.5 their total spaces admit no nonnegatively curved metrics and as usual we can add trivial bundles to make the rank ≥ 4.

Now assume $e(\eta) = 0$. Recall that $p_1(\eta)[S^4]$ is necessarily even and furthermore, any even integer can be realized as $p_1(\xi)[S^4]$ where ξ is a 4-bundle [Mil56]. Thus we can find a 4-bundle ξ' over S^4 with $p_1(\xi')[S^4] = -p_1(\eta)[S^4]$ so that $p_1(\eta \oplus \xi') = 0$. Note that $TS(\eta)$ is stably isomorphic to the pullback of η via the bundle projection $\pi\colon S(\eta) \to S^4$. Setting $\xi = \pi^{\#}\xi'$, we get that $p_1(TS(\eta) \oplus \xi) = 0$. Since $e(\eta) = 0$, the Gysin sequence implies that π induces an isomorphism on H^4, and by the Poincaré duality $H^3(S(\eta)) \neq 0$. Hence by 4.5 there are infinitely many rank 4 bundles over $S(\eta) \times T$ with nonzero p_1 such that their pullback to $S(\eta)$ is ξ. So the proof of 5.9 applies and these bundles admit no nonnegatively curved metrics.

It is interesting to see whether $rank(\xi)$ can be lowered to 3. Recall that an integer k can be realized as $p_1(\xi)[S^4]$ for a 3-bundle over S^4 iff k is a multiple of 4 [Mil56]. Thus, if $p_1(\eta)[S^4]$ is divisible by 4, the argument of the previous paragraph applies and we get infinitely many 3-bundles over $S(\eta) \times T$ with no nonnegatively curved metrics.

Now assume that $p_1(\eta)[S^4] \equiv 2\mathrm{mod}(4)$. We are looking for a 3-bundle ξ over $S(\eta)$ such that $p_1(TS(\eta) \oplus \xi) = 0$. Since $e(\eta) = 0$, the bundle $S(\eta) \to S^4$ has a section s. Setting $\xi' = s^{\#}\xi$, we would get a 3-bundle ξ' over S^4 with $p_1(\eta \oplus \xi') = 0$. In particular, $p_1(\xi')[S^4] \equiv 2\mathrm{mod}(4)$ which is impossible for a 3-bundle.

Thus, the methods of this paper fail here. For instance, we do not have examples of 3-bundles over $S(\eta) \times S^1$ that admit no nonnegatively curved metrics whenever $p_1(\eta)[S^4] \equiv 2\mathrm{mod}(4)$. Note that 4.5 produces many 3-bundles over $S(\eta) \times S^1$ which do not become pullbacks of bundles over $S(\eta)$.

Metastable range, sphere bundles, and surgery. We now describe yet another variation of 5.9. When the method works, it gives a result similar to 5.9 with sometimes lower rank. We showed above that, under certain assumptions on the Pontrjagin classes of $B = C \times T$, there are vector bundles over B whose total spaces admit no nonnegatively curved metric. Now the idea is to replace C by a homotopy equivalent closed manifold C' with "nicer" (e.g. trivial) Pontrjagin classes. Then theorems of this section can be used to produce a vector bundle over $C' \times T$ whose total space admits no nonnegatively curved metric, and can often use it to get a similar bundle over $C \times T$.

Indeed, let $f\colon B \to B'$ be a homotopy equivalence of closed smooth manifolds and let ξ be a vector bundle over B' with total space $E(\xi)$. Assume now that $2\mathrm{rank}(\xi) \geq \dim(B)+3 \geq 5$, that is, we are in the metastable range. By [Hae61], the homotopy equivalence $f\colon B \to E(\xi)$ is homotopic to a smooth embedding $e\colon B \to E(\xi)$. The above inequality implies that $\mathrm{rank}(\xi) \geq 3$, hence by [Sie69, Thm 2.2] $E(\xi)$ is diffeomorphic to the total space of the normal bundle to $E(\nu_e)$. Clearly, $E(\xi)$ is nonnegatively curved iff so is $E(\nu_e)$.

Theorem 5.11. *Let T be a torus and let C be a closed smooth manifold homotopy equivalent to a closed manifold C' such that T and C' satisfy the assumptions of* 5.3 *or* 5.8*. Then, in each rank $> 1 + \dim(C \times T)/2$, there exist infinitely many vector bundles over $B = C \times T$ whose total spaces are not nonnegatively curved.*

Proof. By 5.3 or 5.8 we can find a bundle ξ over $C' \times T$ whose total space $E(\xi)$ is not nonnegatively curved in any rank $> 2i$. Assume now that the rank is $> 1 + \dim(C \times T)/2$ (note that $1 + \dim(C \times T)/2 > 2i$ because $H^{4i}(C \times T, C) \neq 0$. This puts us in the in metastable range so the homotopy equivalence $f \times \mathrm{id}\colon C \times T \to C' \times T \hookrightarrow E(\xi)$ is homotopic to an embedding whose normal bundle has total space diffeomorphic to $E(\xi)$. Of course, the total space of this normal bundle is not nonnegatively curved. By varying ξ (or, rather, the Pontrjagin class of ξ), we get infinitely many such examples. □

One way to replace C by a manifold C' with "nicer" Pontrjagin classes is by surgery. Namely, assume $\pi_1(C) = 1$ and let τ be a vector bundle over C so that τ and TC are stably fiber homotopy equivalent. Then, if the surgery obstruction vanishes (which always happens if $\dim(C)$ is odd [Bro72, II.3.1]), then there is a closed smooth manifold C' and a homotopy equivalence $f\colon C' \to C$ such that TC' is stably isomorphic to $f^{\#}\tau$.

In general, it is not easy to decide when a given vector bundle, such as TC, is stably fiber homotopy equivalent to a bundle with "nicer" Pontrjagin classes. However, each bundle with "nice" Pontrjagin classes is usually stably fiber homotopy equivalent to infinitely many different bundles.

Indeed, recall that two vector bundles are stably fiber homotopy equivalent if the corresponding spherical fibrations are stably equivalent. The stable equivalence spherical fibrations over C (or any finite simply connected cell complex) are in one-to-one correspondence with $[C, BSG]$. The Whitney sum gives BSG

and BSO an H-group structure, and the natural map $BSO \to BSG$ that assigns to a vector bundle the corresponding spherical fibration induces a group homomorphism $[C, BSO] \to [C, BSG]$. After tensoring with rational the group $[C, BSO]$ becomes $\oplus_{i>0} H^{4i}(C, \mathbb{Q})$ while $[C, BSG]$ becomes the trivial group. In particular, if $\oplus_{i>0} H^{4i}(C, \mathbb{Q}) \neq 0$, each stable fiber homotopy equivalence class contains infinitely many vector bundles in any rank $\geq \dim(C)$.

For example, let C be the total space of a sphere bundle over closed simply-connected manifold V associated with a vector bundle η. (Due to [GW00] such manifolds could be a good source of nonnegatively curved manifolds.) The bundle TC is the pullback of $TV \oplus \eta$ via the bundle projection $C \to V$, hence TC is stably fiber homotopy trivial whenever so are TV and η. This construction gives many manifolds with stably fiber homotopy trivial tangent bundles.

6. Sphere bundles with no metric of nonnegative curvature

It was shown in [GW00] that the total space of the sphere bundle associated with the normal bundle to the soul has a nonnegatively curved metric. Thus, potentially, sphere bundles provide a good source of closed nonnegatively curved manifolds.

Theorem 6.1. *For $k > 0$, let $E \to F$ be a k-sphere Serre fibration over a flat manifold F with nonzero rational Euler class. Let P be closed smooth manifold such that there is a map $P \to E$ that induces an isomorphism of fundamental groups. Then P admits no metric of nonnegative Ricci curvature.*

Proof. Arguing by contradiction, assume that P admits a metric of nonnegative Ricci curvature. Pass to a finite cover $\tilde{P} \to P$ so that $\tilde{P}$ is diffeomorphic to $C \times T$ where C is simply connected and T is a torus.

Look at the corresponding covers $\tilde{E} \to E$ and $\tilde{F} \to F$. Note that $\pi_1(\tilde{F})$ is free abelian because $\pi_1(\tilde{F})$ is a torsion free group which is the image of a finitely generated abelian group $\pi_1(\tilde{E}) \cong \pi_1(\tilde{P})$. The k-sphere fibration $\tilde{E} \to \tilde{F}$ still has nonzero rational Euler class since since it is a pullback of $E \to F$ and since finite covers induce injective maps on rational cohomology.

First consider the case $k = 1$. The circle fibration $\tilde{E} \to \tilde{F}$ induces an epimorphism $\phi \colon \pi_1(\tilde{E}) \to \pi_1(\tilde{F})$ of finitely generated free abelian groups. Therefore, ϕ has a section. Since $\tilde{E}$ is aspherical, this section is induced by a continuous map $\tilde{F} \to \tilde{E}$ which defines a homotopy section of the circle fibration $\tilde{E} \to \tilde{F}$. Thus, the Euler class must be zero which is a contradiction.

Now assume that $k > 1$ so that $\tilde{E} \to \tilde{F}$ induces a π_1-isomorphism. Then the inclusion $T \to (C \times T) = \tilde{P}$ followed by the map $\tilde{P} \to \tilde{E} \to \tilde{F}$ induces a π_1-isomorphism hence is a homotopy equivalence. Let s be its homotopy inverse.

Then s followed by the inclusion $T \to \tilde{P}$ and the map $\tilde{P} \to \tilde{E}$ is a homotopy section of the fibration $\tilde{E} \to \tilde{F}$. The Euler class then must be zero which gives a contradiction. □

Remark 6.2. The above argument is a special case of the following phenomenon. Suppose we have a Serre fibration $C \to P \to F$ where F is a flat manifold and C is connected and simply-connected. Look at the spectral sequence of this fibration with rational coefficients. Then if there exists a nonzero differential, then P does not admit a nonnegatively curved metric.

Indeed, if P is nonnegatively curved, then a finite cover $\tilde{P}$ of P splits topologically as $M \times T$ where M is simply connected and T is a torus. By naturality we can see that spectral sequence of the pullback fibration $C \to \tilde{P} \to T$ also has a nonzero differential. Since the universal cover of $\tilde{P}$ is homotopy equivalent to both M and C, they are homotopy equivalent to each other. In particular $\dim H^*(M) = \dim H^*(C)$ and hence $\dim H^*(\tilde{P}) = \dim H^*(M) \otimes H^*(T) = \dim(H^*(C) \otimes H^*(T))$. On the other hand if there is a nonzero differential we should have that $\dim H^*(\tilde{P}) < \dim(H^*(C) \otimes H^*(T))$ which is a contradiction.

Theorem 6.3. *Let $E(\xi)$ be the total space of a vector bundle ξ over a closed smooth manifold B and let $S(\xi) \to B$ be the associated sphere bundle. Assume that ξ has zero rational Euler class and there exists a polynomial Q in rational Pontrjagin classes such that $Q(TE(\xi)) \neq 0$ and $Q(T\tilde{E}(\xi)) = 0$ for the universal cover $\pi\colon \tilde{E}(\xi) \to E(\xi)$. Then $S(\xi)$ admits no metric of nonnegative Ricci curvature.*

Proof. First, we introduce several notations. Let $q\colon \tilde{B} \to B$ be the universal covering and $j\colon S(\xi) \to E(\xi)$ be the inclusion. Also denote by π and i the bundle projection and the zero section of ξ, respectively.

Since $S(\xi) \subset E(\xi)$ is an oriented codimension one hypersurface, $TS(\xi)$ is stably isomorphic to $j^{\#}TE(\xi)$, hence $Q(TS(\xi)) = j^*Q(TE(\xi))$. Also $TE(\xi)$ is isomorphic to $(i\circ\pi)^{\#}TE(\xi)$ since $i\circ\pi$ is homotopic to the identity of $TE(\xi)$. We get $Q(TS(\xi)) = j^*\pi^*Q(i^{\#}TE(\xi))$. By assumption $Q(TE(\xi))$, and hence $Q(i^{\#}TE(\xi))$ is nonzero. Also $j^*\pi^* = (\pi\circ j)^*$ where $\pi\circ j\colon S(\xi) \to B$ is the bundle projection. It follows from the Gysin sequence that $\pi\circ j$ is injective in cohomology because the kernel of $(\pi\circ j)^*$ consists of the cup-multiples of the Euler class which is zero by assumption. Thus, $Q(TS(\xi)) \neq 0$.

On the other hand, the inclusion $S(q^{\#}\xi) \hookrightarrow E(q^{\#}\xi)$ takes $Q(TE(q^{\#}\xi)) = 0$ to $Q(TS(q^{\#}\xi))$. Hence, $Q(TS(q^{\#}\xi)) = 0$ and we are in position to apply the theorem 3.4. □

Corollary 6.4. *Let ξ be a bundle over a flat manifold F with associated sphere bundle $S(\xi)$ and let C be a closed smooth simply-connected manifold. If $C \times$*

$S(\xi)$ admits a metric of nonnegative Ricci curvature, then ξ becomes trivial in a finite cover.

Proof. By 4.4 it suffices to show that $e(\xi) = 0$ and $p(\xi) = 1$. Vanishing of $e(\xi)$ follows from 6.1. Vanishing of all Pontrjagin classes follows exactly as in the proof of 5.1 where instead of referring to 3.4 we use 6.3. □

7. The classification of nonnegatively curved vector bundles over $S^1 \times S^3$

In this section we prove the theorem 1.4. Note that the converse of 1.4 is trivially true, i.e. both the trivial bundle and the product of the trivial bundle over S^3 and Möbius band line bundle over S^1 are nonnegatively curved.

Proof of 1.4. Any vector bundle over a 4-complex is the Whitney sum of a trivial bundle and a bundle of rank ≤ 4, hence it suffices to consider the bundles of rank k at most 4.

First, assume that ξ is orientable. Let $q\colon \mathbb{R} \times S^3 \to S^1 \times S^3$ be the universal cover of $S^1 \times S^3$. Then since any vector bundle over S^3 is trivial we have that $q^{\#}(\xi)$ is trivial. In particular, $p_1(q^{\#}(\xi)) = e(q^{\#}(\xi)) = 0$. Therefore, according to 3.4. the classes $p_1(\xi)$ and $e(\xi)$ vanish. Thus, it suffices to prove the following.

Lemma 7.1. *Let η be an orientable vector bundle over $S^1 \times S^3$ such that $p_1(\eta) = e(\eta) = 0$. Then η is trivial.*

Proof. Since $H^1(S^1 \times S^3, \mathbb{Z}/2\mathbb{Z}) = 0 = H^2(S^1 \times S^3, \mathbb{Z})$, any rank one or rank two orientable bundle over $S^1 \times S^3$ is trivial.

Assume that η is an orientable bundle of rank 4. Let $f\colon S^1 \times S^3 \to BSO(4)$ denote a classifying map for η, i.e $\eta \cong f^*\gamma^4$ where γ^4 is the universal 4-bundle over $BSO(4)$. The first four homotopy groups of $BSO(4)$ are as follows: $\pi_0(BSO(4)) = 0, \pi_1(BSO(4)) = 0, \pi_2(BSO(4)) = \mathbb{Z}/2\mathbb{Z}, \pi_3(BSO(4)) = 0$ and $\pi_4(BSO(4)) = \mathbb{Z} \oplus \mathbb{Z}$. Consider the standard product cell decomposition of $S^1 \times S^3$ coming from canonical cell decompositions $S^1 = e^0 \bigcup e^1$ and $S^3 = e^0 \bigcup e^3$. Then the 3-skeleton of $S^1 \times S^3$ is the wedge $S^1 \bigvee S^3$. Since any orientable vector bundle over S^1 or S^3 is trivial, $f|_{S^1 \bigvee S^3}$ is homotopic to a point and therefore by the homotopy extension property we can assume that f send $S^1 \bigvee S^3$ to a point to begin with. In other words, f can be written as a composition $f = \bar{f} \circ \pi$ where π is the factorization map $\pi\colon S^1 \times S^3 \to S^1 \times S^3/(S^1 \bigvee S^3) \cong S^4$. Since π induces an isomorphism on H^4, the bundle $\bar{f}^*(\gamma^4)$ has zero Euler and Pontrjagin classes. It is a well known that a bundle over S^4 with zero Euler and Pontrjagin classes is trivial. (Indeed, the map $(e, p_1)\colon \pi_4(BSO(4)) \to \mathbb{Z} \oplus \mathbb{Z}$ which associates to a 4-bundle over S^4 its Euler and Pontrjagin classes is a rational homotopy equivalence. Then the induced map on π_4 has finite, and hence trivial, kernel because $\pi_4(BSO(4)) \cong \mathbb{Z} \times \mathbb{Z}$.) Thus $\bar{f}$, and hence f, is nullhomotopic.

A very similar argument shows that any orientable 3-bundle over $S^1 \times S^3$ with zero first Pontrjagin class is trivial. Again, everything can be reduced to 3-bundles over S^4 with zero p_1. The map $p_1\colon BSO(3) \to K(\mathbb{Z}, 4)$ is a rational homotopy equivalence, in particular, the induced map on π_4 has finite, and hence trivial, kernel because $\pi_4(BSO(3)) \cong \mathbb{Z}$. Hence, only the trivial 3-bundle over S^4 has zero p_1. □

Now suppose that ξ is not orientable and its total space admits a metric of nonnegative curvature. Since $\mathbb{Z}$ has a unique subgroup of index 2 the orientation double cover for ξ is given by the map $\pi_S = (z \to z^2) \times \mathrm{id}\colon S^1 \times S^3 \to S^1 \times S^3$. Then the pullback $\pi_S^{\#}(\xi)$ is orientable and also admits a metric of nonnegative curvature. By above, the pullback $\pi_S^{\#}(\xi)$ is trivial. The following lemma completes the proof of 1.4 in the nonorientable case. □

Lemma 7.2. *Let η be a nonorientable rank k bundle over $S^1 \times S^3$ whose orientation lift is a trivial bundle. Then η is isomorphic to the product $\mu^1 \times \epsilon^{k-1}$ of the Möbius band line bundle μ^1 over S^1 and a trivial rank $(k-1)$-bundle ϵ^{k-1} over S^3.*

Proof. Since $H^1(S^1 \times S^3, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, there is only one nonorientable line bundle over $S^1 \times S^3$, namely, $\mu^1 \times \epsilon^0$.

Case of rank four. Let $f\colon S^1 \times S^3 \to BO(4)$ be the classifying map for η and $f_0\colon S^1 \times S^3 \to BO(4)$ be the classifying map for $\mu^1 \times \epsilon^3$. We want to show that these maps are homotopic. The same argument as in the proof of 7.1 shows that f and f_0 are homotopic on the 3-skeleton. Let us show that this homotopy can be extended over the 4-cell.

Let $\pi_B\colon BSO(4) \to BO(4)$ be the canonical double cover. Then each of the maps $f \circ \pi_S$ and $f_0 \circ \pi_S$ lifts to a map $\tilde{f}\colon S^1 \times S^3 \to BSO(4)$ which is the classifying map for $\pi_S^* \eta$. In other words, we have the following commutative diagram

$$\begin{array}{ccc} S^1 \times S^3 & \xrightarrow{\tilde{f}} & BSO(4) \\ \downarrow \pi_S & & \downarrow \pi_B \\ S^1 \times S^3 & \xrightarrow{f} & BO(4) \end{array}$$

By construction, the map $\tilde{f}$ is equivariant under the action of the group of deck transformations $\mathbb{Z}/2\mathbb{Z}$ where the nontrivial element $i \in \mathbb{Z}/2\mathbb{Z}$ acts on $S^1 \times S^3$ by the formula $(z, q) \mapsto (-z, q)$ and acts on $BSO(4)$ by reversing orientations of 4-planes.

Clearly the maps f and f_0 are homotopic iff the maps $\tilde{f}$ and $\tilde{f}_0$ are equivariantly homotopic. By above we can assume that $\tilde{f}$ and $\tilde{f}_0$ are equivariantly homotopic on the 3-skeleton of $S^1 \times S^3$. Next we compute the equivariant cohomology group $H^4_{eq}(S^1 \times S^3, \{\pi_4(BSO(4))\})$ that contains the obstruction for extending the homotopy over the 4-skeleton and show that in our situation the obstruction has to vanish.

In order to explicitly describe equivariant cochains we have to identify the action of $\mathbb{Z}/2\mathbb{Z}$ on $\pi_4(BSO(4))$. Recall that $\pi_4(BSO(4))$ classifies the isomorphism classes of orientable 4-bundles over S^4. On the other hand, $\pi_4(BSO(4)) \cong \pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$ where the last isomorphism can be described explicitly as $(m, n) \mapsto (q \to q^n \cdot v \cdot q^m)$ where we identify $\mathbb{R}^4$ with the quaternions $\mathbb{H}$ and S^3 with the set of unit quaternions q. According to [Mil56], the classes $p_1, e \in H^4(S^4, \mathbb{Z}) \cong \mathbb{Z}$ of the bundle (m, n) are given by $p_1(m, n) = 2(m - n)$ and $e(m, n) = m + n$. The action of i on $BSO(4)$ sends the canonical oriented 4-bundle γ^4 to $-\gamma^4$ (i.e the same bundle with its orientation reversed). Therefore, $i^*(p_1(\gamma^4)) = p_1(\gamma^4)$ and $i^*(e(\gamma^4)) = -e(\gamma^4)$, and hence the action of i on $\pi_3(SO(4))$ is given by $i(m, n) = (-n, -m)$.

Now once the action is identified, a straightforward computation shows that $H^4_{eq}(S^1 \times S^3, \{\pi_4(BSO(4))\}) \cong (\mathbb{Z} \oplus \mathbb{Z})/\text{diagonal} \cong \mathbb{Z}$. Let $O_4 \in H^4_{eq}(S^1 \times S^3, \{\pi_4(BSO(4))\})$ be the obstruction for the equivariant extension of the homotopy between $\tilde{f}$ and $\tilde{f}_0$ over the 4-skeleton. It remains to show that O_4 vanishes. The double cover π_S induces a homomorphism

$$\pi_S^*\colon\ H^4_{eq}(S^1 \times S^3, \{\pi_4(BSO(4))\}) \to H^4_{eq}(S^1 \times S^3, \pi_S^{\#}\{\pi_4(BSO(4))\})$$

where the last group is equal to $H^4(S^1 \times S^3, \pi_4(BSO(4))$ because the pullback bundle of coefficients $\pi_S^{\#}\{\pi_4(BSO(4))\}$ is trivial. We claim that this map is injective. (Indeed, since both groups are isomorphic to $\mathbb{Z}$, it suffices to show that the map is nonzero. If π_S^* were zero, the orientation lift of any nonorientable 4-bundle over $S^1 \times S^3$ would be trivial which is certainly not the case since there exist nonorientable bundles over $S^1 \times S^3$ with nonzero p_1. An example of such a bundle is the Whitney sum of a nontrivial line bundle and the pullback of a 3-bundle over S^4 with nonzero p_1 via a degree one map $S^1 \times S^3 \to S^4$.) Since both $f \circ \pi_S$ and $f_0 \circ \pi_S$ are null homotopic we know that $\pi_S^*(O_4) = 0$ and hence O_4 vanishes.

Case of rank three and two. Again, the classifying maps f and f_0 are homotopic on the 3-skeleton and one has to compute the obstruction O_4 to extending the homotopy over the 4-skeleton.

If the rank is three, $\mathbb{Z}/2\mathbb{Z}$ action on the coefficient group is trivial and the obstruction group $H^4_{eq}(S^1 \times S^3, \{\pi_4(BSO(3))\})$ reduces to $H^4(S^1 \times S^3, \mathbb{Z})$. As before we have $\pi_S^*(O_4) = 0$ and since in this case π_S^* is clearly injective, we conclude that O_4 vanishes. In the rank two case the obstruction is always zero simply because $\pi_4(BSO(2)) = 0$. □

Proposition 7.3. *The total space of any vector bundle over $S^1 \times S^2$ has a complete metric of nonnegative curvature such that the zero section is a soul.*

Proof. Since all vector bundles over S^1 and S^2 admit nonnegatively curved metric such that the zero sections are souls, it suffices to show that any vector bundle over $S^1 \times S^2$ is isomorphic to the product of a bundle over S^1 and a bundle over S^2.

First of all observe that two rank k vector bundles over $S^1 \times S^2$ are isomorphic iff their restrictions to the two-skeleton $S^1 \vee S^2$ are isomorphic. Indeed, we only need to extend the homotopy of the classifying maps $S^1 \times S^2 \to BO(k)$ from $S^1 \vee S^2$ to the remaining 3-cell. This is always possible since $\pi_3(BO(k)) \cong \pi_2(O(n)) = 0$.

Now let ξ be a vector bundle of rank k over $S^1 \times S^2$ with the classifying map $f\colon S^1 \times S^2 \to BO(k)$.

The case $k = 1$ is obvious because line bundles are classified by w_1 and the inclusion $S^1 \hookrightarrow S^1 \times S^2$ induces an isomorphism on $H^1(\ ,\mathbb{Z}/2\mathbb{Z})$ so that any line bundle over $S^1 \times S^2$ is a pullback of a bundle over S^1. Similarly, if $k = 2$ and ξ is orientable, then ξ is completely determined by its Euler class. Since the inclusion $S^2 \hookrightarrow S^1 \times S^2$ induces an isomorphism on $H^2(\ ,\mathbb{Z})$, we conclude that ξ is a pullback of a bundle over S^2.

Assume that $k \geq 3$. Since $\pi_2(BSO(k)) = \mathbb{Z}/2\mathbb{Z}$, there are exactly two k-bundles over S^2, namely the trivial bundle and the Whitney sum of a trivial bundle and a 2-bundle with nonzero w_2. Let κ be a 2-bundle over S^2 that has the same w_2 as the restriction of ξ to S^2 and let λ be a line bundle over S^1 with the same w_1 as the restriction of ξ to S^1. Finally, let g be the classifying map for the the Whitney sum of $\kappa \times \lambda$ and the trivial bundle of rank $(k-3)$. By construction the restrictions of f and g to the two-skeleton $S^1 \vee S^2$ are homotopic as needed.

Finally, suppose that $k = 2$ and ξ is not orientable. Note that the orientable two-fold cover $\tilde{\xi}$ of ξ has zero Euler class. (Indeed, since S^2 represents the generator of $H_2(S^1 \times S^2, \mathbb{Z})$ it suffices to show that the intersection number of S^2 and the zero section of $\tilde{\xi}$ inside the total space $E(\tilde{\xi})$ is zero. To compute the intersection number put S^2 in the general position to the zero section of ξ and then look at the the preimage of the manifolds inside $E(\tilde{\xi})$. The covering action of $\mathbb{Z}/2\mathbb{Z}$ on $E(\tilde{\xi})$ preserves the orientation on the base and changes the orientation of the total space. Thus, points of intersection come in pairs: one with plus sign and the other with minus sign. So the intersection number is zero.) This implies that the restriction of ξ to S^2 has zero Euler class, and so

$\xi|_{S^2}$ is a trivial bundle. By above ξ is isomorphic to the product of $\xi|_{S^1}$ and the rank zero bundle over S^2. □

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