PINCHING ESTIMATES FOR NEGATIVELY CURVED MANIFOLDS WITH NILPOTENT FUNDAMENTAL GROUPS

IGOR BELEGRADEK AND VITALI KAPOVITCH

ABSTRACT. Let M be a complete Riemannian metric of sectional curvature within $[-a^2, -1]$ whose fundamental group contains a k-step nilpotent subgroup of finite index. We prove that $a \geq k$ answering a question of M. Gromov. Furthermore, we show that for any $\epsilon > 0$, the manifold M admits a complete Riemannian metric of sectional curvature within $[-(k+\epsilon)^2, -1]$.

1. INTRODUCTION

If the fundamental group of a complete pinched negatively curved manifold is amenable, it must be finitely generated and virtually nilpotent [BS87, Bow93, BGS85]. In this paper we relate the nilpotency degree of the group to the pinching of the negatively curved metric.

Theorem 1.1. *Let M be complete Riemannian manifold with sectional curvature satisfying $-a^2 \leq \sec(M) \leq -1$. If Γ is a k-step nilpotent subgroup of $\pi_1(M)$, then $a \geq k$. In particular, if $a \in [1, 2)$, then Γ is abelian.*

If the cohomological dimension $\mathrm{cd}(\Gamma)$ of Γ equals to $\dim(M) - 1$, which if $\dim(M) > 2$ is equivalent to assuming that Γ acts cocompactly on horospheres, Theorem 1.1 follows from the proof of Gromov's theorem of almost flat manifolds (see [BK81, Corollary 1.5.2]), by combining the commutator estimate in almost flat horosphere quotients with the displacement estimate coming from the exponential convergence of geodesics.

More recently, Gromov sketched in [Gro91, p.309] a proof of the more general estimate

$$a \geq \frac{k}{r+1} \quad \text{for} \quad r = \left[\frac{\dim(M) - 1 - \mathrm{cd}(G)}{2} \right],$$

where $[x]$ denotes the largest integer satisfying $\leq x$. If $k \leq r+1$, the estimate gives no information, so Gromov asked [Gro91, p.309] whether it can be improved to an estimate that is nontrivial for all $\mathrm{cd}(G) < \dim(M)$. Theorem 1.1

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provides a satisfying answer that involves no dimension assumptions whatsoever. The proof of Theorem 1.1 follows the original Gromov's idea in [BK81], except that the commutator estimate is run in a "central" orbit of an N-structure given by the collapsing theory of J. Cheeger, K. Fukaya, and Gromov [CFG92]. In [BK] we proved the following classification theorem:

Theorem 1.2. [BK] *A smooth manifold M with amenable fundamental group admits a metric of pinched negative curvature if and only if it is diffeomorphic to the Möbius band, or to the product of a line and the total space a flat Euclidean vector bundle over a compact infranilmanifold.*

The "if" direction in Theorem 1.2 involves an explicit warped product construction of a negatively pinched metric on the product of $\mathbb{R}$ and the total space of a flat Euclidean bundle over a closed infranilmanifold. By improving this warped product construction, we show that the pinching bounds provided by Theorem 1.1 are essentially optimal.

Theorem 1.3. *If M be a pinched negatively curved manifold such that $\pi_1(M)$ has a k-step nilpotent subgroup of finite index, then M admits a complete Riemannian metric of $\sec(M) \in [-(k+\epsilon)^2, -1]$ for any $\epsilon > 0$.*

The metric constructed in Theorem 1.3 has cohomogeneity one, specifically $M/\mathrm{Iso}(M)$ is diffeomorphic to $\mathbb{R}$ (with the only exception when M is the Möbius band equipped with a hyperbolic metric).

We do not know whether M in Theorem 1.3 always admits a complete metric with $\sec(M) \in [-k^2, -1]$. This does happen for $k = 1$, since as we show in [BK] any complete pinched negatively curved manifolds with virtually abelian fundamental group admits a complete hyperbolic metric.

Another way to phrase the optimality of Theorem 1.1 is via the concept of pinching. Given a smooth manifold M, we define $\mathrm{pinch}^{\mathrm{diff}}(M)$ to be the infimum of $a^2 \geq 1$ such that M admits a complete Riemannian metric of $-a^2 \leq \sec(M) \leq -1$. If M admits no complete metric of pinched negative curvature, it is convenient to let $\mathrm{pinch}(M)^{\mathrm{diff}} = +\infty$. We then define $\mathrm{pinch}^{\mathrm{top}}(M)$ to be the infimum of all $\mathrm{pinch}^{\mathrm{diff}}(N)$ where N is homeomorphic to M, and define $\mathrm{pinch}^{\mathrm{hom}}(M)$ to be the infimum of $\mathrm{pinch}^{\mathrm{diff}}(N)$'s where N is manifold with $\dim(N) = \dim(M)$ that is homotopy equivalent to M. Of course, $\mathrm{pinch}^{\mathrm{diff}}(N) \geq \mathrm{pinch}^{\mathrm{top}}(M) \geq \mathrm{pinch}^{\mathrm{hom}}(N) \geq 1$.

In general, the pinching invariants are hard to estimate and even harder to compute (see [Gro91] and [Bel01, Section 5] for surveys). Combining Theorems 1.1–1.3, we compute the invariants in case $\pi_1(M)$ is virtually nilpotent.

Corollary 1.4. *If M be a pinched negatively curved manifold such that $\pi_1(M)$ has a k-step nilpotent subgroup of finite index, then $pinch^{\mathrm{diff}}(M) = pinch^{\mathrm{top}}(M) = pinch^{\mathrm{hom}}(M) = k^2$.*

2. Proof of Theorem 1.1

A Riemannian metric is called A-*regular* if $A = \{A_i\}$ is a sequence of nonnegative reals such that the norm of the curvature tensor satisfies $||\nabla^i R|| \leq A_i$. We call a metric *regular* if it is is A-regular for some A. The collapsing theory works best for regular metrics, and the Ricci flow can be used to deform any metric with bounded sectional curvature to a complete Riemannian metric that is close to the original metric in uniform C^1 topology, has almost the same sectional curvature bounds, and is regular. (This fact has been known to some experts, but the first written account only recently appeared in [Kap]). Thus we fix an arbitrary $\delta > 0$ and replace the given metric on M by a nearby A-regular metric g with $\sec_g \in [-(a+\delta)^2, -1]$, and then prove that $a + \delta \geq k$, which would imply $a \geq k$ because δ is arbitrary.

Since the Riemannian covering of (M, g) corresponding to $\Gamma \leq \pi_1(M)$ has the same curvature bounds as (M, g), we can assume that $\pi_1(M) = \Gamma$. Denote the universal cover of M by X. If $k = 1$, all we assert is $a \geq 1$ which is trivially true, so we assume from now on that $k > 1$. Then Γ fixes a unique point at infinity of the universal cover X of M; let $c(t)$ be a ray asymptotic to the point. Since $\sec(X)$ is bounded below, the family $(X, c(t), \Gamma)$ has a subsequence $(X, c(t_i), \Gamma)$ that converges in the equivariant GH-topology topology to $(X_\infty, c_\infty, \Gamma_\infty)$. Since $\sec(X)$ is also bounded above and X has infinite injectivity radius, the convergence $(X, c(t_i)) \to (X_\infty, c_\infty)$ is in fact in $C^{1,\alpha}$ topology. Then the quotients $(X/\Gamma, p_i)$ converge in pointed GH-topology to $(X_\infty/\Gamma_\infty, p_\infty)$, where p_i, p_∞ are the projections of $c(t_i)$, c_∞, respectively.

We now review the main results of [CFG92] as they apply to our situation; we refer to [CFG92] for terminology. Fix ϵ, λ with $0 < \epsilon \ll 1 \ll \lambda$. By [CFG92, Theorems 1.3, 1.7, Proposition 7.21], there are positive constants ρ, κ, ν, σ, depending only on n, ϵ, A such that for each large i, the manifold M carries an N-structure $\mathcal{N}_i$ and an $\mathcal{N}_i$-invariant (ρ, κ)-round metric g_i that is ϵ-close to g in uniform C^λ-topology. Furthermore, there exists an orbit O_i of $\mathcal{N}_i$ such that
(i) the metric on O_i induced by g_i has $\operatorname{diam}(O_i) \to 0$ as $i \to \infty$,
(ii) p_i lies in the ρ-neighborhood V_i of O_i,
(iii) the normal injectivity radius of O_i is $\geq \rho$,
(iv) the norm of the second fundamental form of O_i is $\leq \nu$, and $|\sec(O_i)| \leq \sigma$.
(v) if $\tilde{V}_i \to V_i$ is the Riemannian universal cover, then $\tilde{V}_i$ admits a isometric effective action of a connected nilpotent Lie group G_i that acts transitively on

the preimage $\tilde{O}_i$ of O_i under $\tilde{V}_i \to V_i$, and intersects $\pi_1(V_i) \cong \pi_1(O_i)$ in a normal subgroup that is cocompact in G_i and has index $\leq \kappa$ in $\pi_1(V_i)$.

The above results are stated in [CFG92] in a different form, and their proofs are often omitted or merely sketched, so for reader's convenience we briefly explain in the appendix how to deduce (i)-(iv). For (v) see [CFG92, pp.364–365].

Now we show that the inclusion $V_i \to M$ is π_1-surjective for all large i. Indeed, let $\bar{V}_i$ be a connected component of the preimage of V_i under the cover $X \to M$, and as before let $\tilde{V}_i$, $\tilde{O}_i$ be the universal covers of V_i, O_i, respectively. Fix $\tilde{q}_i \in \tilde{O}_i$, and its projections, $\bar{q}_i \in \bar{V}_i$ and $q_i \in O_i \subset V_i$. By (i)-(ii) the sequence q_i subconverges to some $q_\infty \in X_\infty / \Gamma_\infty$, hence for any $\gamma \in \Gamma$, we have $d(\gamma(\bar{q}_i), \bar{q}_i) \to 0$ as $i \to \infty$. So since Γ is finitely generated, if i is sufficiently large, then (i)-(ii) implies that $\bar{V}_i$ contains the images of $\bar{q}_i$ under some finite generating set S of Γ. By (iii) we see that $\bar{V}_i$ contains the geodesic segment $[\bar{q}_i, s(\bar{q}_i)]$ with $s \in S$, whose projection to $V_i \subset M$ represent the generator of $\pi_1(M, q_i) \cong \Gamma$ corresponding to s.

Hence the surjection $\pi_1(O_i) \to \Gamma$ takes $\pi_1(O_i) \cap G_i$ onto a normal subgroup of Γ of index $\leq \kappa$. The intersection of all normal subgroups of Γ of index $\leq \kappa$ is a subgroup Γ_0 of finite index $\leq n\kappa^2$ where $n = \dim M$. (In fact, $|\Gamma : \Gamma_0| \leq \kappa \cdot \nu_\kappa$ where ν_κ is the number of normal subgroups of index $\leq \kappa$. Since Γ is nilpotent of $\mathrm{cd}(\Gamma) < n$, it can be generated by $< n$ elements, so there is a surjection from a rank n free group F_n onto Γ, and ν_κ equals to the number of normal subgroups of F_n of index $\leq \kappa$, i.e. the number of elements in $\mathrm{Hom}(F_n, \mathbb{Z}_\kappa)$, which is at most $n\kappa$.)

Denote $d(\bar{q}_i, \gamma(\bar{q}_i))$ by d_γ. Below this notation is used for different distance functions, and each time we specify which metric we use.

Since $|\Gamma : \Gamma_0| < \infty$, the nilpotency degree of Γ_0 is k. Thus there are $\gamma_j \in \Gamma_0$, $j = 1, \ldots, k$ satisfying

$$[\gamma_1, [\gamma_2, [\gamma_3, [\ldots [\gamma_{k-1}, \gamma_k]...] = \gamma \neq 1.$$

Since Γ_0 lies in the image of $\pi_1(O_i) \to \Gamma$, we can think of each γ_j as acting on $\bar{O}_i \subset X$, where $\bar{O}_i$ is the preimage of O_i under the cover $X \to M$. Note that one can choose γ_j's so that in the intrinsic metric on $\bar{O}_i$ induced by g_i we have $d_{\gamma_j} \leq 2n\kappa^2 \cdot \mathrm{diam}(O_i)$. (Indeed, the Γ_0-action on $\bar{O}_i \subset X$ has a fundamental domain F_i of diameter $\leq n\kappa^2 \cdot \mathrm{diam}(O_i)$. Then Γ_0 is generated by $S = \{s \in \Gamma_0 : s(\bar{F}_i) \cap \bar{F}_i \neq \emptyset\}$, and each element of S has displacement at most $2n\kappa^2 \cdot \mathrm{diam}(O_i)$. Then there is a nontrivial k-fold commutator formed by elements of S, because otherwise the identity $[a, bc] = [a, b] \cdot [b, [a, c]] \cdot [a, c]$ implies that any k-fold commutator in Γ_0 is trivial, so its nilpotency degree is $< k$). In particular, for the intrinsic metric induced on O_i by g_i the displacements of γ_j's satisfy $d_{\gamma_j} \to 0$ as $i \to 0$.

By (i) and (iv) we see that each O_i with intrinsic metric induced by g_i is almost flat, so the commutator estimate of [BK81, Proposition 3.5 (iii), Theorem 2.4.1 (iii)] for the intrinsic metric on O_i induced by g_i gives

$$(2.1) \qquad d_\gamma \leq c \prod_j d_{\gamma_j},$$

where the constant c depends only on n, a.

By Rauch comparison for Jacobi fields, the normal exponential map is bi-Lipschitz on the ρ-neighborhood in the normal bundle to O_i, with Lipschitz constants depending on a, n, ρ. Hence the nearest point projection of the ρ-tubular neighborhood of $\bar{O}_i$ onto $\bar{O}_i$ is K-Lipschitz for $K = K(a, n, \rho)$, so any g_i-geodesic of length $\leq 2\rho$ with endpoints on $\bar{O}_i$ is projected by the nearest point projection to a curve of length $\leq 2\rho K$. Since the intrinsic displacements of γ_j's are $< 2\rho$ for all large i, the estimate (2.1) holds with a different c, for the distance function of the extrinsic metric g_i, and again c only depends on n, a, ϵ, λ.

Finally, since the distance functions of g and g_i are bi-Lipschitz on $B_1(p_i))$, we get the same estimate (2.1) for the original metric g, with c depending on n, a, ϵ, k, λ.

For the rest of the proof we work with displacements in metric g. Passing to a subsequence of p_i's, we can find j such that $d_{\gamma_j} \geq d_{\gamma_l}$ for all l, i. Taking logs we get

$$\ln d_\gamma \leq \ln C + \ln d_{\gamma_1} + ... + \ln d_{\gamma_k} \leq \ln C + k \ln d_{\gamma_j}$$

Since $\ln d_{\gamma_j} < 0$ and $\lim_{i \to \infty} d_{\gamma_j} = 0$, we deduce

$$\limsup_{t \to \infty} \frac{\ln d_\gamma}{\ln d_{\gamma_j}} \geq \limsup_{t \to \infty} \frac{\ln C}{\ln d_{\gamma_j}} + k = k$$

On the other hand, by exponential convergence of geodesic rays, for any two elements of Γ, and in particular for γ, γ_j we get

$$\limsup_{t \to \infty} \frac{\ln d_\gamma}{\ln d_{\gamma_j}} \leq a + \delta$$

so $a + \delta \geq k$, which completes the proof.

Remark 2.2. The weaker conclusion $a \geq k-1$ can be obtained by the following easier argument that does not use collapsing theory. The collapsing theory was used in the above proof to get the commutator estimate (2.1), which is a combination of the two independent estimates in [BK81], namely:
(a) an upper bound on the displacement of the commutator of two elements in terms of their displacements and rotational parts [BK81, Corollary 2.4.2 (i)] that only uses bounded curvature assumption, and

(b) an upper bound of the rotational part of γ_j by a constant multiple of d_{γ_j} that uses almost flatness [BK81, Proposition 3.5 (i)]).

An alternative way to get (b) in our case is via the rotation homomorphism $\phi\colon \Gamma \to O(n)$, introduced by B. Bowditch [Bow93], which is the holonomy of a Γ-invariant flat connection on X. A key property of ϕ is that $\phi(\gamma)$ approximates the rotational part of any $\gamma \in \Gamma$ with error $\le d_\gamma$. Now since any nilpotent subgroup of $O(n)$ is abelian, ϕ must have a kernel of nilpotence degree $k-1$. Hence, there is a $(k-1)$-fold commutator in Γ whose entries lie in the kernel of ϕ, and hence their rotational parts are bounded by their displacements. Repeating the argument at the end of the proof of Theorem 1.1 for this commutator, we get $a \ge k-1$.

3. Infranilmanifolds are horosphere quotients

Let G be a simply-connected nilpotent Lie group acting on itself by left translations, and let K be a compact subgroup of $\mathrm{Aut}(G)$, so that the semidirect product $G \rtimes K$ acts on G by affine transformations. The quotient of G by a discrete torsion free subgroup of $G \rtimes K$ is called an *infranilmanifold*. We showed in [BK] that any pinched negatively curved manifold with amenable fundamental group is either the Möbius band or product of an infranilmanifold with $\mathbb{R}$, and conversely, each of these manifolds admits an explicit warped product metrics of pinched negative curvature.

This section contains a slight improvement of the warped product construction, that yields Theorem 1.3. Consider the product of the above $G \rtimes K$-action on G with the trivial $G \rtimes K$-action on $\mathbb{R}$. For the $G \rtimes K$-action on $G \times \mathbb{R}$, we prove the following.

Theorem 3.1. *If G has nilpotence degree k, then for any $\epsilon > 0$, $G \times \mathbb{R}$ admits a complete $G \rtimes K$-invariant Riemannian metric of sectional curvature within $[-(k+\epsilon)^2, -1]$.*

Proof. The Lie algebra $L(G)$ can be written as

$$L(G) = L_1 \supset L_2 \supset \cdots \supset L_k \supset L_{k+1} = 0$$

where $L_{i+1} = [L_1, L_i]$. Note that $[L_i, L_j] \subset L_{i+j+1}$. Indeed, assume $i \le j$ and argue by induction on i. The case $i = 1$ is obvious and the induction step follows from the Jacobi identity and the induction hypothesis, because $[L_i, L_j] = [[L_1, L_{i-1}], L_j]$ lies in

$$\mathrm{span}([[L_{i-1}, L_j], L_1], [[L_1, L_j], L_{i-1}]) \subset \mathrm{span}([L_{i+j}, L_1], [L_{j+1}, L_{i-1}]) = L_{i+j+1}$$

The group K preserves each L_i, so we can choose a K-invariant inner product $\langle\ ,\ \rangle_0$ on L. Let

$$F_i = \{X \in L_i\colon \langle X, Y \rangle_0 = 0 \text{ for } Y \in L_{i+i}\}.$$

Then $L = F_1 \oplus \cdots \oplus F_k$. Define a new K-invariant inner product $\langle \; , \; \rangle_r$ on L by $\langle X, Y \rangle_r = h_i(r)^2 \langle X, Y \rangle_0$ for $X, Y \in F_i$, and $\langle X, Y \rangle_r = 0$ if $X \in F_i$, $Y \in F_j$ for $i \neq j$, where h_i are some positive function defined below. This defines a $G \rtimes K$-invariant Riemannian metric g_r on G.

Let $\alpha_i = i$ with $i = 1, \cdots, k$ and $a = k$. Given $\rho > 0$, we define the warping function h_i to be a positive, smooth, strictly convex, decreasing function that is equal to $e^{-\alpha_i r}$ if $r \geq \rho$, and is equal to e^{-ar} if $r \leq -\rho$; such a function exists since $a \geq a_i$ for each i. Thus $h_i' < 0 < h_i''$, and the functions $\frac{h_i'}{h_i}$, $\frac{h_i''}{h_i}$ are uniformly bounded away from 0 and ∞.

Define the warped product metric on $G \times \mathbb{R}$ by $g = s^2 g_r + dr^2$, where $s > 0$ is a constant; clearly g is a complete $G \rtimes K$-invariant metric. A straightforward tedious computation (mostly done e.g. in [BW]) yields for g-orthonormal vector fields $Y_s \in F_s$ that

$$\langle R_g(Y_i, Y_j)Y_j, Y_i \rangle_g = \tfrac{1}{s^2} \langle R_{g_r}(Y_i, Y_j)Y_j, Y_i \rangle_{g_r} - \frac{h_i' h_j'}{h_i h_j},$$

$$\langle R_g(Y_i, Y_j)Y_l, Y_m \rangle_g = \tfrac{1}{s^2} \langle R_{g_r}(Y_i, Y_j)Y_l, Y_m \rangle_{g_r} \quad \text{if } \{i, j\} \neq \{l, m\},$$

$$\langle R_g(Y_i, \tfrac{\partial}{\partial r})\tfrac{\partial}{\partial r}), Y_i \rangle_g = -\frac{h_i''}{h_i}, \qquad \langle R_g(Y_i, \tfrac{\partial}{\partial r})\tfrac{\partial}{\partial r}), Y_j \rangle_g = 0 \quad \text{if } i \neq j,$$

$$\langle R_g(\tfrac{\partial}{\partial r}, Y_i)Y_j, Y_l \rangle_g = \left(\frac{h_j'}{2h_j} + \frac{h_l'}{2h_l} \right) \left(\langle [Y_j, Y_i], Y_l \rangle_g + \langle [Y_i, Y_l], Y_j \rangle_g + \langle [Y_j, Y_l], Y_i \rangle_g \right).$$

Since $[L_i, L_j] \subset L_{i+j+1}$, we have for $Z = \sum_{i=1}^k Z_i$ and $W = \sum_{j=1}^k W_j$ with $Z_i, W_i \in F_i$

$$|[Z, W]|_{g_r} \leq \sum_{ij} |[Z_i, W_j]|_{g_r} \leq \sum_{ij} \sum_{s > i+j} h_s |[Z_i, W_j]|_0$$

The above choice of a_i's implies that if $r \geq \rho$, then $\sum_{s > i+j} h_s \leq k h_i h_j$. Also $|[Z_i, W_j]|_0 \leq C |Z_i|_0 |W_j|_0$ where C only depends on the structure constants of L, so that we conclude

$$|[Z, W]|_{g_r} \leq C k |Z_i|_0 |W_j|_0 \sum_{ij} h_i h_j \leq C k^2 |Z|_{g_r} |W|_{g_r}.$$

It follows that if $r \geq \rho$, then the norm of the curvature tensor of g_r is bounded in terms of C, k [CE75, Proposition 3.18]. The same conclusions trivially hold for $r \leq -\rho$, because then g_r is the rescaling of g_0 by a constant $e^{-ar} > 1$, and also for $r \in [-\rho, \rho]$ by compactness, since g_r is left-invariant and depends continuously of r. Hence $\langle R_g(Y_i, Y_j)Y_l, Y_m \rangle_g \to 0$ as $s \to \infty$ if $\{i, j\} \neq \{l, m\}$.

Also $\langle R_g(\tfrac{\partial}{\partial r}, Y_i)Y_j, Y_l \rangle_g \to 0$ as $s \to \infty$, because

$$|\langle [Y_j, Y_i], Y_l \rangle_g| = s^2 |\langle [Y_j, Y_i], Y_l \rangle_{g_r}| \leq s^2 C k^2 |Y_j|_{g_r} |Y_i|_{g_r} |Y_l|_{g_r} \leq C k^2 / s,$$

where the last inequality holds since $s |Y|_{g_r} = 1$ for any g-unit vector Y.

It follows that as $s \to \infty$, then R_g uniformly converges to a tensor $\bar{R}$ whose nonzero components are

$$\bar{R}(Y_i, Y_j, Y_j, Y_i) = -\frac{h_i' h_j'}{h_i h_j} \quad \text{and} \quad \bar{R}\left(Y_i, \frac{\partial}{\partial r}, \frac{\partial}{\partial r}, Y_i\right) = -\frac{h_i''}{h_i}.$$

Thus g has pinched negative curvature for all large s. Finally, we show that for any $\epsilon > 0$ there exists ρ such that $\sec_g \in [-(k+\epsilon)^2, -1]$. Note that

$$\frac{h_i'}{h_i} = \ln(h_i)' \qquad \text{and} \qquad \frac{h_i''}{h_i} = \ln(h_i)'' + (\ln(h_i)')^2.$$

By construction $|\ln(h_i)'| \leq k$. Also let ρ be large enough, so that one can choose h_i on $[-\rho, \rho]$ to satisfy $|\ln(h_i)''| \ll \epsilon$. Then for all sufficiently large s, the sectional curvature of g is within $[-(k+\epsilon)^2, -1]$. $\qquad\square$

Proof of Theorem 1.3. By [BK] if a pinched negatively curved manifold contains has a virtually k-step nilpotent fundamental group, then it is diffeomorphic to the quotient of $G \times \mathbb{R}$ by a discrete torsion free subgroup of $G \rtimes K$. Thus we are done by Theorem 3.1. $\qquad\square$

Appendix A. On collapsing theory

The purpose of this appendix is to outline the proof of the claims (i)-(iv) made in the proof of Theorem 1.1. Some details can be found in [CFG92].

Since g is regular, so is the corresponding metric $\tilde{g}$ on the frame bundle FM. The balls $(FB_1(x), \tilde{g})$ form an $O(n)$-GH-precompact family, where $FB_1(x)$ denotes the frame bundle over the unit ball $B_1(x)$, $x \in M$. By [Fuk88] the closure of the family consists of regular Riemannian manifolds. So for an arbitrary sequence $p_i \in M$, the manifolds $(FB_1(p_i), \tilde{g})$ subconverge in $O(n)$-GH-topology to a pointed regular Riemannian manifold (Y, y).

By the local version of Fukaya's fibration theorem for some sequence $\delta_i > 0$ satisfying $\delta_i \to 0$ as $i \to \infty$, there exists for each large i an $O(n)$-equivariant δ_i-almost Riemannian submersion $FB_1(p_i) \to Y$ with nilmanifolds as fibers, which is also an $O(n)$-δ_i-Hausdorff approximation. Furthermore, each $FB_1(p_i)$ carries an $O(n)$-invariant N-structure $\tilde{\mathcal{N}}_i$ whose orbits are the nilmanifold fibers of the above submersion, and because of the $O(n)$-invariance, the structure descends to an N-structure $\mathcal{N}_i$ on $B_1(p_i)$. By [CFG92, Proposition 7.21] $FB_1(p_i)$ carries a metric $\tilde{g}_i$ that is ϵ-close to $\tilde{g}$ in C^λ-topology, and is both $O(n)$-invariant and $\tilde{\mathcal{N}}_i$-invariant. Hence $\tilde{g}_i$ induces unique Riemannian submersion metrics $\bar{g}_i$ on Y, and g_i on $B_1(x)$.

To see (ii)-(iv), note that if $l \leq \lambda - 2$, then $\|\nabla^l R_{\bar{g}_i}\|$ is bounded independently of i, so the sequence $\bar{g}_i$ is precompact in $C^{\lambda-2}$-topology. Then by [PT99,

Lemma 2.7] $\bar{g}_i$ is precompact in $O(n)$-$C^{\lambda-2}$-topology, i.e. after pulling back by self-diffeomorphisms of Y, the metrics smoothly subconverge and share the same isometric $O(n)$-action. Thus there exists $\rho > 0$ such that for each large i, the point $y \in (Y, \bar{g}_i)$ lies in a ρ-neighborhood of an $O(n)$-orbit that has normal injectivity radius $\geq \rho$. The preimage O_i of the $O(n)$-orbit under the Riemannian submersion $(FB_1(p_i), \tilde{g}_i) \to (Y, \bar{g}_i)$ satisfies (ii)-(iii). Finally, (iii) implies the second fundamental form bound in (iv), which by Gauss formula gives a bound on $|\sec(O_i)|$.

To see (i) note that the $\tilde{g}$-diameter of any orbit of $\tilde{\mathcal{N}}_i$ is $\leq \delta_i$, so since $\tilde{g}$ and $\tilde{g}_i$ are bi-Lipschitz, the $\tilde{g}_i$-diameter of any orbit of $\tilde{\mathcal{N}}_i$ tends to zero as $i \to \infty$, and the same holds for orbits of $\mathcal{N}_i$ because $FB_1(p_i) \to B_1(p_i)$ is distance non-increasing. Finally, the ambient diameter bound implies the intrinsic diameter bound, because Rauch comparison for Jacobi fields gives bounds on bi-Lipschitz constants of the normal exponential map of O_i, and in particular, the Lipschitz constant of the nearest point projection of the ρ-tubular neighborhood of O_i onto O_i depends only on a, n, ρ, ϵ, λ.

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IGOR BELEGRADEK, SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, AT-
LANTA, GA 30332-0160

E-mail address: ib@math.gatech.edu

VITALI KAPOVITCH, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA
BARBARA, CA 93110

E-mail address: vitali@math.ucsb.edu