

Ricci curvature and the fundamental group

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Structure of the talk

- 1 Background on lower sectional curvature bounds
 - Toponogov comparison
 - Short basis.
 - Bishop-Gromov volume comparison
 - Gromov-Hausdorff convergence
 - Equivariant Gromov-Hausdorff convergence
 - Yamaguchi's fibration theorem
 - Semiconcave functions
 - Topological results for sectional curvature bounded below

- 2 Background in Ricci curvature bounded below
 - Bochner's formula
 - Segment Inequality
 - Stability
 - Structure of limit spaces of manifolds with lower Ricci curvature bounds
 - Submetries

Structure of the talk

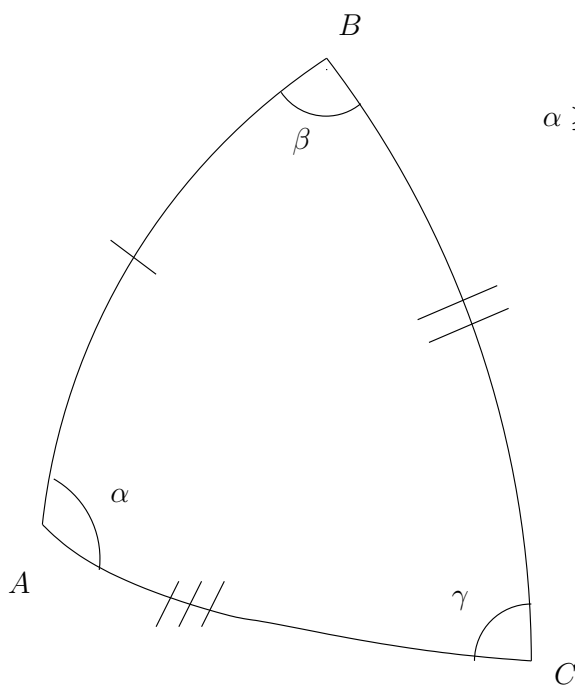
- 3 Main Results
- 4 Finite generation of fundamental groups
 - Gap Lemma
 - Finite generation
- 5 Maps which are on all scales close to isometries.
 - Properties and examples
 - The Rescaling Theorem.
- 6 Margulis Lemma.
 - Idea of the proof of the Margulis Lemma
 - The Induction Theorem for C-Nilpotency

Theorem (Toponogov comparison)

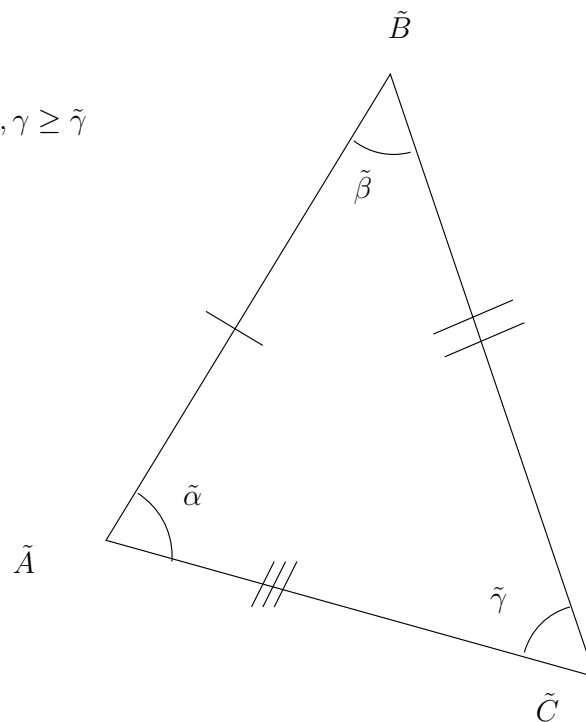
Let (M^n, g) be a complete Riemannian manifold of $K_{sec} \geq 0$. Let ΔABC be a geodesic triangle in M and let $\Delta \tilde{A}\tilde{B}\tilde{C}$ be a comparison triangle in \mathbb{R}^2 , i.e. $|AB| = |\tilde{A}\tilde{B}|$, $|AC| = |\tilde{A}\tilde{C}|$, $|CB| = |\tilde{C}\tilde{B}|$.
Then $\alpha \geq \tilde{\alpha}$, $\beta \geq \tilde{\beta}$, $\gamma \geq \tilde{\gamma}$.

$M^n, K \geq 0$

$\mathbb{R}^2, K = 0$



$$\alpha \geq \tilde{\alpha}, \beta \geq \tilde{\beta}, \gamma \geq \tilde{\gamma}$$



Gromov's short basis

Ricci curvature
and the
fundamental
group

Vitali Kapovitch

Background on
lower sectional
curvature bounds

Toponogov
comparison

Short basis.

Bishop-Gromov
volume
comparison

Gromov-
Hausdorff
convergence

Equivariant
Gromov-
Hausdorff
convergence

Yamaguchi's
fibration
theorem

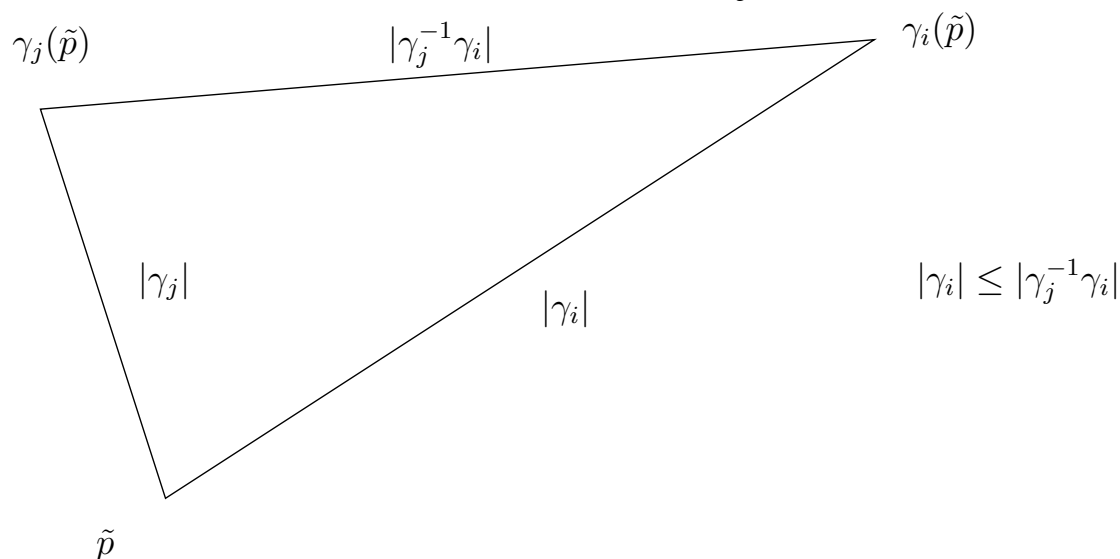
Semiconcave
functions

Topological
results for
sectional
curvature
bounded below

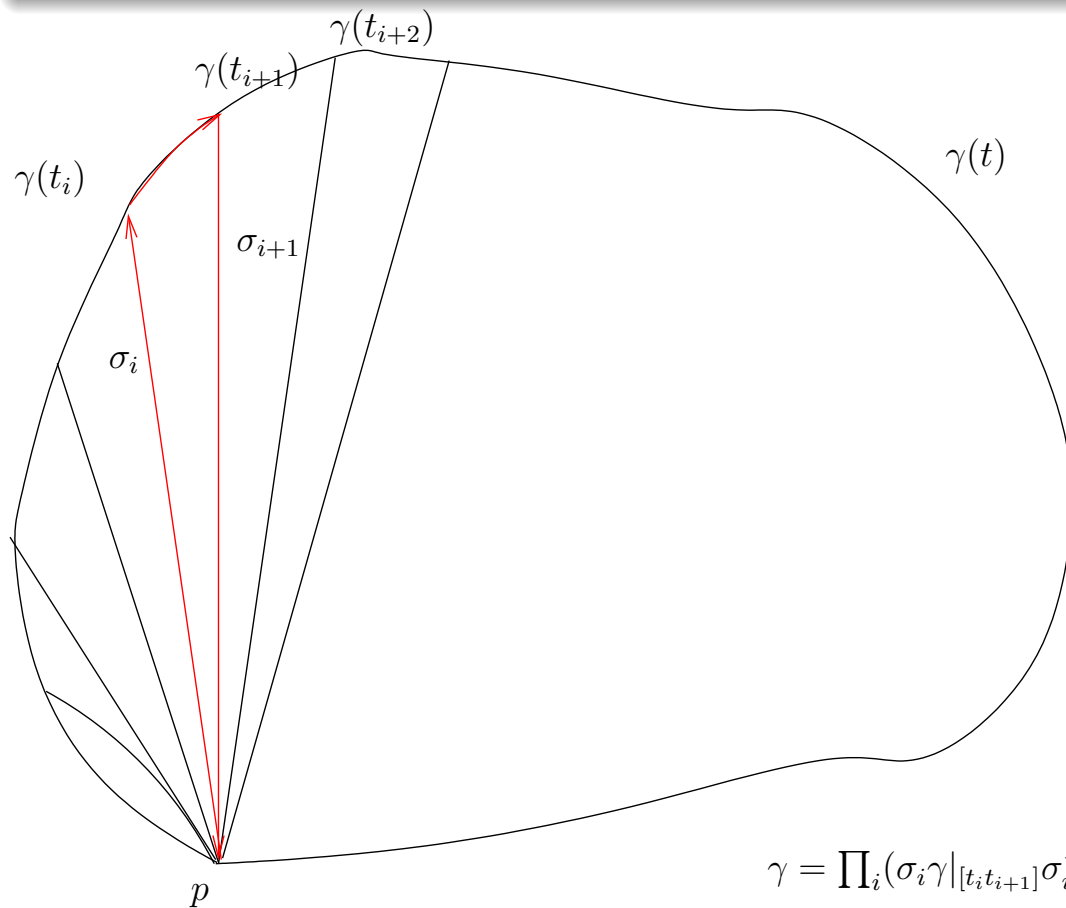
Background in
Ricci curvature
bounded below

Given a complete Riemannian manifold (M, g) and a point $p \in M$ let $\Gamma = \pi_1(M)$ acting on the universal cover \tilde{M} and let \tilde{p} be a lift of p . For $\gamma \in \Gamma$ we will refer to $|\gamma| := d(\tilde{p}, \gamma(\tilde{p}))$ as the norm of γ . Choose $\gamma_1 \in \Gamma$ with the minimal norm in Γ . Next choose γ_2 to have minimal norm in $\Gamma \setminus \langle \gamma_1 \rangle$. On the i -th step choose γ_i to have minimal norm in $\Gamma \setminus \langle \gamma_1, \gamma_2, \dots, \gamma_{i-1} \rangle$. The sequence $\{\gamma_1, \gamma_2, \dots\}$ is called a *short basis* of Γ at p . In general, the number of elements of a short basis can be infinite.

For any $i > j$ we have $|\gamma_i| \leq |\gamma_j^{-1} \gamma_i|$.



If M is closed then $|\gamma_i| \leq 2 \text{diam } M$ for all i and $\{\gamma_1, \gamma_2, \dots\}$ is finite.



$$L(\sigma_i) \leq D$$

$$L(\gamma|_{[t_i, t_{i+1}]}) \leq \varepsilon$$

$$L(\sigma_i \gamma|_{[t_i, t_{i+1}]} \sigma_{i+1}^{-1}) \leq 2D + \varepsilon$$

$$\gamma = \prod_i (\sigma_i \gamma|_{[t_i, t_{i+1}]} \sigma_{i+1}^{-1})$$

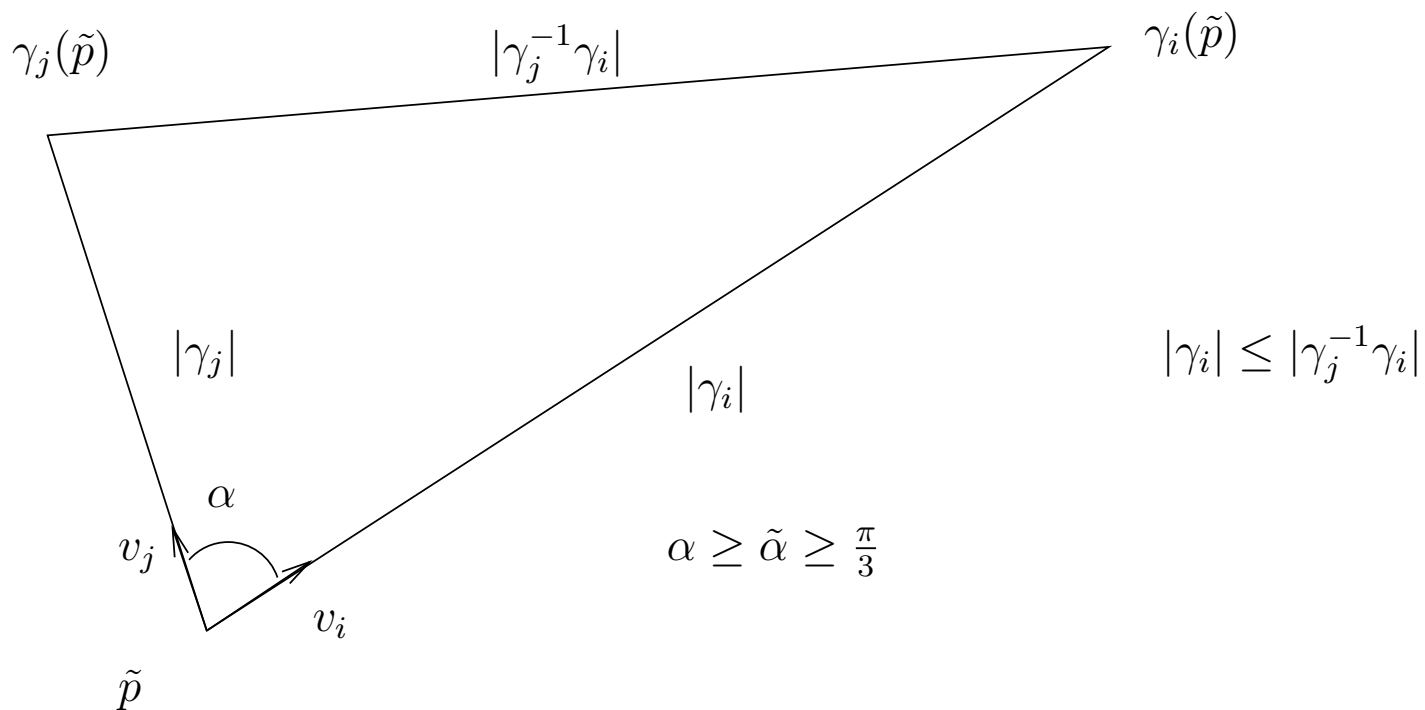
Theorem (Gromov)

Let (M^n, g) have $\text{sec} \geq k$, $\text{diam} \leq D$. Then $\pi_1(M)$ can be generated by $\leq C(n, k, D)$ elements.

Proof.

Let $\gamma_1, \dots, \gamma_i, \dots$ be a short basis of $\pi_1(M)$. Let $v_i \in T_{\tilde{p}}\tilde{M}$ be the direction of a shortest geodesic from \tilde{p} to $\gamma_i(\tilde{p})$. Then for any $i \neq j$ by above we have that the angle $\angle v_i v_j = \alpha \geq \tilde{\alpha} \geq \pi/3$. This means that the vectors $v_1, v_2, \dots \in \mathbb{S}^{n-1}$ are at least $\pi/3$ -separated which immediately implies the result. \square

$$K \geq 0$$



Bishop-Gromov volume comparison

Ricci curvature
and the
fundamental
group

Vitali Kapovitch

Background on
lower sectional
curvature bounds

Toponogov
comparison

Short basis.

**Bishop-Gromov
volume
comparison**

Gromov-
Hausdorff
convergence

Equivariant
Gromov-
Hausdorff
convergence

Yamaguchi's
fibration
theorem

Semiconcave
functions

Topological
results for
sectional
curvature
bounded below

Background in
Ricci curvature
bounded below

Theorem (Bishop-Gromov's Relative Volume Comparison)

Suppose M^n has $\text{Ric}_M \geq (n-1)k$. Then

$$(1) \quad \frac{\text{Vol}(\partial B_r(p))}{\text{Vol}(\partial B_r^k(0))} \text{ and } \frac{\text{Vol}(B_r(p))}{\text{Vol}(B_r^k(0))} \text{ are nonincreasing in } r.$$

In particular,

$$(2) \quad \text{Vol}(B_r(p)) \leq \text{Vol}(B_r^k(0)) \quad \text{for all } r > 0,$$

$$(3) \quad \frac{\text{Vol}(B_r(p))}{\text{Vol}(B_R(p))} \geq \frac{\text{Vol}(B_r^k(0))}{\text{Vol}(B_R^k(0))} \quad \text{for all } 0 < r \leq R,$$

and equality holds if and only if $B_r(p)$ is isometric to $B_r^k(0)$. Here $B_r^k(0)$ is the ball of radius r in the n -dimensional simply connected space of constant curvature k .

Note that this implies that if the volume of a big ball has a lower bound, then all smaller balls also have lower volume bounds

Idea of the proof for $K_{\text{sec}} \geq 0$

Ricci curvature
and the
fundamental
group

Vitali Kapovitch

Background on
lower sectional
curvature bounds

Toponogov
comparison

Short basis.

Bishop-Gromov
volume
comparison

Gromov-
Hausdorff
convergence

Equivariant
Gromov-
Hausdorff
convergence

Yamaguchi's
fibration
theorem

Semiconcave
functions

Topological
results for
sectional
curvature
bounded below

Background in
Ricci curvature
bounded below

Consider the following *contraction* map $f: B_R(p) \rightarrow B_r(p)$. Given $x \in B_R(p)$ let $\gamma: [0, 1] \rightarrow M$ be a shortest geodesic from p to x . Define $f(x) = \gamma(\frac{r}{R})$. At points where the geodesics are not unique we choose any one. By Toponogov comparison we have that

$$|f(x)f(y)| \geq \frac{r}{R}|xy|$$

Therefore

$$\text{Vol}B_r(p) \geq \text{Vol}f(B_R(p)) \geq \left(\frac{r}{R}\right)^n \text{Vol}B_R(p)$$

Ricci curvature and the fundamental group

Vitali Kapovitch

Background on lower sectional curvature bounds

Toponogov comparison

Short basis.

Bishop-Gromov volume comparison

Gromov-Hausdorff convergence

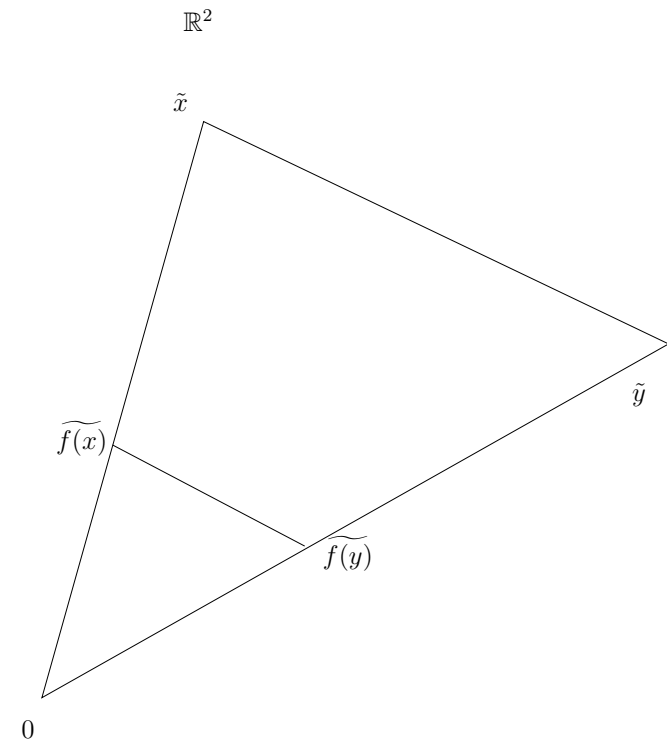
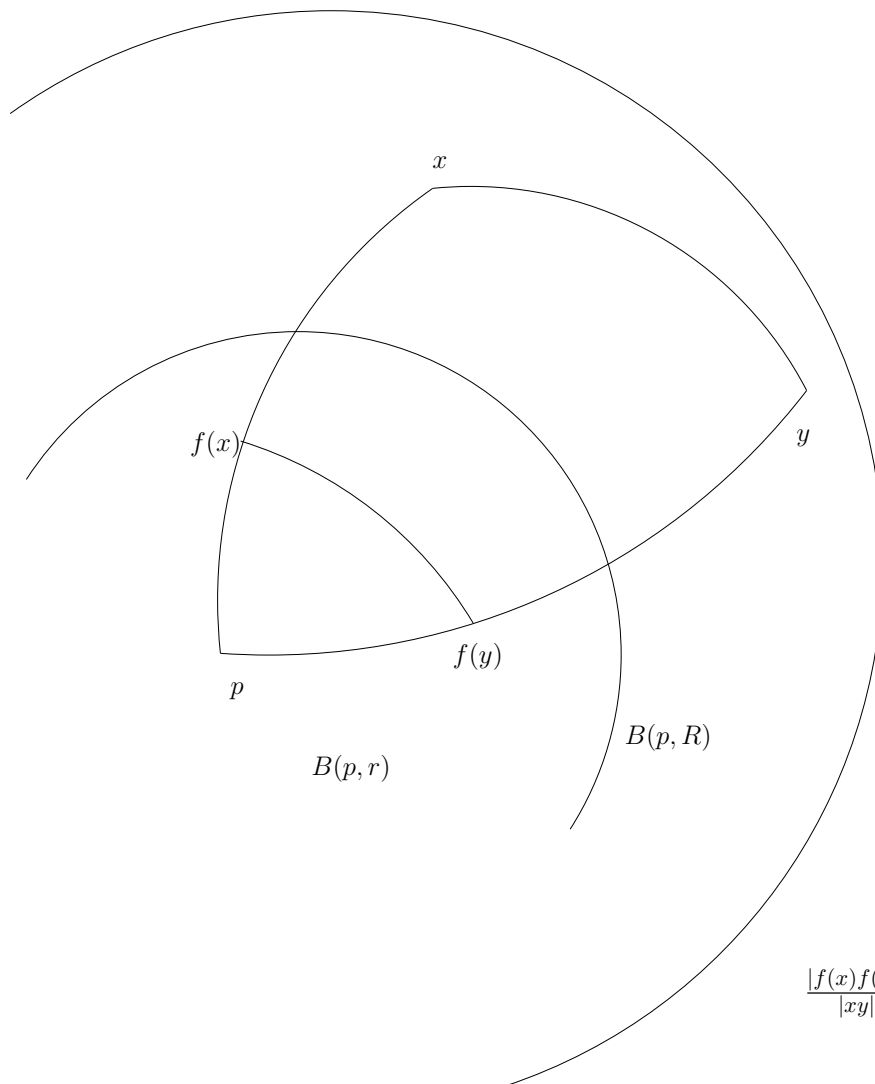
Equivariant Gromov-Hausdorff convergence

Yamaguchi's fibration theorem

Semiconcave functions

Topological results for sectional curvature bounded below

Background in Ricci curvature bounded below



$$\frac{|f(x)f(y)|}{|xy|} \geq \frac{|f(\tilde{x})f(\tilde{y})|}{|\tilde{x}\tilde{y}|} = \frac{r}{R}$$

Corollary (δ -separated net bound)

Let (M^n, g) have $\text{Ric} \geq k(n-1)$. Let $p \in M, 0 < \delta < R/2$.
Suppose x_1, \dots, x_N is a δ -separated net in $B_R(p)$. Then

$$N \leq C(n, R, \delta)$$

Proof.

By Bishop-Gromov we have $\text{Vol}B_{\delta/2}(x_i) \geq c(n, R, \delta)\text{Vol}B_{2R}(p)$.
Since the balls $B_{\delta/2}(x_i)$ are disjoint we have
 $\text{Vol}B_{2R}(p) \geq \sum_i \text{Vol}B_{\delta/2}(x_i) \geq N \cdot c(n, R, \delta)\text{Vol}B_{2R}(p)$ and the
result follows. □

Ricci curvature and the fundamental group

Vitali Kapovitch

Background on lower sectional curvature bounds

Toponogov comparison

Short basis.

Bishop-Gromov volume comparison

Gromov-Hausdorff convergence

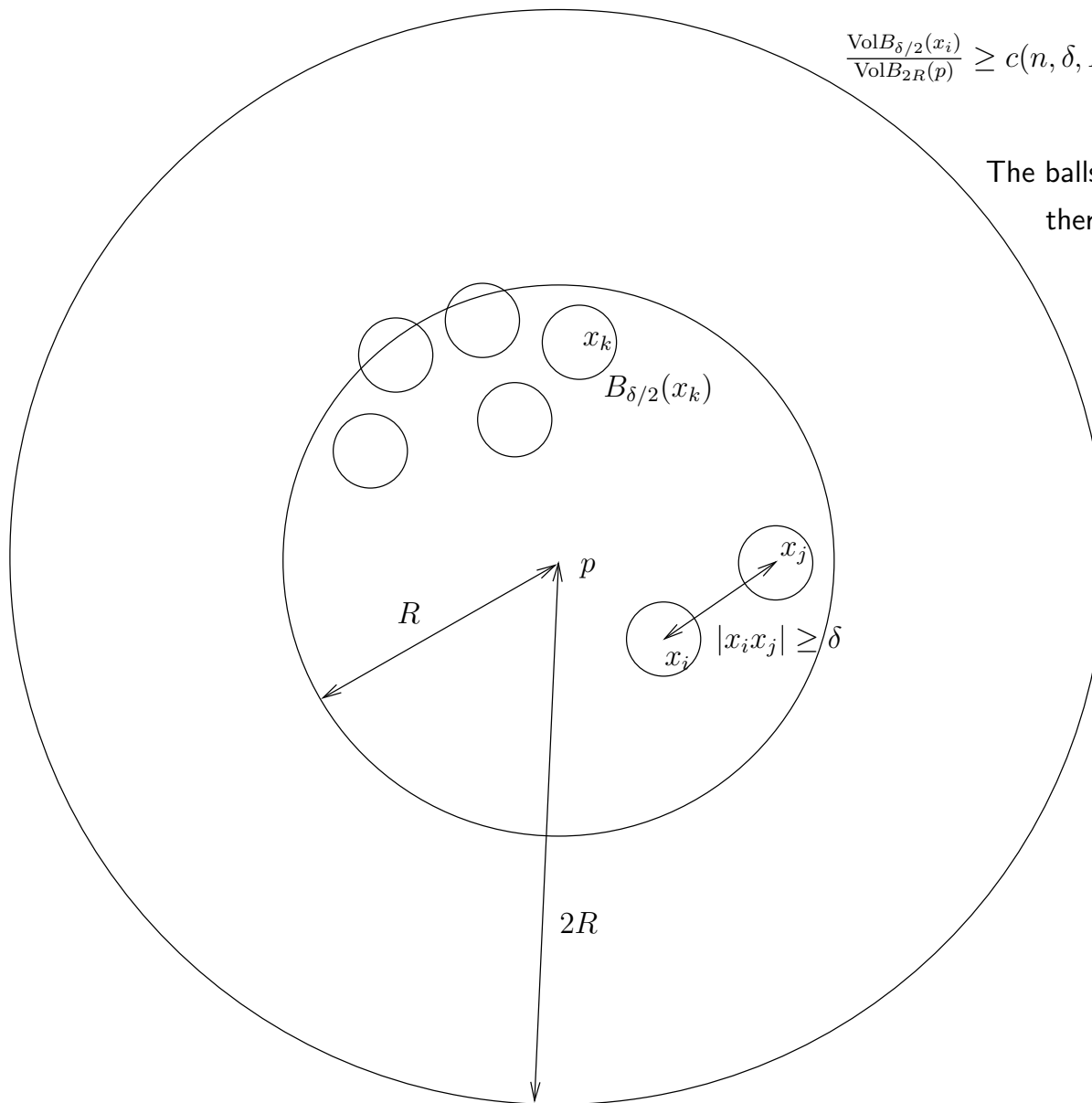
Equivariant Gromov-Hausdorff convergence

Yamaguchi's fibration theorem

Semiconcave functions

Topological results for sectional curvature bounded below

Background in Ricci curvature bounded below



$$\frac{\text{Vol}B_{\delta/2}(x_i)}{\text{Vol}B_{2R}(p)} \geq c(n, \delta, R)$$

The balls $B_{\delta/2}(x_i)$ are disjoint therefore there can only be so many of them.

Definition

Let X, Y be two compact inner metric spaces. A map $f: X \rightarrow Y$ is called an ϵ -Hausdorff approximation if

- $||f(p)f(q)| - |pq|| \leq \epsilon$ for any $p, q \in X$;
- For any $y \in Y$ there exists $p \in X$ such that $|f(p)y| \leq \epsilon$

We define the Gromov-Hausdorff distance between X and Y as $d_{G-H}(X, Y) = \inf \epsilon$ such that there exist ϵ -Hausdorff approximations from X to Y and from Y to X .

Gromov-Hausdorff distance turns out to be a distance on the set of isometry classes of compact inner metric spaces.

Remark

If $f: X \rightarrow Y$ is an ϵ -Hausdorff approximation then there exist $g: Y \rightarrow X$ which is a 2ϵ -Hausdorff approximation. In particular, $d_{G-H}(X, Y) \leq 2\epsilon$.

Definition

Let $X_i \xrightarrow[i \rightarrow \infty]{G-H} X$. Suppose Γ_i is a closed subgroup of $\text{Isom}(X_i)$ and Γ is a closed subgroup of $\text{Isom}(X)$. We say that (X_i, Γ_i) converges to (X, Γ) in equivariant Gromov-Hausdorff topology if there is $\varepsilon_i \rightarrow 0$ such that

- For any $g \in \Gamma$ there is $g_i \in \Gamma_i$ which is ε_i -close to g ;
- For any $g_i \in \Gamma_i$ there is $g \in \Gamma$ which is ε_i -close to g .

Remark

One can similarly define pointed (equivariant) Gromov-Hausdorff convergence of pointed proper spaces.

The following observation of Gromov is crucial.

Theorem (Gromov)

Let \mathfrak{M} be a class of compact inner metric spaces satisfying the following property. There exists a function $N: (0, \infty) \rightarrow (0, \infty)$ such that

for any $\delta > 0$ and any $X \in \mathfrak{M}$ there are at most $N(\delta)$ points in X with pairwise distances $\geq \delta$.

Then \mathfrak{M} is precompact in the Gromov-Hausdorff topology.

By the Corollary on δ -separated net bound this immediately implies

Corollary

The class $\mathfrak{M}_{\text{Ric}}(n, k, D)$ of complete n -manifolds with $\text{Ric} \geq k(n - 1)$, $\text{diam} \leq D$ is precompact in the the Gromov-Hausdorff topology.

The class of complete pointed n -manifolds (M^n, p) with $\text{Ric} \geq k(n - 1)$ is precompact in the pointed Gromov-Hausdorff topology.

Remark

The Corollary on δ -separated net bound also easily implies that limit points of $\mathfrak{M}_{\text{Ric}}(n, k, D)$ have Hausdorff dimension $\leq n$.

Theorem (Yamaguchi)

Let $M_i^n \rightarrow N^m$ as $i \rightarrow \infty$ in Gromov-Hausdorff topology. where $\text{sec}(M_i^n) \geq k$ and N is a smooth manifold. Then for all large i there exists a fiber bundle $F_i \rightarrow M_i^n \rightarrow N$.

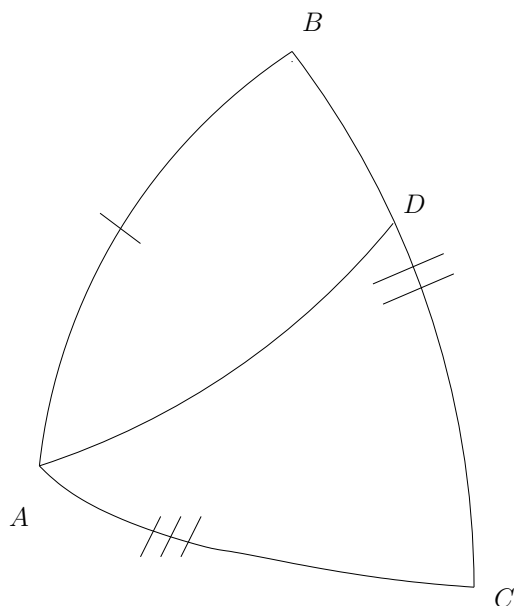
Example

Consider S^3 with the metric g_ε given by $(S^3 \times S_\varepsilon^1)/S^1$ where S_ε^1 is the circle of radius ε and S^1 acts on $S^3 \times S_\varepsilon^1$ diagonally by the Hopf action on S^3 and by rotations on S_ε^1 . Then (S^3, g_ε) has $\text{sec} \geq 0$ and (S^3, g_ε) Gromov-Hausdorff converges to the round S^2 of radius $\frac{1}{2}$ as $\varepsilon \rightarrow 0$.

Theorem (Toponogov comparison alternate formulation)

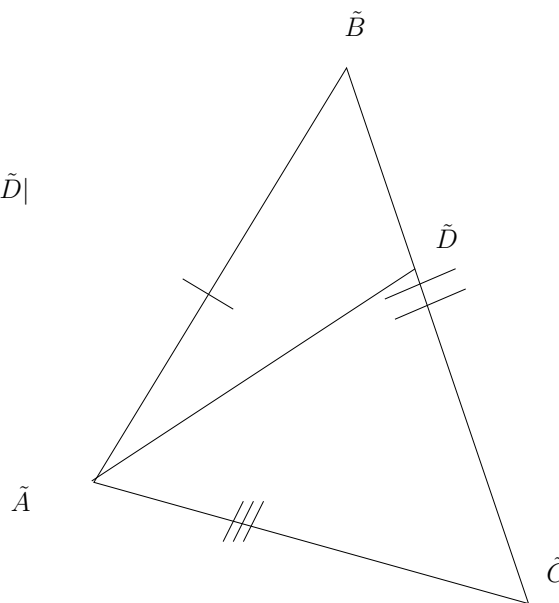
Let (M^n, g) be a complete Riemannian manifold of $K_{sec} \geq 0$. Let ΔABC be a geodesic triangle in M and let $\Delta \tilde{A}\tilde{B}\tilde{C}$ be a comparison triangle in \mathbb{R}^2 . Let D be a point on the side BC and let \tilde{D} be the point on the side $\tilde{B}\tilde{C}$ with $|BD| = |\tilde{B}\tilde{D}|$ and $|CD| = |\tilde{C}\tilde{D}|$. Then $|AD| \geq |\tilde{A}\tilde{D}|$.

$M^n, K \geq 0$



$$|AD| \geq |\tilde{A}\tilde{D}|$$

$\mathbb{R}^2, K = 0$



How concave is that?

Ricci curvature
and the
fundamental
group

Vitali Kapovitch

Background on
lower sectional
curvature bounds

Toponogov
comparison

Short basis.

Bishop-Gromov
volume
comparison

Gromov-
Hausdorff
convergence

Equivariant
Gromov-
Hausdorff
convergence

Yamaguchi's
fibration
theorem

Semiconcave
functions

Topological
results for
sectional
curvature
bounded below

Background in
Ricci curvature
bounded below

This means that distance function to a point is more concave than the distance function to a point in the space of constant curvature.
How concave is that?

Definition

Define $\text{md}_k(r)$ by the formula

$$\text{md}_k(r) = \begin{cases} \frac{r^2}{2} & \text{if } r = 0 \\ \frac{1}{k}(1 - \cos(\sqrt{k}r)) & \text{if } k > 0 \\ \frac{1}{k}(1 - \cosh(\sqrt{|k|}r)) & \text{if } k < 0 \end{cases}$$

Then

$$\text{md}_k(0) = 0, \text{md}'_k(0) = 1 \text{ and } \text{md}''_k + k\text{md}_k \equiv 1$$

Let $f(x) = \text{md}_k(|xp|)$ where $p \in \mathbb{S}_k^n$ - simply connected space form of constant curvature k we have $\text{Hess}_x f = (1 - kf(x))\text{Id}$. In particular, for any unit speed geodesic $\gamma(t)$ we have that

$$f(\gamma(t))'' + kf(\gamma(t)) = 1$$

Note that for $k = 0$ this means that $\text{Hess}_x f = \text{Id}$ and

$$f(\gamma(t))'' = 1$$

Theorem (Toponogov restated)

Let M^n have $\text{sec} \geq k$ and $p \in M$. Let $f(x) = \text{md}_k(|xp|)$.

Then

$$\text{Hess}_x f \leq (1 - kf(x))\text{Id}$$

and

$$f(\gamma(t))'' + kf(\gamma(t)) \leq 1$$

For any unit speed geodesic γ .

These inequalities can be understood in the barrier sense or in the following sense.

Definition

A function $f: M \rightarrow \mathbb{R}$ is called λ -concave if for any unit speed geodesic $\gamma(t)$ we have

$$f(\gamma(t)) + \frac{\lambda t^2}{2} \text{ is concave}$$

Why do we care?

Ricci curvature
and the
fundamental
group

Vitali Kapovitch

Background on
lower sectional
curvature bounds

Toponogov
comparison
Short basis.
Bishop-Gromov
volume
comparison
Gromov-
Hausdorff
convergence
Equivariant
Gromov-
Hausdorff
convergence
Yamaguchi's
fibration
theorem

Semiconcave
functions

Topological
results for
sectional
curvature
bounded below

Background in
Ricci curvature
bounded below

This means that manifolds with $\text{sec} \geq k$ naturally possess a LOT of semiconcave functions. Why do we care? They provide a useful technical tools.

Specifically, the following simple observation is of crucial importance.

Theorem

Gradient flow of a concave function f on a complete Riemannian manifold M^n is 1-Lipshitz.

Proof.

Let $p, q \in M$ and let $\gamma: [0, d] \rightarrow M$ be a unit speed geodesic with $\gamma(0) = p, \gamma(d) = q$. Here $d = |pq|$. Let ϕ_t be the gradient flow of f . Then

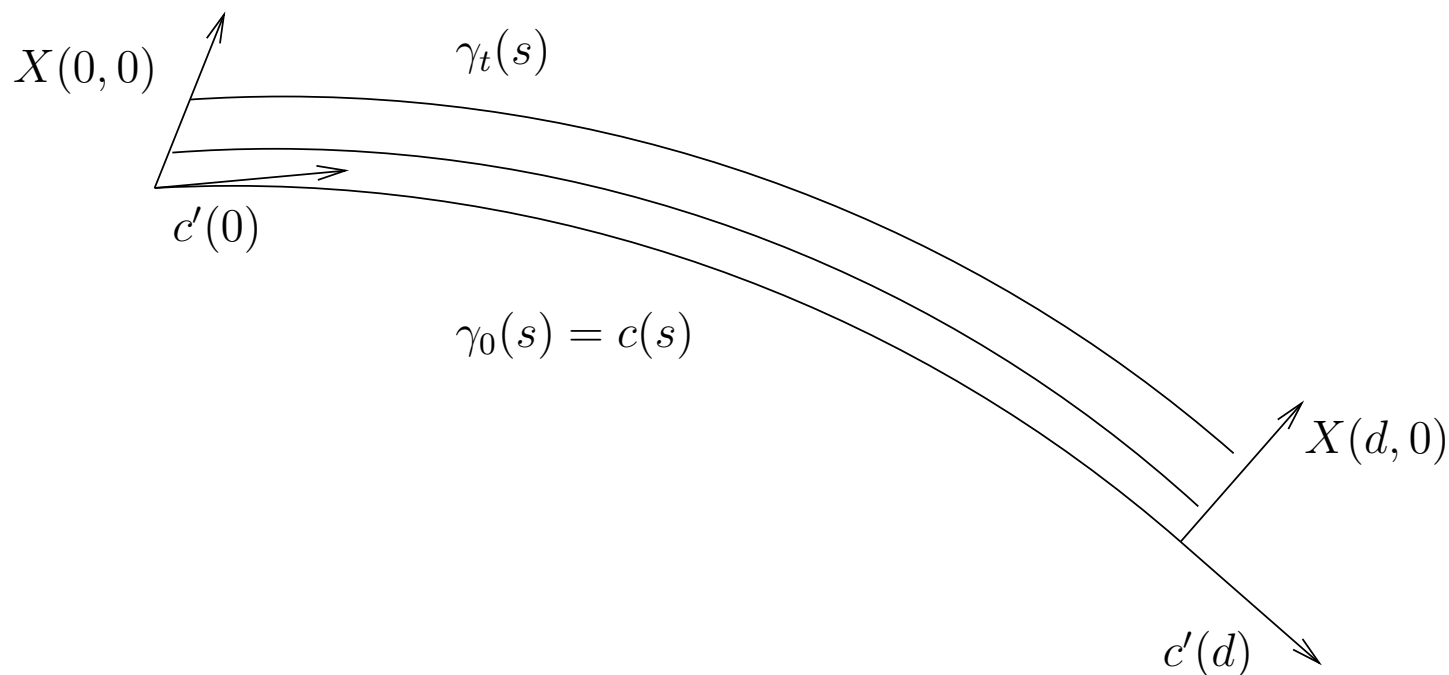
$$|\phi_t(p)\phi_t(q)|'_+(0) \leq L(\phi_t(\gamma))'(0)$$

Aside. Recall

First variation formula

Let $\gamma: [0, d] \times (-\varepsilon, \varepsilon) \rightarrow M^n$ be a smooth family of curves with $c(s) = \gamma(s, 0)$ being a unit speed geodesic. Let $X = \frac{\partial}{\partial t} \gamma(s, t)$ be the variation vector field.

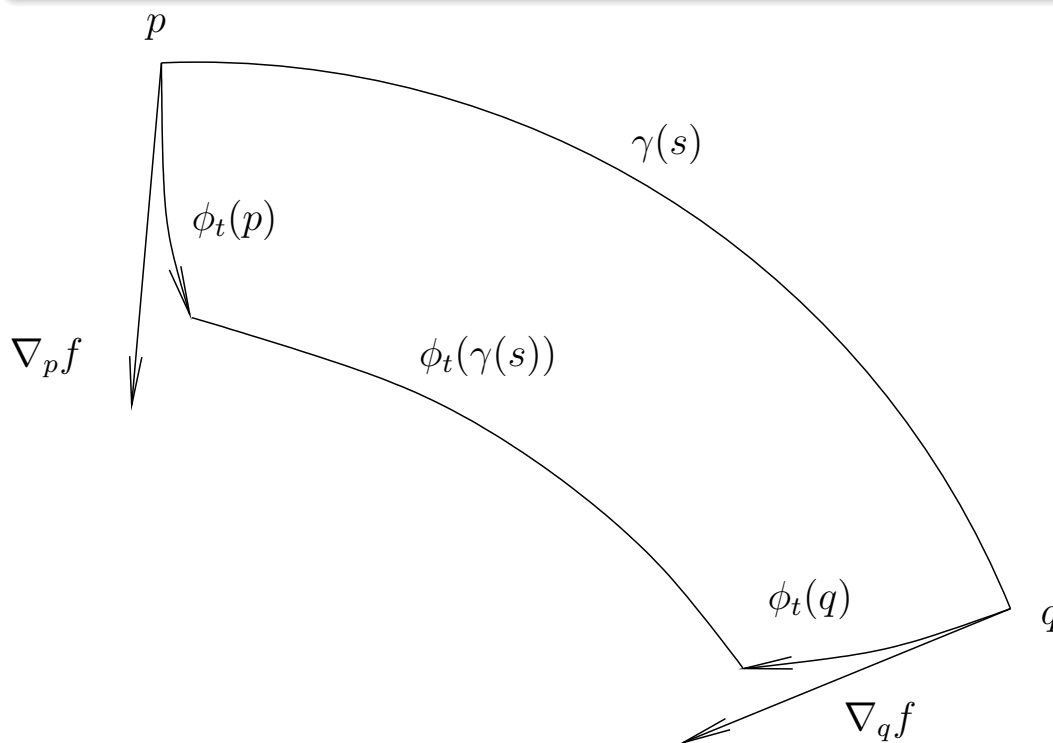
Then $L(\gamma_t)'(0) = \langle X(d, 0), c'(d) \rangle - \langle X(0, 0), c'(0) \rangle$



Applying first variation formula gives

$$L(\phi_t(\gamma))'(0) = \langle \nabla_q f, \gamma'(d) \rangle - \langle \nabla_p f, \gamma'(0) \rangle = f(\gamma(d))' - f(\gamma(0))' \leq 0$$

since $f(\gamma(s))$ is concave. □



A similar argument shows that if f is λ -concave then ϕ_t is $e^{\lambda t}$ -Lipschitz. Yamaguchi's Fibration Theorem and gradient flows of semi-concave functions are key technical tools for proving topological results about manifolds with lower sectional curvature bounds.

Theorem (Almost splitting theorem)

Let $(M_i^n, p_i) \xrightarrow[i \rightarrow \infty]{G-H} (X, p)$ where $\sec_{M_i} \geq -\frac{1}{i}$. Suppose X contains a line.

Then X is isometric to $Y \times \mathbb{R}$ for some metric space Y .

Remark

If X^n is an Alexandrov space of $\text{curv} \geq 0$ and X contains a line then $X \cong Y \times \mathbb{R}$ for some Y .

If $\sec_{M^n} \geq 0$ and M is closed then one can use the Splitting Theorem to show that $\tilde{M} \cong \mathbb{R}^k \times K$ where K is compact. This can be used to show

Theorem (Cheeger-Gromoll)

If $\sec_{M^n} \geq 0$ and M is closed then a finite cover of M is diffeomorphic to $T^k \times K$ where K is simply connected.

Theorem (Gromov)

Let M^n have $\sec \geq k$, $\text{diam} \leq D$. Then

$$\sum_{i=0}^n \beta_i(M) \leq C(n, k, D)$$

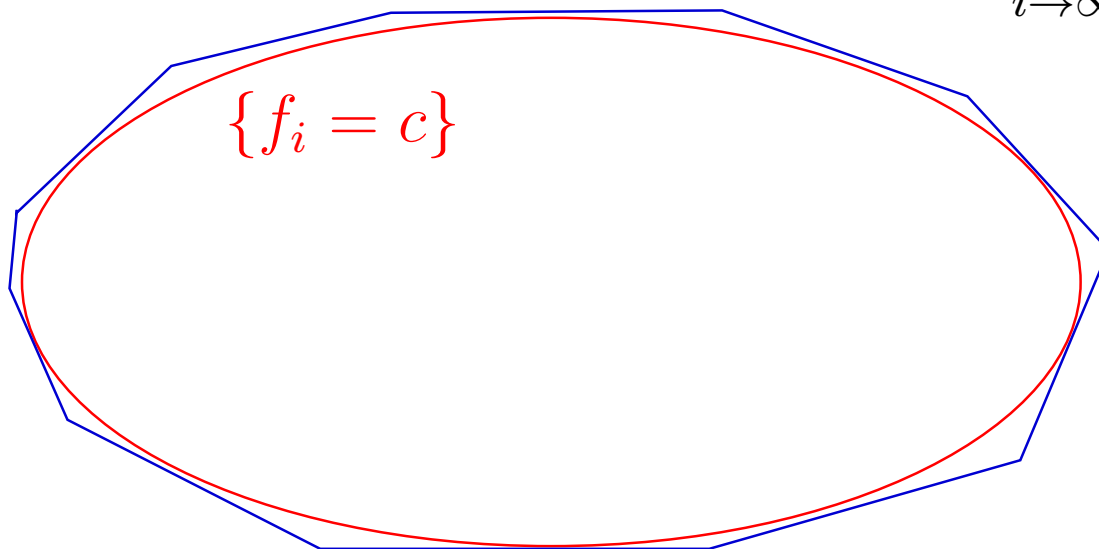
Theorem (Perelman's stability theorem)

Let M_i^n be a sequence of compact manifolds with $\sec \geq k$ Gromov-Hausdorff converging to X where $\dim X = n$. Then M_i is homeomorphic to X for all large i .

Example

Let $\sec N \geq k$ and let $f: N \rightarrow \mathbb{R}$ be convex. Let $f_i: N \rightarrow \mathbb{R}$ be smooth convex functions converging to f . Let c be any value in the range of f different from $\max f$. Then $\{f_i = c\}$ is a smooth manifold of $\sec \geq k$ and $\{f_i = c\}$ Gromov-Hausdorff converges to $\{f = c\}$ with respect to intrinsic metrics.

$$\{f = c\} \quad \{f_i = c\} \xrightarrow[i \rightarrow \infty]{G-H} X = \{f = c\}$$



Corollary (Grove-Petersen-Wu)

The class of n -manifolds with $\sec \geq k$, $\text{diam} \leq D$, $\text{Vol} \geq v$ contains $\leq C(n, k, D, v)$ homeomorphism types.

Definition

A closed smooth manifold M is called almost nonnegatively curved if it admits a sequence of Riemannian metrics $\{g_i\}$ on M whose sectional curvatures and diameters satisfy

$$\sec(M, g_i) \geq -1/i \quad \text{and} \quad \text{diam}(M, g_i) \leq 1/i.$$

Example

Let N^3 be the space of real 3×3 of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

N^3 is a nilpotent Lie group. Let $\Gamma = N \cap \mathrm{SL}(3, \mathbb{Z})$. Then $M^3 = N/\Gamma$ admits almost nonnegative sectional curvature. But it does not admit nonnegative sectional curvature because Γ is not virtually abelian.

Theorem (*C*-Nilpotency Theorem for π_1 , K-Petrinin-Tuschmann, 2006)

Let M be an almost nonnegatively curved m -manifold. Then $\pi_1(M)$ is $C(m)$ -nilpotent, i.e., $\pi_1(M)$ contains a nilpotent subgroup of index at most $C(m)$.

Bochner's formula

Ricci curvature
and the
fundamental
group

Vitali Kapovitch

Background on
lower sectional
curvature bounds

Background in
Ricci curvature
bounded below

Bochner's
formula

Segment
Inequality

Stability

Structure of
limit spaces

Submetries

The most basic tool in studying manifolds with Ricci curvature bounds is Bochner's formula.

Theorem (Bochner's formula)

For a smooth function f on a complete Riemannian manifold (M^n, g) ,

$$\frac{1}{2}\Delta|\nabla f|^2 = |\text{Hess}f|^2 + \langle \nabla f, \nabla(\Delta f) \rangle + \text{Ric}(\nabla f, \nabla f).$$

Bochner's formula applied to distance functions can be used to prove

Theorem (Global Laplacian Comparison)

Let $\text{Ric}_{M^n} \geq (n-1)k$, $p \in M$ and let $f(x) = \text{md}_k(|xp|)$. Then

$$\Delta_x f \leq (1 - kf(x))n$$

in the weak sense.

Recall

Theorem (Toponogov restated)

Let M^n have $\text{sec} \geq k$ and $p \in M$. Let $f(x) = \text{md}_k(|xp|)$.
Then

$$\text{Hess}_x f \leq (1 - kf(x))\text{Id}$$

By applying the Bochner formula to $f = \log u$ with an appropriate cut-off function and looking at the maximum point one has Cheng-Yau's gradient estimate for harmonic functions.

Theorem (Gradient Estimate, Cheng-Yau 1975)

Let $\text{Ric}_{M^n} \geq (n-1)k$ on $B_{R_2}(p)$ and $u: B_{R_2}(p) \rightarrow \mathbb{R}$ satisfying $u > 0, \Delta u = 0$. Then for $R_1 < R_2$, on $B_{R_1}(p)$,

$$(4) \quad \frac{|\nabla u|}{u} \leq c(n, k, R_1, R_2).$$

Bishop-Gromov volume comparison

Ricci curvature
and the
fundamental
group

Vitali Kapovitch

Background on
lower sectional
curvature bounds

Background in
Ricci curvature
bounded below

Bochner's
formula

Segment
Inequality

Stability

Structure of
limit spaces

Submetries

Theorem (Bishop-Gromov's Relative Volume Comparison)

Suppose M^n has $\text{Ric}_M \geq (n-1)k$. Then

$$(5) \quad \frac{\text{Vol}(\partial B_r(p))}{\text{Vol}(\partial B_r^k(0))} \text{ and } \frac{\text{Vol}(B_r(p))}{\text{Vol}(B_r^k(0))} \text{ are nonincreasing in } r.$$

In particular,

$$(6) \quad \text{Vol}(B_r(p)) \leq \text{Vol}(B_r^k(0)) \quad \text{for all } r > 0,$$

$$(7) \quad \frac{\text{Vol}(B_r(p))}{\text{Vol}(B_R(p))} \geq \frac{\text{Vol}(B_r^k(0))}{\text{Vol}(B_R^k(0))} \quad \text{for all } 0 < r \leq R,$$

and equality holds if and only if $B_r(p)$ is isometric to $B_r^k(0)$. Here $B_r^k(0)$ is the ball of radius r in the n -dimensional simply connected space of constant curvature k .

Segment Inequality

Ricci curvature
and the
fundamental
group

Vitali Kapovitch

Background on
lower sectional
curvature bounds

Background in
Ricci curvature
bounded below

Bochner's
formula

Segment
Inequality

Stability

Structure of
limit spaces

Submetries

Theorem (Cheeger–Colding Segment inequality)

Given n and r_0 there exists $\tau = \tau(n, r_0)$ such that the following holds. Let $\text{Ric}(M^n) \geq -(n-1)$ and let $g: M \rightarrow \mathbb{R}^+$ be a nonnegative function. Then for $r \leq r_0$

$$\int_{B_r(p) \times B_r(p)} \left[\int_0^{|xy|} g(\gamma_{x,y}(t)) \right] d\mu_x d\mu_y \leq \tau \cdot r \cdot \int_{B_{2r}(p)} g(q) d\mu_q$$

where $\gamma_{x,y}$ denotes a minimal geodesic from x to y .

Ricci curvature
and the
fundamental
group

Vitali Kapovitch

Background on
lower sectional
curvature bounds

Background in
Ricci curvature
bounded below

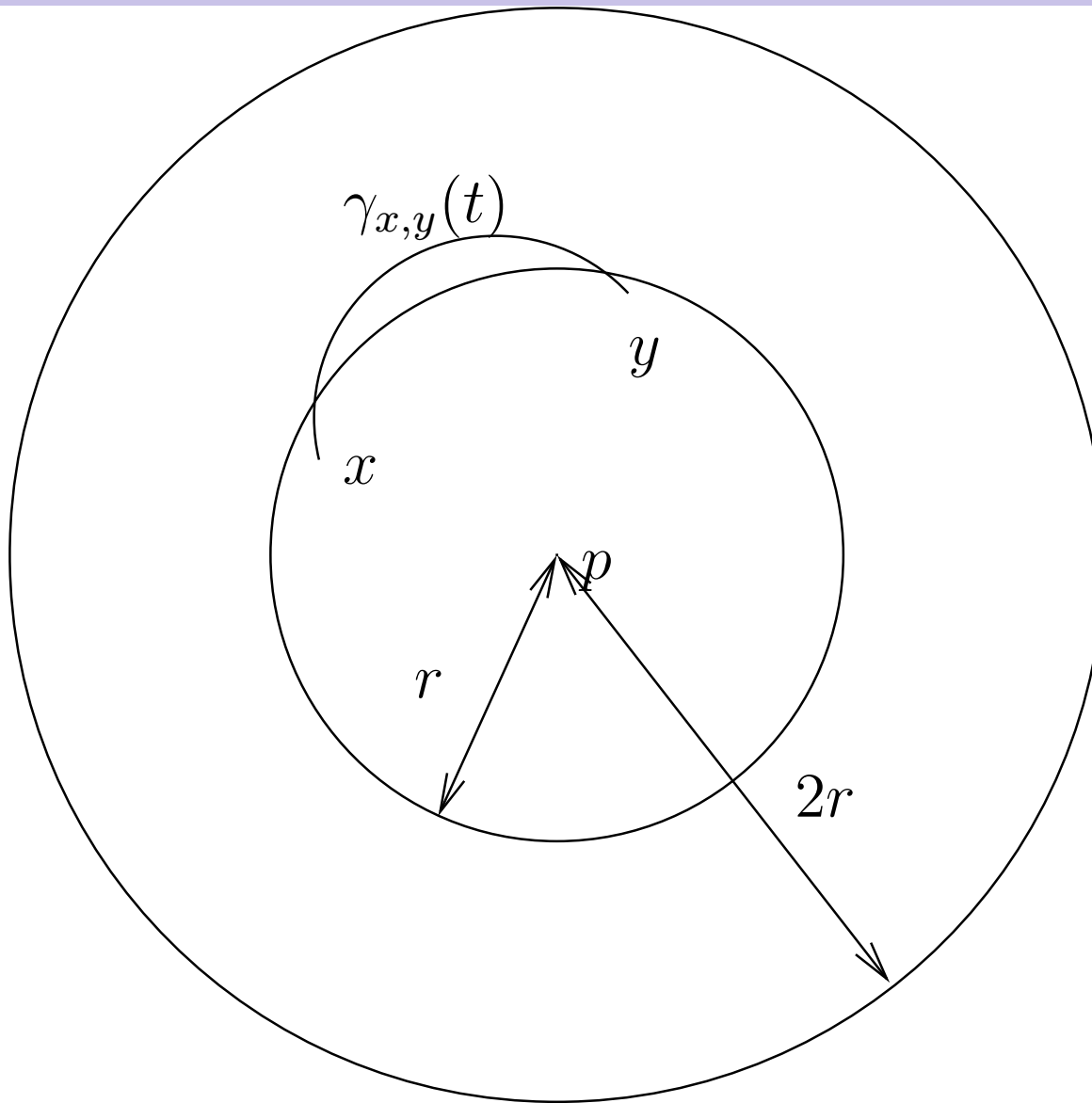
Bochner's
formula

**Segment
Inequality**

Stability

Structure of
limit spaces

Submetries



Comparison to lower sectional curvature bounds

Ricci curvature
and the
fundamental
group

Vitali Kapovitch

Background on
lower sectional
curvature bounds

Background in
Ricci curvature
bounded below

Bochner's
formula

Segment
Inequality

Stability

Structure of
limit spaces

Submetries

- Toponogov Comparison does not hold
- Fibration Theorem does not hold
- L^2 -Toponogov comparison holds for long thin triangles (Colding)
- Almost splitting theorem holds

Theorem (Cheeger–Colding Almost splitting theorem)

Let $(M_i^n, p_i) \xrightarrow[i \rightarrow \infty]{G-H} (X, p)$ with $\text{Ric}(M_i) \geq -\frac{1}{i}$. Suppose X has a line. Then X splits isometrically as $X \cong Y \times \mathbb{R}$.

As far as I know, this crucial property does not hold for any synthetic definition of metric spaces with lower Ricci curvature bounds. For example, Banach spaces satisfy such definitions but fail this property.

- Topological stability theorem aka Perelman does not hold

Example

There exists a sequence of metrics on $K3$ with $\text{Ric} \equiv 0$ converging to T^4/\mathbb{Z}_2 where the \mathbb{Z}_2 action is given by complex conjugation $(z_1, z_2, z_3, z_4) \mapsto (\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4)$. The quotient T^4/\mathbb{Z}_2 is not a topological manifold as neighbourhoods of fixed points are homeomorphic to cones over $\mathbb{R}P^3$.

Theorem (Volume and topological stability. Colding, Cheeger-Colding)

Let $M_i^n \xrightarrow[i \rightarrow \infty]{G-H} N^n$ with $\text{Ric}(M_i) \geq k(n-1)$ and N is a closed smooth manifold. Then

- $\text{Vol}M_i \rightarrow \text{Vol}N$.
- For all large i Hausdorff approximations $M_i \rightarrow N$ are close to diffeomorphisms.

Corollary

Let $(M_i^n, p_i) \rightarrow (\mathbb{R}^n, 0)$ with $\text{Ric}_{M_i} > -1$. Then $B_R(p_i)$ is contractible in $B_{R+\varepsilon}(p_i)$ for all $i \geq i_0(R, \varepsilon)$.

Theorem

Suppose $(M_i^n, p_i) \rightarrow (\mathbb{R}^k, 0)$ with $\text{Ric}_{M_i} \geq -1/i$. Then there exist harmonic functions $b_1^i, \dots, b_k^i: B_2(p_i) \rightarrow \mathbb{R}$ such that

$$|\nabla b_j^i| \leq C(n) \text{ for all } i \text{ and } j$$

and

$$\int_{B(p_i, 1)} \sum_{j, l} |\langle \nabla b_j^i, \nabla b_l^i \rangle - \delta_{j, l}| + \sum_j \|\text{Hess}_{b_j^i}\|^2 d\mu \rightarrow 0 \text{ as } i \rightarrow \infty.$$

Moreover, the maps $\Phi^i = (b_1^i, \dots, b_k^i): M_i \rightarrow \mathbb{R}^k$ provide ε_i -Gromov-Hausdorff approximations between $B_1(p_i)$ and $B_1(0)$ with $\varepsilon_i \rightarrow 0$.

Ricci curvature
and the
fundamental
group

Vitali Kapovitch

Background on
lower sectional
curvature bounds

Background in
Ricci curvature
bounded below

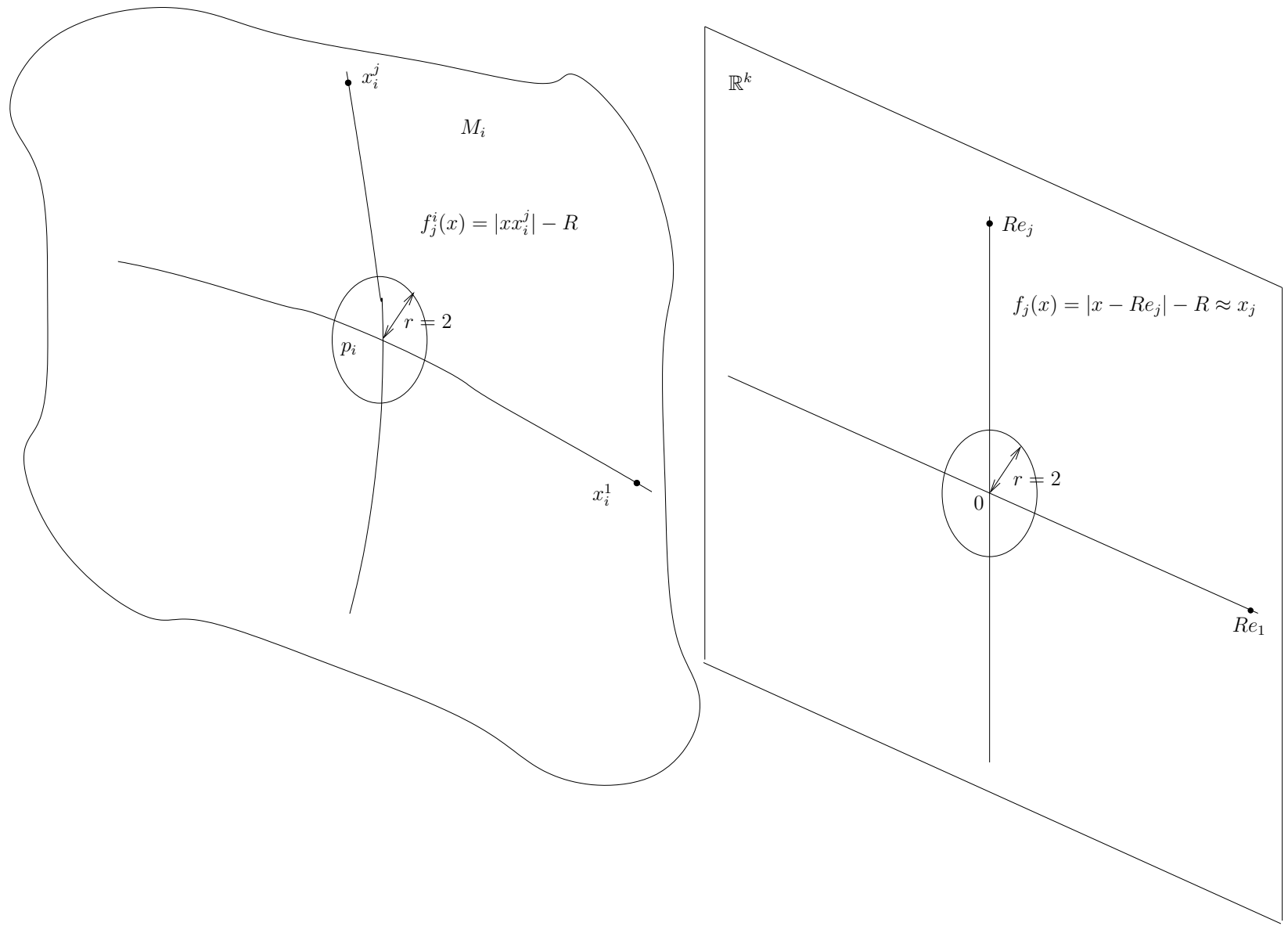
Bochner's
formula

Segment
Inequality

Stability

Structure of
limit spaces

Submetries



The functions b_j^i in the above theorem are constructed as follows. Fix a large R . Approximate Busemann functions f_j in \mathbb{R}^k given by $f_j = d(\cdot, Re_j) - R$ are lifted to M_i using Hausdorff approximations to corresponding functions f_j^i . Here e_j is the j -th coordinate vector in the standard basis of \mathbb{R}^k . The functions b_j^i are obtained by solving the Dirichlet problem on $B(p_i, 2)$ with $b_j^i|_{\partial B(p_i, 2)} = f_j^i|_{\partial B(p_i, 2)}$. Note that $|f_j^i(x)| \leq 3$ on $B(p_i, 2)$ and therefore $|\nabla b_j^i|$ is uniformly bounded on $B(p_i, 1)$ by Chen-Yau gradient estimate.

Lemma (Weak type 1-1 inequality)

Suppose (M^n, g) has $\text{Ric} \geq -(n-1)$ and let $f: M \rightarrow \mathbb{R}$ be a nonnegative function. Define $Mx f(p) := \sup_{r \leq 1} \int_{B_r(p)} f$. Then the following holds

- (a) If $f \in L^\alpha(M)$ with $\alpha \geq 1$ then $Mx f$ is finite almost everywhere.
- (b) If $f \in L^1(M)$ then $\text{Vol}\{x \mid Mx f(x) > c\} \leq \frac{C(n)}{c} \int_M f$ for any $c > 0$.
- (c) If $f \in L^\alpha(M)$ with $\alpha > 1$ then $Mx f \in L^\alpha(M)$ and $\|Mx f\|_\alpha \leq C(n, \alpha) \|f\|_\alpha$.

If $f \in L^\alpha(M)$ with $\alpha > 1$ then we have pointwise

$$Mx((Mx f)^\alpha)(x) \leq C(n, \alpha) Mx(f^\alpha)(x).$$

What is known about limit spaces?

Ricci curvature
and the
fundamental
group

Vitali Kapovitch

Background on
lower sectional
curvature bounds

Background in
Ricci curvature
bounded below

Bochner's
formula

Segment
Inequality

Stability

Structure of
limit spaces

Submetries

Recall that the class of pointed n -manifolds with $\text{Ric} \geq k(n-1)$ is precompact in pointed Gromov–Hausdorff topology. Let (X, q) be a limit point of this class.

Definition

A *tangent cone*, $T_p X$, at $p \in X$ is the pointed Gromov-Hausdorff limit of a sequence of the rescaled spaces $(\lambda_i X, p)$, where $\lambda_i \rightarrow \infty$ as $i \rightarrow \infty$.

Tangent cones need not be unique and need not be metric cones.

Definition

A point, $y \in X$, is called *regular* if for some k , every tangent cone at y is isometric to \mathbb{R}^k .

Theorem (Cheeger-Colding)

The set of regular points in X has full measure and in particular is dense.

Definition

A map $\pi: X \rightarrow Y$ is called a **submetry** if $\pi(B_r(x)) = B_r(\pi(y))$ for any $p \in X, r > 0$.

Example

Let G act on X by isometries. Then $\pi: X \rightarrow X/G$ is a submetry.

- Submetries are 1-Lipschitz.
- Fibers of submetries are equidistant.

Exercise

Let X be proper and let $\pi: X \rightarrow Y$ be a submetry. Let $\gamma(t)$ be a shortest geodesic in Y and let $p \in X$ be a lift of $\gamma(0)$, i.e.

$$\pi(p) = \gamma(0).$$

Then there exists a lift of γ starting at p , i.e. there exists a geodesic $\tilde{\gamma}$ in X such that $\tilde{\gamma}(0) = p$ and $\pi(\tilde{\gamma}(t)) = \gamma(t)$.

Ricci curvature
and the
fundamental
group

Vitali Kapovitch

Background on
lower sectional
curvature bounds

Background in
Ricci curvature
bounded below

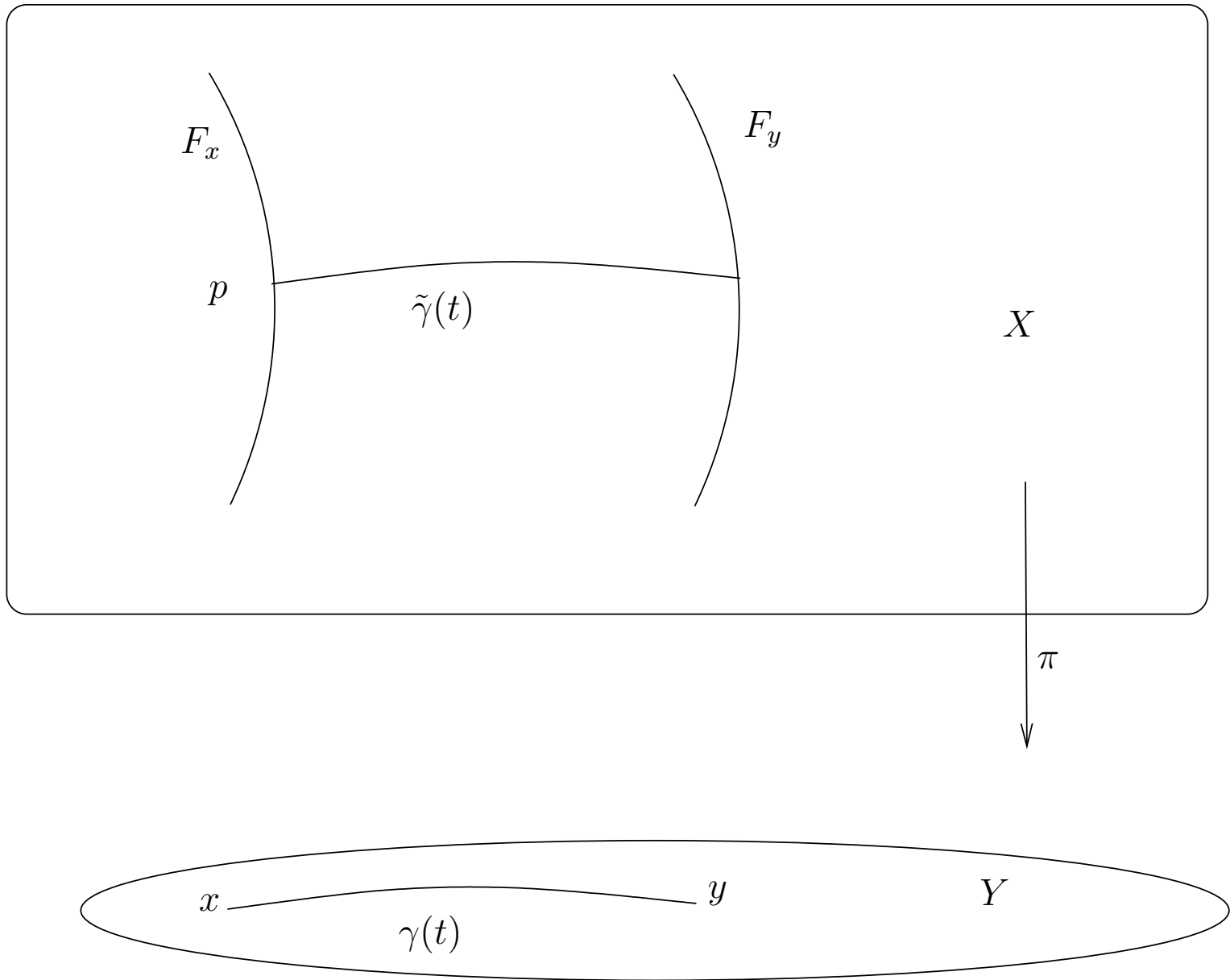
Bochner's
formula

Segment
Inequality

Stability

Structure of
limit spaces

Submetries



Fundamental group results

Ricci curvature
and the
fundamental
group

Vitali Kapovitch

Background on
lower sectional
curvature bounds

Background in
Ricci curvature
bounded below

Bochner's
formula

Segment
Inequality

Stability

Structure of
limit spaces

Submetries

Sectional curvature

If $\sec M > \delta > 0$ then $\pi_1(M)$ is finite. (Myers)

If $\sec M \geq 0$ and M is compact then $\pi_1(M)$ is virtually abelian. (Cheeger-Gromoll)

If $\sec M^n \geq k$, $\text{diam} \leq D$ then $\pi_1(M)$ is generated by $\leq C(n, k, D)$ elements.

Ricci curvature

If $\text{Ric}M > \delta > 0$ then $\pi_1(M)$ is finite. (Myers)

If $\text{Ric}M \geq 0$ and M is compact then $\pi_1(M)$ is virtually abelian. (Cheeger-Gromoll)

???

Fundamental group results

Ricci curvature
and the
fundamental
group

Vitali Kapovitch

Background on
lower sectional
curvature bounds

Background in
Ricci curvature
bounded below

Bochner's
formula

Segment
Inequality

Stability

Structure of
limit spaces

Submetries

If M^n admits almost nonnegative sectional curvature then $\pi_1(M)$ is $C(n)$ -nilpotent. (KPT)

Conjecture (Gromov): If M^n admits almost nonnegative Ricci curvature then $\pi_1(M)$ is virtually nilpotent.

If M^n admits almost nonnegative sectional curvature and $\pi_1(M)$ is finite then $\frac{\text{diam } \tilde{M}}{\text{diam } M} \leq C(n)$.
Fukaya-Yamaguchi (incorrect proof)

???

If $\text{sec}_{M^n} \geq 0$ then $\pi_1(M)$ is finitely generated.

Conjecture (Milnor): If $\text{Ric}_{M^n} \geq 0$ then $\pi_1(M)$ is finitely generated.

joint with B. Wilking

Ricci curvature
and the
fundamental
group

Vitali Kapovitch

Main Results

Finite generation
of fundamental
groups

Maps which are
on all scales close
to isometries.

Margulis Lemma.

Theorem (Finite generation of fundamental group)

Given n, k, D there exists $C(n, k, D)$ such that for any n -manifold with $\text{Ric} \geq -k(n-1)$ and $\text{diam}(M, g) \leq D$, the fundamental group $\pi_1(M)$ can be generated by at most C elements.

We call a generator system b_1, \dots, b_n of a group N a nilpotent basis if the commutator $[b_i, b_j]$ is contained in the subgroup $\langle b_1, \dots, b_{i-1} \rangle$ for $1 \leq i < j \leq n$. Having a nilpotent basis of length n implies in particular that N is nilpotent of rank $(N) \leq n$.

Theorem (Generalized Margulis Lemma)

In each dimension n there are positive constants $C(n)$, $\varepsilon(n)$ such that the following holds for any complete n -dimensional Riemannian manifold (M, g) with $\text{Ric} > -(n - 1)$ on a metric ball $B_1(p) \subset M$. The image of the natural homomorphism

$$\pi_1(B_\varepsilon(p), p) \rightarrow \pi_1(B_1(p), p)$$

contains a nilpotent subgroup N of index $\leq C(n)$. Moreover, N has a nilpotent basis of length at most n .

We will also show that equality in this inequality can only occur if M is homeomorphic to an infranilmanifold.

joint with B. Wilking

Ricci curvature
and the
fundamental
group

Vitali Kapovitch

Main Results

Finite generation
of fundamental
groups

Maps which are
on all scales close
to isometries.

Margulis Lemma.

Corollary (π_1 of Ricci almost nonnegatively curved manifolds)

Let (M, g) be a compact manifold with $\text{Ric} > -(n - 1)$ and $\text{diam}(M) \leq \varepsilon(n)$ then $\pi_1(M)$ contains a nilpotent subgroup N of index $\leq C(n)$. Moreover, N has a nilpotent basis of length $\leq n$.

Conjecture (Milnor)

If M^n is open with $\text{Ric} \geq 0$ then $\pi_1(M)$ is finitely generated.

Corollary

Let (M, g) be an open n -manifold with nonnegative Ricci curvature. Then $\pi_1(M)$ contains a nilpotent subgroup N of index $\leq C(n)$ such that any finitely generated subgroup of N has a nilpotent basis of length $\leq n$.

joint with B. Wilking

Ricci curvature
and the
fundamental
group

Vitali Kapovitch

Main Results

Finite generation
of fundamental
groups

Maps which are
on all scales close
to isometries.

Margulis Lemma.

Milnor: if $\text{Ric}_M \geq 0$ and $\pi_1(M)$ is finitely generated then it has polynomial growth.

Gromov: If Γ is finitely generated and has polynomial growth then it is virtually nilpotent.

Problem

Rule out M^n with $\text{Ric} \geq 0$ and $\pi_1(M) \cong \mathbb{Q}$.

Theorem (Compact Version of the Margulis Lemma)

Given n and D there are positive constants ε_0 and C such that the following holds: If (M, g) is a compact n -manifold M with $\text{Ric} > -(n - 1)$ and $\text{diam}(M) \leq D$, then there is $\varepsilon \geq \varepsilon_0$ and a normal subgroup $N \triangleleft \pi_1(M)$ such that for all $p \in M$:

- 1 *the image of $\pi_1(B_{\varepsilon/1000}(p), p) \rightarrow \pi_1(M, p)$ contains N ,*
- 2 *the index of N in the image of $\pi_1(B_{\varepsilon}(p), p) \rightarrow \pi_1(M, p)$ is $\leq C$
and*
- 3 *N is a nilpotent group which has a nilpotent basis of length $\leq n$.*

Theorem (Finiteness of π_1 mod nilpotent subgroup)

For each $D > 0$ and each dimension n there are finitely many groups F_1, \dots, F_k such that the following holds: If M is a compact n -manifold with $\text{Ric} > -(n-1)$ and $\text{diam}(M) \leq D$, then there is a nilpotent normal subgroup $N \triangleleft \pi_1(M)$ with a nilpotent basis of length $\leq n-1$ and $\text{rank}(N) \leq n-2$ such that $\pi_1(M)/N \cong F_i$ for suitable i .

Theorem (Diameter Ratio Theorem)

*For n and D there is a \tilde{D} such that any compact manifold M with $\text{Ric} \geq -(n-1)$ and $\text{diam}(M) = D$ satisfies:
If $\pi_1(M)$ is finite, then the diameter of the universal cover \tilde{M} of M is bounded above by \tilde{D} .*

Lemma (Product Lemma)

Let M_i be a sequence of complete manifolds with $\text{Ric}_{M_i} > -\varepsilon_i \rightarrow 0$ satisfying

- 1 $\text{Ric}_{M_i} > -\varepsilon_i \rightarrow 0$
- 2 for every i and $j = 1, \dots, k$ there are harmonic functions $b_j^i: M_i \rightarrow \mathbb{R}$ which are L -Lipschitz and fulfill

$$\int_{B(p_i, R)} \sum_{j, l=1}^k |\langle \nabla b_j^i, \nabla b_l^i \rangle - \delta_{jl}| + \sum_{j=1}^k \|\text{Hess}_{b_j^i}\|^2 d\mu \rightarrow 0$$

For all $R > 0$

Then (M_i, p_i) subconverges in the pointed Gromov–Hausdorff topology to a metric product $(\mathbb{R}^k \times X, p_\infty)$ for some metric space X . Moreover, (b_1^i, \dots, b_k^i) converges to the projection onto the Euclidean factor.

Proof.

The main problem is to prove this in the case of $k = 1$. Put $b_i = b_1^i$. After passing to a subsequence we may assume that (M_i, p_i) converges to some limit space (Y, p_∞) . We also may assume that b_i converges to an L -Lipschitz map $b_\infty : Y \rightarrow \mathbb{R}$.

Step 1. b_∞ is 1-Lipschitz.

Indeed, fix $x, y \in B_R(p)$ and a small $\delta \ll |xy|$. Let $x_i, y_i \in M_i$ be a sequence of points converging to x and y respectively. By the Segment Inequality we have that

$$\int_{B_\delta(x_i) \times B_\delta(y_i)} \left(\int_{\gamma_{z_1, z_2}} \left| \|\nabla b_i\| - 1 \right| \right) \leq \tau(R, \delta, n) \cdot \int_{B_R(p_i)} \left| \|\nabla b_i\| - 1 \right| \rightarrow 0$$

as $i \rightarrow \infty$.

Therefore, for some $z_1 \in B_\delta(x_i), z_2 \in B_\delta(y_i)$ we have

$$\int_{\gamma_{z_1, z_2}} ||\nabla b_i| - 1| \leq h_i \rightarrow 0$$

Therefore

$$|b_i(z_1) - b_i(z_2)| \leq |z_1 z_2| + h_i$$

and

$$\begin{aligned} |b_i(x_i) - b_i(y_i)| &\leq 2L\delta + |b_i(z_1) - b_i(z_2)| \leq \\ &\leq 2L\delta + |z_1 z_2| + h_i \leq 4L\delta + |x_i y_i| + h_i \end{aligned}$$

Passing to the limit

$$|b_\infty(x) - b_\infty(y)| \leq 2L\delta + |xy|$$

Since $\delta > 0$ was arbitrary this gives the claim.

Ricci curvature
and the
fundamental
group

Vitali Kapovitch

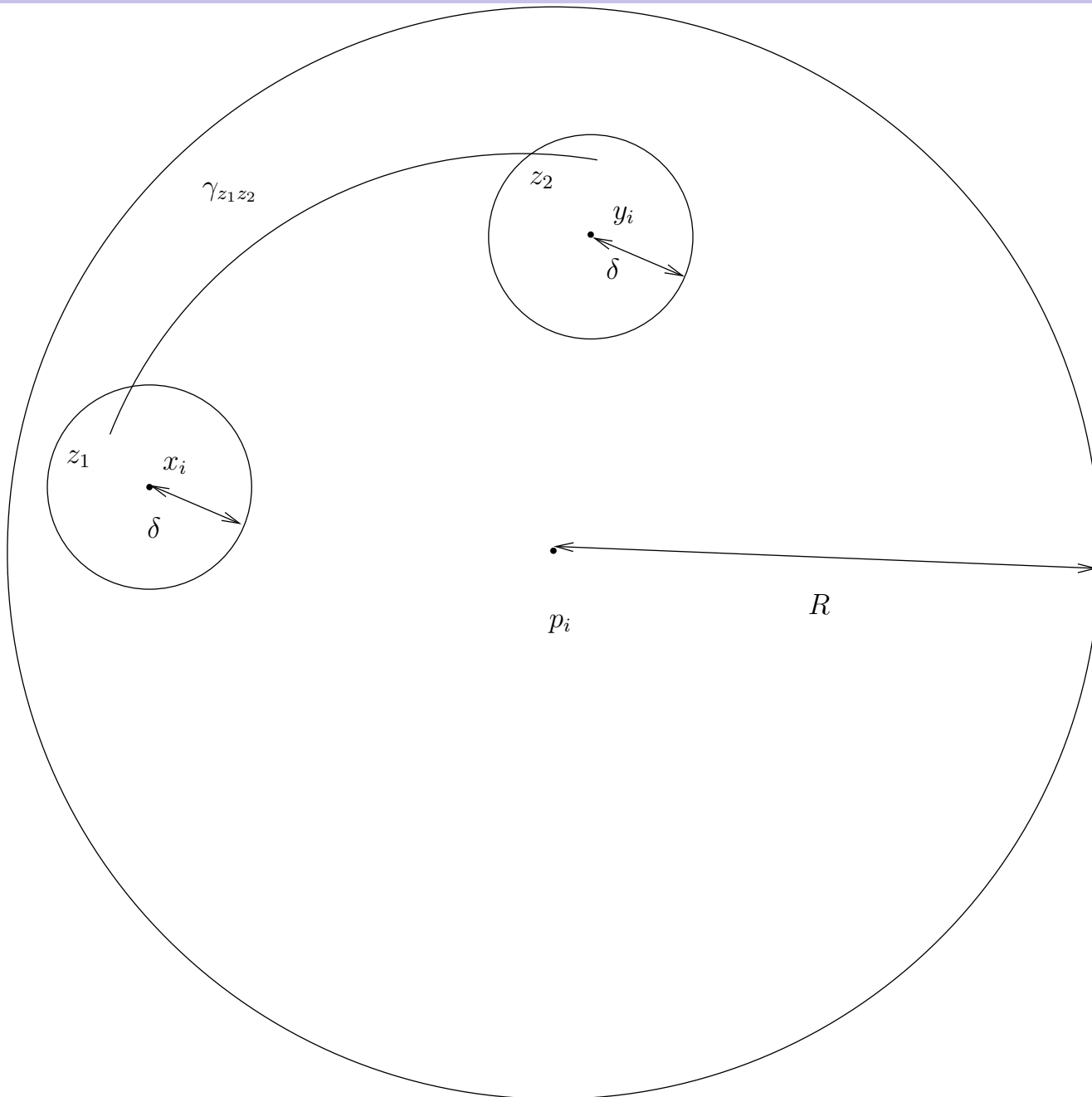
Main Results

Finite generation
of fundamental
groups

Gap Lemma
Finite generation

Maps which are
on all scales close
to isometries.

Margulis Lemma.



Step 2. b_∞ is a submetry.

Fix $r > 0, t_0 > 0$. Let ϕ_t^i be the gradient flow of b_i on M_i .

$$\int_{B_r(p_i)} (b^i(\phi_{t_0}^i(x)) - b^i(x)) = \int_{B_r(p_i)} \int_0^{t_0} \frac{d}{dt} b^i(\phi_t^i(x)) =$$

$$\int_{B_r(p_i)} \int_0^{t_0} |\nabla b_i(\phi_t(x))|^2 = \int_0^{t_0} \int_{B(p_i,1)} |\nabla b_i(\phi_t(x))|^2 =$$

$$\int_0^{t_0} \int_{\phi_t(B_r(p_i))} |\nabla b_i(x)|^2 = t_0 \pm \varepsilon_i$$

where $\varepsilon_i \rightarrow 0$.

b^i is harmonic and hence ϕ_t^i is measure preserving.

For most points $x \in B_r(p_i)$

$$(8) \quad b^i(\phi_{t_0}^i(x)) - b^i(x) = t_0 \pm \varepsilon_i$$

By the first variation formula

$$\begin{aligned} \int_{B(p_i,1)} |\phi_{t_0}^i(x)x| &\leq \int_{B(p_i,1)} \int_0^{t_0} |\nabla b_i(\phi_t(x))| = \\ &= \int_0^{t_0} \int_{\phi_t(B(p_i,1))} |\nabla b_i(\phi_t(x))| = t_0 \pm \varepsilon_i \end{aligned}$$

by the same argument as above. This means that for most $x \in B(p_i, 1)$ we have

$$(9) \quad \phi_{t_0}^i(x) \in B_{t_0+\varepsilon_i}(x)$$

Combining (8) and (9) we get that in the limit space Y

$$\phi_{t_0}^\infty(x) \in B_{t_0}(x) \quad \text{and} \quad b_\infty(\phi_{t_0}^\infty(x)) - b_\infty(x) = t_0$$

i.e. b is a submetry. □

Ricci curvature
and the
fundamental
group

Vitali Kapovitch

Main Results

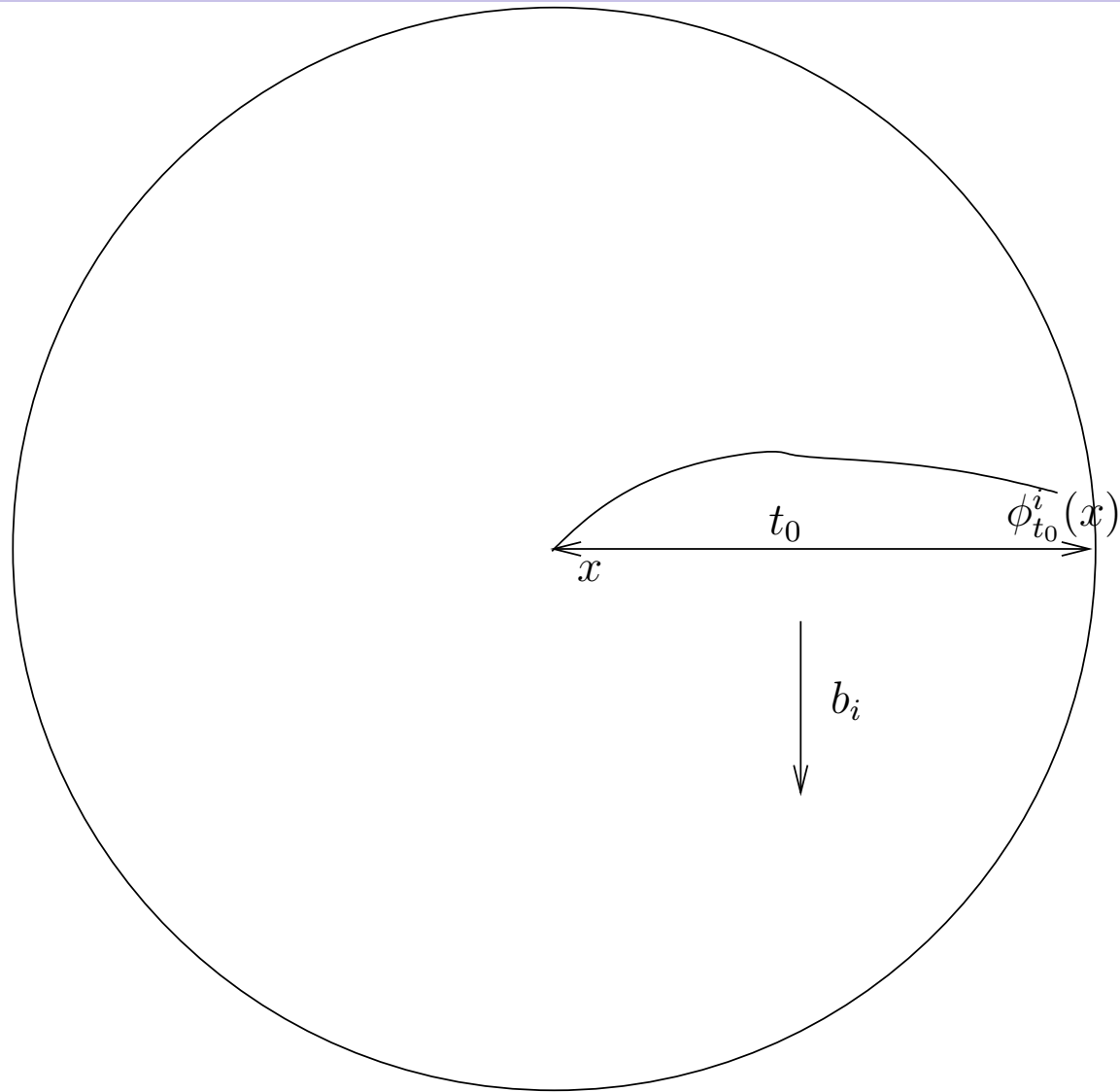
Finite generation
of fundamental
groups

Gap Lemma

Finite generation

Maps which are
on all scales close
to isometries.

Margulis Lemma.



$$b_i(x)$$

$$b_i(\phi_{t_0}^i(x)) = b_i(x) + t_0 \pm \varepsilon_i$$

Lemma A

Let $(Y_i, G_i, p_i) \xrightarrow[i \rightarrow \infty]{G-H} (Y_\infty, G_\infty, p_\infty)$.

Let $G_i(\varepsilon)$ denote the subgroup generated by those elements that displace p_i by at most ε , $i \in \mathbb{N} \cup \{\infty\}$. Suppose $G_\infty(\varepsilon) = G_\infty(\frac{a+b}{2})$ for all $\varepsilon \in (a, b)$ and $0 \leq a < b$.

Then there is some sequence $\varepsilon_i \rightarrow 0$ such that $G_i(\varepsilon) = G_i(\frac{a+b}{2})$ for all $\varepsilon \in (a + \varepsilon_i, b - \varepsilon_i)$.

Proof

Suppose on the contrary we can find $g_i \in G_i(\varepsilon_2) \setminus G_i(\varepsilon_1)$ for fixed $\varepsilon_1 < \varepsilon_2 \in (a, b)$. Without loss of generality $d(p_i, g_i p_i) \leq \varepsilon_2$. Because of $g_i \notin G_i(\varepsilon_1)$ it follows that for any finite sequence of points $p_i = x_1, \dots, x_h = g_i p_i \in G_i \star p_i$ there is one j with $d(x_j, x_{j+1}) \geq \varepsilon_1$. Clearly this property carries over to the limit and implies that $g_\infty \in G_\infty(\varepsilon_2)$ is not contained in $G((\varepsilon_1 + a)/2)$ – a contradiction. □

Ricci curvature
and the
fundamental
group

Vitali Kapovitch

Main Results

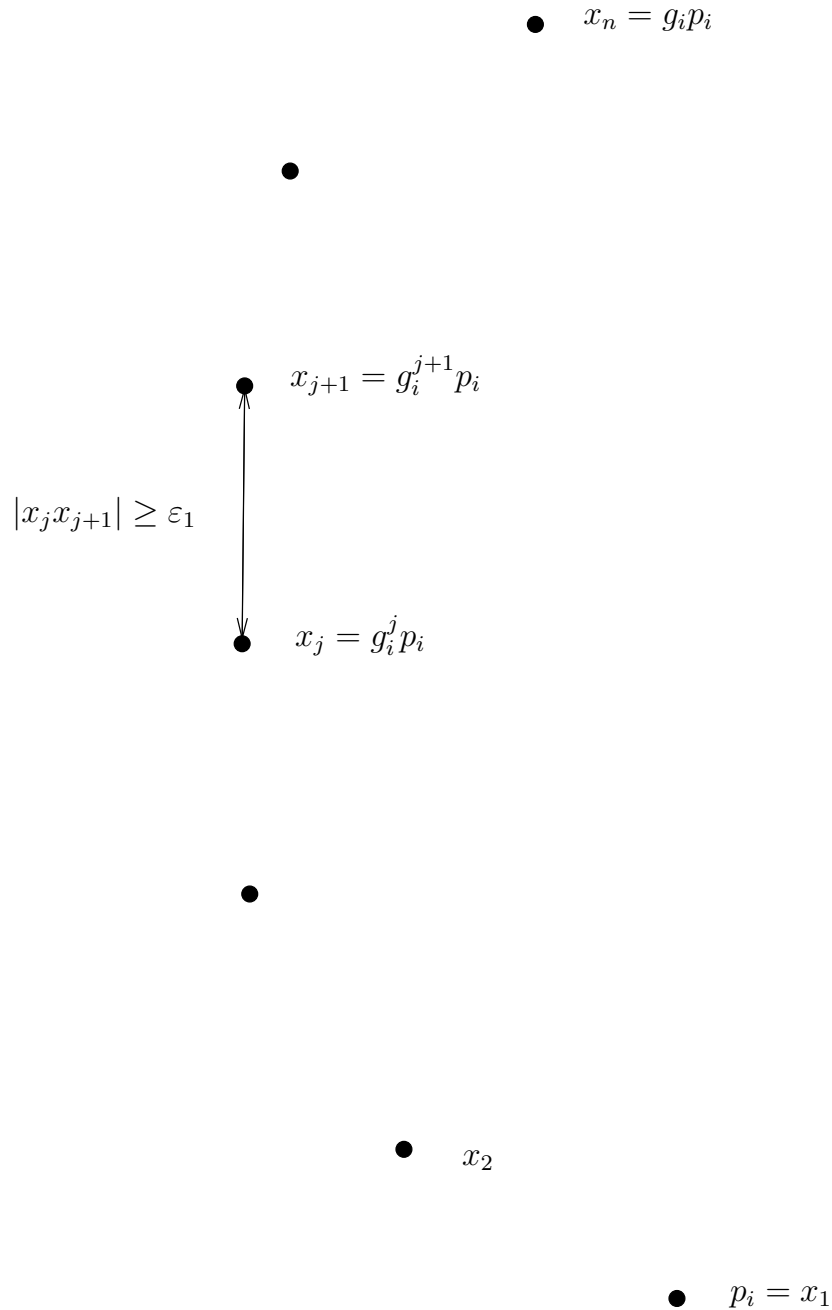
Finite generation
of fundamental
groups

Gap Lemma

Finite generation

Maps which are
on all scales close
to isometries.

Margulis Lemma.



Lemma B

Suppose (M_i^n, q_i) converges to $(\mathbb{R}^k \times K, q_\infty)$ where $\text{Ric}_{M_i} \geq -1/i$ and K is compact. Assume the action of $\pi_1(M_i)$ on the universal cover $(\tilde{M}_i, \tilde{q}_i)$ converges to a limit action of a group G on some limit space (Y, \tilde{q}_∞) .

Then $G(r) = G(r')$ for all $r, r' > 2 \text{diam}(K)$.

Proof

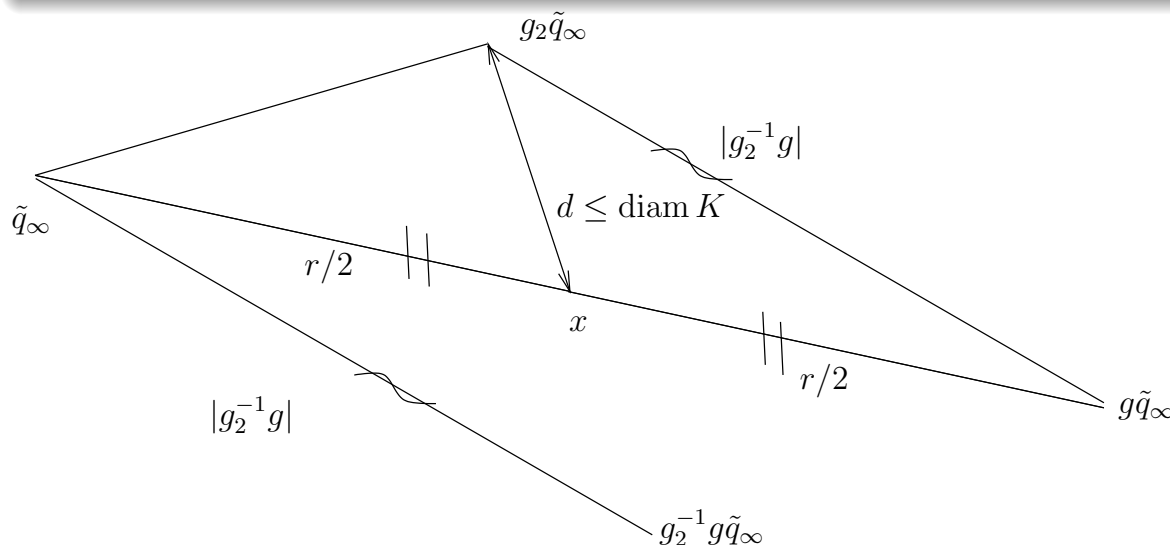
Since Y/G is isometric to $\mathbb{R}^k \times K$, it follows that there is a submetry $\sigma: Y \rightarrow \mathbb{R}^k$. It is immediate from the splitting theorem that this submetry has to be linear, that is, for any geodesic c in Y the curve $\sigma \circ c$ is affine linear. Hence we get a splitting $Y = \mathbb{R}^k \times Z$ such that G acts trivially on \mathbb{R}^k and on Z with compact quotient K . We may think of \tilde{q}_∞ as a point in Z . For $g \in G$ consider a mid point $x \in Z$ of \tilde{q}_∞ and $g\tilde{q}_\infty$. Because $Z/G = K$ we can find $g_2 \in G$ with $d(g_2\tilde{q}_\infty, x) \leq \text{diam}(K)$.

Clearly

$$d(\tilde{q}_\infty, g_2 \tilde{q}_\infty) \leq \frac{1}{2} d(\tilde{q}_\infty, g \tilde{q}_\infty) + \text{diam}(K)$$

$$d(\tilde{q}_\infty, g_2^{-1} g \tilde{q}_\infty) = d(g_2 \tilde{q}_\infty, g \tilde{q}_\infty) \leq \frac{1}{2} d(\tilde{q}_\infty, g \tilde{q}_\infty) + \text{diam}(K).$$

This proves $G(r) \subset G(r/2 + \text{diam}(K))$ and the lemma follows.



Lemma (Gap Lemma)

Suppose we have a sequence of manifolds (M_i, p_i) with a lower Ricci curvature bound converging to some limit space (X, p_∞) and suppose that the limit point p_∞ is regular. Then there is a sequence $\varepsilon_i \rightarrow 0$ and a number $\delta > 0$ such that the following holds. If $\gamma_1, \dots, \gamma_{l_i}$ is a short basis of $\pi_1(M_i, p_i)$ then either $|\gamma_j| \geq \delta$ or $|\gamma_j| < \varepsilon_i$.

Moreover, if the action of $\pi_1(M_i)$ on the universal cover $(\tilde{M}_i, \tilde{p}_i)$ converges to an action of the limit group G on (Y, \tilde{p}_∞) , then the orbit $G \star \tilde{p}_\infty$ is locally path connected.

This means there is a gap in lengths of short generators. They are either very short or longer than δ .

Proof

Idea. By rescaling reduce to the case $X \cong \mathbb{R}^k$. Apply Lemma B (with $K = \{pt\}$) to conclude $G(r) = G(r')$ for any $r, r' > 0$. Apply Lemma A to get the result. \square

Finite generation of π_1

Ricci curvature
and the
fundamental
group

Vitali Kapovitch

Main Results

Finite generation
of fundamental
groups

Gap Lemma

Finite generation

Maps which are
on all scales close
to isometries.

Margulis Lemma.

Theorem (Finite generation of π_1)

Given n and R there is a constant C such that the following holds. Suppose (M^n, g) has $\text{Ric} \geq -(n-1)$, $p \in M$, and $\pi_1(M, p)$ is generated by loops of length $\leq R$.

Then there is a point $q \in B_{R/2}(p)$ such that any Gromov short generator system of $\pi_1(M, q)$ has at most C elements.

Observation

If $\text{Ric}_{M^n} \geq -(n-1)$ then the number of short generators of $\pi_1(M, p)$ with $0 < r_1 < |\gamma_i| < r_2$ is bounded above by $C(n, r_1, r_2)$. This immediately follows from Bishop-Gromov volume comparison.

Proof of Finite generation theorem

Arguing by contradiction we get a sequence (M_i^n, p_i) satisfying

- $\text{Ric}_{M_i} \geq -(n-1)$.
- For all $q_i \in B_1(p_i)$ the number of short generators of $\pi_1(M_i, q_i)$ of length ≤ 4 is larger than 2^i .

By precompactness we may assume that (M_i, p_i) converges to some limit space (X, p_∞) . We put

$$\dim(X) = \max\{k \mid \text{there is a regular } p \in B_{1/4}(p_\infty) \text{ with } T_x X \cong \mathbb{R}^k\}$$

Reverse induction on $\dim(X)$. Base of induction. $\dim(X) \geq n + 1$. It is well known that this can not happen so there is nothing to prove. Induction step.

Step 1. For any contradicting sequence (M_i, p_i) converging to (X, p_∞) there is a new contradicting sequence converging to $(\mathbb{R}^{\dim(X)}, 0)$.

Can assume p_∞ is regular. Then use the observation above to find a slow rescaling $\lambda_i \rightarrow \infty$ such that after passing to a subsequence $(\lambda_i M_i, p_i) \rightarrow (\mathbb{R}^{\dim(X)}, 0)$ and the number of short generators of length ≤ 4 in $\pi_1(\lambda_i M_i, q_i)$ is still $\geq 2^i$ for any $q_i \in B_1^{\lambda_i M_i}(p_i) = B_{\lambda_i}^{M_i}(p_i)$.

Step 2. If there is a contradicting sequence converging to \mathbb{R}^k , then we can find a contradicting sequence converging to a space whose dimension is larger than k . WLOG $\text{Ric}_{M_i} \geq -1/i$.

By Cheeger-Colding we can find harmonic functions $(b_1^i, \dots, b_k^i): B_1(q_i) \rightarrow \mathbb{R}^k$ with

$$\int_{B_1(q_i)} \sum_{j,l=1}^k |\langle \nabla b_l^i, \nabla b_j^i \rangle - \delta_{lj}| + \|\text{Hess}(b_l^i)\|^2 = \varepsilon_i^2 \rightarrow 0$$

and

$$|\nabla b_j^i| \leq C(n).$$

By the weak (1,1) inequality we can find $z_i \in B_{1/2}(q_i)$ with

$$\int_{B_r(z_i)} \sum_{j,l=1}^k |\langle \nabla b_l^i, \nabla b_j^i \rangle - \delta_{lj}| + \|\text{Hess}(b_l^i)\|^2 \leq C\varepsilon_i \rightarrow 0$$

for all $r \leq 1/4$.

By the Product Lemma, for *any* sequence $\mu_i \rightarrow \infty$ the spaces $(\mu_i M_i, z_i)$ subconverge to a metric product $(\mathbb{R}^k \times Z, z_\infty)$ for some Z depending on the rescaling.

Choose $r_i \leq 1$ maximal with the property that there is $y_i \in B_{r_i}(z_i)$ such that the short generator system of $\pi_1(M_i, y_i)$ contains one generator of length r_i . By the Gap Lemma, $r_i \rightarrow 0$

By the Product Lemma, (N_i, z_i) subconverges to a product $(\mathbb{R}^k \times Z, z_\infty)$. Lemma B implies that Z can not be a point and the claim is proved. \square

For a map $f: X \rightarrow Y$ between metric spaces we define the

$$dt_r^f(p, q) = \min\{r, |d(p, q) - d(f(p), f(q))|\}.$$

we call $dt_r^f(p, q)$ *the distortion on scale r* .

Maps close to isometries on all scales

Ricci curvature
and the
fundamental
group

Vitali Kapovitch

Main Results

Finite generation
of fundamental
groups

Maps which are
on all scales close
to isometries.

Properties and
examples

The Rescaling
Theorem.

Margulis Lemma.

Definition

Let (M_i, p_i^1) and (N_i, p_i^2) be two sequences of complete Riemannian manifolds with $\text{Ric}_{M_i} > -C$ for some C , all i and $j = 1, 2$. We say that a sequence of diffeomorphisms $f_i: M_i \rightarrow N_i$ is **close to isometry on all scales** if the following holds:

There exist $R_0 > 0$, sequences $r_i \rightarrow \infty$, $\varepsilon_i \rightarrow 0$ and subsets $B_{2r_i}(p_i^j)' \subset B_{2r_i}(p_i^j)$ ($j = 1, 2$) satisfying

- $\text{Vol}(B_1(q) \cap B_{2r_i}(p_i^j)') \geq (1 - \varepsilon_i)\text{Vol}(B_1(q))$ for all $q \in B_{r_i}(p_i^j)$.
- For all $p \in B_{r_i}(p_i^1)'$, all $q \in B_{r_i}(p_i^2)'$ and all $r \in (0, 1]$ we have

$$\int_{B_r(p) \times B_r(p)} dt_r^{f_i}(x, y) d\mu(x) d\mu(y) \leq r\varepsilon_i \text{ and}$$
$$\int_{B_r(q) \times B_r(q)} dt_r^{f_i^{-1}}(x, y) d\mu(x) d\mu(y) \leq r\varepsilon_i.$$

- There are subsets $S_i^j \subset B_1(p_i^j)$ with $\text{Vol}(S_i^j) \geq \frac{1}{2}\text{Vol}(B_1(p_i^j))$ ($j = 1, 2$) and $f(S_i^1) \subset B_{R_0}(p_i^2)$ and $f^{-1}(S_i^2) \subset B_{R_0}(p_i^1)$.

Properties

- if $f_i: (M_i, p_i^1) \rightarrow (N_i, p_i^2)$ is close to isometry on all scales and $(M_i, p_i^1) \xrightarrow[i \rightarrow \infty]{G-H} (X, p^1)$, $(N_i, p_i^2) \xrightarrow[i \rightarrow \infty]{G-H} (Y, p^2)$ Then f_i converges in a weakly measured sense to an isometry $f: X \rightarrow Y$.
- If $f_i: M_i \rightarrow N_i$ and $g_i: N_i \rightarrow P_i$ is close to isometry on all scales then $f_i \circ g_i$ is also close to isometries on all scales.

The main source of such maps are gradient flows of modified distance functions with small L^2 norms of Hessians. If $b: M \rightarrow \mathbb{R}$ is a modified harmonic distance function let $X = \nabla b$ and let ϕ_t be the gradient flow.

Let $p, q \in M$ and let $\gamma: [0, d] \rightarrow M$ be a unit speed geodesic with $\gamma(0) = p, \gamma(d) = q$. Here $d = |pq|$. Then

$$|\phi_t(p)\phi_t(q)|'_+(0) \leq L(\phi_t(\gamma))'(0) \leq \int_{\gamma} |\nabla X| = \int_{\gamma} |\text{Hess}_b|$$

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fundamental
group

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Main Results

Finite generation
of fundamental
groups

Maps which are
on all scales close
to isometries.

Properties and
examples

The Rescaling
Theorem.

Margulis Lemma.

If $\int_{B_R(p_i)} |\text{Hess}_{b_i}|^2 \rightarrow 0$ we can use the weak 1-1 inequality, the Segment inequality and the fact that ϕ_t^i is measure preserving to show that ϕ_t^i is close to isometry on all scales.

Proposition (Main example)

Let $(M_i, g_i) \xrightarrow[i \rightarrow \infty]{G-H} (\mathbb{R}^k \times Y, p_\infty)$ satisfy $\text{Ric}_{M_i} > -1/i$.

Then for each $v \in \mathbb{R}^k$ there is a sequence of diffeomorphisms $f_i: [M_i, p_i] \rightarrow [M_i, p_i]$ close to isometries on all scales which converges in the weakly measured sense to an isometry f_∞ of $\mathbb{R}^k \times Y$ that acts trivially on Y and by $w \mapsto w + v$ on \mathbb{R}^k .

Moreover, f_i is isotopic to the identity and there is a lift $\tilde{f}_i: [\tilde{M}_i, \tilde{p}_i] \rightarrow [\tilde{M}_i, \tilde{p}_i]$ of f_i to the universal cover which is also close to isometry on all scales

Remark

This is EASY for manifolds with $\text{sec} \geq -1/i$ using gradient flows of distance functions.

Theorem (Rescaling Theorem)

Let $(M^n, p_i) \xrightarrow[i \rightarrow \infty]{G-H} (\mathbb{R}^k, 0)$ for some $k < n$ where $\text{Ric}_{M_i} > -1/i$.

Then after passing to a subsequence we can find a compact metric space K with $\text{diam}(K) = 10^{-n^2}$, a sequence of subsets

$G_1(p_i) \subset B_1(p_i)$ with $\frac{\text{Vol}(G_1(p_i))}{\text{Vol}(B_1(p_i))} \rightarrow 1$ and $\lambda_i \rightarrow \infty$ such that

- ① For all $q_i \in G_1(p_i)$ the isometry type of the limit of any convergent subsequence of $(\lambda_i M_i, q_i)$ is given by the metric product $\mathbb{R}^k \times K$.
- ② For all $a_i, b_i \in G_1(p_i)$ we can find a sequence of diffeomorphisms

$$f_i: [\lambda_i M_i, a_i] \rightarrow [\lambda_i M_i, b_i]$$

close to isometries on all scales such that f_i is isotopic to the identity. Moreover, for any lift $\tilde{a}_i, \tilde{b}_i \in \tilde{M}_i$ of a_i and b_i to the universal cover \tilde{M}_i we can find a lift \tilde{f}_i of f_i such that

$$\tilde{f}_i: [\lambda_i \tilde{M}_i, \tilde{a}_i] \rightarrow [\lambda_i \tilde{M}_i, \tilde{b}_i]$$

are close to isometries on all scales as well.

Finally, if $\pi_1(M_i, p_i)$ is generated by loops of length $\leq R$ for all i , then we can find $\varepsilon_i \rightarrow 0$ such that $\pi_1(M_i, q_i)$ is generated by loops of length $\leq \frac{1+\varepsilon_i}{\lambda_i}$ for all $q_i \in G_1(p_i)$.

The Rescaling Theorem serves as Ricci curvature substitute for the Fibration theorem.

Idea of the proof of the Rescaling Theorem

Let $b_i: M_i \rightarrow \mathbb{R}^k$ be the modified harmonic distance functions
Put

$$h_i = \sum_{j,l=1}^k | \langle \nabla b_j^i, \nabla b_l^i \rangle - \delta_{j,l} | + \sum_j \| \text{Hess}_{b_j^i} \|^2.$$

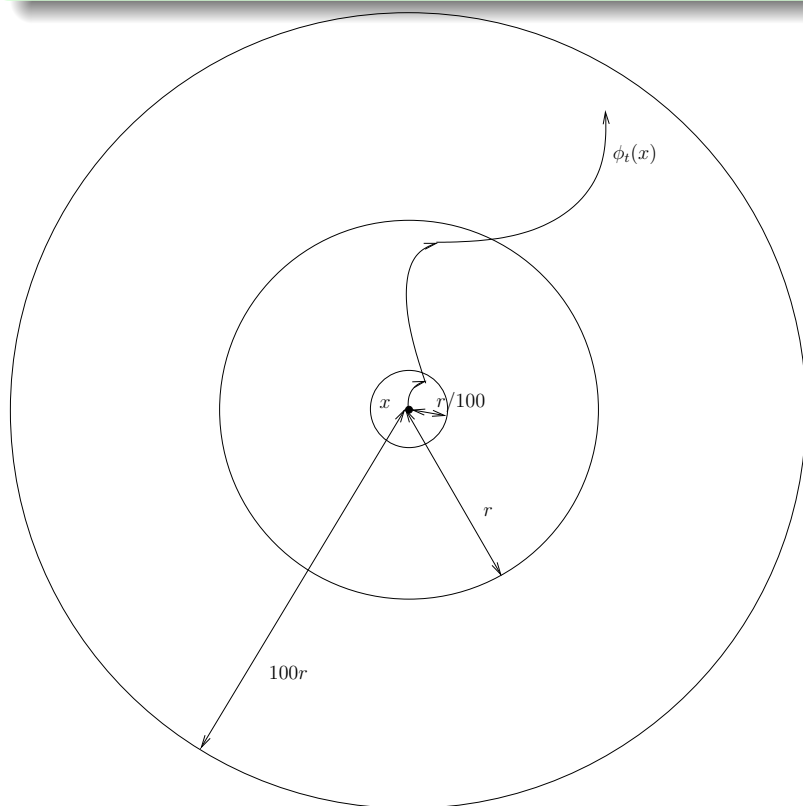
After passing to a subsequence by Cheeger–Colding we have

$$\int_{B_1(p_i)} h_i \leq \varepsilon_i^2 \text{ with } \varepsilon_i \rightarrow 0.$$

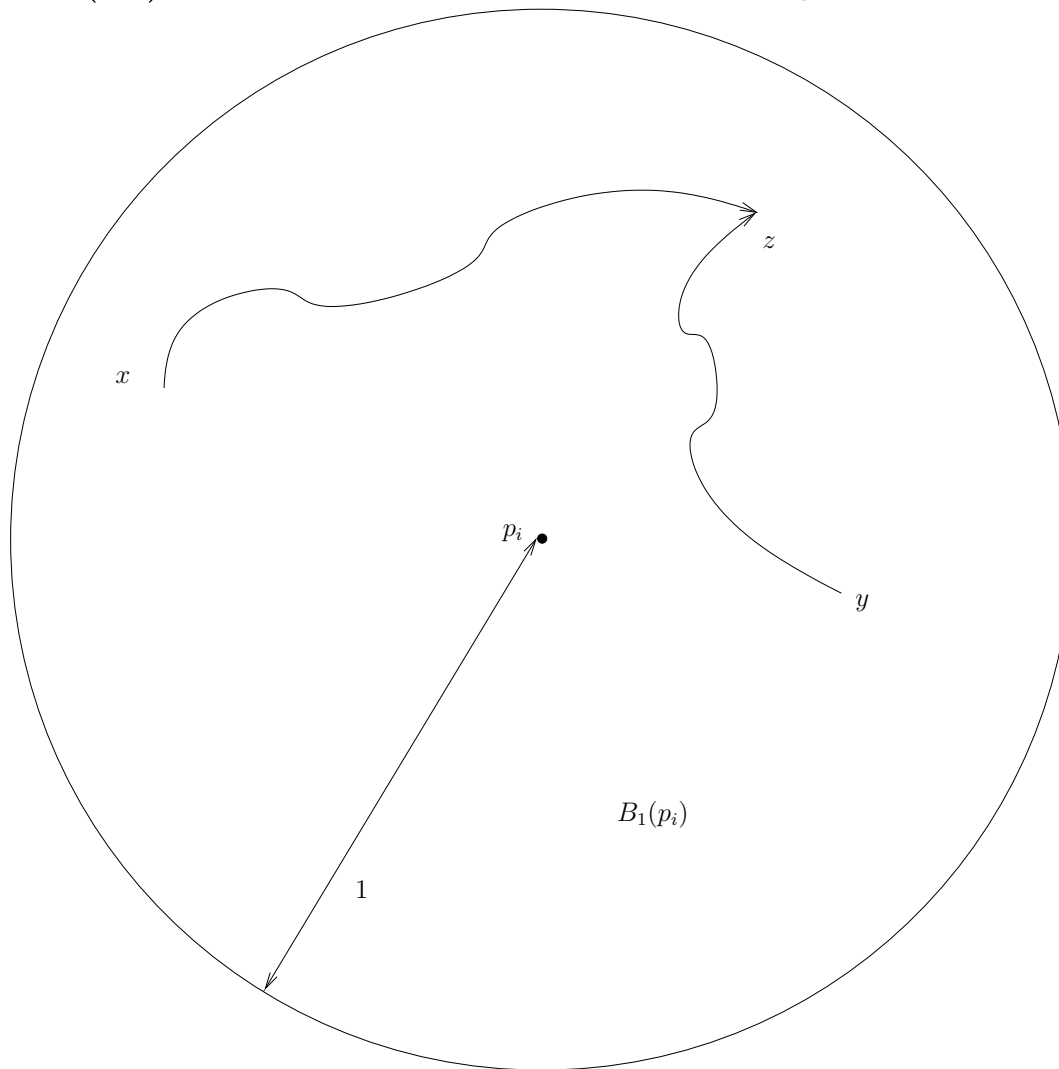
$$(10) \quad G_1(p_i) := \{x \in B_1(p_i) \mid \text{Mx } h_i(x) \leq \varepsilon_i\}.$$

We will call elements of $G_1(p_i)$ "good points" in $B_1(p_i)$.

Most points in $B_1(p_i)$ are good by the weak 1-1 inequality.
Let x be a good point. Use induction on the size of the ball to show that one can get from x to most points in $B_r(x)$ with $r \leq 1$ by composing gradient flows of appropriate modified distance functions which produce maps close to isometries on all scales.
Induction step: If true for $r/100$ then true for r .



Given good points $x, y \in B_1(p_i)$ can connect them to most points in $B_1(p_i)$ and hence to each other by composition.



Idea of the proof of the Margulis Lemma.

Ricci curvature
and the
fundamental
group

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Main Results

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of fundamental
groups

Maps which are
on all scales close
to isometries.

Margulis Lemma.

Idea of the proof
of the Margulis
Lemma

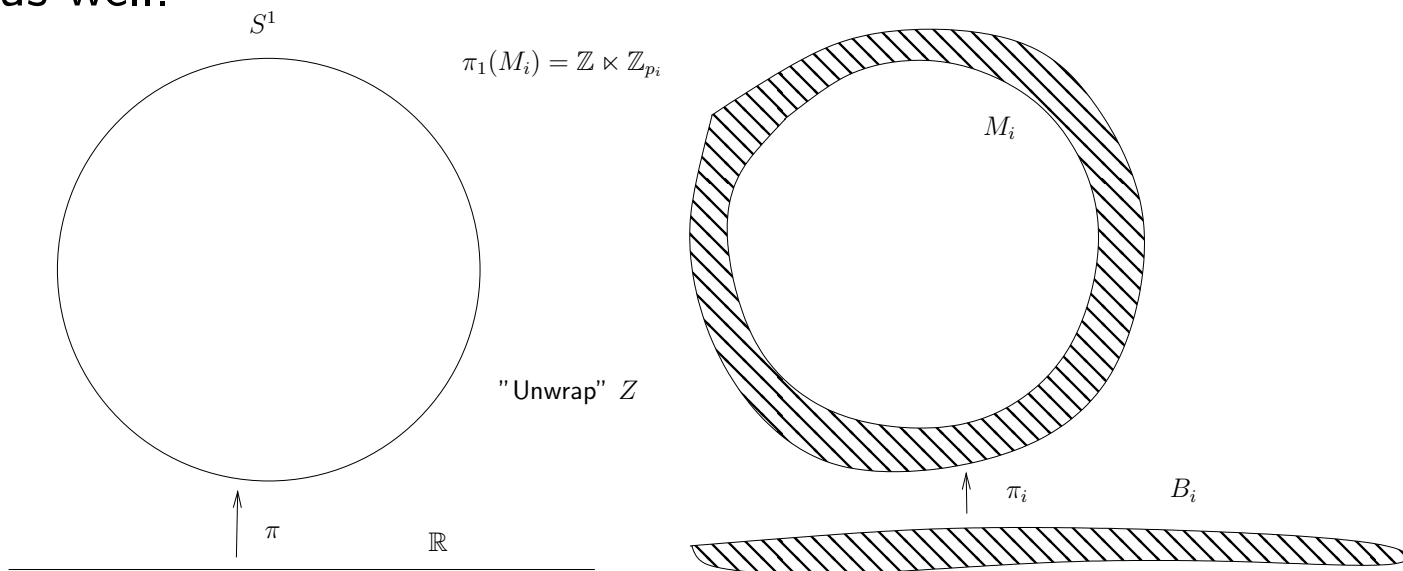
The Induction
Theorem for
C-Nilpotency

Let $p_i \rightarrow \infty$ be a sequence of odd primes and let $\Gamma_i := \mathbb{Z} \ltimes \mathbb{Z}_{p_i}$ be the semidirect product where the homomorphism $\mathbb{Z} \rightarrow \text{Aut}(\mathbb{Z}_{p_i})$ maps $1 \in \mathbb{Z}$ to φ_i given by $\varphi_i(z + p_i\mathbb{Z}) = 2z + p_i\mathbb{Z}$.

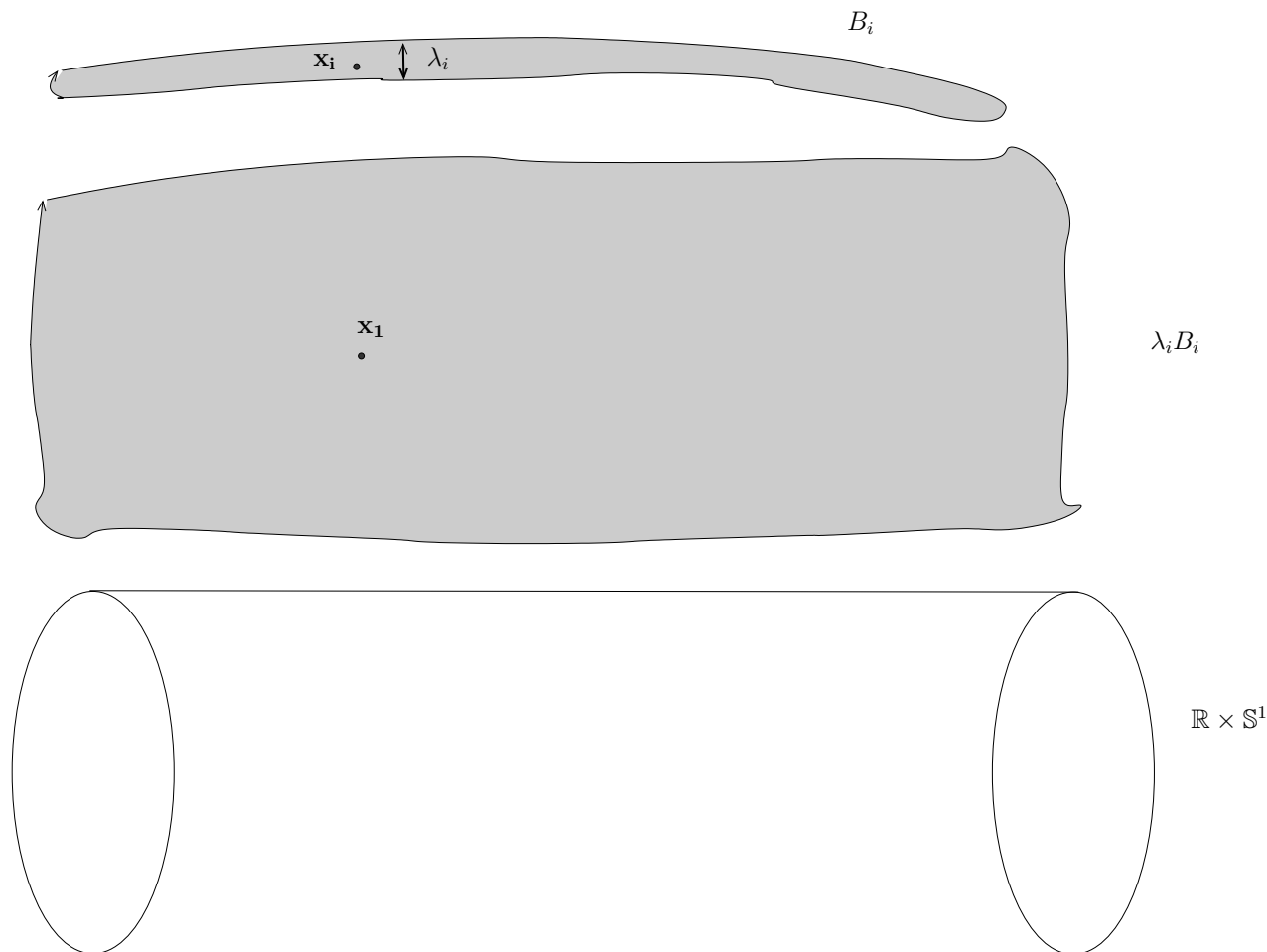
Suppose, contrary to the Margulis Lemma, we have a sequence M_i^n with $\text{Ric} > -1/i$ and $\text{diam}(M_i) = 1$ and fundamental group Γ_i .

A typical problem would be that $M_i \xrightarrow[i \rightarrow \infty]{G-H} S^1$ and $\tilde{M}_i \xrightarrow[i \rightarrow \infty]{G-H} \mathbb{R}$.

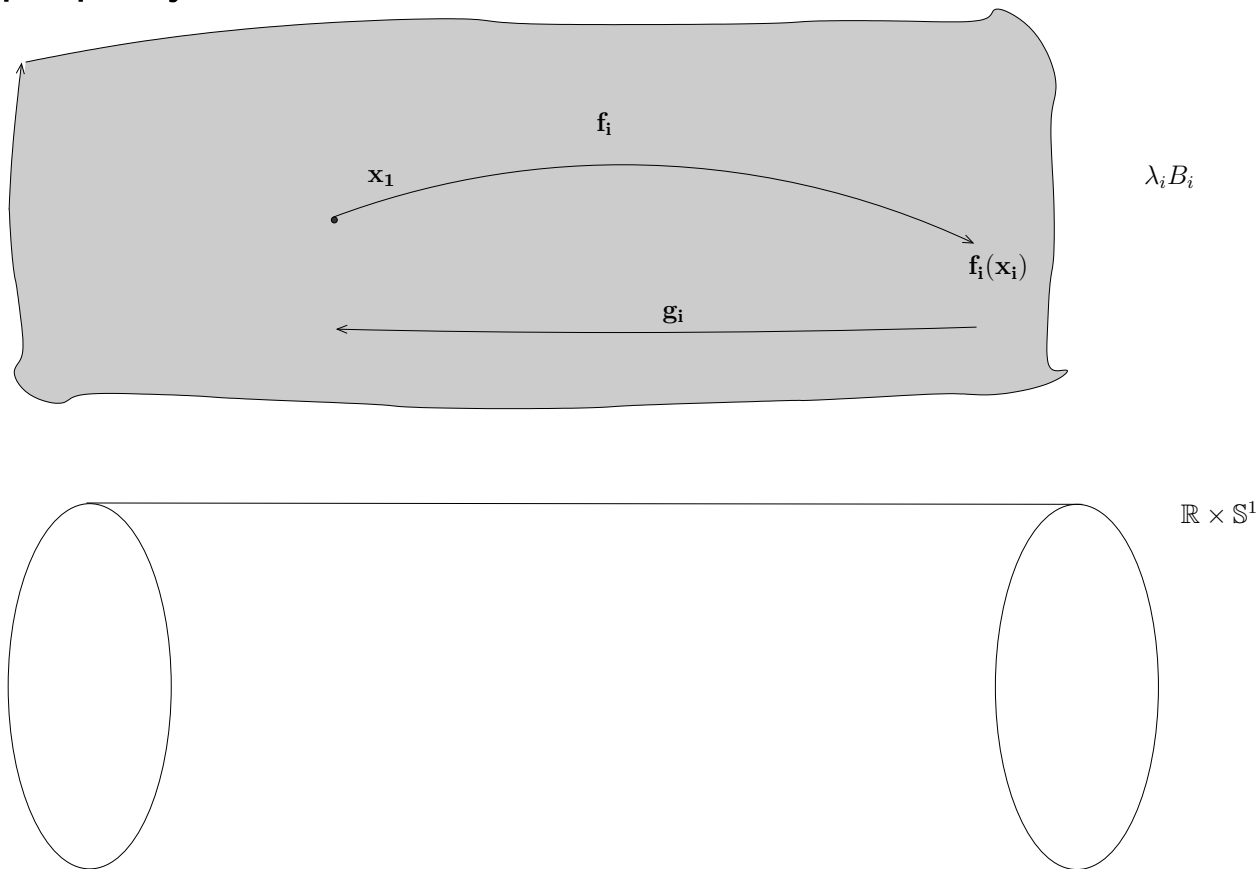
We then replace M_i by $B_i = \tilde{M}_i / \mathbb{Z}_{p_i}$ and in order not to lose information we endow B_i with the deck transformation $f_i: B_i \rightarrow B_i$ representing a generator of $\Gamma_i / \mathbb{Z}_{p_i} \cong \mathbb{Z}$. Then B_i will converge to \mathbb{R} as well.



Then one can find $\lambda_i \rightarrow \infty$ such that the rescaled sequence $\lambda_i B_i$ converges to $\mathbb{R} \times K$ with K being compact but not equal to a point. Suppose for illustration that $\lambda_i B_i \rightarrow \mathbb{R} \times S^1$ and $\lambda_i \tilde{M}_i \rightarrow \mathbb{R}^2$ and that the action of \mathbb{Z}_{p_i} converges to a discrete action of \mathbb{Z} on \mathbb{R}^2 .



The maps $f_i: \lambda_i B_i \rightarrow \lambda_i B_i$ do not converge, because typically f_i would map a base point x_i to some point $y_i = f_i(x_i)$ with $d(x_i, y_i) = \lambda_i \rightarrow \infty$ with respect to the rescaled distance. We use gradient flows of modified distance functions to construct diffeomorphisms $g_i: [\lambda_i B_i, y_i] \rightarrow [\lambda_i B_i, x_i]$ with the zooming in property.



The composition $f_{new,i} := f_i \circ g_i: [\lambda_i B_i, x_i] \rightarrow [\lambda_i B_i, x_i]$ also has the zooming in property and thus converges to an isometry of the limit.

Moreover, a lift $\tilde{f}_{new,i}: \lambda_i \tilde{M}_i \rightarrow \lambda_i \tilde{M}_i$ of $f_{new,i}$ has the zooming in property, too. Since g_i can be chosen isotopic to the identity, the action of $\tilde{f}_{new,i}$ on the deck transformation group $\mathbb{Z}_{p_i} = \pi_1(B_i)$ by conjugation remains unchanged.

On the other hand, the \mathbb{Z}_{p_i} -action on \tilde{M}_i converges to a discrete \mathbb{Z} -action on \mathbb{R}^2 and $\tilde{f}_{new,i}$ converges to an isometry $\tilde{f}_{new,\infty}$ of \mathbb{R}^2 normalizing the \mathbb{Z} -action. This implies that $\tilde{f}_{new,\infty}^2$ commutes with

Theorem (Induction Theorem)

Suppose (M_i^n, p_i) satisfies

- ① $\text{Ric}_{M_i} \geq -1/i$;
- ② There is some $R > 0$ such that $\pi_1(B_R(p_i)) \rightarrow \pi_1(M_i)$ is surjective for all i ;
- ③ $(M_i, p_i) \xrightarrow[i \rightarrow \infty]{G-H} (\mathbb{R}^k \times K, (0, p_\infty))$ where K is compact.

Suppose in addition that we have k sequences

$f_i^j : [\tilde{M}_i, \tilde{p}_i] \rightarrow [\tilde{M}_i, \tilde{p}_i]$ which are close to isometries on all scales where \tilde{p}_i is a lift of p_i , and which normalize the deck transformation group acting on \tilde{M}_i , $j = 1, \dots, k$.

Then there exists a positive integer C such that for all sufficiently large i , $\pi_1(M_i)$ contains a nilpotent subgroup $N \triangleleft \pi_1(M_i)$ of index at most C such that N has an $(f_i^j)^{C!}$ -invariant ($j = 1, \dots, k$) cyclic nilpotent chain of length $\leq n - k$, that is:

We can find $\{e\} = N_0 \triangleleft \dots \triangleleft N_{n-k} = N$ such that $[N, N_h] \subset N_{h-1}$ and each factor group N_{h+1}/N_h is cyclic. Furthermore, each N_h is invariant under the action of $(f_i^j)^{C!}$ by conjugation and the induced automorphism of N_h/N_{h+1} is the identity.

We argue by contradiction. After passing to a subsequence we can assume $(\tilde{M}_i, \Gamma_i, \tilde{p}_i) \xrightarrow[i \rightarrow \infty]{G-H} (\mathbb{R}^k \times \mathbb{R}^l \times \tilde{K}, G, (0, \tilde{p}_\infty))$, where \tilde{K} is compact and the action of G on the first \mathbb{R}^k factor is trivial.

Structure of the Proof

- 1 Without loss of generality we can assume that K is not a point. If $K = \{pt\}$ use the Rescaling Theorem to get a contradicting sequence with $K \neq \{pt\}$.
- 2 WLOG we can assume that f_i^j converges in the measured sense to the identity map of the limit space $\mathbb{R}^k \times \mathbb{R}^l \times \tilde{K}$, $j = 1, \dots, k$. Use gradient flows of harmonic functions and the fact that $\text{Isom}(\tilde{K})$ is compact so that a high power of any element is close to identity.

- 3 By the Gap Lemma there is an $\varepsilon > 0$ and $\varepsilon_i \rightarrow 0$ such that $\Gamma_i(\varepsilon) = \Gamma_i(\varepsilon_i)$ for large i . Then WLOG $[\Gamma_i : \Gamma_i(\varepsilon)] \rightarrow \infty$.
If not can assume after passing to a bounded cover that $\Gamma_i = \Gamma_i(\varepsilon) = \Gamma_i(\varepsilon_i)$.
Then use Rescaling Theorem to get a contradicting sequence converging to a space which splits $\mathbb{R}^{k'}$ with $k' > k$.
- 4 Show that after passing to a bounded cover, we can assume that $\Gamma_i(\varepsilon) \triangleleft \Gamma_i$ is normal in Γ_i .
- 5 The limit group G acts on \mathbb{R}^l cocompactly by ρ . Since $[\Gamma_i : \Gamma_i(\varepsilon)] \rightarrow \infty$ the $\rho(G)/\rho(G)_0$ is *infinite* and virtually abelian.
One can "unwrap" one \mathbb{Z} from it and find corresponding subgroups $\Gamma'_i \leq \Gamma_i$ with Γ_i/Γ'_i cyclic of order going to infinity. Replace M_i by $M'_i = \tilde{M}_i/\Gamma'_i$ and add one new f_i^{k+1} given by the deck transformation generating Γ_i/Γ'_i . Then $M'_i \rightarrow X$ which splits off \mathbb{R}^{k+1} and we can use the induction assumption.