

# METRIC-MEASURE BOUNDARY AND GEODESIC FLOW ON ALEXANDROV SPACES

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ABSTRACT. We relate the existence of many infinite geodesics on Alexandrov spaces to a statement about the average growth of balls. We deduce that the geodesic flow exists and preserves the Liouville measure in several important cases. The developed analytic tool has close ties to integral geometry.

## 1. INTRODUCTION

**1.1. Motivation and application.** The following structural question in the theory of Alexandrov spaces was formulated in a slightly different way in [PP96] and remained open since then.

- Are there “many” infinite geodesics on any Alexandrov space without boundary?

We address this question and obtain an affirmative answer in several cases, by developing a new analytic tool. This tool has a formal similarity with classical integral geometry and might be interesting in its own right beyond the realm of Alexandrov geometry.

In particular, we prove the existence of such infinite geodesics in the most classical examples of non-smooth Alexandrov spaces:

**THEOREM 1.1.** *Let  $X$  be the boundary of a convex body in  $\mathbb{R}^{n+1}$ . Then almost any direction in the tangent bundle  $TX$  is the starting direction of a unique infinite geodesic on  $X$ . The geodesic flow is defined almost everywhere and preserves the Liouville measure.*

Even the existence of a *single* infinite geodesic has apparently not been known, even in the case  $n = 2$ , [Zam92]. Our result might appear a bit surprising, since on *most* convex surfaces *most* points in the sense of Baire categories are not inner points of any geodesic, due to [Zam82].

**1.2. The geodesic flow and mm-boundary.** On a smooth manifold with boundary the geodesic flow is not defined for all times and the amount of geodesics terminating at the boundary in a given time interval depends on the size of this boundary. In the following definition

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we capture the size of the boundary by estimating the average volumes of small balls and their deviations from the corresponding volumes in the Euclidean space.

Let  $(X, d)$  be a locally compact separable metric space and let the Hausdorff dimension  $n$  of  $X$  be a natural number. Let  $\mu$  be a Radon measure on  $X$  and assume that closed bounded subsets of  $X$  have finite  $\mu$ -measure. For  $x \in X$  and  $r > 0$  denote by  $B_r(x)$  the open metric ball of radius  $r$  around  $x$ . Consider the Borel function  $b_r : X \rightarrow [0, \infty)$ ,

$$(1.1) \quad b_r(x) := \mu(B_r(x)) .$$

We define the *deviation measure*  $v_r$  of  $X$  as a signed Radon measure  $v_r$  on  $X$ , absolutely continuous with respect to  $\mu$ , given by

$$(1.2) \quad v_r = \left(1 - \frac{b_r}{\omega_n r^n}\right) \cdot \mu .$$

Here  $\omega_n$  is the volume of the  $n$ -dimensional unit Euclidean ball. A signed Radon measure, here and below, is an element of the space  $\mathcal{M}(X)$  dual to the space of compactly supported continuous functions  $C_c(X)$ . We consider  $\mathcal{M}(X)$  with the topology of weak convergence.

In all examples and applications below,  $\mu$  is the  $n$ -dimensional Hausdorff measure  $\mathcal{H}^n = \mathcal{H}_X^n$  on  $X$ , always normalized so that it coincides with the Lebesgue measure on  $\mathbb{R}^n$ .

The curve  $r \mapsto v_r \in \mathcal{M}(X)$  controls in a very rough integral sense the deviation of  $X$  from  $\mathbb{R}^n$ . The Taylor coefficients of  $v_r$  at 0 (if they are defined) can be interpreted as integral curvatures, in analogy with integral geometry. The following example is well known.

*Example 1.2.* Let  $X^n$  be a smooth Riemannian manifold with Riemannian volume  $\mu = \mathcal{H}^n$ . Then, for  $r \rightarrow 0$ , the measures  $v_r/r^2$  converge in  $\mathcal{M}(X)$  to  $\frac{1}{6(n+2)} \cdot \text{scal} \cdot \mu$ , where *scal* denotes the scalar curvature of  $X$ .

The next example can be obtained by computations in local coordinates. Since it is not needed in the sequel, we omit the details.

*Example 1.3.* Let  $X$  be a smooth  $n$ -dimensional Riemannian manifold with boundary  $\partial X$ . Then, for  $r \rightarrow 0$ , the measures  $v_r/r$  converge in  $\mathcal{M}(X)$  to  $c_n \cdot \mathcal{H}_{\partial X}^{n-1}$ , for some constant  $c_n > 0$  depending only on  $n$ .

This example suggests to view the first Taylor coefficient of  $v_r$  as the “boundary” of the metric-measure space  $(X, d, \mu)$ , which we make more precise by the following central definition of this paper.

**Definition 1.4.** Let  $(X, d, \mu)$  be a metric measure space as above. Let  $v_r$  be the deviation measure of  $X$ , as in (1.2). We say that  $X$  has

locally finite metric-measure boundary, abbreviated as mm-boundary, if the family of signed Radon measures

$$\left\{ \frac{\nu_r}{r} ; 0 < r \leq 1 \right\}$$

is uniformly bounded. If  $\lim_{r \rightarrow 0} \nu_r/r = \nu$  in  $\mathcal{M}(X)$ , we call  $\nu$  the mm-boundary of  $X$ . If  $\nu = 0$  we say that  $X$  has trivial mm-boundary.

We refer to Subsection 1.7 and Section 8 below for a discussion of examples and questions, and state now our central result connecting mm-boundaries to the existence of infinite geodesics in Alexandrov spaces:

**THEOREM 1.5.** *Let  $X$  be an Alexandrov space. If  $X$  has trivial mm-boundary, then almost each direction of the tangent bundle  $TX$  is the starting direction of an infinite geodesic. In this case, the geodesic flow preserves the Liouville measure on  $TX$ .*

**1.3. Size of the mm-boundary in Alexandrov spaces.** The next theorem shows that, similarly to Example 1.3, the topological boundary is closely related to the mm-boundary in Alexandrov spaces.

**THEOREM 1.6.** *Let  $X^n$  be an  $n$ -dimensional Alexandrov space. Then  $X$  has locally finite mm-boundary. If  $\nu = \lim \frac{\nu_{s_j}}{s_j}$ , for a sequence  $s_j \rightarrow 0$ , then  $\nu$  is a Radon measure and the following holds true.*

- (1) *There is a Borel set  $A_0$  with  $\mathcal{H}^n(X \setminus A_0) = \nu(A_0) = 0$ .*
- (2) *If the topological boundary  $\partial X$  is non-empty then  $\nu \geq c \cdot \mathcal{H}_{\partial X}^{n-1}$ , for a positive constant  $c$  depending only on  $n$ .*
- (3) *If the topological boundary  $\partial X$  is empty then  $\nu(A) = 0$ , for any Borel subset  $A \subset X$  with  $\mathcal{H}^{n-1}(A) < \infty$ .*

We strongly believe that an Alexandrov space with empty topological boundary  $\partial X$  has a trivial mm-boundary, which would solve the question about the existence of infinite geodesics. We could prove this conjecture in two special cases:

**THEOREM 1.7.** *Let  $X^n$  be a convex hypersurface in  $\mathbb{R}^{n+1}$  or let  $X$  be a two-dimensional Alexandrov space without boundary. Then  $X$  has trivial mm-boundary.*

In combination with Theorem 1.5 this proves Theorem 1.1. The two-dimensional case will follow from a much stronger result discussed in the next subsection. Another proof of the two-dimensional case could be derived from the statement about convex hypersurfaces and Alexandrov's embedding theorems.

1.4. **mm-curvature.** Motivated by Example 1.2 one can naively hope that the second Taylor coefficient at 0 of the map  $v_r: r \mapsto \mathcal{M}(X)$  describes the scalar curvature of the space.

**Definition 1.8.** Let  $X, v_r$  be as in Definition 1.4 above. If the family  $v_r/r^2, r \leq 1$  is uniformly bounded then we say that  $X$  has locally finite mm-curvature. If the measures  $v_r/r^2$  converge to a measure  $\nu$ , we call  $\nu$  the mm-curvature of  $X$ .

Clearly, local finiteness of mm-curvature as defined above implies that the mm-boundary is trivial. Thus, the following result proves Theorem 1.7 in the 2-dimensional case.

**THEOREM 1.9.** *Let  $X$  be a 2-dimensional Alexandrov space without boundary. Then  $X$  has locally finite mm-curvature.*

This finiteness result holds true in the much greater generality of surfaces with bounded integral curvature in the sense of Alexandrov–Zallgaler–Reshetnyak [Res93], [AZ67], see Section 4 below.

Note, however, that the mm-curvature in Theorem 1.9 does not need to coincide with the “curvature measure” as defined in [AZ67], even in the case of a cone, cf. Example 1.14. In particular, this shows that the mm-curvatures in 2-dimensional Alexandrov spaces are not stable under Gromov–Hausdroff convergence.

*Remark 1.10.* N. Lebedeva and the third named author have found in [LP17] a “scalar curvature measure” on all smoothable Alexandrov spaces. There is a hope, supported by our proof of Theorem 1.9, that a better understanding of this “stable curvature measure” will lead to some control of the mm-boundary and mm-curvature discussed here.

1.5. **Analogy with the Killing–Lipschitz curvatures and the proof of Theorem 1.5.** For a compact convex body or compact smooth submanifold  $M$  in  $\mathbb{R}^n$ , one classically considers the volumes  $w(r) = \mathcal{H}^n(B_r(M))$  of distance tubes  $B_r(M)$  around  $M$ . It turns out that  $w(r)$  is a polynomial, at least for small  $r$ , and the coefficients of  $w$ , called the Killing–Lipschitz curvatures of  $M$ , are given by integrals over some intrinsically defined curvature terms. Moreover, these coefficients can be localized and considered as measures on  $M$ .

To make the formal similarity with our approach to mm-boundary and mm-curvature more transparent, we observe that (at least for a smooth  $n$ -dimensional manifold  $M$ ) the number  $\int_M \mathcal{H}^n(B_r(x)) \cdot d\mathcal{H}^n(x)$  can be interpreted as the  $\mathcal{H}^{2n}$ -measures of the distance tubes  $B_{\frac{r}{\sqrt{2}}}(\Delta)$  around the diagonal  $\Delta$  in the Cartesian product  $M \times M$ .

The natural interpretation of the tangent bundle of  $M$  as the normal bundle of the diagonal  $\Delta$  in  $M \times M$  gives a connection between the measure theoretical properties of the tubes around  $M$  and the dynamical properties of the geodesic flow. We clarify this abstract statements by explaining the main idea of our proof of Theorem 1.5 in the well-known case of a complete smooth Riemannian manifold  $X = M$ . In this case the existence of geodesics is trivial. Thus, we just sketch a new proof of the classical fact that the geodesic flow  $\phi$  preserves the Liouville measure  $\mathcal{M}$  on  $TM$ . This proof is stable and rough enough to be transferred to the singular situation,

Let  $\phi_t: TM \rightarrow TM$  be the geodesic flow for time  $t$ . Denote by  $\pi: TM \rightarrow M$  the tangent bundle of  $M$ . Define the map  $E: TM \rightarrow M \times M$  by

$$E(v) = (\pi(v), \pi(\phi_1(v))).$$

By construction,  $E(-\phi_1(v)) = J(E(v))$ , where  $J$  is the involution of  $M \times M$  which switches the coordinates. Since  $J$  preserves  $\mathcal{H}^{2n}$  on  $M \times M$  and  $v \rightarrow -v$  preserves the Liouville measure  $\mathcal{M}$  on  $TM$ , the statement that  $\phi$  is measure preserving hinges upon the smallness of measure-distortion of the map  $E: (TM, \mathcal{M}) \rightarrow (M \times M, \mathcal{H}^{2n})$  close to the 0-section. In the present case of a Riemannian manifold, this property of  $\phi_1$  is expressed by the fact that the differential of  $E$  is the identity (after suitable identifications). In the general case of Alexandrov spaces, we similarly observe that the “infinitesimal” deviation (via the canonical map  $E$ ) between  $J$  being measure preserving (which we know) and  $\phi_1$  being measure preserving (which is what we want to show) is expressed precisely as the triviality of the mm-boundary.

**1.6. Stability and relation with quasi-geodesics.** Many Alexandrov spaces, for instance all convex hypersurfaces, appear naturally as Gromov-Hausdorff limits of smooth Riemannian manifolds. However, the properties of the geodesic flow, mm-boundaries and mm-curvature are unstable under limit operations, cf. the discussion at the end of Subsection 1.4. Thus, there is no hope to deduce Theorem 1.9, Theorem 1.7 or Theorem 1.1 by a direct limiting argument.

For instance, being a geodesic is a local notion, not preserved under limits. However, any limit of geodesics in a non-collapsed limit of Alexandrov spaces is a so called quasi-geodesic, a curve sharing many properties with geodesics. Quasi-geodesics have been investigated in [PP96], cf. [Pet07] and the references therein. It was shown that any direction is the starting direction of an infinite quasi-geodesic. One motivation for the present paper was an attempt to prove Liouville’s theorem for the “quasi-geodesic flow”, see Subsection 3.6 below.

**1.7. Examples.** The estimates of the mm-boundary and mm-curvature are quite involved even in quite simple situations. We collect a few instructive examples. The examples are not needed in the sequel and we omit the somewhat tedious computations.

*Example 1.11.* Let  $X$  be a Lipschitz manifold and  $\mu = \mathcal{H}^n$ . Then  $\lim_{r \rightarrow 0} v_r = 0$ . Moreover,  $\mathcal{H}^n$  is the only measure with this property.

*Example 1.12.* Let  $X$  be a Riemannian manifold with a Lipschitz continuous metric. Then  $X$  has trivial mm-boundary.

*Example 1.13.* If  $X$  is a manifold with two-sided bounded curvature in the sense of Alexandrov then its mm-curvature is well-defined and absolutely continuous with respect to the Hausdorff measure.

*Example 1.14.* Let  $X$  be the flat cone  $CS_\rho$  over a circle of length  $\rho$ . Then  $X$  has finite mm-curvature. The mm-curvature is a Dirac measure of mass  $m(\rho)$  concentrated in the origin. If  $\alpha = 2\pi - \rho$  denotes the defect of  $X$  then the classical curvature measure is the Dirac measure of total mass  $\frac{\alpha}{6}$ . On the other hand  $m(\rho) = \frac{\alpha}{6} + f(\alpha)$ , where  $f(\alpha)$  is a non-zero function which has at the origin the form  $f(\alpha) = O(\alpha^3)$ .

*Example 1.15.* Let  $X$  be a finite  $n$ -dimensional simplicial complex with an intrinsic metric  $d$ . Assume that the restriction of  $d$  to each simplex is given by a (sufficiently) smooth Riemannian metric. Then  $X$  has a finite boundary  $\nu$ , concentrated on the  $(n - 1)$ -skeleton  $X^{n-1}$ .

*Example 1.16.* Assume that  $X$  as in the last example is a pseudo-manifold. Then  $X$  has finite mm-curvature.

**1.8. Structure of the paper.** After preliminaries collected in Section 2, we prove Theorem 1.5 in Section 3 along the lines sketched above. In Sections 4, 5 and 7 we prove the remaining Theorems 1.9, 1.7, 1.6, respectively. The proofs of these theorems all rely on a decomposition of the space into a regular and a singular part, with a quantitative estimate of the size of the singular part. Finally, on the regular part we estimate the mm-curvature and mm-boundary through other canonical measures on these spaces. In the case of surfaces, this comparison measure is the classical curvature measure, in the case of convex hypersurfaces, this comparison measure is the mean curvature. Finally, in the case of a general Alexandrov space, the comparison is given by the derivative of the metric tensor expressed in DC-coordinates, [Per95]. The needed control of the ball growth in terms of these measures is given by a theorem of M. Bonk and U. Lang in the case of surfaces and follows from classical convex geometry in the case of hypersurfaces. The analytical comparison result needed for Alexandrov spaces

is established in Section 6. In the final Section 8 we collect a number of comments and open questions which naturally arose during the work on this paper.

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## 2. PRELIMINARIES

**2.1. Metric spaces.** We refer to [BBI01] for basics on metric spaces. The distance between points  $x, y$  in a metric space  $X$  will be denoted by  $d(x, y)$ . By  $B_r(x)$  we will denote the open metric ball of radius  $r$  around a point  $x$ . For  $A \subset X$  we denote by  $B_r(A)$  the open  $r$ -neighborhood  $B_r(A) = \cup_{x \in A} B_r(x)$ .

A *minimizing geodesic*  $\gamma$  in a metric space  $X$  is a map  $\gamma : \mathbb{I} \rightarrow X$  defined on an interval  $\mathbb{I}$  such that for some number  $\lambda \geq 0$  and all  $t, s \in \mathbb{I}$

$$d(\gamma(t), \gamma(s)) = \lambda \cdot |t - s|.$$

In particular, we allow  $\gamma$  to have any constant velocity  $\lambda \geq 0$ . A *geodesic* is a curve  $\gamma : \mathbb{I} \rightarrow X$  such that its restriction to a small neighborhood of any point in  $\mathbb{I}$  is a minimizing geodesic. Note that a geodesic is a curve of constant velocity.

**2.2. Metric measure spaces.** We refer to [Fed69] and [EG15] for basics on measure theory needed here.

Let  $X$  be a locally compact separable metric space. A *Radon measure* on  $X$  is a measure on  $X$  for which all compact subsets are measurable and have finite measure. Any Radon measure is canonically contained in the dual space  $\mathcal{M}(X)$  of the topological vector space  $\mathcal{C}_c(X)$  of compactly supported continuous functions on  $X$ . All elements in  $\mathcal{M}(X)$  are called *signed Radon measures*. For any signed Radon measure  $\mu$  there exist unique Radon measures  $\mu^\pm$  such that  $\mu = \mu^+ - \mu^-$ . In such a situation, we will denote by  $|\mu|$  the Radon measure  $\mu^+ + \mu^-$  and call it the *total variation* of  $\mu$ .

A family  $\mathcal{F}$  of signed Radon measures on  $X$  is *uniformly bounded* if for any compact subset  $K \subset X$  there exist a constant  $C(K) > 0$  such that  $|\mu|(K) \leq C(K)$  for any  $\mu \in \mathcal{F}$ . Any uniformly bounded sequence of signed measures  $\mu_i$  has a convergent subsequence.

We will repeatedly use the following simple

**Lemma 2.1.** *Let  $X$  be a metric space with two Radon measures  $\mu$  and  $\nu$ . Let  $r > 0$  be arbitrary and let  $A \subset X$  be a Borel subset. Then*

$$\int_A \mu(B_r(x)) \cdot d\nu(x) \leq \int_{B_r(A)} \nu(B_r(x)) \cdot d\mu(x) .$$

*Proof.* Due to Fubini's theorem the left hand side is the volume of

$$S = \{(y, x) \in X \times X \mid y \in A, d(y, x) < r\}$$

with respect to the product measure  $\nu \otimes \mu$ . And on the right hand side of the inequality is the volume of the larger set

$$T = \{(y, x) \in X \times X \mid x \in B_r(A), d(y, x) < r\}$$

with respect to the same measure. □

**2.3. Alexandrov spaces.** We are assuming that the reader is familiar with basic theory of Alexandrov spaces and refer to [BGP92] as a good introduction to the subject. Throughout this paper, an Alexandrov space will denote a complete, locally compact, geodesic metric space of finite Hausdorff dimension and of curvature bounded from below by some  $\kappa \in \mathbb{R}$ . For Alexandrov spaces, an upper index will indicate the Hausdorff dimension; that is,  $X^n$  denotes an  $n$ -dimensional Alexandrov space, equipped with the  $n$ -dimensional Hausdorff measure  $\mathcal{H}^n$ .

The set of starting directions of geodesics starting at a given point  $x \in X$  carries a natural metric, whose completion is the *tangent space*  $T_x = T_x X$  of  $X$  at the point  $x$ . It is again an  $n$ -dimensional Alexandrov space of non-negative curvature. Moreover, it is the Euclidean cone over the space  $\Sigma_x$  of unit directions. The Euclidean cone structure defines multiplications by positive scalars  $\lambda \geq 0$  on  $T_x X$ . The origin of the cone  $T_x X$  is denoted by  $0 = 0_x$ . Elements of  $T_x X$  are called *tangent vectors* at  $x$ . For  $v \in T_x X$  the *norm*  $|v|$  of  $v$  is the distance of  $v$  from the origin  $0_x$ .

Geodesics in  $X$  do not branch and any two geodesics identical starting vectors coincide. For  $x \in X$  the *exponential map*  $\exp_x$  is defined as follows. Let  $D_x$  denote the set of all vectors  $v \in T_x X$  for which there exists an (always unique) *minimizing geodesic*  $\gamma_v : [0, 1] \rightarrow X$  with starting direction  $v$ . The exponential map is defined on  $D_x$  as

$$\exp_x(v) = \gamma_v(1) .$$

For any  $r > 0$ , the map  $\exp_x$  sends  $D_x \cap B_r(0_x) \subset T_x X$  surjectively onto  $B_r(x) \subset X$ . Moreover, for a constant  $C = C(\kappa) \geq 0$  and all  $r < \frac{1}{C}$  the map  $\exp_x : D_x \cap B_r(0_x) \rightarrow B_r(x)$  is  $(1 + C \cdot r^2)$ -Lipschitz continuous.

By the theorem of Bishop-Gromov, the volume  $b_r(x) = \mathcal{H}^n(B_r(x))$  is bounded from above by the corresponding volume in the space of



constant curvature  $\kappa$ . In particular,  $b_r(x) \leq \omega_n \cdot r^n + C \cdot r^{n+2}$  for all  $r \leq \frac{1}{C}$ , where the constant  $C$  can be chosen as before. Thus, the deviation measures  $v_r$  from (1.2) satisfy

$$v_r \geq -Cr^2\mathcal{H}^n,$$

for all sufficiently small  $r$ . Here and in the previous paragraph, one can set  $C = 0$  if  $\kappa \geq 0$ .

Denote by  $X_{reg}$  the set of all points  $x \in X$  with  $T_xX$  isometric to  $\mathbb{R}^n$ . The set  $X_{reg}$  has full  $\mathcal{H}^n$ -measure in  $X$ . Any inner point of any geodesic starting on  $X_{reg}$  is contained in  $X_{reg}$ , [Pet98].

The topological boundary  $\partial X$  of  $X$  can be defined as the closure of the set of all points  $x \in X$  with  $T_xX$  isometric to a Euclidean half-space. Up to a subset of Hausdorff dimension  $n - 2$ ,  $\partial X$  is an  $(n - 1)$ -dimensional Lipschitz manifold.

**2.4. Volume and Bi-lipschitz maps.** It is easy to get an estimate on  $v_r$  on parts which are Lipschitz manifolds and where one can control the biLipschitz constants. Let  $\mu = \mathcal{H}^n$  be a Radon measure on the metric space  $X$ . Let  $U \subset X$  and  $V \subset \mathbb{R}^n$  be open and assume that there is an  $(1 + \delta)$ -biLipschitz map  $f: U \rightarrow V$ . Let  $A \subset U$  be given with  $B_{(1+\delta)r}(A) \subset U$  and  $B_{(1+\delta)r}(f(A)) \subset V$ . Then, for all  $x \in A$ ,

$$(2.1) \quad (1 + \delta)^{-2n} \leq \frac{b_r(x)}{\omega_n \cdot r^n} \leq (1 + \delta)^{2n}.$$

Therefore, if  $\delta$  is sufficiently small,  $|v_r|(A) \leq 3n \cdot \delta \cdot \mathcal{H}^n(A)$ .

### 3. LIOUVILLE MEASURE AND GEODESICS

**3.1. Tangent bundle and Liouville measure.** Let  $X$  be an  $n$ -dimensional Alexandrov space. Denote by  $TX$  the disjoint union of the tangent spaces at different points,

$$TX = \bigsqcup_{x \in X} T_xX.$$

Let  $\pi: TX \rightarrow X$  be the canonical footpoint projection, so  $\pi(T_xX) = \{x\}$  for any  $x \in X$ . For a subset  $K \subset X$  denote by  $TK$  the inverse image  $\pi^{-1}(K) = \cup_{x \in K} T_xX$ . Given  $r > 0$ , denote by  $T^rK$  the set of all vectors in  $TK$  of norm smaller than  $r$ .

The Riemannian structure on the set of regular points discussed in [OS94] (cf. [KMS01], [Per95]) provides  $TX_{reg}$  with a canonical structure of a Euclidean vector bundle over  $X_{reg}$ . In this topology, for any sequence of geodesics  $\gamma_i$  in  $X_{reg}$  converging to a geodesic  $\gamma$ , the starting directions of  $\gamma_i$  converge to the starting direction of  $\gamma$ .

On the Euclidean vector bundle  $TX_{reg}$  over  $X_{reg}$  we have a canonical measure, which locally coincides with the product measure of  $\mathcal{H}_X^n$  and the Lebesgue measures on the fibers. More precisely, it is the unique Borel measure  $\mathcal{M}$  on  $TX_{reg}$  such that for any Borel set  $A \subset TX_{reg}$

$$\mathcal{M}(A) = \int_X \mathcal{H}^n(A \cap T_x X) \cdot d\mathcal{H}^n(x) .$$

We extend  $\mathcal{M}$  to a measure on  $TX$  by setting  $\mathcal{M}(TX \setminus TX_{reg})$  to be 0. By definition, a subset  $A \subset TX$  is  $\mathcal{M}$ -measurable if and only if there exists a Borel subset  $A' \subset A \cap TX_{reg}$  such that for  $\mathcal{H}^n$ -almost all  $x \in X$  the intersection  $(A \setminus A') \cap T_x X$  has  $\mathcal{H}^n$ -measure zero in  $T_x X$ .

For any  $\lambda > 0$  the multiplication by  $\lambda$  on  $T_x X$  sends  $\mathcal{M}$  to  $\lambda^n \cdot \mathcal{M}$ . The canonical involution  $I: TX_{reg} \rightarrow TX_{reg}$ , defined by  $I(v) = -v$ , preserves  $\mathcal{M}$ , since it preserves the Lebesgue measure in each tangent space.

**3.2. Geodesic flow.** In order to define the geodesic flow  $\phi$  on a maximal subset  $\mathcal{F}$  of  $TX \times \mathbb{R}$  we proceed as follows. For any  $v \in T_x X$  we set  $\phi_0(v) = v$ . If no geodesic starts in the direction of  $v$ , the value  $\phi_t(v)$  will not be defined for  $t \neq 0$ . If such a geodesic  $\gamma_v$  exists, then  $\gamma_v$  can be uniquely extended to a maximal possible half-open interval  $\gamma_v: [0, a) \rightarrow X$ . For  $t \geq a$  the value  $\phi_t(v)$  will not be defined. For  $0 < t < a$  we set  $\phi_t(v)$  to be  $\gamma_v^+(t) \in T_{\gamma_v(t)} X$ , the starting direction of  $\gamma_v: [t, a) \rightarrow X$  at  $\gamma_v(t)$ .

Whenever, the geodesic  $\gamma_v: [0, a) \rightarrow X$  extends to an (again uniquely defined, maximal) geodesic  $\gamma_v: (b, a) \rightarrow X$  for some  $b < 0$  then we define  $\phi_t(v)$  for  $b < t < 0$  to be  $\gamma_v^+(t)$  as above.

For  $\lambda > 0$ , for  $t, s \in \mathbb{R}$  and  $v \in T_x X$  we have

$$\phi_t(\lambda \cdot v) := \lambda \cdot \phi_{\lambda \cdot t}(v) \quad \text{and} \quad \phi_{t+s}(v) = \phi_t(\phi_s(v)) ,$$

whenever the right hand side is defined. The partial flow  $\phi$  preserves the norm of tangent vectors. Since inner point of geodesics starting in  $X_{reg}$  are contained in  $X_{reg}$ , the set  $TX_{reg}$  is invariant under the flow  $\phi$ .

By construction, the domain of the definition of the geodesic flow almost includes the domain of the definition of the exponential map. More precisely, consider the set

$$D = \bigcup_{x \in X} D_x \subset TX;$$

that is, the set of all vectors  $v \in TX$  for which  $\exp_{\pi(v)}(v)$  is defined. Note that  $\lambda \cdot D \subset D$  for any  $0 \leq \lambda \leq 1$ . Moreover, for all  $v \in D$  and all  $\lambda < 1$  the geodesic flow  $\phi_1(\lambda v)$  is defined and  $\pi(\phi_1(\lambda v)) = \exp_{\pi(v)}(\lambda v)$ .

Thus, for  $\mathcal{M}$ -almost all  $v \in D$  we have the following

- $v \in TX_{reg}$ ,
- $\phi_1(v) \in TX_{reg}$  is defined,
- $w = -\phi_1(v) \in D$  and

$$(3.1) \quad (\pi(w), \exp(w)) = (\exp(v), \pi(v)) \in X \times X.$$

**3.3. Measurability.** In order to use measure-theoretic arguments, we need to ensure that the partial geodesic flow  $\phi : \mathcal{F} \subset (TX \times \mathbb{R}) \rightarrow TX$  on its domain of definition  $\mathcal{F}$  is measurable.

For any geodesic  $\gamma : \mathbb{I} \rightarrow X$  defined on a compact interval  $\mathbb{I}$  there exists some  $k > 0$  such that the restriction of  $\gamma$  to any subinterval of length  $\frac{1}{k}$  is a (minimizing) geodesic. For bounded length of  $\mathbb{I}$ , bounded velocity of the curves and bounded  $k$  any sequence of such geodesics contained in a relatively compact set contains a subsequence converging to a geodesic. From this observation we can easily write  $\mathcal{F} \cap (TX_{reg} \times \mathbb{R})$  as a countable union of subsets  $\mathcal{F}_k$  closed in  $TX_{reg} \times \mathbb{R}$ , such that the restriction  $\phi : \mathcal{F}_k \rightarrow TX_{reg}$  is continuous.

Therefore, the intersection  $\mathcal{F} \cap (TX_{reg} \times \mathbb{R})$  is a Borel subset and the geodesic flow  $\phi : \mathcal{F} \cap (TX_{reg} \times \mathbb{R}) \rightarrow TX_{reg}$  is a Borel-measurable map, see also Subsection 3.6 below.

**3.4. Liouville property.** Denote by  $\mathcal{G}$  the set of all vectors  $v \in TX$  such that  $\phi_t(v)$  is defined for all  $t \in \mathbb{R}$ . By definition,  $\mathcal{G}$  contains the 0-section, it is invariant under multiplications by any  $\lambda > 0$  and it is invariant under the geodesic flow  $\phi$ . Moreover,  $\mathcal{G} \cap TX_{reg}$  is invariant under the involution  $I(v) = -v$ .

**Definition 3.1.** We say that an Alexandrov space  $X$  has the *Liouville property* if  $TX \setminus \mathcal{G}$  has zero Liouville measure and if for any  $t \in \mathbb{R}$  the geodesic flow  $\phi_t : \mathcal{G} \rightarrow \mathcal{G}$  preserves the Liouville measure.

The Liouville property can be checked infinitesimally as follows.

**Lemma 3.2.** *An Alexandrov space  $X$  does not have the Liouville property if and only if there is a compact subset  $K \subset X$ , a positive number  $\epsilon$  and a sequence of positive numbers  $r_m \rightarrow 0$  with the following property. For every  $m$ , there exists a Borel subset  $A_m \subset T^{r_m}K$  such that*

$$(3.2) \quad \epsilon \cdot r_m^{n+1} \leq \mathcal{M}(A_m) - \mathcal{M}(\phi_1(A_m)) .$$

By an abuse of notations we denote in (3.2) and below by  $\phi_1(A_m)$  the set of all  $\phi_1(v)$ ,  $v \in A_m$ , for which  $\phi_1(v)$  is defined.

*Proof.* If at least one  $r_m$  with the above property exists, then  $X$  does not have the Liouville property by definition.

Assume on the other hand, that  $X$  does not have the Liouville property. Then, by homogeneity of the geodesic flow,  $\phi_1$  is not defined

almost everywhere on  $TX$  or it does not preserve the measure  $\mathcal{M}$ . Thus, we find a compact subset  $K_1 \subset X_{reg}$  and (by homogeneity) a Borel subset  $A \subset T^1 K_1$  such that either  $\phi_1$  is not defined on almost all of  $A$  or  $\mathcal{M}(A) > \mathcal{M}(\phi(A))$ . Replacing  $A$  by a smaller subset, if necessary, we find in both cases some  $\epsilon > 0$ , such that

$$\epsilon < \mathcal{M}(A) - \mathcal{M}(\phi_1(A)) .$$

Since

$$\phi_1(A) = 2 \cdot \phi_2\left(\frac{1}{2} \cdot A\right) = 2 \cdot \left(\phi_1 \circ \phi_1\left(\frac{1}{2}A\right)\right) ,$$

we deduce

$$\begin{aligned} \left(\frac{1}{2}\right)^n \cdot \epsilon &\leq \mathcal{M}\left(\frac{1}{2} \cdot A\right) - \mathcal{M}\left(\phi_1\left(\frac{1}{2} \cdot A\right)\right) + \\ &\quad + \mathcal{M}\left(\phi_1\left(\frac{1}{2} \cdot A\right)\right) - \mathcal{M}\left(\phi_1\left(\phi_1\left(\frac{1}{2} \cdot A\right)\right)\right) . \end{aligned}$$

Thus, taking either  $A_{\frac{1}{2}} := \frac{1}{2} \cdot A$  or  $A_{\frac{1}{2}} := \phi_1\left(\frac{1}{2} \cdot A\right)$  we infer

$$\epsilon \cdot \left(\frac{1}{2}\right)^{n+1} < \mathcal{M}(A_{\frac{1}{2}}) - \mathcal{M}(\phi_1(A_{\frac{1}{2}})) .$$

The set  $A_{\frac{1}{2}}$  constructed above is contained in  $T^{\frac{1}{2}}K_{\frac{1}{2}}$ , where  $K_{\frac{1}{2}} = B_{\frac{1}{2}}(K_1)$ .

Iterating the above procedure we obtain, for  $r_m = \frac{1}{2^m}$ , a subset  $A_{r_m} \subset T^{r_m}K_m$  where  $K_m = B_{r_m}(K_{m-1})$  and such that (3.2) holds true. The claim follows now since all  $K_m$  are contained in the set  $B_1(K_1)$ , whose closure is compact, by completeness of  $X$ .  $\square$

*Remark 3.3.* The completeness of the space  $X$  is used in the proof of Theorem 1.5 only once, namely in the last line of the above proof.

**3.5. Relation with the mm-boundary.** Before proceeding to the proof of Theorem 1.5, we interpret the deviation measures  $v_r$  from (1.2) in suitable geometric terms. Let  $K \subset X$  be measurable and let  $r > 0$  be arbitrary. Since  $\mathcal{H}^n(X \setminus X_{reg}) = 0$ , we have

$$\mathcal{M}(T^r K) = \int_K \omega_n \cdot r^n = \omega_n \cdot r^n \cdot \mathcal{H}^n(K) .$$

Denote now by  $U^r(K)$  the set of all pairs  $(x, y) \in X \times X$  with  $x \in K$  and  $d(x, y) < r$ . By Fubini's theorem the set  $U^r(K)$  is  $\mathcal{H}^n \otimes \mathcal{H}^n = \mathcal{H}^{2n}$  measurable and we have

$$\mathcal{H}^{2n}(U^r(K)) = \int_K b_r(x) \cdot d\mathcal{H}^n(x) .$$

Taking both equations together, we see that the signed measure  $v_r$  expresses the difference between  $\mathcal{H}^{2n}$  and  $\mathcal{M}$ . More precisely,

$$(3.3) \quad v_r(K) = \frac{1}{\omega_n \cdot r^n} \left( \mathcal{M}(T^r K) - \mathcal{H}^{2n}(U^r(K)) \right).$$

Now we can prove the following result, reformulating Theorem 1.5.

**THEOREM 3.4.** *If an Alexandrov space  $X$  has trivial mm-boundary then it has the Liouville property.*

*Proof.* Assume that  $X$  does not have the Liouville property and consider the compact subset  $K \subset X$ , the positive numbers  $\epsilon, r_m$  and the Borel subsets  $A_m \subset T^{r_m} K$  provided by Lemma 3.2 above.

Let  $Y$  be the closure of  $B_1(K)$ . Let again  $D \subset TX$  be the set of all vectors at which the exponential map is defined. For  $r > 0$ , denote by  $N^r$  the intersection of  $D$  with  $T^r Y$  and consider the "total exponential map"  $E: N^r \rightarrow X \times X$  given by

$$E(v) = (\pi(v), \pi(\exp(v))).$$

As above, let  $U^r = U^r(Y)$  be the set of all pairs  $(y, x) \in X \times X$  with  $y \in Y$  and  $d(x, y) < r$ . Then

$$(3.4) \quad E(N^r) = U^r.$$

Moreover, for any fixed  $x \in Y$ , the restriction of  $E$  to  $D_x \cap N^r$  is a  $(1 + Cr^2)$ -Lipschitz continuous map from  $D_x \subset T_x X$  onto the set  $U^r \cap (\{x\} \times X)$ . Thus, for all sufficiently small  $r$ , and any Borel subset  $S \subset D_x \cap N^r$ , we have

$$\mathcal{H}^n(E(S)) \leq (1 + 4nCr^2) \cdot \mathcal{H}^n(S).$$

Using the definition of the measure  $\mathcal{M}$  and Fubini's formula for the product measure  $\mathcal{H}^{2n} = \mathcal{H}^n \otimes \mathcal{H}^n$  on  $X \times X$  we obtain for any  $\mathcal{M}$ -measurable subset  $S$  of  $N^r$

$$(3.5) \quad \mathcal{H}^{2n}(E(S)) \leq (1 + 4nCr^2) \cdot \mathcal{M}(S).$$

Due to (3.3), the triviality of the mm-boundary of  $X$  implies

$$\lim_{r \rightarrow 0} \frac{1}{r^{n+1}} |\mathcal{M}(T^{<r} Y) - \mathcal{H}^{2n}(U^r)| = 0.$$

Thus, up to terms of higher order, the map  $E$  does not increase the measure of subsets, but the total mass of the image coincide with the total mass of the target. Therefore,  $E$  is measure preserving up to terms of higher order. More precisely, combining the last two inequalities we obtain for every  $\delta > 0$  the existence of some  $s > 0$  with the following property. For all  $0 < r < s$  and all measurable subsets  $S \subset N^r$  we have

$$(3.6) \quad \mathcal{M}(T^r Y) - \mathcal{M}(N^r) < \delta r^{n+1} \text{ and } |\mathcal{H}^{2n}(E(S)) - \mathcal{M}(S)| < \delta r^{n+1}.$$

For any measurable subset  $S \subset N^r \cap TK$  we now claim

$$(3.7) \quad |\mathcal{H}^{2n}(E(S)) - \mathcal{M}(\phi_1(S))| < 2\delta r^{n+1} .$$

In order to prove (3.7), let  $S^+$  be the subset of all vectors  $v \in S$  for which  $\phi_1(v)$  exists and is contained in  $TX_{reg}$ . For all  $v \in S^+$ , we have  $-\phi_1(v) \in N^r$  and, due to (3.1),

$$E(-\phi_1(v)) = J(E(v)) .$$

The involution  $I(v) = -v$  is  $\mathcal{M}$ -preserving on  $TX_{reg}$ . And the involution  $J: X \times X \rightarrow X \times X$  given by  $J(x, y) = (y, x)$  preserves  $\mathcal{H}^{2n}$ . Therefore, from (3.6) we deduce

$$(3.8) \quad |\mathcal{H}^{2n}(E(S^+)) - \mathcal{M}(\phi_1(S^+))| < \delta r^{n+1} .$$

On the other hand, by construction,

$$\mathcal{M}(S \setminus S^+) = 0 \text{ and } \phi_1(S \setminus S^+) \cap TX_{reg} = \emptyset .$$

Hence, applying (3.6), we see

$$|\mathcal{H}^{2n}(E(S)) - \mathcal{H}^{2n}(E(S^+))| < \delta r^{n+1} \text{ and } \mathcal{M}(\phi_1(S \setminus S^+)) = 0 .$$

Together with (3.8) this finishes the proof of (3.7).

Coming back to our subsets  $A_m \subset T^{<r_m}K$  we have

$$\epsilon \cdot r_m^{n+1} \leq \mathcal{M}(A_m) - \mathcal{M}(\phi_1(A_m)) \leq \mathcal{M}(A_m) - \mathcal{M}(\phi_1(A_m \cap N^r)) .$$

Setting  $S_m = A_m \cap N^{r_m}$  we estimate the right hand side from above by

$$|\mathcal{M}(A_m) - \mathcal{M}(S_m)| + |\mathcal{M}(S_m) - \mathcal{H}^{2n}(E(S_m))| + |\mathcal{H}^{2n}(E(S_m)) - \mathcal{M}(\phi_1(S_m))| .$$

Applying (3.6) and (3.7) this expression is bounded above by  $4\delta r_m^{n+1}$ , for all  $m$  large enough. Taking the inequalities together we obtain,  $\epsilon \cdot r_m^{n+1} < 4\delta r_m^{n+1}$  for all  $m$  large enough. Since  $\delta$  is arbitrary small, this provides a contradiction and finishes the proof of Theorem 3.4.  $\square$

**3.6. Quasi-geodesics.** Finally, we discuss some relations with quasi-geodesics, referring the reader to [Pet07] for the basic properties of such curves. Recall, that whenever a unit speed minimizing geodesic  $\gamma_v : [0, a] \rightarrow X$  start at a point  $x$  in the direction  $v$  then this is the unique quasi-geodesic defined on the interval  $[0, a]$ , [PP96], p.8, thus the same statement is also true for (local) geodesics  $\gamma_v$ .

Using this and the fact that a limit of quasi-geodesics is a quasi-geodesic, it is not difficult to conclude that the partial geodesic flow  $\phi : \mathcal{F} \cap TX_{reg} \rightarrow TX_{reg}$  defined above is continuous, slightly strengthening Subsection 3.3.

As in Subsection 3.1, we have a canonical measure  $\mathcal{M}_1$  on the unit tangent bundle  $\Sigma X \subset TX$  of  $X$ , which we also call the Liouville measure. Whenever  $X$  has the Liouville property, then the geodesic flow

is defined  $\mathcal{M}_1 \otimes \mathcal{H}^1$ -almost everywhere on  $\Sigma X \times \mathbb{R}$  and preserves  $\mathcal{M}_1$ . In this case for  $\mathcal{M}_1$ -almost each unit direction there exists exactly one quasi-geodesic starting in this direction.

Let now  $X$  be an Alexandrov space with topological boundary  $\partial X$  and let  $Z$  be the doubling  $X \cup_{\partial X} X$ , which is an Alexandrov space without boundary, [Per91]. Quasi-geodesics in  $X$  are exactly the projections of the quasi-geodesics in  $Z$  under the canonical folding  $f : Z \rightarrow X$ . From this we deduce that if  $Z$  has the Liouville property, then  $\mathcal{M}_1$ -almost each direction  $v \in \Sigma X$  is the starting direction of a unique infinite quasi-geodesic in  $X$ . Moreover, in this case, the corresponding quasi-geodesic flow preserves  $\mathcal{M}_1$ .

Finally, as an application of Theorem 1.5 and Theorem 1.6 we see that the above assumptions are fulfilled whenever the complement  $X \setminus \partial X$  has trivial mm-boundary. Indeed, in this case the mm-boundary of  $Z$  must be concentrated on  $\partial X \subset Z$ , hence it must be trivial by Theorem 1.6,(3).

#### 4. SURFACES WITH BOUNDED INTEGRAL CURVATURE IN THE SENSE OF ALEXANDROV

**4.1. Preparations.** We assume here some familiarity with the theory of surfaces with bounded integral curvature, cf. [AZ67] and [Res93]. Such a surface  $X$  is a locally geodesic metric space, homeomorphic to a two-dimensional surface. It has Hausdorff dimension 2 and the Hausdorff measure  $\mathcal{H}^2$  is a Radon measure on  $X$ . There is another canonical signed Radon measure on  $X$ , the so called curvature measure  $\Omega$ , [Res93, Section 8]. Unlike when talking about Alexandrov spaces *we will not assume that  $X$  is complete*. We will derive Theorem 1.9 as a consequence of the following local and non-exact version of a theorem of M. Bonk and U.Lang, [BL03], relating the curvature measure  $\Omega$  to the volume of balls.

**Lemma 4.1.** *There exists some  $\delta_0 > 0$  with the following property. Let  $X$  be a surface with bounded integral curvature and let  $\Omega \in \mathcal{M}(X)$  be its curvature measure. Let  $x \in X$  be a point and let  $r > 0$  be such that  $\bar{B}_r(x)$  is compact,  $|\Omega|(B_r(x)) < \delta_0$  and  $\bar{B}_r(x) \subset U$  where  $U$  is an open subset of  $X$  homeomorphic to  $\mathbb{R}^2$ . Then*

$$\left| 1 - \frac{b_r(x)}{\pi r^2} \right| \leq 3 \cdot |\Omega|(B_r(x)) .$$

*Proof.* Set  $\delta = |\Omega|(B_r(x))$ . By continuity, it is sufficient to prove that  $\left| 1 - \frac{b_s(x)}{\pi s^2} \right| \leq 3 \cdot \delta$  for any  $s < r$ . Using approximations of the metric on  $U$  by polyhedral metrics [Res93, Theorem 8.4.3, Theorem 8.1.9], it

is sufficient to prove this statement for polyhedral metrics. Thus we assume from now on that  $X$  is polyhedral and homeomorphic to  $\mathbb{R}^2$ .

*Claim:* There exists a complete polyhedral surface  $\hat{X}$  homeomorphic to a plane, which contains a copy of  $B_s(x)$  and such that the curvature measure  $\hat{\Omega}$  of  $\hat{X}$  satisfies  $|\hat{\Omega}|(\hat{X}) < 3\delta$ . Once the claim is proven and  $3\delta < 2\pi$ , [BL03] provides us an  $L$ -biLipschitz map  $f : \hat{X} \rightarrow \mathbb{R}^2$  with  $L \leq 1 + \frac{3\delta}{2\pi - 3\delta}$ . For sufficiently small  $\delta$ , we would only need to apply (2.1) to finish the proof of the lemma.

It remains to prove the existence of  $\hat{X}$ . We take some  $r > t > s$  and consider the compact metric ball  $\bar{B}_t(x) \subset B_r(x)$ . We may assume that the boundary  $S_t = S_t(x)$  of  $\bar{B}_t(x)$  does not contain singular points of  $X$ . Due to [Res93, Theorem 9.1, Theorem 9.3], the boundary  $S_t$  is a (piecewise smooth) Jordan curve, once  $\delta_0 < 2\pi$ , and the negative part  $\kappa^-$  of the geodesic curvature  $\kappa$  of  $S_t$  satisfies  $|\kappa^-|(S_t) \leq \delta$ . Since  $U$  is homeomorphic to a plane this implies that  $\bar{B}_t(x)$  is homeomorphic to a closed disk  $\bar{D}^2$  in  $\mathbb{R}^2$ .

We find a polygonal Jordan curve  $\Gamma$  in  $B_t(x)$  approximating  $S_t$  such that the negative part of the geodesic curvature of  $\Gamma$  is smaller than  $2\delta$ . Consider the closed Jordan domain  $Y$  bounded by  $\Gamma$ , which can be assumed to contain  $B_s(x)$ . Now we glue to  $Y$  along any edge of  $\Gamma$  a flat half-strip. The boundary of the arising polyhedral surface consists of pairs of rays  $\gamma_i^\pm$  emanating from the vertices  $V_1, \dots, V_k$  of  $\Gamma$ . The rays  $\gamma_i^\pm$  enclose an angle equal to  $2\pi - \alpha_i$ , where  $\pi - \alpha_i$  is the angle of  $\Gamma$  at  $V_i$  measured in  $Y$ . In order to finish the construction of  $\hat{X}$  we glue a flat sector of angle  $\alpha_i$  between  $\gamma_i^\pm$ , if  $\alpha_i > 0$  and we glue  $\gamma_i^\pm$  together if  $\alpha_i \leq 0$ . Since  $Y$  was a polyhedral disc, the arising space  $\hat{X}$  is a complete polyhedral plane. All of the singularities of  $\hat{X}$  are contained in  $B_s(x) \cup \{V_1, \dots, V_k\}$ . Moreover, by construction, the curvature measure  $\hat{\Omega}$  of  $\hat{X}$  satisfies

$$\hat{\Omega}(V_i) = \min\{0, \alpha_i\} .$$

We deduce,

$$|\hat{\Omega}|(\hat{X}) = |\Omega(B_s(x))| + |\hat{\Omega}|(\Gamma) \leq \delta + |\kappa^-|(\Gamma) < 3\delta .$$

This finishes the proof of the claim and of Lemma 4.1.  $\square$

**4.2. Local finiteness of mm-curvature.** Now we are in position to prove the following generalization of Theorem 1.9.

**THEOREM 4.2.** *Let  $X$  be an Alexandrov surface with integral curvature bounds. Then, equipped with the Hausdorff measure  $\mathcal{H}^2$ , the space  $X$  has locally finite mm-curvature.*



*Proof.* Let again  $\Omega$  denote the curvature measure of  $X$ . Let  $\delta_0 > 0$  be sufficiently small and satisfy the conclusion of Lemma 4.1. The statement of Theorem 4.2 is local, so we need to prove it only in a small neighborhood of any point. Thus we may and will assume that there is a point  $x_0 \in X$  such that  $|\Omega|(X \setminus \{x_0\}) < \delta_0$  and that  $X$  is homeomorphic to a plane.

Let  $A \subset X$  be a compact subset. Choose some  $\epsilon > 0$  such that the closure of  $B_{2\epsilon}(A)$  in  $X$  is compact and such that, for any  $0 < 2r < \epsilon$  the inequality  $\mathcal{H}^2(B_{2r}(x_0)) < \frac{1}{\epsilon}r^2$  holds true, cf. [Res93, Lemma 8.1.1].

Let  $r < \epsilon$  be arbitrary. For any  $x \in B_{2r}(x_0)$  we have

$$b_r(x) = \mathcal{H}^2(B_r(x)) \leq \mathcal{H}^2(B_{3r}(x_0)) \leq \frac{1}{\epsilon}r^2 .$$

For any  $x \notin B_r(x_0)$  we have  $|\Omega|(B_r(x)) < \delta_0$ . Thus, by Lemma 4.1,

$$\left|1 - \frac{b_r(x)}{\pi r^2}\right| \leq 3|\Omega|(B_r(x)) .$$

For the deviation measures  $v_r$  from (1.2) we estimate:

$$|v_r|(A \cap B_{2r}(x_0)) \leq |v_r|(B_{2r}(x_0)) \leq \left(1 + \frac{1}{\epsilon}\right) \cdot \mathcal{H}^2(B_{2r}(x_0)) \leq \left(1 + \frac{1}{\epsilon}\right) \cdot \frac{1}{\epsilon}r^2 .$$

On the other hand,

$$\begin{aligned} |v_r|(A \setminus B_{2r}(x_0)) &\leq \int_{A \setminus B_{2r}(x_0)} 3 \cdot |\Omega|(B_r(x)) \cdot d\mathcal{H}^2(x) \leq \\ &\leq 3 \int_{B_r(A \setminus B_{2r}(x_0))} \mathcal{H}^2(B_r(x)) \cdot d|\Omega|(x) , \end{aligned}$$

where we have used Lemma 2.1 in the last step. For any  $x$  contained in the domain of integration of the last integral, we have  $\mathcal{H}^2(B_r(x)) = b_r(x) \leq 2\pi r^2$ , by Lemma 4.1, once  $\delta_0$  has been chosen to be sufficiently small. We deduce  $|v_r|(A \setminus B_{2r}(x_0)) \leq 6\pi\delta_0 \cdot r^2$ .

Thus, for some constant  $C = C(\epsilon)$  and all  $r < \epsilon$ , we obtain

$$|v_r|(A) \leq C \cdot r^2 .$$

This finishes the proof of the theorem.  $\square$

## 5. CONVEX HYPERSURFACE

In this section we are going to prove Theorem 1.1. The proof will follow from Theorem 1.5 by comparing the mm-boundary with the mean curvature measure on convex hypersurfaces. It is possible to deduce the theorem without a reference to Theorem 1.6, from Lemma 5.2 alone, but using our main technical result Theorem 1.6 shortens the proof.

All results in this section are local. For simplicity, we nevertheless assume all convex hypersurfaces to be closed in  $\mathbb{R}^{n+1}$ . They will always be equipped with the induced intrinsic metric, not with the restricted distance function from  $\mathbb{R}^{n+1}$ . We assume in here some familiarity with the basics of the theory of convex functions and convex geometry.

**5.1. Mean curvature.** Let  $X^n$  be a convex hypersurface in  $\mathbb{R}^{n+1}$ . Recall that there exists a Radon measure  $K$  on  $X$ , called the mean curvature measure, which has the following properties, cf. [Sch93], [Fed59]. For smooth hypersurfaces  $X$ , we have  $K = \kappa \cdot \mathcal{H}^n$ , where  $\kappa$  is the usual mean curvature function of  $X$ . The mean curvature measure is stable under Hausdorff convergence of convex hypersurfaces in  $\mathbb{R}^{n+1}$ . If the hypersurface is rescaled by  $\lambda$ , the mean curvature  $K$  is rescaled by  $\lambda^{n-1}$ .

We call a point  $x$  in the convex hypersurface  $X$  a smooth point if there is a unique supporting hyperplane of  $X$  at this point. For any smooth point  $x \in X$ , any sequence  $x_j \in X$  converging to  $x$  and any sequence of positive numbers  $t_j$  converging to 0, the sequence of convex hypersurfaces  $X_j$  obtained from  $X$  by the dilatation by the factor  $\frac{1}{t_j}$  centered at the point  $x_j$  converges to the tangent hyperplane of  $X$  at  $x$ . Using stability of the mean curvature measures  $K$ , vanishing of  $K$  on flat hyperplanes and the behavior of  $K$  under rescalings gives us the following conclusion.

**Lemma 5.1.** *Let  $X$  be a convex hypersurface in  $\mathbb{R}^{n+1}$ . Let  $A$  be a compact set of smooth points in  $X$  and let  $\delta > 0$  be arbitrary. Then there exists some  $t > 0$  such that for any  $y \in B_t(A)$  and any  $0 < r < t$  we have  $K(B_r(y)) \leq \delta \cdot r^{n-1}$ .*

Thus, the following lemma applies to all small balls in a neighborhood of any smooth point.

**Lemma 5.2.** *There exist numbers  $\delta_0, C > 0$  depending only on  $n$  with the following property. Let  $X$  be a convex hypersurface in  $\mathbb{R}^{n+1}$ . Let  $x \in X$  be a point and  $r > 0$  be such that the mean curvature  $K$  satisfies  $K(B_{6r}(x)) < \delta \cdot r^{n-1}$  with  $\delta < \delta_0$ . Then*

$$(5.1) \quad \left| 1 - \frac{b_r(x)}{\omega_n r^n} \right| < C \cdot \delta \cdot K(B_{6r}(x)) \cdot r^{1-n} .$$

*Proof.* By rescaling, it suffices to prove the existence of  $\delta_0, C > 0$  such that the lemma holds true for  $r = 1$ . By approximation, it is enough to prove the result for smooth convex hypersurfaces.

We fix a sufficiently small  $\epsilon_0 > 0$ . The mean curvature vanishes on  $B_6(x)$  if and only if  $B_6(x)$  is contained in a flat hyperplane. Due to the

stability of  $K$  under convergence, the ball  $U = B_5(x) \subset X$  must be arbitrary close to a flat hyperplane in  $\mathbb{R}^{n+1}$ , once  $\delta_0$  has been chosen small enough. Thus, we may assume that the tangent hyperplanes to points in  $U$  are  $\epsilon_0$ -close to the tangent space  $W = T_x X \subset \mathbb{R}^{n+1}$ . Therefore,  $U$  is a graph  $U = \{(x, f(x))\}$  of a convex function  $f : V \rightarrow \mathbb{R}$  defined on an open subset  $V \subset W$ . Moreover,  $V$  contains the ball of radius 4 in  $W$  around  $x$ . Let  $O$  be the ball of radius 2 in  $W$  around  $x$ . Set

$$a := \sup\{|\nabla f(y)| ; y \in O\}.$$

If  $\delta_0$  is small enough, then  $a < \epsilon_0$ . The orthogonal projection  $P : U \rightarrow V$  is 1-Lipschitz and the restriction of the inverse  $P^{-1}$  to  $O$  has Lipschitz constant

$$\sqrt{1 + a^2} \leq 1 + a^2 \leq 1 + \epsilon_0^2.$$

Applying (2.1) we only need to prove that  $a < C \cdot \delta$  for a constant  $C$ .

Denote by  $|D^2 f|$  the largest eigenvalue of the Hessian  $D^2 f$ . Since  $f$  is convex and  $\epsilon_0$  small enough, the mean curvature  $\kappa(x)$  at the point  $(x, f(x))$  of the graph  $U$  of  $f$  satisfies  $\kappa(x) \geq \frac{n}{2} \cdot |D^2 f|$ . Hence, the conclusion follows from the following statement.

*Claim:* Let  $f : B \rightarrow \mathbb{R}$  be a smooth convex function on the open ball  $B = B_4(0) \subset \mathbb{R}^n$ . If  $f(0) = |\nabla f(0)| = 0$  then, for some  $C = C(n) > 0$ ,

$$\sup_{y \in B_2(0)} |\nabla f(y)| \leq C \cdot \int_B |D^2 f|.$$

By convexity, it is enough to find some  $C = C(n) > 0$  with

$$(5.2) \quad \sup_{y \in B_3(0)} |f(y)| \leq C \cdot \int_B |D^2 f|,$$

cf. [EG15, Theorem 6.7].

In order to verify (5.2), we can multiply the function  $f$  by a constant and assume that  $f$  takes its maximum on the closed ball  $\bar{B}_3(0)$  at the point  $y_0$  and  $f(y_0) = 1$ . Convexity of  $f$  implies that  $|y_0| = 3$ . Since  $f(0) = 0$  and  $f$  is convex, we must have  $f(y) \leq \frac{1}{3}$  for all  $y \in B_1(0)$ .

By convexity and the choice of  $y_0$ , the restriction of  $f$  to the supporting hyperplane  $H$  of  $\bar{B}_3(0)$  at  $y_0$  is bounded from below by 1. Consider the ball  $S$  of radius  $\frac{1}{2}$  in  $H$  around  $y_0$ . For any point  $z \in S$  consider the restriction

$$f_z(t) = f(z - \frac{t}{3}y_0), t \in [0, 6]$$

to the segment of length 6 starting at  $z$  orthogonal to  $H$ . Then

$$f_z(0) \geq 1 ; f_z(3) \leq \frac{1}{3} ; f_z(6) \geq 0.$$

Thus for some  $t \in (0, 3)$  we have  $f'_z(t) \leq -\frac{2}{9}$  and for some  $t \in (3, 6)$  we have  $f'_z(t) \geq -\frac{1}{9}$ . Therefore

$$\int_0^6 f''_z(t) dt \geq \frac{1}{9}.$$

Integrating over  $S$  we obtain by Fubini's theorem a uniform positive lower bound on  $\int_B |D^2 f|$ . This finishes the proof of (5.2), of the claim and of Lemma.  $\square$

**5.2. The proof.** Now we can provide the proof of

**THEOREM 5.3.** *Let  $X = X^n$  be a convex hypersurface in  $\mathbb{R}^{n+1}$ . Then it has trivial mm-boundary.*

Together with Theorem 1.9 this finishes the proof of Theorem 1.7. In combination with Theorem 1.5 it also finishes the proof of Theorem 1.1.

*Proof.* Since  $X$  has locally finite mm-boundary by Theorem 1.6, it suffices to prove that any limit measure  $\nu$  of a sequence  $\frac{1}{r_j} \nu_{r_j}$  for  $r_j \rightarrow 0$  must be the zero measure. Let us fix such a limit measure  $\nu$ . Due to Theorem 1.6,  $\nu(A) = 0$  for any Borel subset  $A \subset X$  with  $\mathcal{H}^{n-1}(A) < \infty$ . Let  $Y \subset X$  be the set of smooth points of  $X$ . The complement  $X \setminus Y$  is a countable union of subsets with finite  $(n-1)$ -dimensional Hausdorff measure, [Sch93, Theorem 1.4], therefore  $\nu(X \setminus Y) = 0$ . Therefore, it is sufficient to prove  $\nu(A) = 0$  for any compact subset  $A \subset Y$ .

Let  $A$  be such a compact subset and let  $\delta > 0$  be an arbitrary sufficiently small number. Consider a positive  $1 > t > 0$  provided by Lemma 5.1. Let  $U$  be the open set  $B_t(A)$ . Making  $t$  smaller, if needed, we may assume that  $K(B_{7t}(A))$  is bounded by a fixed constant  $C_1$  independent of  $\delta$ .

Let  $0 < r < t$  be arbitrary. Then, by Lemma 5.2, for any  $x \in U$  we have (5.1). Therefore,

$$\begin{aligned} |\nu_r|(U) &\leq \int_U C \cdot \delta \cdot r^{1-n} \cdot K(B_{6r}(y)) \cdot d\mathcal{H}^n(y) \leq \\ &\leq C \cdot \delta \cdot r^{1-n} \cdot \int_{B_{7t}(A)} \mathcal{H}^n(B_{6r}(y)) \cdot dK(y) \leq C \cdot \delta \cdot r^{1-n} \cdot (6r)^n \cdot K(B_{7t}(A)), \end{aligned}$$

where we have used Lemma 2.1 in the second and the theorem of Bishop-Gromov in the last inequality. Hence

$$|\nu|(A) \leq |\nu|(U) \leq C \cdot \delta \cdot 6^n \cdot K(B_7(A)).$$

Since  $\delta$  can be chosen arbitrary small, we obtain  $|\nu|(A) = 0$ .

This finishes the proof of the claim and, therefore, of Theorem 5.3.  $\square$

## 6. AN INTEGRAL INEQUALITY FOR RIEMANNIAN METRICS

**6.1. The smooth case.** We start by estimating from above the deviation measure  $v_r$  on a smooth Riemannian manifold in terms of the first derivatives of the metric. We do not know how to prove a similar estimate from below, see Problem 8.3. However, for the applications to Alexandrov spaces discussed in the next section, the estimate from below is a consequence of the theorem of Bishop-Gromov.

For a smooth Riemannian metric  $g$  defined on an open subset  $U \subset \mathbb{R}^n$  we denote by  $|g'| : U \rightarrow [0, \infty)$  the sum  $\sum_{i,j,k} \left| \frac{\partial}{\partial x_k} g_{ij} \right|$ .

**PROPOSITION 6.1.** *There exists a constant  $C = C(n) > 1$  with the following property. Let  $U \subset \mathbb{R}^n$  be an open subset with a smooth Riemannian metric  $g$  which is  $(1 + \frac{1}{C})$ -biLipschitz to the background Euclidean metric. Let  $A \subset U$  be a Borel subset. Let  $r > 0$  be such that  $B_{2r}(A)$  is relatively compact in  $U$ . Then*

$$v_r(A) \leq C \cdot r \cdot \int_{B_{2r}(A)} |g'|.$$

*Proof.* Throughout the proof we will denote by  $C_i$  various (explicit) constants which depend only on  $n$ .

We will use the following notations. By  $|\cdot|$  and  $\mathcal{L}^n$  we denote respectively the norm and the Lebesgue measure on the background  $\mathbb{R}^n$ . For  $x \in U$  we denote by  $g_x$  the Riemannian tensor at the point  $x$  and by  $|\cdot|_x$  the corresponding norm. The Hausdorff measure of the Riemannian metric  $g$  has the form  $u \cdot \mathcal{L}^n$ , with  $u = \sqrt{\det(g_{ij})}$ .

For  $x \in \mathbb{R}^n$ , we consider the function  $K : U \rightarrow [0, \infty)$  given by

$$K(x) = \sup_{|v|_x=1} \left. \frac{d}{dt} \right|_{t=0} |v|_{x+tv}.$$

By smoothness of the determinant and the square root, we find a constant  $C_1$  such that for all  $x \in U$  we have

$$(6.1) \quad |u'(x)| \leq C_1 \cdot |g'(x)| \quad \text{and} \quad K(x) \leq C_1 \cdot |g'(x)|.$$

We fix  $A \subset X$  and  $r > 0$  as in the formulation of the proposition. For  $x \in U$  denote by  $B_x$  the metric ball  $B_r(x)$  in  $U$ . By  $B^x$  we denote the metric ball of radius  $r$  in the Euclidean norm  $|\cdot|_x$ . In this Euclidean metric the ball  $B^x$  has measure

$$\omega_n r^n = u(x) \cdot \int_{B^x} d\mathcal{L}^n.$$

Thus, in order to estimate the deviation measure  $v_r$ , we only need to control the summands on the right of the following inequality:

$$(6.2) \quad \omega_n \cdot r^n - b_r(x) \leq u(x) \cdot \mathcal{L}^n(B^x \setminus B_x) + \int_{B_x} |u(x) - u(y)| \cdot d\mathcal{L}^n(y) .$$

We may assume that the biLipschitz constant  $1 + \frac{1}{C}$  is close enough to 1, so that  $\frac{1}{2} < u < 2$ . Moreover, we may assume  $B_x$  and  $B^x$  are contained in the ball of radius  $\frac{4}{3}r$  around  $x$  with respect to the background Euclidean metric.

In order to bound the first summand, for  $x \in A$  and  $|v|_x = 1$ , we set  $l_v^x$  to be the length of the segment  $[x, x + v]$  in the Riemannian metric  $g$ . Then we compute

$$\begin{aligned} l_v^x - r &= \int_0^r |v|_{x+tv} dt - \int_0^r |v|_x dt \leq \int_0^r \left( \int_0^t K(x + sv) ds \right) dt \leq \\ &\leq \int_0^r \left( \int_0^r K(x + sv) ds \right) dt = r \cdot \int_0^r K(x + sv) ds . \end{aligned}$$

Observe now that the intersection of  $B^x \setminus B_x$  with the ray starting in  $x$  in the direction of  $v$  has  $\mathcal{H}^1$ -measure (with respect to the norm  $|\cdot|_x$ ) at most  $2 \cdot (l_v^x - r)$ , once the biLipschitz constant  $(1 + \frac{1}{C})$  is close enough to 1. Integrating in polar coordinates over the ball  $B^x \subset (\mathbb{R}^n, |\cdot|_x)$  we infer:

$$\begin{aligned} u(x) \cdot \mathcal{L}^n(B^x \setminus B_x) &\leq r^{n-1} \cdot \left( \int_{v \in S_x^{n-1}} (2 \cdot r \cdot \int_0^r K(x + sv) ds) \cdot d\mathcal{H}^{n-1} \right) = \\ &= 2r^n \cdot \int_{B^x} K(y) \cdot |y - x|^{n-1} \cdot u(x) \cdot d\mathcal{L}^n , \end{aligned}$$

where  $S_x^{n-1}$  is the unit sphere in  $(\mathbb{R}^n, |\cdot|_x)$ .

To get a similar estimate of the other summand in (6.2), we only need to recall the following inequality from [EG15, Lemma 4.1], valid for any  $\mathcal{C}^1$  function  $u$  on a Euclidean ball

$$\int_{|x-y|<r} |u(y) - u(x)| \cdot d\mathcal{L}^n(y) \leq C_2 \cdot r^n \cdot \int_{|x-y|<r} |u'(y)| \cdot |y - x|^{1-n} \cdot d\mathcal{L}^n(y) .$$

Taking both estimates together with (6.1), embedding  $B_x$  and  $B^x$  in slightly larger Euclidean balls and using that  $\frac{1}{2} < u < 2$ , we conclude:

$$\omega_n \cdot r^n - b_r(x) \leq C_3 \cdot r^n \cdot \int_{|x-y|<\frac{4}{3}r} |g'(y)| \cdot |y - x|^{1-n} \cdot d\mathcal{L}^n .$$

We divide both sides by  $\omega_n \cdot r^n$  and integrate over  $A$ . Using that the biLipschitz constant is close enough to 1, we see:

$$\begin{aligned} v_r(A) &\leq C_4 \int_A \left( \int_{|x-y| < \frac{4}{3}r} |g'(y)| \cdot |y-x|^{1-n} \cdot d\mathcal{L}^n(y) \right) \cdot d\mathcal{L}^n(x) \leq \\ &\leq C_4 \int_{B_{2r}(A)} \left( \int_{|x-y| < \frac{4}{3}r} |g'(y)| \cdot |y-x|^{1-n} \cdot d\mathcal{L}^n(x) \right) \cdot d\mathcal{L}^n(y) = \\ &= \frac{4}{3} C_4 \int_{B_{2r}(A)} |g'(y)| \cdot r \cdot d\mathcal{L}^n(y), \end{aligned}$$

where we have used Lemma 2.1 in the second inequality. This finishes the proof of Proposition 6.1.  $\square$

**6.2. Functions of bounded variations.** Let  $U$  be an open subset of  $\mathbb{R}^n$ . A function  $f \in L^1(U)$  is of class BV (bounded variation) if its first partial derivatives,  $\frac{\partial f}{\partial x^i}$  (here and below always in the sense of distributions) are signed Radon measures with finite mass  $|\frac{\partial f}{\partial x^i}|(U)$ . We denote by  $[Df]$  the Radon measure  $\sum_i |\frac{\partial f}{\partial x^i}|$  on  $U$ . If  $f : U \rightarrow \mathbb{R}$  is a BV function, which is continuous on a subset  $R \subset U$  with  $\mathcal{H}^{n-1}(U \setminus R) = 0$  then the Radon measure  $[Df]$  vanishes on all Borel subsets  $A \subset U$  with  $\mathcal{H}^{n-1}(A) < \infty$ , [GL80].

Let  $f : U \rightarrow \mathbb{R}$  be of class BV. Then for  $\mathcal{H}^n$ -almost every point  $x \in U$  there exists an affine function  $\hat{f}_x : \mathbb{R}^n \rightarrow \mathbb{R}$ , such that for the BV function  $h_x = f - \hat{f}_x$  we have

$$(6.3) \quad \lim_{r \rightarrow 0} \frac{1}{r^{n+1}} \int_{B_r(x)} |h_x| = 0; \quad \text{and} \quad \lim_{r \rightarrow 0} \frac{1}{r^n} \cdot [Dh_x](B_r(x)) = 0$$

see [EG15, Theorem 6.1 (2),(3)] for the second and the Hölder inequality and [EG15, Theorem 6.1 (1)] for the first inequality.

**6.3. Almost Riemannian metric spaces.** The following definition provides a suitable description of a large part of any Alexandrov space, see Section 7.

Let  $C = C(n)$  be the constant determined in Proposition 6.1. We will call a locally geodesic metric space  $X$  an *almost Riemannian metric space* if it has the following properties.

- (1) There is a Borel subset  $R \subset X$ , called the subset of *regular points* with  $\mathcal{H}^{n-1}(X \setminus R) = 0$ .
- (2) Any minimizing geodesic  $\gamma$  in  $X$  can be approximated by curves  $\gamma_i$  in  $R$ , such that the lengths of  $\gamma_i$  converge to the length of  $\gamma$ .
- (3) For any  $x \in X$ , there is a neighborhood  $U$  of  $x$ , called a *regular chart*, and a biLipschitz map  $\phi : U \rightarrow O$  onto an open subset  $O \subset \mathbb{R}^n$ , with the biLipschitz constant less than  $(1 + \frac{1}{C})$ .

- (4) There a continuous Riemannian tensor  $g_{ij}$  on  $\phi(U \cap R)$  such that  $g_{ij}$  is a function of bounded variation on  $O$  for each  $1 \leq i, j \leq n$ .
- (5) The length of any curve  $\gamma \subset R$  can be computed as the length of  $\phi(\gamma)$  via this Riemannian tensor  $g$ .

For any regular chart  $U$  as above, we set  $\mathcal{N}_0(U)$  to be the Radon measure  $[g']$  on  $U$  given as the sum of the Radon measures  $[Dg_{ij}]$  over the coordinates  $g_{ij}$  of the metric tensor  $g$ . For an almost Riemannian metric space  $X$ , we define an outer measure  $\mathcal{N}$  on  $X$  in the following way. For a subset  $A \subset X$ , we consider all coverings  $A \subset \cup U_i$  by countably many regular charts  $U_i$  and let  $\mathcal{N}(A)$  to be the infimum of the sums  $\sum_i \mathcal{N}_0(U_i)$  over all such coverings. This is indeed an outer measure, which takes finite values on compact subsets. Since  $\mathcal{N}$  satisfies the Caratheodory criterion, [EG15, Theorem 1.9], it is indeed a Radon measure. We will call  $\mathcal{N}$  the *minimal metric derivative measure* on the almost Riemannian metric space  $X$ .

**Lemma 6.2.** *Let  $X^n$  be a almost Riemannian metric space and let  $\mathcal{N}$  be its minimal metric derivative measure. Then  $\mathcal{N}(A) = 0$  for any Borel subset  $A \subset X$  with  $\mathcal{H}^{n-1}(A) < \infty$ . There exists a Borel subset  $C \subset X$  of full  $\mathcal{H}^n$ -measure in  $X$  with  $\mathcal{N}(C) = 0$ , thus  $\mathcal{N}$  is absolutely singular with respect to  $\mathcal{H}^n$ .*

*Proof.* Clearly, both claims are local. Hence we need to verify them only in a regular chart  $U$ , which we identify with its image  $\phi(U) \subset \mathbb{R}^n$ . The first statement follows directly from the continuity of the metric tensor  $g$  on the subset  $U \cap R$  and the result of [GL80] cited above.

In order to verify the second claim we only need to show the following statement, cf. [EG15, Section 1.6]. For almost all  $x \in U$  there is another regular chart  $x \in V$ , such that the derivative measure  $[h']$  of the Riemannian tensor  $h$  in this chart  $V$ , has  $n$ -dimensional density 0 at  $x$ , thus

$$(6.4) \quad \lim_{r \rightarrow 0} \frac{1}{r^n} \cdot [h'](B_r(x)) = 0.$$

Here and below, the ball  $B_r(x)$  over which we integrate can be equally considered with respect to the Euclidean or to the original metric on  $U$ , since both are biLipschitz equivalent. In order to prove (6.4), we follow [Per95, Section 4.2] and consider the Riemannian tensor  $g$  of the original chart  $U$ . Applying (6.3) to the coordinates of  $g$ , we find for  $\mathcal{H}^n$ -almost all  $x \in U$  a smooth symmetric 2-tensor  $\hat{g} = \hat{g}_x$  on  $U$  such that for  $u = g - \hat{g}$  we have:

$$(6.5) \quad \lim_{r \rightarrow 0} \frac{1}{r^{n+1}} \int_{B_r(x)} \|u\| = 0; \quad \text{and} \quad \lim_{r \rightarrow 0} \frac{1}{r^n} \cdot [Du](B_r(x)) = 0.$$



The first statement implies that  $\hat{g}$  is indeed a Riemannian metric in a neighborhood  $U_0$  of  $x$ .

Fix such a point  $x$ , neighborhood  $U_0$  and  $\hat{g}$ . Consider a small neighborhood  $W$  of 0 in  $\mathbb{R}^n$  and let  $\xi : W \rightarrow U$  be the exponential map with respect to the metric  $\hat{g}$ . Then  $\xi(0) = x$ ,  $D\xi(0) = Id$  and the pull-back Riemannian metric  $\hat{h} = \xi^*(\hat{g})$  has zero derivative at 0. Since  $D\xi$  is the identity, the biLipschitz constant of the restriction  $F = \xi^{-1} \circ \phi$  to a sufficiently small neighborhood  $V$  of the point  $x$  is still less than  $(1 + \frac{1}{C})$ . Hence,  $F : V \rightarrow \mathbb{R}^n$  is a regular chart.

The Riemannian tensor  $h$  in this chart equals  $\hat{h} + \xi^*(g - \hat{g})$ . Now,  $D\hat{h}(0) = 0$ , thus (6.4) holds for  $\hat{h}$  instead of  $h$ . For the other summand  $\xi^*(u)$ , the density estimate (6.4) follows from (6.5) and the fact that  $\xi$  is a  $C^2$ -diffeomorphism if  $W$  is sufficiently small. This finishes the proof of Lemma 6.2.  $\square$

**6.4. The upper bound on the deviation measures.** Continuing to denote by  $C = C(n)$  the constant from Proposition 6.1 we show:

**Corollary 6.3.** *Let  $U$  regular chart of an almost Riemannian metric space  $X$ . Identifying  $U$  with its image  $O = \phi(U)$ , let  $g$  be the metric tensor and the measure  $\mathcal{N}_0 = [Dg]$  the derivative of the metric tensor. For any Borel subset  $A \subset U$  and any  $r$  such that  $B_{3r}(A)$  is relatively compact in  $U$  we have*

$$v_r(A) \leq 2 \cdot C \cdot \mathcal{N}_0(B_{3r}(A)).$$

*Proof.* Consider a relatively compact open subset  $V \subset U$ , which contains  $B_{2r}(A)$ . Apply (coordinatewise) the standard mollifying construction to the Riemannian tensor  $g$ . For all small positive  $\epsilon$ , we thus obtain smooth metrics  $g_\epsilon$  on  $V$  with the following properties. The total derivatives  $|g'_\epsilon|$ , considered as measures, satisfy  $|g'_\epsilon| \leq \mathcal{N}_0$  on  $V$ , [Zie89, Theorem 5.3.1]. Since  $g$  is pointwise  $\frac{1}{C}$ -close to the background Euclidean inner product, the same is true for  $g_\epsilon$ . For all sufficiently small  $\epsilon$  the  $2r$ -tubular neighborhood around  $A$  with respect to  $g_\epsilon$  is contained in the  $3r$ -tubular neighborhood around  $A$  with respect to the original distance in  $X$ . Moreover,  $g_\epsilon$  converges to  $g$  pointwise at all points of  $R$ , [Zie89, Theorem 1.6.1].

Denote by  $d_\epsilon$  the distance function induced by  $g_\epsilon$ . From the last statement and the properties (2),(5) in the definition of an almost Riemannian metric space we deduce that

$$\lim_{\epsilon \rightarrow 0} \left( \sup_{x,y \in V, d(x,y) < r} |d_\epsilon(x,y) - d(x,y)| \right) = 0.$$

Finally, the Hausdorff measures of the Riemannian metrics  $g_\epsilon$  converge on  $V$  to the Hausdorff measure of  $V$  with respect to the original metric.

Now the result follows directly from Proposition 6.1 applied to the metrics  $g_\epsilon$ , by letting  $\epsilon$  go to 0.  $\square$

As a consequence of Corollary 6.3, the minimal metric derivative measure bounds from above the deviations measure  $v_r$  on any almost Riemannian metric space:

**Lemma 6.4.** *Let  $X$  be a almost Riemannian metric space with the metric derivative measure  $\mathcal{N}$ . Then for any compact subset  $A \subset X$ , there exists some  $r_0 > 0$  such that for all  $r < r_0$  we have*

$$v_r(A) \leq r \cdot 2(n+2) \cdot C \cdot \mathcal{N}(A).$$

*Proof.* Cover  $A$  by finitely many regular charts  $U_i$  such that  $\sum \mathcal{N}_0(U_i)$  is sufficiently close to  $\mathcal{N}(A)$ . Since the covering dimension of  $X$  is  $n$ , we find a finite covering  $V_j$  of  $A$ , which refines the first covering but has intersection multiplicity less than  $(n+2)$ . Considering each  $V_j$  as a subchart of the corresponding chart  $U_i$  we see that

$$\sum \mathcal{N}_0(V_j) \leq (n+2) \sum \mathcal{N}(A).$$

Consider  $r_0 > 0$  such that for any  $x \in A$  the ball  $B_{4r_0}(x)$  is contained in one of the sets  $V_j$ . Denote by  $A_j$  the set of all such  $x$ . Then  $v_r(A) \leq \sum v_r(A_j)$  and, due to Corollary 6.3,  $v_r(A_j) \leq 2 \cdot C \cdot \mathcal{N}_0(V_j)$ . Combining these inequalities finishes the proof.  $\square$

## 7. ALEXANDROV SPACES

**7.1. Strained points.** Strainers and strainer maps are basic tools for Alexandrov spaces, cf. [BGP92], [OS94], [KMS01] and will play an important role in the proof of Theorem 1.6 below. Rather than recalling the definition, we will just list main properties of the subsets of strained points. We fix a natural number  $n$ . There exist a positive number  $A = A(n)$  such that for all  $0 < r, \delta \leq \frac{1}{A^2}$  and all  $n$ -dimensional Alexandrov space  $X$  of curvature  $\geq -1$  the following holds true.

- (1) The set  $X_{r,\delta}$  of points in  $X$  which have an  $Ar$ -long  $(n, \delta)$ -strainer is open in  $X$ . For  $s < r$ , we have  $X_{r,\delta} \subset X_{s,\delta}$ , [BGP92, 9.7].
- (2) Let a sequence  $(X_i^n, x_i)$  of Alexandrov spaces of curvature  $\geq -1$  converge in the pointed Gromov-Hausdorff topology to an  $n$ -dimensional Alexandrov space  $(X, x)$ . If  $x \in X_{r,\delta}$  then, for all large  $i$ , the point  $x_i$  has an  $Ar$ -long  $(n, \delta)$ -strainer in  $X_i$ .
- (3) Rescaling  $X$  with a constant  $\lambda \geq 1$  sends the subset  $X_{r,\delta}$  to a subset of  $(\lambda X)_{\lambda r, \delta}$  of the rescaled Alexandrov space  $\lambda X$ .

- (4) The union  $X_\delta := \cup_{r>0} X_{r,\delta}$  contains the set  $X_{reg}$  of all regular points of  $X$ . The Hausdorff dimension of the set  $X \setminus (X_\delta \cup \partial X)$  is at most  $n - 2$  [BGP92, 10.6, 10.6.1, 12.8].
- (5) For any point  $x \in X_{r,\delta}$  there are natural distance coordinates  $\phi : B_{3r}(x) \rightarrow \mathbb{R}^n$  which are  $(1 + \epsilon)$ -biLipschitz onto an open subset  $O \subset \mathbb{R}^n$ . Here,  $\epsilon \rightarrow 0$  as  $A \rightarrow \infty$ , [BGP92, 9.4].
- (6) The natural coordinates  $\phi$  can be smoothed out to satisfy the following property, [OS94, Theorem B]. There exists a continuous Riemannian metric  $g$  on  $\phi(X_{reg} \cap B_{3r}(x)) \subset O$  such that for any curve  $\gamma \subset X_{reg} \cap B_{3r}$  its length coincides with the length of  $\phi(\gamma)$  with respect to the Riemannian metric  $g$ .
- (7) The metric tensor  $g$  on a chart  $O$  defined above is of bounded variation on  $O$ , [Per95, 4.2].

The last three statements in the above list together with the density and convexity of the set  $X_{reg}$  of regular points imply:

**Corollary 7.1.** *In the above notations, the subset  $X_\delta \subset X$  is an almost Riemannian metric space, once  $A$  is sufficiently large.*

In fact, the arguments in [Per95, 4.2], provide a slightly more precise version of (7) in the above list:

**Lemma 7.2.** *In the notations above, the constant  $A$  can be chosen large enough, so that the following holds true. The derivative measure  $[g']$  of the Riemannian tensor  $g$  in the canonical distance chart  $O$  satisfies  $[g'](\hat{O}) \leq A \cdot r^{n-1}$ , where  $\hat{O}$  is the image  $\phi(B_{2r}(x)) \subset O = \phi(B_{3r}(x))$ .*

*Proof.* We only sketch the proof, referring to [Per95] for details. First we fix  $r = \frac{1}{A^2}$ .

The fact that  $g$  has bounded variation in the chart  $O$  follows in [Per95, Section 4.2], by writing the coordinates of  $g$  as a universal smooth map  $\Phi(f_1, \dots, f_\alpha)$  of a finite number of distance functions  $f_j$  on  $X$  and their partial derivatives, both expressed in the chart  $\phi$ . It is shown in [Per95, Section 3], that any such distance function  $f_j$  is expressed in the chart  $O$  as a difference of two  $L$ -Lipschitz and  $\lambda$ -concave functions, where  $L, \lambda$  depends only on the semi-concavity of the corresponding distance functions in  $X$ . Since we have fixed  $r > 0$ , these numbers  $\lambda, L$  can be chosen independently of  $X$ . Thus,  $f_j$  can be written in the chart  $O$  as the difference of two convex functions with universal Lipschitz constants  $L'$ . Therefore, for any unit vector  $v \in \mathbb{R}^n$ , we have a uniform bound on the total mass of the Radon measure  $[\frac{\partial^2 f_j}{\partial^2 v}](\hat{O})$ . This implies that all partial second derivatives of  $f$  have uniformly bounded mass on  $\hat{O}$ , cf. [EG15, Theorem 6.8].

From this we deduce a uniform bound  $A'$  on the total mass  $[g'](\hat{O})$ , for the fixed value of  $r_0 = \frac{1}{A^2}$ .

For any  $r < r_0$  we rescale the space by  $\frac{r_0}{r}$ . The total mass of the Riemannian tensor  $g$  is then rescaled by  $(\frac{r_0}{r})^{n-1}$ . Thus,

$$[g'](\hat{O}) \leq A' \cdot (r_0)^{1-n} \cdot r^{n-1}.$$

We finish the proof by replacing  $A$  by  $\max(A, A'r_0^{1-n})$ .  $\square$

Now we use Corollary 6.3 to conclude:

**PROPOSITION 7.3.** *Let  $C = C(n)$ ,  $A = A(n)$  be the constants from Proposition 6.1 and Lemma 7.2. For any point  $x \in X_{r,\delta}$ , any  $s < r$  and any Borel subset  $K \subset B_r(x)$  the deviation measure  $v_s$  satisfies  $v_s(K) \leq 2 \cdot C \cdot A \cdot r^{n-1}$ .*

**7.2. Decomposition in good balls.** Let the constant  $A$  be as above. A ball  $B_r(x)$  in  $X^n$  will be called *good* if  $x \in X_{r,\delta}$ . A ball  $B_r(x)$  in  $X$  will be called *bad* if it is not good. The aim of this subsection is the proof of a controlled covering result, cf. Problem 8.10.

**PROPOSITION 7.4.** *Let  $X^n$  be an  $n$ -dimensional Alexandrov space without boundary. For every compact  $W \subset X$ , every  $\alpha > n - 2$  there exists a positive number  $q = q(W, \alpha) > 0$  with the following property. For every  $x \in W$  and every  $s < 1$  there exists a countable collection of good balls  $B_m = B_{r_m}(x_m) \subset X$  such that*

- (1)  $r_m < s$  for all  $m$ .
- (2)  $\mathcal{H}^n(B_s(x) \setminus (\cup_m B_m)) = 0$ .
- (3)  $\sum_m r_m^\alpha < q \cdot s^\alpha$ .

The proof will be obtained by a recursive application of the following lemma.

**Lemma 7.5.** *There is an integer  $N = N(W, \alpha)$  with the following property. For any  $p \in W$  and  $\rho < 1$  the ball  $B_\rho(p)$  can be covered by at most  $N$  balls  $B_i = B_{r_i}(x_i)$  such that  $r_i < \rho$ , for all  $i$ , and*

$$\sum_{i \in \text{BAD}} r_i^\alpha < \frac{1}{2} \cdot \rho^\alpha,$$

where  $i \in \text{BAD}$  means that  $B_i$  is a bad ball.

*Proof.* Assume the contrary. Thus we can find a sequence of balls  $K_l = B(p_l, \rho_l)$  such that  $p_l \in W$ ,  $\rho_l < 1$  and one needs at least  $l$  balls to cover  $K_l$ , so that the conditions in the lemma are fulfilled.

Taking a subsequence we may assume that the following limit exists in the pointed Gromov-Hausdorff metric.

$$\left(\frac{1}{\rho_m} \cdot X, p_m\right) \xrightarrow{\text{GH}} (Y, p).$$

Since the points  $p_m$  range over a compact subset of  $X$  and  $\rho_m < 1$ , the sequence is non-collapsing, i.e.  $Y$  is an  $n$ -dimensional Alexandrov space. By Perelman's stability theorem,  $\partial Y$  is empty. Therefore,  $S := (Y \setminus Y_\delta) \cap \bar{B}_2(p)$  is a compact set of Hausdorff dimension  $\leq n - 2$ .

By the definition of Hausdorff dimension, we can cover  $S$  by a finite number of balls  $B_i = B_{r_i}(x_i)$  such that

$$\sum_i r_i^\alpha < \left(\frac{1}{2}\right)^\alpha.$$

Any point in the remaining compact set  $K \setminus (\cup_i B_i)$  is contained in  $Y_\delta$ . Therefore a small ball centered at any point of this set is good. By compactness, we can cover  $K \setminus (\cup_i B_i)$  by a finite number of good balls. Let  $N$  be the total number of balls in the obtained covering of  $K$ .

Lifting the constructed covering to  $K_l$ , for all  $l$  large enough, we cover the ball  $K_l$  by at most  $N$  balls satisfying the conditions of the lemma. This contradiction to our assumption finishes the proof of the lemma.  $\square$

Now it is easy to derive

*Proof of Proposition 7.4.* Cover  $B_s(x)$  by  $N$  balls as in Lemma 7.5 and call this covering  $\mathcal{F}_1$ . Now cover every bad ball from the covering  $\mathcal{F}_1$  by at most  $N$  balls provided by Lemma 7.5. Together with the good balls from  $\mathcal{F}_1$  the new balls define a covering  $\mathcal{F}_2$  of  $B_s(x)$ . Proceeding in this way define for each natural number  $k$  a covering  $\mathcal{F}_k$  of  $B_s(x)$ .

Denote by  $g_l^+$  and  $g_l^-$  the sum of  $r_i^\alpha$  over good, respectively bad balls  $B_{r_i}(x_i)$  in the covering  $\mathcal{F}_l$ . Then, by construction,  $g_{l+1}^- < \frac{1}{2}g_l^-$  and  $g_{l+1}^+ \leq g_l^+ + N \cdot g_l^-$ . Therefore,  $g_l^- \leq 2^{-l} \cdot g_1^-$  and  $g_l^+$  is uniformly bounded from above. The volume of the union of bad balls in  $\mathcal{F}_l$  is at most  $g_l^-$  and converges to 0 as  $l$  goes to  $\infty$ .

Let  $\mathcal{F}$  be the set of all good balls  $B_j = B_{r_j}(x_j)$  from all the coverings  $\mathcal{F}_l$ . Then  $\mathcal{H}^n(B_s(x) \setminus (\cup_{\mathcal{F}} B_j)) \leq \lim_{l \rightarrow \infty} g_l^- = 0$ . On the other hand, by construction,  $\sum_{B_j \in \mathcal{F}} r_j^\alpha = \lim_{l \rightarrow \infty} g_l^+ \leq 3N \cdot s^\alpha$ .

Setting  $q = 3 \cdot N$  finishes the proof.  $\square$

**7.3. Final step.** Now we can provide the

*Proof of Theorem 1.6.* Let  $X$  be a fixed  $n$ -dimensional Alexandrov space. By the inequality of Bishop-Gromov, the deviation measures  $v_r$  are uniformly bounded from below by a quadratic term in  $r$ . Thus in order

to control the mm-boundary we only need to bound  $v_r$  from above on balls in  $X$ .

Let the constants  $A, C$  be chosen as above, so that Proposition 7.3 can be applied.

Let us first assume that  $\partial X$  is empty. Let  $W \subset X$  be an arbitrary compact subset. Fix  $\alpha = n - \frac{3}{2}$  and choose the constant  $q$  as in Proposition 7.4. For any  $x \in W$  and  $s < 1$  consider the good balls  $B_i = B_{r_i}(x_i)$  provided by Proposition 7.4 and set  $K' = \cup_i B_i$ . Let  $r < \frac{1}{A^2}$  be sufficiently small. Since  $\mathcal{H}^n(K \setminus K') = 0$  we have  $v_r(K) = v_r(K')$ .

For all  $m$  with  $r < r_m$ , we apply Proposition 7.3 and infer

$$v_r(B_m \cap K) \leq 2 \cdot C \cdot A \cdot r \cdot r_m^{n-1}$$

On the other hand, for  $r_m < r$ , we have

$$v_r(B_m \cap K) \leq \mathcal{H}^n(B_m) \leq 2 \cdot \omega_n \cdot r_m^n < 2 \cdot \omega_n \cdot r \cdot r_m^{n-1}$$

Summing up and using  $r_m^{n-1} < r_m^\alpha$  we obtain

$$(7.1) \quad v_r(K) \leq \sum_m v_r(B_m \cap K) \leq (2 \cdot C \cdot A + 2 \cdot \omega_n) \cdot q \cdot r \cdot s^\alpha.$$

This proves that  $X$  has locally finite mm-boundary. As already mentioned and used above, any signed Radon measure  $\nu$  obtained as a limit of a sequence  $v_{r_j}/r_j$  for some  $r_j \rightarrow 0$  must be non-negative, hence a Radon measure. We fix such  $\nu$ .

Inequality (7.1) implies that  $\nu$  has finite  $\alpha$ -dimensional density at every point of  $X$ , in particular,  $\nu$  vanishes on subsets of Hausdorff dimension  $\leq n - 2$ . Thus  $\nu(X \setminus X_\delta) = 0$ .

Recall that  $X_\delta$  is an almost Riemannian space. Denote, by  $\mathcal{N}$  its minimal metric derivative measure. We extend it to a measure on all of  $X$  (still denoted by  $\mathcal{N}$ ) by setting it to be 0 on  $X \setminus X_\delta$ . By Lemma 6.4 the Radon measure  $\nu$  is absolutely continuous with respect to  $\mathcal{N}$  on compact subsets of  $X_\delta$ . Now (1) and (3) of Theorem 1.6 follow from Lemma 6.2. This finishes the proof in the case  $\partial X = \emptyset$ .

Assume now that  $\partial X \neq \emptyset$  and consider the doubling  $Y = X \cup_{\partial X} X$  of  $X$ . Consider  $X$  as a convex subset of  $Y$  and let  $K \subset X$  be compact. We find a constant  $L > 0$  such that for all sufficiently small  $r > 0$ , we have  $\mathcal{H}^n(K \cap B_{2r}(\partial X)) \leq L \cdot r$ . (This follows, for example, by the coarea formula using the Lipschitz properties of the gradient flow of the distance function  $d(\cdot, \partial X)$  which is semiconcave). On the other hand, for  $x \in K \setminus B_r(\partial X)$ , the volumes of the  $r$ -ball in  $X$  and in  $Y$  coincide. Using that  $Y$  has locally finite mm-boundary, we deduce that  $v_r(K)$  (computed in the space  $X$ ) is bounded from above  $L \cdot r + \tilde{v}_r(K)$ , where  $\tilde{v}_r(K)$  is the deviation measure of  $K$  considered as a subset of  $Y$ . This

implies that  $v_r/r$  is uniformly bounded for  $r \rightarrow 0$ . Thus  $X$  has locally finite mm-boundary as well.

Any limit of a sequence  $v_{r_j}/r_j$  for some  $r_j \rightarrow 0$  must be again non-negative, hence a Radon measure. Outside of  $\partial X$   $\nu$  coincides with the restriction of the corresponding measure defined on  $Y$ . From the corresponding statement about  $Y$  we deduce that  $\nu$  is absolutely singular with respect to  $\mathcal{H}^n$ . Moreover,  $\nu$  vanishes on subsets  $S \subset X \setminus \partial X$  with finite  $\mathcal{H}^{n-1}(S)$ .

It remains to prove (2), i.e. to show that the restriction of  $\nu$  onto  $\partial X$  is at least  $c \cdot \mathcal{H}^{n-1}$  for universal constant  $c = c(n)$ . This statement is local on  $\partial X$  and needs to be verified only in small neighborhoods of points  $x$  whose tangent  $T_x X$  are isometric to flat halfspaces.

We fix such a point  $x \in \partial X$ . We further fix a sufficiently small  $\epsilon > 0$  and find a small neighborhood  $U$  of  $x$  in  $X$  which is  $(1 + \epsilon)$ -biLipschitz to a half-ball in the Euclidean space. Choose an arbitrary  $s > 0$  such that  $B_{2s}(x) \subset U$ . Let  $K = \bar{B}_s(x) \cap \partial X$  be the closed ball of radius  $s$  in  $\partial X$  with respect to the ambient metric. Due to [EG15, Section 1.6], it is sufficient to prove that  $\nu(K) \geq c_0 \cdot s^{n-1}$  for a universal constant  $c_0$  depending only on the dimension.

In order to prove this inequality, we consider any open neighborhood  $V$  of  $K$  in  $X$ . For all small  $r > 0$ , the neighborhood  $V$  contains  $B_{2r}(K)$ . Once  $\epsilon$  has been chosen small enough, the ball  $B_r(z)$  in  $X$  has volume at most  $(1 - k_1) \cdot \omega_n \cdot r^n$ , for any point  $z \in B_{\frac{1}{10}r}(K)$ . Here  $k_1 = k_1(n) > 0$  is a universal constant. Moreover, the set  $B_{\frac{1}{10}r}(K)$  has volume at least  $\frac{1}{20} \cdot r \cdot \omega_{n-1} \cdot s^{n-1}$ . Integrating over  $V$  (and using the inequality of Bishop-Gromov on the complement of  $B_{\frac{1}{10}r}(K)$ ) we deduce:

$$v_r(V) \geq k_1 \cdot \frac{1}{20} \cdot \omega_{n-1} \cdot r \cdot s^{n-1} - k_3 \cdot r^2 ,$$

for some  $k_3$  depending only on the volume of  $V$  and independent of  $r$ . Dividing by  $r$  and letting it go to 0 we obtain  $\nu(V) \geq k_4 \cdot s^{n-1}$ , for a universal constant  $k_4 > 0$ . Since the neighborhood  $V$  of  $K$  was arbitrary, we infer the same inequality for  $K$  instead of  $V$ , finishing the proof.  $\square$

## 8. QUESTIONS AND COMMENTS

**8.1. Manifolds.** The notions of mm-boundary and mm-curvature are very easy to define but difficult to control even in simple situations. For instance, the examples mentioned in the introduction require some

amount of computations and estimates. On the other hand, constructing examples in which the mm-boundary is not locally finite also seems to be highly non-trivial. The first question in this direction is:

**PROBLEM 8.1.** *Construct a compact manifold with a continuous Riemannian metric that does not have finite mm-boundary.*

The following problem is motivated by our approach to Theorem 1.6 in Sections 6, 7.

**PROBLEM 8.2.** *Let  $X$  be an almost Riemannian space. Can the minimal metric derivative measure be non-zero?*

Note that the minimal metric derivative measure must vanish if the metric can be locally defined by a Riemannian tensor of class  $W^{1,1}$ , since the metric derivative measure is absolutely singular with respect to the Hausdorff measure by Lemma 6.2.

The following question is motivated by Lemma 6.4 and potential applications to geodesic flows of spaces with curvature bounded from above, cf. Problem 8.12.

**PROBLEM 8.3.** *Let  $X$  be an almost Riemannian space. Can one use the minimal metric derivative measure in order to control the deviation measures  $v_r$  from below?*

**8.2. Surfaces and hypersurfaces.** The answer to the following question is not trivial in view of Example 1.14.

**PROBLEM 8.4.** *Can one express the mm-curvature of an Alexandrov surface in terms of its curvature measure?*

In view of Theorem 1.9 it is reasonable to expect an affirmative answer to the following question

**PROBLEM 8.5.** *Do convex hypersurfaces of  $\mathbb{R}^n$  have locally finite mm-curvature?*

A natural approach to this question is related to the following conjectural generalization of a theorem of Bonk and Lang [BL03]:

**PROBLEM 8.6.** *Let  $X$  be a convex hypersurface sufficiently close to a flat hyperplane. Can we bound the optimal biLipschitz constant for maps into the Euclidean space in terms of the total scalar curvature?*

Some natural generalizations of our Theorem 5.3 are possible. Probably, slightly refined arguments can be used to prove that any DC-submanifold of a Euclidean space has trivial mm-boundary. Using the embedding theorem of Nash, this would also provide an easy generalization of Theorem 5.3 and Theorem 1.1 to convex hypersurfaces of smooth Riemannian manifolds.



**8.3. Alexandrov geometry and beyond.** As the next generalization of Theorem 1.1, one should study the case of smoothable Alexandrov spaces.

**PROBLEM 8.7.** *Does the mm-boundary vanish in smoothable Alexandrov spaces? Are there relations to scalar curvature measures defined in [LP17]?*

Due to the observation after Problem 8.2, the triviality of mm-boundary would follow from the existence of slightly smoother coordinates than the ones provided by Perelman’s DC-structure.

**PROBLEM 8.8.** *Let  $X$  be an Alexandrov space. Can one introduce coordinates on a neighborhood of the set of regular points, such that the metric is locally given by a Riemannian tensor of class  $W^{1,1}$ ?*

In the two-dimensional case, the answer to this question is “yes” by the work of Reshetnyak.

Due to Theorem 1.5, an affirmative answer to the following question should be expected:

**PROBLEM 8.9.** *Are there connections between the size of the mm-boundary of an Alexandrov space  $X$  and the “average size” of the cut loci of points in  $X$ .*

Should one have a chance to go beyond mm-boundary and towards mm-curvature, one would definitely need to improve the decomposition statement Proposition 7.4, which provides a good geometric control of the size of the set of singular points of an Alexandrov space.

**PROBLEM 8.10.** *Can one replace  $\alpha > n - 2$  by  $\alpha = n - 2$  in the statement of Proposition 7.4?*

It is interesting to understand if our results provide a quantitative version of biLipschitz closedness of small balls to Euclidean balls:

**PROBLEM 8.11.** *Does there exist a constant  $C = C(n)$  such that for any Alexandrov space  $X = X^n$ , any  $x \in X$  and any  $\delta < \frac{1}{C}$  the following holds true. If  $\omega_n r^n - \mathcal{H}^n(B_r(x)) \leq \delta r^n$ , then the ball  $B_{\frac{r}{4}}(x)$  is  $(1 + C(n)\delta)$ -biLipschitz to a Euclidean ball.*

It is natural to look at what happens for curvature bounded above:

**PROBLEM 8.12.** *Can one estimate and use the mm-boundary in geodesically complete spaces with upper curvature bounds to study the geodesic flow?*

Finally, it seems reasonable to expect some generalizations to spaces with Ricci curvature bounds, for instance:

**PROBLEM 8.13.** *Can one control the mm-boundary of noncollapsed limits of Riemannian manifolds with Ricci curvature bounded below? Can one expect something like a geodesic flow in this setting?*

From the work of Cheeger and Naber [CN15] it should follow that on any non-collapsed limit of manifolds with both-sided Ricci curvature bounds, the mm-curvature is locally finite and mm-boundary is zero. Vanishing of the mm-boundary should then imply that the geodesic flow is defined almost everywhere and preserves the Liouville measure by the same argument as in the proof of Theorem 1.5.

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