ON GLUING ALEXANDROV SPACES WITH LOWER RICCI CURVATURE BOUNDS

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Abstract. In this paper we prove that in the class of metric measure space with Alexandrov curvature bounded from below the Riemannian curvature-dimension condition $RCD(K,N)$ with $K \in \mathbb{R}$ & $N \in [1,\infty)$ is preserved under doubling and gluing constructions.

Contents

1. Introduction and Statement of Main Results

A way to construct Alexandrov spaces is by gluing together two or more given Alexandrov spaces along isometric connected components of their intrinsic boundaries. The isometry between the boundaries is understood w.r.t. induced length metric. A special case of this construction is the double space where one glues together two copies of the same Alexandrov space with nonempty boundary. It was shown by Perelman that the double of an Alexandrov space of curvature $\geq k$ is again Alexandrov of curvature $\geq k$. Petrunin later showed [Pet97] that the lower

CK is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Projektnummer 396662902. KTS gratefully acknowledges financial support by the European Union through the ERC-AdG "RicciBounds" and by the DFG through the Excellence Cluster "Hausdorff Center for Mathematics" and through the Collaborative Research Center 1060.

2010 Mathmatics Subject Classification. Primary 53C21, 54E35. Keywords: metric measure space, curvature-dimension condition, gluing construction.

curvature bound is preserved in general for any gluing of two possibly different Alexandrov spaces.

In this article we study Ricci curvature bounds in the sense of Lott, Sturm and Villani for this setup. More precisely, we consider the class of n-dimensional Alexandrov spaces with some lower curvature bound equipped with a Borel measure of the form $\Phi\mathcal{H}^n = \mathrm{m}$ for a semi-concave function $\Phi : X \to [0,\infty)$ such that the corresponding metric measure space (X, d, m) satisfies a curvature-dimension condition $CD(K,N)$ for $K \in \mathbb{R}$ and $N \in [n,\infty)$. Here K does not necessarily coincide with $k(n-1)$. In particular, it's possible that $k < 0$ but $K \geq 0$.

To state our main theorem we recall the following. The Alexandrov boundary of (X, d) is denoted as ∂X equipped with the induced length metric $d_{\partial X}$. We write Σ_p for the space of direction at $p \in X$ that is an Alexandrov space with curvature bounded below by 1. We say $v \in \Sigma_p$ for $p \in \partial X$ is a normal vector at p if $\angle(v, w) = \frac{\pi}{2}$ for any $w \in \partial\Sigma_p$. Here $d_x\Phi_i$ denotes the differential of the semi-concave function Φ at some point $x \in X_i$. We also refer to the remarks after Definition 2.20.

Our main theorem is

Theorem 1.1 (Glued spaces)**.** *For $i = 0, 1$ let X_i be n-dimensional Alexandrov spaces with curvature bounded below and let $\mathrm{m}_{X_i} = \Phi_i\mathcal{H}^n_{X_i}$ be measures where $\Phi_i : X_i \to [0,\infty)$ are semi-concave functions. Suppose there exists an isometry $\mathcal{I} : \partial X_0 \to \partial X_1$ such that $\Phi_0 = \Phi_1 \circ \mathcal{I}$.*

If the metric measure spaces $(X_i, d_{X_i}, \mathrm{m}_i)$ satisfy the curvature-dimension condition $CD^*(K,N)$ for $K \in \mathbb{R}$, $N \in [1,\infty)$ and if

$$d_p\Phi_0(v_0) + d_p\Phi_1(v_1) \leq 0 \ \forall p \in \partial X_i \text{ and any normal vectors } v_i \in \Sigma_p X_i,\ i = 0, 1,$$

then the glued metric measure space $(X_0 \cup_{\mathcal{I}} X_1, (\iota_0)_\# \mathrm{m}_{X_0} + (\iota_1)_\# \mathrm{m}_{X_1}))$ satisfies the reduced curvature-dimension condition $CD^*(K,N)$.

Remark 1.2. If the measures are finite, one can replace in the conclusion of Theorem 1.1 the condition $CD^*(K,N)$ with the full curvature-dimension condition $CD(K,N)$. In this case the two conditions are equivalent [CM16].

Corollary 1.3. *For $i = 0, 1$ let X_i be Alexandrov spaces with curvature bounded below, and let $\mathcal{I} : \partial X_0 \to \partial X_1$ be an isometry. Assume the metric measure spaces $(X_i, d_{X_i}, \mathcal{H}^n_{X_i})$ satisfy the condition $CD^*(K,N)$ for $K \in \mathbb{R}, N \in [1,\infty)$.*

Then the metric measure space $(X_0 \cup_{\mathcal{I}} X_1, \mathcal{H}^n_{X_0 \cup_{\mathcal{I}} X_1})$ satisfies the condition $CD^*(K,N)$.

Remark 1.4. An Alexandrov space with curvature bounded from below is infinitesimally Hilbertian. Therefore it satisfies the condition $CD(K,N)$ (or $CD^*(K,N)$) if only if it satisfies the Riemannian curvature-dimension condition $RCD(K,N)$ (or $RCD^*(K,N)$) (Corollary 2.10).

Remark 1.5. If X_i are convex domains in smooth Riemannian manifolds with lower Ricci curvature bounds, then the statement of Corollary 1.3 is regarded as folklore. A complete proof has been given in [PS18], based on a detailed approximation property derived in [Sch12].

For general noncollapsed RCD spaces there are two natural notions of boundary: one that was introduced by DePhillippis and Gigli in [DPG18] and another by Mondino and the first named author in [KM19]. Conjecturally both notions coincide and the boundary (defined either way) is a closed subset in the ambient space. We make the following conjecture

Conjecture 1.6. *For $i = 0, 1$ let X_i be noncollapsed $RCD(K, n)$ spaces with nonempty boundary ∂X_i. Suppose there exists an isometry $\mathcal{I} : \partial X_0 \to \partial X_1$. Then the glued metric measure space $(X_0 \cup_{\mathcal{I}} X_1, \mathcal{H}^n_{X_0 \cup_{\mathcal{I}} X_1})$ satisfies the condition $RCD(K, n)$.*

As a biproduct of our proof Theorem 1.1 we also obtain the following result that also seems to be new.

Theorem 1.7. *For $i = 0, 1$ let X_i be n-dimensional Alexandrov spaces with curvature bounded below as in the previous theorem, let $X_0 \cup_{\mathcal{I}} X_1$ be the glued Alexandrov spaces and let $\Phi_i : X_i \to \mathbb{R}$, $i = 0, 1$, be semi-concave with $\Phi_0|_{\partial X_0} = \Phi_1|_{\partial X_1}$ such that for any $p \in \partial X_i$ it holds that*

$$d\Phi_0|_p(v_0) + d\Phi_1|_p(v_1) \leq 0 \quad \forall \text{ normal vectors } v_i \in \Sigma_p X_i,\ i = 0, 1.$$

Then $\Phi_0 + \Phi_1 : X_0 \cup_{\mathcal{I}} X_1 \to \mathbb{R}$ is semiconcave.

We say that a function $\Phi : X \to \mathbb{R}$ on an Alexandrov space X is double semiconcave if $\Phi \circ P : \hat{X} \to \mathbb{R}$ is semi-concave in the usual sense where $\hat{X}$ denotes the Alexandrov double space of X and $P : \hat{X} \to X$ is the canoncial map. We give an alternative characterisation of this condition in Lemma 4.1 and Corollary 4.4.

As another consequence of our main theorem we also obtain the following.

Corollary 1.8 (Doubled spaces)**.** *Let X be an n-dimensional Alexandrov space with curvature bounded below, and let $\mathrm{m}_X = \Phi \mathcal{H}^n_X$ be a measure for a double semiconcave function $\Phi : X \to [0, \infty)$. Assume the metric measure space (X, d_X, m_X) satisfies the condition $CD^*(K, N)$ for $K \in \mathbb{R}$ and $N \geq 1$. Then, the double space $(\hat{X}, d_{\hat{X}}, \mathrm{m}_{\hat{X}})$ satisfies the condition $CD^*(K, N)$.*

Let us briefly comment on the statement and the proof of Theorem 1.1 and Corollary 1.3. By Petrunin's glued space theorem one knows that the glued space of two Alexandrov spaces with curvature bounded from below is again an Alexandrov space with the same lower curvature bound. However the intrinsic *best lower Ricci bound* might be different from the Alexandrov curvature bound. So Petrunin's theorem does not imply any of the statements above.

But we can use the improved regularity of the glued space for our purposes. It implies *some* lower Ricci bound that yields a priori information for transport densities and densities along needles in the Cavalletti-Mondino $1D$ localisation procedure. At this point a crucial difficulty appears. It is not known whether geodesics cross the boundary set where the spaces are glued together, only finitely many times. This difficulty does not occur for the double space construction. By symmetry in this case it is known that geodesics in the double space only cross once.

For general glued spaces we overcome this problem by the following strategy. First, given a $1D$ localisation we show that the collection of geodesic that cross the boundary infinitely many times has measure 0 w.r.t. to the corresponding quotient measure. Then, we apply a theorem of Cavalletti and Milman on characterization of synthetic Ricci curvature bounds via $1D$ localisation.

Remark 1.9. As pointed out by Rizzi [Riz18], in the previous theorem one cannot replace the curvature-dimension conditon $CD(K, N)$ for any $K \in \mathbb{R}$ and $N \in (1, \infty)$

with the measure contraction property $MCP(K,N)$ [Stu06, Oht07]. The MCP is a weaker condition that still characterizes lower Ricci curvature bounds for N-dimensional smooth manifolds and is also consistent with lower Alexandrov curvature bounds. For the precise definition we refer to [Stu06, Oht07]. The counterexample in [Riz18] is given by the Grushin half-plane which satisfies $MCP(0,N)$ if and only if $N \geq 4$ while its double satisfies $MCP(0,N)$ if and only if $N \geq 5$.

Another example that is even Alexandrov is provided in the last section of this article (Example 5.1).

Remark 1.10. Let us mention that one can show that Petrunin's gluing theorem holds for gluing n-dimensional Alexandrov spaces along isometric extremal subsets of codimension 1 which do not need to be equal to the whole components of their boundaries (see for instance [Mit16]). For example gluing two triangles having a side of equal length with all adjacent angles to it $\leq \pi/2$ is again an Alexandrov space (a convex quadrilateral). Our results then generalize to this situation as well.

1.1. **Application to heat flow with Dirichlet boundary condition.** The concept of doubling has recently found significant application in the study of the heat flow with Dirichlet boundary conditions. In particular, it allows the use of optimal transportation techniques. As widely known, these techniques are not directly applicable since the Dirichlet heat flow will not preserve masses.

As observed in [PS18], this obstacle can be overcome by looking at the heat flow in the doubled space instead. The latter is accessible to optimal transport techniques and to the powerful theory of metric measure spaces with synthetic Ricci bounds. Moreover, it can always be expressed as a linear combination of the Dirichlet heat flow and the Neumann heat flow on the original space – and vice versa, both the Dirichlet and the Neumann heat flow on the original space can be expressed in terms of the heat flow on the doubled space.

More precisely now, let X be an n-dimensional Alexandrov space with curvature bounded below, and let $\mathrm{m}_X = \Phi \mathcal{H}^n_X$ be a measure for a double semi-concave function $\Phi : X \to [0,\infty)$. Let $(P_t)_{t\geq 0}$ denote the heat semigroup with Neumann boundary conditions on X and let $(P^0_t)_{t\geq 0}$ denote the heat semigroup on $X^0 := X \setminus \partial X$ with Dirichlet boundary conditions with respective generators Δ and Δ^0.

Theorem 1.11. *Assume the metric measure space (X, d_X, m_X) satisfies the condition $CD(K,\infty)$ for $K \in \mathbb{R}$. Then the following gradient estimate of Bakry-Emery type*

$$\big|\nabla P^0_t f\big| \leq e^{-Kt}\, P_t \big|\nabla f\big| \quad \text{a.e. on } X^0 \tag{1}$$

and the following Bochner inequality hold true

$$\frac{1}{2}\Delta \left|\nabla f\right|^2 - \left\langle \nabla f, \nabla \Delta^0 f\right\rangle \geq K \left|\nabla f\right|^2 \tag{2}$$

weakly on X^0 for all sufficiently smooth f on X. (Note that in the latter estimate, two different Laplacians appear and in the former, two different heat semigroups.) More precisely, (1) *holds for all $f \in W^{1,2}_0(X^0)$, the form domain for the Dirichlet Laplacian. And* (2) *is rigorously formulated as*

$$\frac{1}{2}\int_{X^0} \Delta\varphi \left|\nabla f\right|^2 d\mathrm{m} - \iint_{X^0} \varphi \left\langle \nabla f, \nabla \Delta^0 f\right\rangle d\mathrm{m} \geq K \int_{X^0} \varphi \left|\nabla f\right|^2 d\mathrm{m}$$

for all $f \in D(\Delta^0)$ with $\Delta^0 f \in W^{1,2}_0(X^0)$ and all nonnegative $\varphi \in D(\Delta^0)$ with $\varphi, \Delta^0\varphi \in L^\infty$.

Proof. Both estimates follow from Corollary 1.8 and [PS18], Thm. 1.26. For the readers' convenience, let us briefly recall the main argument. The estimates for the Dirichlet heat semigroup and Dirichlet Laplacian are direct consequences of analogous estimates for the heat semigroup $(\hat{P}_t)_{t\geq 0}$ and Laplacian $\hat{\Delta}$ on the doubled space

$$\hat{X} := X \cup X' \Big/_{\partial X = \partial X'}$$

obtained by gluing X and a copy of it, say X', along their common boundary $\partial X \sim \partial X'$. Then Dirichlet and Neumann heat semigroups on X can be expressed in terms of the heat semigroup on $\hat{X}$ as

$$P^0_t f = \hat{P}_t(f - f'), \qquad P_t f = \hat{P}_t(f + f')$$

for any given bounded, measurable $f : X \to \mathbb{R}$ where f is extended to $\hat{X}$ by putting $f := 0$ on $\hat{X} \setminus X$ and where $f' : \hat{X} \to \mathbb{R}$ is defined as $f'(x') := f(x)$ if $x' \in X'$ denotes the mirror point of $x \in X$. Then the gradient estimate for $\hat{P}_t$ on $\hat{X}$ obviously implies that

$$|\nabla P^0_t f| = |\nabla \hat{P}_t(f - f')| \leq e^{-Kt}\, \hat{P}_t|\nabla(f - f')| = e^{-Kt}\, P_t|\nabla f|$$

for every $f \in W^{1,2}_0(X^0)$.

Actually, (1) is stated in [PS18] only for functions $f \in W^{1,2}_0(X^0)$ which in addition satisfy $f, |\nabla f| \in L^1$. But any $f \in W^{1,2}_0(X^0)$ can be approximated in $W^{1,2}$-norm by compactly supported Lipschitz functions f_n (which in particular satisfy $f_n, |\nabla f_n| \in L^1$). Hence, $P_t|\nabla f|$ is the L^2-limit of $P_t|\nabla f_n|$ and the claim follows by passing to a suitable subsequence which leads to a.e.-convergence. □

We outline the remaining content of the article. In section 2 we recall preliminaries and basics on optimal transport, Ricci curvature for metric measure spaces, Alexandrov spaces, gluing of Alexandrov spaces and 1D localisation technique. We also state a new result by Cavalletti and Milman on characterizing the Ricci curvature bounds via $1D$ localisation.

In section 3 we will give two application of the $1D$ localisation technique. The first application shows that almost all geodesics avoid set of $\mathcal{H}^n$-measure 0 in the boundary in the glued space. The second application shows that given a $1D$ localisation w.r.t. an arbitrary 1-Lipschitz function, geodesics that are tangential to the boundary have measure 0 w.r.t. the corresponding quotient measure.

In section 4 we use the results of the previous section to prove Theorem 1.7.

In section 5 we prove the glued space theorem applying the results we obtained in section 3 and section 4.

Acknowledgments. The authors want to thank Anton Petrunin for helpful conversations on gluing spaces and other topics.

2. Preliminaries

2.1. **Curvature-dimension condition.** Let (X, d) be a complete and separable metric space equipped with a locally finite Borel measure m. We call a triple (X, d, m) a metric measure space.

A geodesic is a length minimizing curve $\gamma : [a, b] \to X$. We denote the set of constant speed geodesics $\gamma : [a, b] \to X$ with $\mathcal{G}^{[a,b]}(X)$ equipped with the topology of uniform convergence and set $\mathcal{G}^{[0,1]}(X) =: \mathcal{G}(X)$. For $t \in [a,b]$ the evaluation map $e_t : \mathcal{G}^{[a,b]}(X) \to X$ is defined as $\gamma \mapsto \gamma(t)$ and e_t is continuous.

A set of geodesics $F \subset \mathcal{G}(X)$ is said to be *non-branching* if $\forall \epsilon \in (0,1)$ the map $e_{[0,\epsilon]}|_F$ is one to one.

The set of (Borel) probability measure is denoted with $\mathcal{P}(X)$, the subset of probability measures with finite second moment is $\mathcal{P}^2(X)$, the set of probability measures in $\mathcal{P}^2(X)$ that are m-absolutely continuous is denoted with $\mathcal{P}^2(X, \mathrm{m})$ and the subset of measures in $\mathcal{P}^2(X, \mathrm{m})$ with bounded support is denoted with $\mathcal{P}^2_b(X, \mathrm{m})$.

The space $\mathcal{P}^2(X)$ is equipped with the L^2-Wasserstein distance W_2. A dynamical optimal coupling is a probability measure $\Pi \in \mathcal{P}(\mathcal{G}(X))$ such that $t \in [0,1] \mapsto (e_t)_\#\Pi$ is a W_2-geodesic in $\mathcal{P}^2(X)$. The set of dynamical optimal couplings $\Pi \in \mathcal{P}(\mathcal{G}(X))$ between $\mu_0, \mu_1 \in \mathcal{P}^2(X)$ is denoted with $\mathrm{OptGeo}(\mu_0, \mu_1)$.

A metric measure space (X, d, m) is called *essentially nonbranching* if for any pair $\mu_0, \mu_1 \in \mathcal{P}^2(X, \mathrm{m})$ any $\Pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ is concentrated on a set of nonbranching geodesics.

Definition 2.1. For $\kappa \in \mathbb{R}$ we define $\cos_\kappa : [0, \infty) \to \mathbb{R}$ as the solution of

$$v'' + \kappa v = 0 \quad v(0) = 1 \ \ \& \ \ v'(0) = 0.$$

$\sin_\kappa$ is defined as solution of the same ODE with initial value $v(0) = 0$ & $v'(0) = 1$. That is

$$\cos_\kappa(x) = \begin{cases} \cosh(\sqrt{|\kappa|}x) & \text{if } \kappa < 0 \\ 1 & \text{if } \kappa = 0 \\ \cos(\sqrt{\kappa}x) & \text{if } \kappa > 0 \end{cases} \quad \sin_\kappa(x) = \begin{cases} \frac{\sinh(\sqrt{|\kappa|}x)}{\sqrt{|\kappa|}} & \text{if } \kappa < 0 \\ x & \text{if } \kappa = 0 \\ \frac{\sin(\sqrt{\kappa}x)}{\sqrt{\kappa}} & \text{if } \kappa > 0 \end{cases}$$

Let π_κ be the diameter of a simply connected space form $\mathbb{S}^2_k$ of constant curvature κ, i.e.

$$\pi_\kappa = \begin{cases} \infty & \text{if } \kappa \le 0 \\ \frac{\pi}{\sqrt{\kappa}} & \text{if } \kappa > 0 \end{cases}$$

For $K \in \mathbb{R}$, $N \in (0, \infty)$ and $\theta \ge 0$ we define the *distortion coefficient* as

$$t \in [0,1] \mapsto \sigma^{(t)}_{K,N}(\theta) = \begin{cases} \frac{\sin_{K/N}(t\theta)}{\sin_{K/N}(\theta)} & \text{if } \theta \in [0, \pi_{K/N}), \\ \infty & \text{otherwise.} \end{cases}$$

Note that $\sigma^{(t)}_{K,N}(0) = t$. Moreover, for $K \in \mathbb{R}$, $N \in [1, \infty)$ and $\theta \ge 0$ the *modified distortion coefficient* is defined as

$$t \in [0,1] \mapsto \tau^{(t)}_{K,N}(\theta) = \begin{cases} \theta \cdot \infty & \text{if } K > 0 \text{ and } N = 1, \\ t^{\frac{1}{N}} \left[\sigma^{(t)}_{K,N-1}(\theta)\right]^{1-\frac{1}{N}} & \text{otherwise} \end{cases}$$

where our convention is $0 \cdot \infty = 0$. It holds that

$$\tau^{(t)}_{K,N}(\theta) \ge \sigma^{(t)}_{K,N}(\theta). \tag{3}$$

Definition 2.2 ([Stu06, LV09, BS10])**.** A metric measure space (X, d, m) satisfies the *curvature-dimension condition* $CD(K, N)$ for $K \in \mathbb{R}$, $N \in [1, \infty)$ if for every pair $\mu_0, \mu_1 \in \mathcal{P}_b^2(X, \mathrm{m})$ there exists an L^2-Wasserstein geodesic $(\mu_t)_{t\in[0,1]}$ and an optimal coupling π between μ_0 and μ_1 such that

$$S_N(\mu_t|\,\mathrm{m}) \le -\int \left[\tau_{K,N}^{(1-t)}(\theta)\rho_0(x)^{-\frac{1}{N}} + \tau_{K,N}^{(t)}(\theta)\rho_1(y)^{-\frac{1}{N}}\right] d\pi(x,y) \tag{4}$$

where $\mu_i = \rho_i d\,\mathrm{m}$, $i = 0, 1$, and $\theta = d(x, y)$.

We say (X, d, m) satisfies the *reduced curvature-dimension condition* $CD^*(K, N)$ for $K \in \mathbb{R}$ and $N \in (0, \infty)$ if we replace the coefficients $\tau_{K,N}^{(t)}(\theta)$ with $\sigma_{K,N}^{(t)}(\theta)$.

Remark 2.3. By the inequality (3) the condition $CD(K, N)$ always implies the condition $CD^*(K, N)$ and the latter is equivalent to a local version of $CD(K, N)$. Under the assumptions that (X, d, m) is essentially nonbranching and m is finite Cavalletti and Milman [CM16] prove that $CD(K, N)$ and $CD^*(K, N)$ are equivalent (compare with Theorem 2.27 below).

Definition 2.4. A metric measure space (X, d, m) satisfies the Riemannian curvature-dimension condition $RCD(K, N)$ (or $RCD^*(K, N)$) if it satisfies the condition $CD(K, N)$ (or $CD^*(K, N)$) and is infinitesimally Hilbertian, that is the corresponding Cheeger energy is quadratic.

Remark 2.5. Since $RCD(K, N)$ and $RCD^*(K, N)$ spaces are essentially non-branching, the two conditions are equivalent provided m is finite (compare with Remark 2.15 in [KK17].

2.2. **Alexandrov spaces.** In the following we introduce metric spaces with Alexandrov curvature bounded from below. For an introduction to this subject we refer to [BBI01].

Definition 2.6. We define $\mathrm{md}_\kappa : [0, \infty) \to [0, \infty)$ as the solution of

$$v'' + \kappa v = 1 \quad v(0) = 0 \;\;\&\;\; v'(0) = 0.$$

More explicitly

$$\mathrm{md}_\kappa(x) = \begin{cases} \frac{1}{\kappa}\left(1 - \cos_\kappa x\right) & \text{if } \kappa \ne 0, \\ \frac{1}{2}x^2 & \text{if } \kappa = 0. \end{cases}$$

Definition 2.7. Let (X, d) be a complete geodesic metric space. We say (X, d) has curvature bounded below by $\kappa \in \mathbb{R}$ in the sense of Alexandrov if for any unit speed geodesic $\gamma : [0, l] \to X$ such that

$$d(y, \gamma(0)) + l + d(\gamma(l), y) < 2\pi_k, \tag{5}$$

it holds that

$$[\mathrm{md}_\kappa(d_y \circ \gamma)]'' + \mathrm{md}_\kappa(d_y \circ \gamma) \le 1. \tag{6}$$

If (X, d) has curvature bounded from below for some $k \in \mathbb{R}$ in the sense of Alexandrov, we say that (X, d) is an Alexandrov space.

Remark 2.8. Alexandrov spaces are non-branching.

Theorem 2.9 (Petrunin, [Pet11])**.** *Let (X, d) be an n-dimensional Alexandrov space with curvature bounded from below by k. Then, $(X, d, \mathcal{H}_X^n)$ satisfies the condition $CD(k(n-1), n)$.*

Corollary 2.10. *Let (X,d) be an n-dimensional Alexandrov space with curvature bounded from below by κ. Then, the metric measure space $(X,d,\mathcal{H}^n_X)$ satisfies the condition $RCD(\kappa(n-1),n)$.*

Proof. The statement is known. Here, we give a straightforward argument for completeness. It is enough to show that Cheeger energy is quadratic.

It is known that a doubling condition and a 1-1 Poincaré inequality hold for Alexandrov spaces. Hence, we can follow the same argument as in [KK17, Section 6]. It is known [BGP92] that for $\mathcal{H}^n$-a.e. points $x \in X$ the tangent cone T_pX is isometric to $\mathbb{R}^n$, and for a Lipschitz function the differential exists and is linear $\mathcal{H}^n$-a.e. [Che99, Theorem 8.1]. This implies the Cheeger energy is quadratic by the same argument as in [KK17, Section 6]. □

Let (X,d) be an n-dimensional Alexandrov space. We denote with T_pX the unique blow up tangent cone at $p \in X$. The tangent cone T_pX coincides with the metric cone $C(\Sigma_p)$ where Σ_p is the space of directions at p equipped with the angle metric. The definition of the angle metric is as follows. The angle $\angle(\gamma^1,\gamma^2)$ between two geodesics γ^i, $i=1,2$, with $\gamma^1(0)=\gamma^2(0)=p$ and parametrized by arclength is defined by the formula

$$\cos\angle(\gamma^1,\gamma^2) = \lim_{s,t\to 0}\frac{s^2+t^2-d(\gamma^1(s),\gamma^2(t))}{2st}.$$

Then, the space of directions Σ_pX is given as the metric completion of S_pX via $\angle$ where S_pX is the space of geodesics starting in p. We refer to [BBI01] for details. One can show that $(\Sigma_pX,\angle)$ is an $(n-1)$-dimensional Alexandrov space with curvature bounded below by 1. We say $p \in X$ is a regular point if $\Sigma_pX = \mathbb{S}^{n-1}$. We denote the set of regular points with X^{reg}. As was mentioned above $\mathcal{H}^n$-almost every point $p \in X$ is regular. A theorem of Petrunin [Pet98] is the next statment. If $\gamma:[a,b]\to X$ is a geodesic such that there exists $t_0 \in [a,b]$ with $\gamma(t_0) = X^{reg}$ then $\gamma(t_0) \in X^{reg}$ for all $t \in [a,b]$.

One can define the boundary $\partial X \subset X$ of X via induction over the dimension. One says that $p \in X$ is a boundary point if $\partial\Sigma_pX \neq \emptyset$. ∂X denotes the set of all boundary points, and we call ∂X the boundary of X.

Let $p \in X$ be a boundary point, that is $\partial\Sigma_pX \neq \emptyset$. We say $v \in \Sigma_pX$ is a normal vector in p if $\angle(v,w) = \frac{\pi}{2}$ for any $w \in \partial\Sigma_pX$.

Theorem 2.11 (Perelman [Per93], [PP93, Lemma 4.3])**.** *For any point in an n-dimensional Alexandrove space there exists an arbitrary small, closed, geodesically convex neighborhood.*

2.3. **Gluing.** Let (X_0,d_{X_0}) and (X_1,d_{X_1}) be complete, n-dimensional Alexandrov spaces with non-empty boundaries ∂X_0 and ∂X_1 equipped with their intrinsic distances $d_{\partial X_0}$ and $d_{\partial X_1}$ respectively. Let $\mathcal{I}:\partial X_0 \to \partial X_1$ be an isometry.

The topological glued space of X_0 and X_1 along their boundaries w.r.t. $\mathcal{I}$ is defined as the quotient space $X_0\dot\cup X_1/R$ of the disjoint union $X_0\dot\cup X_1$ where

$$x \sim_R y \text{ if and only if } \mathcal{I}(x)=y \text{ if } x\in\partial X_0, y\in\partial X_1, \text{ and } x=y \text{ otherwise.}$$

The equivalence relaiton R induces a pseudo distance on $X_0\dot\cup X_1 = X$ as follows. First, we introduce an extended metric d on $X_0\dot\cup X_1$ via $d(x,y)=d_{X_i}(x,y)$ if $x,y\in X_i$ for some $i\in\{0,1\}$ and $d(x,y)=\infty$ otherwise. Then, for $x,y \in X_0\dot\cup X_1$

we define

$$\hat{d}(x,y) = \inf \sum_{i=0}^{k-1} d(p_i, q_i)$$

where the infimum runs over all collection of tuples $\{(p_i, q_i)\}_{i=0,\dots,k-1} \subset X \times X$ for some $k \in N$ such that $q_i \sim_R p_{i+1}$, for all $i = 0, \dots, k-1$ and $x = p_0, y = q_k$. One can show that $x \sim_R y$ if and only if $\hat{d}(x,y) = 0$ if X_0 and X_1 are Alexandrov spaces. The glued space between (X_0, d_{X_0}) and (X_1, d_{X_1}) w.r.t. $\mathcal{I} : \partial X_0 \to \partial X_1$ is the metric space defined as

$$X_0 \cup_{\mathcal{I}} X_1 := (X_0 \cup X_1 / R, \hat{d}).$$

In the following we denote the glued space as (Z, d_Z), and boundary ∂X_0 with its intrinsic metric with (Y, d_Y). In the case when $X_0 = X_1 = X$ and $\mathcal{I} = \mathrm{id}_{\partial X}$, we call $X \cup_{\mathcal{I}} X =: \hat{X}$ the double space of X.

Remark 2.12. For every point $p \in X_i \backslash Y$, $i = 0, 1$, there exists $\epsilon > 0$ such that $B_\epsilon(p) \subset X_i$ and $d_Z|_{B_\epsilon(p) \times B_\epsilon(p)} = d_{X_i}|_{B_\epsilon(p) \times B_\epsilon(p)}$.

Theorem 2.13 (Petrunin, [Pet97]). *Let (X_0, d_{X_0}) and (X_1, d_{X_0}) be n-dimensional Alexandrov spaces with nonempty boundary and curvature bounded from below by k. Let $\mathcal{I} : \partial X_0 \to \partial X_1$ be an isometry w.r.t. the induces intrinsic metrics. Then, $X_0 \cup_{\mathcal{I}} X_1$ is an Alexandrov space with curvature bounded from below by k.*

Remark 2.14. The special case of a double space was proven first by Perelman [Per].

Remark 2.15. By symmetry of the construction one can see that geodesics in the double space $\hat{X}$ of an Alexandrov space X connecting points in $\hat{X} \backslash \partial X$ intersect with the boundary at most once, and the restriction of the double metric to $X \backslash \partial X$ coincides with d_X. This observation was crucial in Perelman's proof of the double theorem. However, in the general case of glued spaces it's not clear if geodesics connecting points in $Z \backslash Y$ intersect Y at most finitely many times. This creates an extra difficulty in the proof of Petrunin's theorem and also in the proof of Theorem 1.1.

Let us recall some additional facts about the glued space Z [Pet97]. Since the boundary $Y \subset X_0$ is an extremal subset in X_0, the following holds. Consider the blow up tangent cone $\lim_{\epsilon \to 0}(X_0, \frac{1}{\epsilon} d_{X_0}, p) = T_p X_0$ for $p \in Y$. Then, $\lim_{\epsilon \to 0}(Y, \frac{1}{\epsilon} d_Y, p) = T_p Y$ w.r.t. the intrinsic metric d_Y on Y is equal to $C(\partial \Sigma_p X_0) = \partial C(\Sigma_p X_0)$.

It follows that $\partial \Sigma_p X_0$ is isometric to $\partial \Sigma_p X_1$ via an isometry $\mathcal{I}'$ that arises as blow up limit of $\mathcal{I}$.

Then it also follows from Petrunin's proof of the glued space theorem that $T_p Z = T_p X_0 \cup_{\mathcal{I}'} T_p X_1$ and $\Sigma_p Z = \Sigma_p X_0 \cup_{\mathcal{I}'} \Sigma_p X_1$.

If $p \in Y$ is a regular point in the glued space Z, that is $\Sigma_p Z = \mathbb{S}^{n-1}$, it follows by maximality of the volume of $\mathbb{S}^{n-1}$ in the class of Alexandrov spaces with curvature bounded below by 1 that $\Sigma_p X_0 = \mathbb{S}^{n-1}_+$ and $\Sigma_p X_0 = \mathbb{S}^{n-1}_-$ where $\mathbb{S}^{n-1}_{+/-}$ denote the lower and upper half sphere respectively, and $\Sigma_p Y = \partial \Sigma_p X_0 = \mathbb{S}^{n-2}$. In particular, the north pole N in $\mathbb{S}^{n-1}_+$ is the unique normal vector a $p \in Y \subset X_0$, and the south pole $S \in \mathbb{S}^{n-1}_-$ is the unique normal vector a $p \in Y \subset X_1$.

2.4. **Semi-concave functions.** We recall a few basic facts about concave functions following [Pla02].

A function $u : [a, b] \to \mathbb{R}$ is called concave if the segment between any pair of points lies below the graph. If u is concave, u is lower semi continuous and continuous on (a, b). The secant slope $\frac{u(s)-u(t)}{s-t}$ is a decreasing function in s and t. It follows that the right and left derivative

$$\frac{d^+}{dr}u(r) = \lim_{h\downarrow 0} \frac{u(r+h) - u(r)}{h} \quad \& \quad \frac{d^-}{dr}u(r) = \lim_{h\downarrow 0} \frac{u(r-h) - u(r)}{-h}$$

exist in $\mathbb{R} \cup \{\infty\}$ and $\mathbb{R} \cup \{-\infty\}$ respectively for all $r \in [a, b]$ with values in $\mathbb{R}$ if $r \in (a, b)$. Moreover $\frac{d^+}{dr}u(r) \leq \frac{d^-}{dr}u(r)$ and $\frac{d^{+/-}}{dr}u(r)$ are decreasing in r. If $\frac{d^+}{dr}u(a) < \infty$ ($\frac{d^-}{dr}u(b) > -\infty$), u is continuous in a (in b).

Let $u : [a, b] \to (0, \infty)$ satisfy

$$u \circ \gamma(t) \geq \sigma_\kappa^{(1-t)}(|\dot{\gamma}|)u \circ \gamma(0) + \sigma_\kappa^{(t)}(|\dot{\gamma}|)u \circ \gamma(1) \tag{7}$$

for any constant speed geodesic $\gamma : [0, 1] \to [a, b]$. It follows that u is lower semi continuous and continuous on (a, b).

Definition 2.16. Let $f : [a, b] \to \mathbb{R}$ be continuous on (a, b), and let $F : [a, b] \to \mathbb{R}$ such that $F'' = f$ on (a, b). For a function $u : [a, b] \to \mathbb{R}$ we write $u'' \leq f$ on (a, b) if $u - F$ is concave on (a, b).

We say a function $u : (0, \theta) \to \mathbb{R}$ is λ-concave if $u'' \leq \lambda$. We say u is semiconcave if for any $r \in (0, \theta)$ we can find $\epsilon > 0$ and $\lambda \in \mathbb{R}$ such that u is λ-concave on $(r - \epsilon, r + \epsilon)$.

If u satisfies (7) for every constant speed geodesic $\gamma : [0, 1] \to [a, b]$, then one can check that

$$u'' + ku \leq 0 \text{ on } (a, b)$$

in the sense of the previous definition. We note that (7) implies that u is continuous on (a, b), and $U(t) = \int_a^b g(s, t)u(s)ds$ satisfies $U'' = -u$ on (a, b) where $g(s, t)$ is the Green function of the interval $[a, b]$.

On the other hand we have the next lemma.

Lemma 2.17. *Let $u : [a, b] \to \mathbb{R}$ be lower semi-continuous and continuous on (a, b) such that $u'' + ku \leq 0$ on (a, b) in the sense of the definition above.*

Then u satisfies (7) for every constant speed geodesic $\gamma : [0, 1] \to [a, b]$.

Proof. We sketch the proof. If $u'' + ku \leq 0$ then $u - kU$ is concave. In particular, it follows for $\phi \in C_c^2((a, b))$, $\phi \geq 0$, that

$$0 \geq \int (u + kU)\phi'' dt = \int u\phi'' + k \int u\phi dt$$

by the distributional characterisation of convexity (see [Sim11]). Hence, u satisfies $u'' + ku \leq 0$ in distributional sense, and therefore (7) follows by [EKS15, Lemma 2.8]. ∎

Lemma 2.18. *If u satisfies (7) for every constant speed geodesic $\gamma : [0, 1] \to [a, b]$ of length less than $\theta < b - a$, then u satisfies (7).*

If $u : [a,b] \to \mathbb{R}$ satifies (7), it is is semi-concave, therefore locally Lipschitz on (a,b) and hence differentiable $\mathcal{L}^1$-almost everywhere. Moreover, the right and left derivative also exist in this case and satisfy $\frac{d^+}{dr}u(r) \leq \frac{d^-}{dr}u(r)$ with equality if and only if u is differentiable in r. $\frac{d^{+/-}}{dr}u$ is continuous from the right/left. Since u is locally semi concave, the second derivative u'' exists $\mathcal{L}^1$-almost everywhere.

The following Lemma can be found in [Pla02] (Lemma 113).

Lemma 2.19. *Consider $u : (a,b) \to \mathbb{R}$ continuous such that $u'' \leq -ku$ on (a,c) and on (c,b) for some $c \in (a,b)$. Then $u'' \leq -ku$ on (a,b) if and only if*

$$\frac{d^-}{dr}u(c) \geq \frac{d^+}{dr}u(c).$$

Definition 2.20. Let (X,d) be an n-dimensional Alexandrov space and $\Omega \subset X$. A function $f : \Omega \to \mathbb{R}$ is λ-concave if f is **locally Lipschitz** and $f \circ \gamma : [0, \mathrm{L}(\gamma)] \to \mathbb{R}$ is λ-concave for every constant speed geodesic $\gamma : [0, \mathrm{L}(\gamma)] \to \Omega$. A function $f : X \to \mathbb{R}$ is semi-concave if for every $p \in X$ there exists a neighborhood $U \ni p$ such that $f|_U$ is λ-concave for some real λ.

We say a function $f : X \to \mathbb{R}$ is double semi-concave if the function $f \circ P : \hat{X} \to \mathbb{R}$ is semi-concave where the $P : \hat{X} \to X$ is the projection map form the double space $\hat{X}$ to X. If $\partial X = \emptyset$, concavity and double concavity coincide.

In [Pet07] Petrunin defines concavity as double concavity.

Let X be an Alexandrov space and let $f : X \to \mathbb{R}$ be locally Lipschitz. Then, the limit

$$\lim_{r \downarrow 0} \frac{f \circ \gamma(r) - f \circ \gamma(0)}{r} = \frac{d^+}{dr}(f \circ \gamma)(0) =: df_p(\dot{\gamma}) =: df(\dot{\gamma}) \in \mathbb{R}$$

exists for every geodesic $\gamma : [0, \theta] \to X$ parametrized by arc length with $\gamma(0) = p$, and for every $p \in X$. We call $df : T_pX \to \mathbb{R}$ the differential of f.

The differential df_p on T_pX can be equivalently defined as limit of the sequence $\frac{1}{\epsilon}(f - f(p)) : (\frac{1}{\epsilon}X, p) \to \mathbb{R}$. This limit is understood in the sense of Gromov's Arzela-Ascoli theorem (see for instance [Sor04]. It also makes sense for functions that are just locally Lipschitz but the differential is not unique in this case. Note that since Alexandrov spaces are nonbranching, under GH convergence of Alexandrov spaces every geodesic in the limit is a limit of geodesics in the sequence. Therefore it follows that $df_p : T_pX \to \mathbb{R}$ is Lipschitz for Lipschitz functions, and also concave if f is semiconcave. This in turn implies that $v = df_p : \Sigma_p \to \mathbb{R}$ satisfies $v'' + v \leq 0$ along geodesics in Σ_p.

2.5. $1D$ **localisation of generalized Ricci curvature bounds.** In this section we will recall the localisation technique introduced by Cavalletti and Mondino. The presentation follows Section 3 and 4 in [CM17]. We assume familarity with basic concepts in optimal transport.

Let (X, d, m) be a locally compact metric measure space that is essentially nonbranching. We assume that $\operatorname{supp} \mathrm{m} = X$.

Let $u : X \to \mathbb{R}$ be a 1-Lipschitz function. Then

$$\Gamma_u := \{(x,y) \in X \times X : u(x) - u(y) = d(x,y)\}$$

is a d-cyclically monotone set, and one defines $\Gamma_u^{-1} = \{(x,y) \in X \times X : (y,x) \in \Gamma_u\}$. If $\gamma \in \mathcal{G}^{[a,b]}(X)$ for some $[a,b] \subset \mathbb{R}$ such that $(\gamma(a), \gamma(b)) \in \Gamma_u$ then $(\gamma(t), \gamma(t)) \in \Gamma_u$

for $a < t \leq s < b$. It is therefore natural to consider the set G of unit speed transport geodesics $\gamma : [a,b] \to \mathbb{R}$ such that $(\gamma(t), \gamma(s)) \in \Gamma_u$ for $a \leq t \leq s \leq b$.

The union $\Gamma \cup \Gamma^{-1}$ defines a relation R_u on $X \times X$, and R_u induces a *transport set with endpoints*

$$\mathcal{T}_u := P_1(R_u \backslash \{(x,y) : x = y\}) \subset X$$

where $P_1(x,y) = x$. For $x \in \mathcal{T}_u$ one defines $\Gamma_u(x) := \{y \in X : (x,y) \in \Gamma_u\}$, and similar $\Gamma_u^{-1}(x)$ as well as $R_u(x) = \Gamma_u(x) \cup \Gamma_u^{-1}(x)$. Since u is 1-Lipschitz, Γ_u, Γ_u^{-1} and R_u are closed as well as $\Gamma_u(x), \Gamma_u^{-1}(x)$ and $R_u(x)$.

The *transport set without branching* $\mathcal{T}_u^b$ associated to u is then defined as

$$\mathcal{T}_u^b = \{x \in \mathcal{T}_u : \forall y, z \in R_u(x) \Rightarrow (y,z) \in R_u\}$$

$\mathcal{T}_u$ and $\mathcal{T}_u \backslash \mathcal{T}_u^b$ are σ-compact, and $\mathcal{T}_u^b$ and $R_u \cap \mathcal{T}_u^b \times \mathcal{T}_u^b$ are Borel sets. In [Cav14] Cavalletti shows that R_u restricted to $\mathcal{T}_u^b \times \mathcal{T}_u^b$ is an equivalence relation. Hence, from R_u one obtains a partition of $\mathcal{T}_u^b$ into a disjoint family of equivalence classes $\{X_\gamma\}_{\gamma \in Q}$. Moreover, $\mathcal{T}_u^b$ is also σ-compact.

Every X_γ is isometric to some interval $I_\gamma \subset \mathbb{R}$ via an isometry $\gamma : I_\gamma \to X_\gamma$. $\gamma : I_\gamma \to X$ extends to a geodesic that is arclength parametrized and that we also denote γ defined on the closure $\overline{I}_\gamma$ of I_γ. We set $\overline{I}_\gamma = [a_\gamma, b_\gamma]$.

The set of equivalence classes Q has a measurable structure such that $\mathfrak{Q} : \mathcal{T}_u^b \to Q$ is a measurable map. We set $\mathfrak{q} := \mathfrak{Q}_\# \mathrm{m}|_{\mathcal{T}_u^b}$.

Recall that a measurable section of the equivalence relation R on $\mathcal{T}_u^b$ is a measurable map $s : \mathcal{T}_u^b \to \mathcal{T}_u^b$ such that $R_u(s(x)) = R_u(x)$ and $(x,y) \in R_u$ implies $s(x) = s(y)$. In [Cav14, Proposition 5.2] Cavalletti shows there exists a measurable section s of R on $\mathcal{T}_u^b$. Therefore, one can identify the measurable space Q with $\{x \in \mathcal{T}_u^b : x = s(x)\}$ equipped with the induced measurable structure and we can see $\mathfrak{q}$ as a Borel measure on X. By inner regularity there exists a σ-compact set $Q' \subset X$ such that $\mathfrak{q}(Q \backslash Q') = 0$ and in the following we will replace Q with Q' without further notice. We parametrize $\gamma \in Q$ such that $\gamma(0) = s(x)$. In particular, $0 \in (a_\gamma, b_\gamma)$.

Now, we assume that (X, d, m) is an essentially non-branching $CD^*(K,N)$ space for $K \in \mathbb{R}$ and $N \geq 1$. The following lemma is Theorem 3.4 in [CM17].

Lemma 2.21. *Let (X,d,m) be an essentially non-branching $CD^*(K,N)$ space for $K \in \mathbb{R}$ and $N \in (1,\infty)$ with $\operatorname{supp} \mathrm{m} = X$ and $\mathrm{m}(X) < \infty$. Then, for any 1-Lipschitz function $u : X \to \mathbb{R}$, it holds $\mathrm{m}(\mathcal{T}_u \backslash \mathcal{T}_u^b) = 0$.*

For $\mathfrak{q}$-a.e. $\gamma \in Q$ it was proved in [CM16] (Theorem 7.10) that

$$R_u(x) = \overline{X_\gamma} \supset X_\gamma \supset (R_u(x))^\circ \quad \forall x \in \mathfrak{Q}^{-1}(\gamma).$$

where $(R_u(x))^\circ$ denotes the relative interiour of the closed set $R_u(x)$.

Theorem 2.22. *Let (X,d,m) be a compact geodesic metric measure space with $\operatorname{supp} \mathrm{m} = X$ and m finite. Let $u : X \to \mathbb{R}$ be a 1-Lipschitz function, let $(X_\gamma)_{\gamma \in Q}$ be the induced partition of $\mathcal{T}_u^b$ via R_u, and let $\mathfrak{Q} : \mathcal{T}_u^b \to Q$ be the induced quotient map as above. Then, there exists a unique strongly consistent disintegration $\{\mathrm{m}_\gamma\}_{\gamma \in Q}$ of $\mathrm{m}|_{\mathcal{T}_u^b}$ w.r.t. $\mathfrak{Q}$.*

Define the ray map

$$g : \mathcal{V} \subset Q \times \mathbb{R} \to X \quad \text{via} \quad \operatorname{graph}(g) = \{(\gamma, t, x) \in Q \times \mathbb{R} \times X : \gamma(t) = x\}$$

By definition $\mathcal{V} = g^{-1}(\mathcal{T}_u^b)$. The map g is Borel measurable, $g(\gamma, \cdot) = \gamma : (a_\gamma, b_\gamma) \to X$ is a geodesic, $g : \mathcal{V} \to \mathcal{T}_u^b$ is bijective and its inverse is given by $g^{-1}(x) = (\mathfrak{Q}(x), \pm d(x, \mathfrak{Q}(x)))$.

Theorem 2.23. *Let (X, d, m) be an essentially non-branching $CD^*(K, N)$ space with* $\operatorname{supp} \mathrm{m} = X$*, $\mathrm{m}(X) < \infty$, $K \in \mathbb{R}$ and $N \in (1, \infty)$.*

Then, for any 1-Lipschitz function $u : X \to \mathbb{R}$ there exists a disintegration $\{\mathrm{m}_\gamma\}_{\gamma \in Q}$ of m *that is strongly consistent with R_u^b.*

Moreover, for $\mathfrak{q}$-a.e. $\gamma \in Q$, m_γ is a Radon measure with $\mathrm{m}_\gamma = h_\alpha \mathcal{H}^1|_{X_\alpha}$ and $(X_\gamma, d_{X_\gamma}, \mathrm{m}_\gamma)$ verifies the condition $CD(K, N)$.

More precisely, for $\mathfrak{q}$-a.e. $\gamma \in Q$ it holds that

$$h_\gamma(c_t)^{\frac{1}{N-1}} \geq \sigma_{K/N-1}^{(1-t)}(|\dot{c}|) h_\gamma(\gamma_0)^{\frac{1}{N-1}} + \sigma_{K/N-1}^{(t)}(|\dot{c}|) h_\gamma(\gamma_1)^{\frac{1}{N-1}} \tag{8}$$

for every geodesic $c : [0, 1] \to (a_\gamma, b_\gamma)$.

Remark 2.24. The property (8) yields that h_γ is locally Lipschitz continuous on (a_γ, b_γ) [CM17, Section 4], and that $h_\gamma : \mathbb{R} \to (0, \infty)$ satifies

$$\frac{d^2}{dr^2} h_\gamma^{\frac{1}{N-1}} + \frac{K}{N-1} h_\gamma^{\frac{1}{N-1}} \leq 0 \text{ on } (a_\gamma, b_\gamma) \text{ in distributional sense.}$$

2.6. Characterization of curvature bounds via $1D$ localisation.

Definition 2.25. Let (X, d_X, m_X) be an essentially non-braching metric measure space with $\mathrm{m}(X) = 1$, let $K \in \mathbb{R}$ and $N \geq 1$, and let $u : X \to \mathbb{R}$ be a 1-Lipschitz function. We say that (X, d_X, m_X) satisfies the condition $CD_u^1(K, N)$ if there exist subsets $X_\gamma \subset X$, $\gamma \in Q$, such that

(i) There exists a disintegration of $\mathrm{m}_{\mathcal{T}_u}$ on $(X_\gamma)_{\gamma \in Q}$:

$$\mathrm{m}|_{\mathcal{T}_u} = \int_{X_\gamma} \mathrm{m}_\gamma \, d\mathfrak{q}(\gamma) \quad \text{with } \mathrm{m}_\gamma(X_\gamma) = 1 \text{ for } \mathfrak{q}\text{-a.e. } \gamma \in Q.$$

(ii) For $\mathfrak{q}$-a.e. $\gamma \in Q$ the set X_γ is the image $\mathrm{Im}(\gamma)$ of a geodesic $\gamma : I_\gamma \to X$ for an interval $I_\gamma \subset \mathbb{R}$.

(iii) The metric measure space $(X_\gamma, d_{X_\gamma}, \mathrm{m}_\gamma)$ satisfies the condition $CD(K, N)$.

The metric measure space (X, d_X, m_X) satisfies the condition $CD_{Lip}^1(K, N)$ if it satisfies the condition $CD_u^1(K, N)$ for any 1-Lipschitz function $u : X \to \mathbb{R}$.

Remark 2.26. From the previous subsection it is immediatly clear that the condition $CD(K, N)$ implies the condition $CD_{Lip}^1(K, N)$.

The following theorem will play an important role in the proof of our gluing result.

Theorem 2.27 (Cavalletti-Milman)**.** *If an essentially non-branching metric measure space (X, d_X, m_X) satisfies the condition $CD_{Lip}^1(K, N)$ for $K \in \mathbb{R}$ and $N \in [1, \infty)$ then it satisfies the condition $CD^*(K, N)$.*

If m *is a finite measure it even satisfies $CD(K, N)$.*

Remark 2.28. Taking into account a disintegration result for σ-finite measures in [CM18, Section 3.1] it should be possible to prove the previous theorem also for σ-finite measures. Consequently in our main theorem we would then be able to replace the condition CD^* with the condition CD in general.

3. Applying $1D$ localisation

3.1. **First application.** Let X_0 and X_1 be n-dimensional Alexandrov spaces, let $\mathcal{I} : \partial X_0 \to \partial X_1$ be an isometry as in Theorem 2.13. Let Z be the glued space, set $\partial X_0 = \partial X_1 =: Y$ and recall that $(Z, d_Z, \mathcal{H}^n)$ satisfies $CD(k(n-1), n)$ for some $k \in \mathbb{R}$. We consider continuous functions Φ_0 and Φ_1 on X_0 and X_1 respectively such that $\Phi_0|_{\partial X_0} = \Phi_1|_{\partial X_1}$, and we define $\Phi_Z : X_0 \cup_{\mathcal{I}} X_1 \to \mathbb{R}$ by

$$\Phi_Z(x) = \begin{cases} \Phi_0(x) & \text{if } x \in X_0, \\ \Phi_1(x) & \text{otherwise.} \end{cases}$$

Lemma 3.1. *Let X be an n-dimensional Alexandrov space with $Y = \partial X \neq \emptyset$. Let $\Phi : X \to \mathbb{R}$, $i = 0, 1$ be semi-concave. Then $\Phi|_Y : Y \to \mathbb{R}$ is differentiable $\mathcal{H}^{n-1}$-a.e. meaning that Y contains a subset A such that $\mathcal{H}^{n-1}(Y \backslash A) = 0$ and for every $a \in A$ it holds that $T_aY \cong \mathbb{R}^{n-1}$ and $d\Phi \colon T_aY \to \mathbb{R}$ is linear.*

Proof. One says a point $p \in Y = \partial X$ is boundary regular if $T_pY = \mathbb{R}^{n-1}$. Since Y is the boundary of an n-dimensional Alexandrov space X, it is $\mathcal{H}^{n-1}$-rectifiable. Even stronger, the set of boundary regular points $\mathcal{R}(Y)$ in Y has full $\mathcal{H}^{n-1}_Y$-measure and for any $\epsilon > 0$ one can cover $\mathcal{R}(Y)$ by $(1+\epsilon)$-biLipschitz coordinate maps $F_i^\epsilon : U_i \to V_i \subset \mathbb{R}^{n-1}$ where each V_i is open. The maps F_i^ϵ are given by standard strainer coordinates centered at boundary regular points. By the metric version of Rademacher's theorem due to Cheeger [Che99] it follows that $\Phi|_Y$ is differentiable $\mathcal{H}^{n-1}$-a.e. .

Let us give another, self-contained argument that does not rely on Cheeger's theorem.

Without loss of generality we can assume that Φ is L-Lipscjhitz for some finite $L > 0$. For $0 < \epsilon < 1$ we consider the maps F_i^ϵ. Let us drop the superscript ϵ for a moment. In particular, each coordinate component F_i^j, $j = 1, \ldots, n-1$, of F_i is a semiconcave function on U_i and admits a differential $dF_i^j|_p$ in the sense of Alexandrov spaces at every point $p \in U_i$. Since F_i and F_i^{-1} are $(1+\epsilon)$-biLipschitz, one has that $(1+\epsilon)|v| \leq |dF_i^j|_p(v)| \leq (1+\epsilon)|v|$ for every $p \in U_i$ and every $v \in T_pX$.

The function $\Phi \circ F_i^{-1} : V_i \to \mathbb{R}$ is $2L$-Lipschitz and therefore differentiable $\mathcal{L}^{n-1}$-a.e. by the standard Rademacher theorem. So we can choose a set of full measure $W_i^\epsilon = W_i$ in U_i such that $\forall p \in F_i(W_i) \subset V_i$ the point $p \in W_i$ is regular and the function $\Phi \circ F_i^{-1}$ is differentiable at $F_i(p)$.

The chain rule for Alexandrov space differentials yields

$$d\Phi|_p = d(\Phi \circ F_i^{-1})|_{F_i(p)} \circ DF_i|_p \quad \forall p \in W_i \tag{9}$$

where $DF_i|_p = (dF_i^1|_p, \ldots, dF_i^{n-1}|_p)$.

We obtain by (9) that for all $p \in W_i$ and for any $\epsilon > 0$ the Alexandrov differential $d\Phi|_p : \mathbb{R}^{n-1} \mapsto \mathbb{R}$ of Φ at p is the composition of a $2L$-Lipschitz linear map $A^\epsilon = d(\Phi \circ F_i^{-1})|_{F_i(p)} : \mathbb{R}^{n-1} \mapsto \mathbb{R}$ and 1-homogeneous map $B^\epsilon = DF_i|_p : \mathbb{R}^{n-1} \mapsto \mathbb{R}^{n-1}$ that is ϵ-close to an isometry on a the unit ball around the origin (we have identified T_pY with $\mathbb{R}^{n-1}$ in the above).

Let us consider $\epsilon_n = \frac{1}{n}$ and let $p \in \bigcap_{n \in \mathbb{N}} \bigcup W_i^{\frac{1}{n}}$. After eventually choosing a subsequence $A^{\frac{1}{n}} \to A$ for a linear map A and $B^{\frac{1}{n}} \to B$ for an isometry B. Hence $d\Phi_p = A \circ B$ is linear. Since $\bigcap_{n \in \mathbb{N}} \bigcup W_i^{\frac{1}{n}}$ has full $\mathcal{H}^{n-1}$-measure this yields the claim.

□

Proposition 3.2. *Let Z be the glued space, let $Y=\partial X_i\subset Z$. Let $N\subset Y$ such that $\mathcal{H}^{n-1}(N)=0$. Let $(X_\gamma)_{\gamma\in Q}$ be the 1D localisation of $m=\mathcal{H}^n_Z$ w.r.t. the 1-Lipschitz function $u=d(x_1,\cdot)$ for $x_1\in B_\eta(x_1)\subset X_1$ and $\eta>0$. Let $(Q,\mathfrak{q})$ and $\mathfrak{Q}:\mathcal{T}^b_u\to Q$ be the corresponding quotient space and the quotient map. Then*

$$\mathcal{H}^n(X_0\cap\mathfrak{Q}^{-1}(\{\gamma\in Q:\exists t\in[0,L(\gamma))\text{ s.t. }\gamma(t)\in N\}))=0.$$

Proof. The property that $\mathcal{H}^{n-1}(N)=0$ is equivalent to the following statement. For any $\epsilon>0$ and any $\delta\in(0,\eta/4)$ there exist $(r_i)_{i\in\mathbb{N}}$ with $r_i\in(0,\delta)$ and $x_i\in Z$, $i\in\mathbb{N}$, such that

$$N\subset\bigcup_{i\in\mathbb{N}}B_{r_i}(x_i)\quad\text{and}\quad\sum_{i\in\mathbb{N}}(r_i)^{n-1}\leq\epsilon. \tag{10}$$

We set $Q_i'=\{\gamma\in\mathcal{G}(X):\exists t\in[0,1]\text{ s.t. }\gamma(t)\in B_r(x_i)\}$ and $Q_i=Q_i'\backslash\bigcup_{j=1}^{i-1}Q_j$. W.l.o.g. we assume that $\mathfrak{q}(Q_i)>0$ for any $i\in\mathbb{N}$. We will prove that

$$\mathcal{H}^n\left(X_0\cap\bigcup_{i\in\mathbb{N}}\mathfrak{Q}^{-1}(Q_i)\right)\leq C\epsilon$$

for some constant $C=C(k,n)$. This implies the claim of the proposition.

Let us fix $i\in\mathbb{N}$. There exists $Q_i^\dagger$ with $\mathfrak{q}(Q_i)=\mathfrak{q}(Q_i^\dagger)$ such that m_γ admits a density h_γ w.r.t. $\mathcal{H}^1$ and $(X_\gamma,\mathrm{m}_\gamma)$ is $CD(k(n-1),n)$ for all $\gamma\in Q_i^\dagger$. In particular, if $J\subset I_\gamma$ and $J_t=tb_\gamma+(1-t)J$, then the Brunn-Minkowski inequality implies

$$\mathrm{m}_\gamma(J_t)\geq C(k,n)t^n\,\mathrm{m}_\gamma(J).$$

We pick $J=\gamma^{-1}(X_0)$. Let $D=\operatorname{diam}Z$ and choose $s\in\mathbb{N}$ such that $\frac{D}{s-1}\leq r_i\leq\frac{D}{s}$. We decompose J into intervals $(J^{l,s})_{l=1,\dots,s}$ such that $|J^{l,s}|\leq\frac{D}{s}$. Moreover, there exists $t_l\in(0,1)$ such that $J^{l,s}_{t_l}\subset\gamma^{-1}(B_{2r_i}(x_i))$. Since $r_i\leq\delta<\eta/4$ and since $B_\eta(x_1)\subset X_1$, we have $B_{\eta/2}(x_1)\cap B_{2r_i}(x_i)=\emptyset$. Therefore $t_l\geq\frac{\eta}{2D}$. Hence

$$\frac{1}{r_i}\,\mathrm{m}_\gamma(B_{2r_i}(x_i))\geq\frac{s}{D}\,\mathrm{m}_\gamma(J^{l,s}_{t_l})\geq\sum_{l=1}^{s}C(k,n)t_l^n\,\mathrm{m}_\gamma(J^{l,s})\geq C(k,n,D)\eta^n\,\mathrm{m}_\gamma(J).$$

Integration w.r.t. $\mathfrak{q}$ on Q_i yields

$$\hat{C}(k,n)r_i^{n-1}\geq\frac{1}{r_i}\mathcal{H}^n(B_{2r_i}(x_i))\geq C(k,n,D,\eta)\mathcal{H}^n(X_0\cap\mathfrak{Q}^{-1}(Q_i)).$$

After summing up w.r.t. $i\in\mathbb{N}$ together with (10) and since $\{Q_i\}_{i\in\mathbb{N}}$ are disjoint, we obtain the claim and we proved the proposition. □

3.2. **Second application.** Let $u:X\to\mathbb{R}$ be a 1-Lipschitz function, let $(\mathrm{m}_\gamma)_{\gamma\in Q}$ be the induced disintegration of $\mathcal{H}^n$. We pick a subset $\hat{Q}$ of full $\mathfrak{q}$ measure in Q such that $R_u(x)=\overline{X_\gamma}$ for all $x\in X_\gamma$. By abuse of notation we write $\hat{Q}=Q$ and $\mathcal{T}_u=\mathfrak{Q}^{-1}(\hat{Q})$.

We say that a unit speed geodesic $\gamma:[a,b]\to X$ is tangent to Y if there exists $t_0\in[a,b]$ such that $\gamma(t_0)\in Y$ and $\dot\gamma(t_0)\in T_pY$. We define

$$Q^\dagger:=\left\{\gamma\in Q:\#\gamma^{-1}(Y)<\infty\right\}.$$

Lemma 3.3. *If $\gamma\in Q\backslash Q^\dagger$, then γ is tangent to Y.*

Proof. If $\gamma \in Q\backslash Q^{\dagger}$, then $\#\gamma^{-1}(Y) = \infty$. Hence, after taking a subseqence one can find a strictly monotone sequence $t_i \in [a_\gamma, b_\gamma]$ such that $\gamma(t_i) \in Y$, $t_i \to t_0 \in [a_\gamma, b_\gamma]$ and $\gamma(t_i) \to \gamma(t_0) \in Y$. In a blow up of Y around $\gamma(t_0)$ the sequence $\gamma(t_i)$ converges to the the velocity vector of γ at t_0. Hence, we conclude that $\dot\gamma(t_0) \in T_{\gamma(t_0)}Y$ and γ is tangent to Y. □

For $U \subset X$ open we write

$$\mathcal{H}^n(\mathcal{T}_u \cap U) = \int_Q \mathrm{m}_\gamma(U) d\mathfrak{q}(\gamma) = \int_{g^{-1}(U)} h_\gamma(r) dr \otimes d\mathfrak{q}(\gamma)$$

where $g : \mathcal{V} \subset \mathbb{R} \times Q \to \mathcal{T}_u^b$ is the ray map defined in Subsection 2.5. We also note that $(r, \gamma) \in \mathcal{V} \mapsto h_\gamma(r)$ is measurable.

Remark 3.4. Let $B \subset \mathbb{R} \times Q$ be measurable. Then $g(B \times Q) =: \mathcal{B} \subset X$ is a measurable subset since g is a Borel isomorphism. Then $\mathcal{B} \cap X_\gamma$ is measurable w.r.t. the induced measurable structure and by Fubini's theorem the map

$$\gamma \in Q \mapsto L(\gamma|_{\gamma^{-1}(\mathcal{B})})$$

is measurable. We can apply this for the case when $B = (-\infty, 0) \times Q$. It follows that $\gamma \in Q \mapsto a_\gamma = L(\gamma|_{\gamma^{-1}(g((-\infty,0)\times Q))}) \in \mathbb{R}$ is measurable. Similar for b_γ.

Remark 3.5. Consider the map $\Phi_t : \mathbb{R} \times Q \to \mathbb{R} \times Q$, $\Phi_t(r, q) = (tr, q)$ for $t > 0$. Then, it is clear that $\Phi_t(\mathcal{V}) = \mathcal{V}_t$ is a measurable subset of $\mathcal{V}$ for $t \in (0, 1]$. Moreover $g(\mathcal{V}_t) = \mathcal{T}_{u,t}^b$ is a measurable subset of $\mathcal{T}_u^b$ such that $X_\gamma \cap \mathcal{T}_{u,t}^b = tX_\gamma \subset (a_\gamma, b_\gamma)$. If $t \in (0, 1)$, then $\mathcal{H}^n(\mathcal{T}_u^b \backslash \mathcal{T}_{u,t}^b) > 0$.

Again by Fubinis theorem $U \cap X_\gamma \cap \mathcal{T}_{u,t}^b = U \cap tX_\gamma$ is measurable in X_γ for $\mathfrak{q}$-a.e. $\gamma \in Q$ and the map

$$L_{U,t} : \gamma \in Q \mapsto \mathrm{L}(\gamma|_{(ta_\gamma, tb_\gamma)\cap\gamma^{-1}(U)}) = \int 1_{U\cap tX_\gamma} d\mathcal{L}^1$$

is measurable. We note that the set $(ta_\gamma, tb_\gamma) \cap \gamma^{-1}(U)$ might not be an interval.

Let $Y \subset X$ and consider $U_\epsilon = B_\epsilon(Y)$ for $\epsilon > 0$. For $s \in \mathbb{N}$ and $t \in (0, 1]$ we define

$$C_{\epsilon,s,t} = \left\{\gamma \in Q : \mathrm{L}(\gamma|_{\gamma^{-1}(U_\epsilon)\cap(ta_\gamma, tb_\gamma)}) > \epsilon s\right\}.$$

Further, we set

$$C_{s,t} = \bigcup_{\epsilon>0} \bigcap_{\epsilon'\leq\epsilon} C_{\epsilon',s,t} = \{\gamma \in Q : \liminf_{\epsilon\to0} \mathrm{L}(\gamma|_{\gamma^{-1}(U_\epsilon)\cap(ta_\gamma, tb_\gamma)})/\epsilon \geq s\}$$

and

$$C_t = \bigcap_{s\in\mathbb{N}} C_{s,t} = \{\gamma \in Q : \lim_{\epsilon\to0} \mathrm{L}(\gamma|_{\gamma^{-1}(U_\epsilon)\cap(ta_\gamma, tb_\gamma)})/\epsilon = \infty\}.$$

Lemma 3.6. *Let $0 < t \leq 1$ and let $\gamma \in Q$. If $\gamma|_{[ta_\gamma, tb_\gamma]}$ is tangent to Y, then $\gamma \in C_t$.*

Proof. For the proof we ignore $t \in (0, 1]$ and consider $\gamma|_{[a_\gamma, b_\gamma]}$.

Let $\gamma \in Q$ be tangent to Y. Assume $\gamma \notin C$. Then there exists a sequence $(\epsilon_i)_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty} L(\gamma|_{\gamma^{-1}(B_{\epsilon_i}(Y))\cap(a_\gamma, b_\gamma)})/\epsilon_i = C \in [0, \infty)$. By assumption there exists $t_0 \in [a_\gamma, b_\gamma]$ such that $\gamma(t_0) \in Y$. Hence $t_0 \in \gamma^{-1}(B_{\epsilon_i}(Y)) \cap [a_\gamma, b_\gamma]$. There exists a maximal interval I^{ϵ_i} contained in $\gamma^{-1}(B_{\epsilon_i}(Y))\cap[a_\gamma,b_\gamma]$ such that $t_0\in I^{\epsilon_i}$. After taking another subsequence we still have

$$\infty>\lim_{i\to\infty}\frac{L(\gamma|_{\gamma^{-1}(B_{\epsilon_i}(Y))\cap(a_\gamma,b_\gamma)})}{\epsilon_i}\geq\lim_{i\to\infty}\frac{L(\gamma|_{I^{\epsilon_i}\cap(a_\gamma,b_\gamma)})}{\epsilon_i}=:C'\geq 0$$

and $L(\gamma|_{I^{\epsilon_i}\cap(a_\gamma,b_\gamma)})=:L_i\to 0$. We set $\gamma_i=\gamma|_{I^{\epsilon_i}\cap(a_\gamma,b_\gamma)}$. Since I^{ϵ_i} is maximal such that $\mathrm{Im}(\gamma_i)\subset B_{\epsilon_i}(Y)$, we have $\sup_{y\in Y,t\in I^{\epsilon_i}}d(y,\gamma(t))\geq\epsilon_i$. In the rescaled space $(Z,\frac{1}{L_i}d_Z)$ the geodesic γ_i is a geodesic of length 1 and

$$\sup_{y\in Y,t\in I_{\epsilon_i}}\frac{1}{L_i}d(y,\gamma_i(t_0))\geq C'/2$$

for $i\in\mathbb{N}$ suffienciently large. By the proof of Petrunin's glued space theorem we know that $(Y,\frac{1}{L_i}d|_Y)$ converges in GH sense to $T_{\gamma(t_0)}Y$. Therefore, for $\epsilon_i\to 0$ a sequence of points $\gamma_i(t_i)$ converges to $v\in T_{\gamma(t_0)}Z$ such that $\sup_{w\in T_{\gamma(t_0)}Y}\angle(w,v)\geq C'/2$. This is a contradiction since the tangent vector $\dot\gamma(t_0)\in T_{\gamma(t_0)}Y$ and tangent vector of geodesics in Alexandrov spaces are well-defined and unique.

Hence, for any sequence $\epsilon_i\to 0$ it follow that $\frac{L_i}{\epsilon_i}\to\infty$ and therefore $\gamma\in C_t$. □

Corollary 3.7. *Let $\gamma\in Q$. If $\gamma|_{(a_\gamma,b_\gamma)}$ is tangent to Y, then $\gamma\in C=\bigcup_{t\in(0,1)}C_t$.*

Lemma 3.8. *Let $\mathfrak{q}$ be associated to u. Then $\mathfrak{q}(Q\cap\bigcup_{t\in(0,1)}C_t)=0$.*

Proof. It is clearly enough to show that for any $t\in(0,1)$ it holds that $\mathfrak{q}(Q\cap C_t)=0$. Therefore in the following we work with a fixed t.

We recall that $a_\gamma<0<b_\gamma$, $\gamma\in Q\mapsto a_\gamma,b_\gamma$ are measurable and $Q=\bigcup_{l\in\mathbb{N}}\{l\geq|b_\gamma|,|a_\gamma|\geq\frac{1}{l}\}$. It is obviously enough to prove the lemma for $Q^l=\{l\geq|a_\gamma|,|b_\gamma|\geq\frac{1}{l}\}$ for arbitrary $l\in\mathbb{N}$. Therefore we fix $l\in N$ and replace Q with Q^l. By abuse of notation we will drop the superscript l for the rest of the proof. By rescaling the whole space with $4l$ we can assume that $4\leq|a_\gamma|,|b_\gamma|\leq 4l^2$ for each $\gamma\in Q$.

Let $C_{\epsilon,s,t}$ be defined as before for $\epsilon\in(0,\epsilon_0)$ and $s\in\mathbb{N}$.

We pick $\gamma\in C_{\epsilon,s,t}$ and consider $\gamma^{-1}(B_\epsilon(Y))\cap(ta_\gamma,tb_\gamma)=:I_{\gamma,\epsilon}$. We set $L(\gamma|_{I_{\gamma,\epsilon}})=:L^\epsilon$.

We observe that

$$4l^2\geq(1-t)|a_\gamma|\geq(1-t)4,\quad 4l^2\geq(1-t)|b_\gamma|\geq(1-t)4.$$

We pick $r\in I_{\gamma,\epsilon}$ and $\tau\in(a_\gamma,ta_\gamma)\cup(tb_\gamma,b_\gamma)$. Theorem 2.23 implies that $([a_\gamma,b_\gamma],h_\gamma dr)$ satisfies the condition $CD(k(n-1),n)$. Then, the following estimate holds (c.f. [CM17, Inequality (4.1)])

$$\begin{aligned}h_\gamma(r)&\geq\frac{\sin_k^{n-1}((r-a_\gamma)\wedge(b_\gamma-r))}{\sin_k^{k-1}((\tau-a_\gamma)\wedge(b_\gamma-\tau))}h_\gamma(\tau)\\&\geq\frac{\sin_k^{n-1}((1-t)4)}{\sin_k^{n-1}4l^2}h_\gamma(\tau)=C(k,n,t,l)h_\gamma(\tau).\end{aligned}$$

for a universal constant $C(k,n,t,l)$. We take the mean value w.r.t. $\mathcal{L}^1$ on both sides and obtain

$$\frac{1}{L^\epsilon}\int_{I_{\gamma,\epsilon}}h_\gamma d\mathcal{L}^1\geq C(k,n,t,l)\frac{1}{4l^2}\int_{(a_\gamma,ta_\gamma)\cup(tb_\gamma,b_\gamma)}h_\gamma d\mathcal{L}^1.$$

Hence, after integrating w.r.t. $\mathfrak{q}$ on $C_{\epsilon,s,t}$ and taking into account $\frac{1}{\epsilon s} \geq \frac{1}{L^\epsilon}$ by definition of $C_{\epsilon,s,t}$, it follows

$$\begin{aligned}\frac{1}{\epsilon s}\mathcal{H}^n(B_\epsilon(Y)) &\geq \frac{1}{s\epsilon}\int_{C_{\epsilon,s,t}} \mathrm{m}_\gamma(B_\epsilon(Y))d\mathfrak{q}(\gamma)\\ &\geq \frac{1}{L^\epsilon}\int_{C_{\epsilon,s,t}}\int_{I_{\gamma,\epsilon}} h_\gamma d\mathcal{L}^1 d\mathfrak{q}(\gamma)\\ &\geq \hat{C}\int_{C_{\epsilon,s,t}}\int_{(a_\gamma,ta_\gamma)\cup(tb_\gamma,b_\gamma)} h_\gamma d\mathcal{L}^1 d\mathfrak{q}(\gamma)\\ &\geq \hat{C}\int_{C_{\epsilon,s,t}} \mathrm{m}_\gamma(\mathcal{T}^b_u\backslash\mathcal{T}^b_{u,t})d\mathfrak{q}(\gamma)\end{aligned}$$

where $\hat{C} = \frac{1}{2l}C(k,n,t,l)$. It is known that $\mathcal{H}^n(B_\epsilon(Y)) \leq \epsilon M$ for some constant $M > 0$ provided $\epsilon > 0$ is sufficiently small. This follows from semiconcavity of the boundary distance function in Alexandrov spaces, Lipschitz continuity of the induced gradient flow and the coarea formula. Hence

$$\frac{M}{s} \geq C(K,N,k,t)\int_{C_{\epsilon,s,t}} \mathrm{m}_\gamma(\mathcal{T}^b_u\backslash\mathcal{T}^b_{u,t})d\mathfrak{q}(\gamma).$$

If we take limit for $\epsilon \to 0$, we obtain

$$\frac{M}{s} \geq C(K,N,k,t)\int_{C_{s,t}} \mathrm{m}_\gamma(\mathcal{T}^b_u\backslash\mathcal{T}^b_{u,t})d\mathfrak{q}(\gamma).$$

Finally, for $s \to \infty$ it follows

$$0 = \int_{C_t} \mathrm{m}_\gamma(\mathcal{T}^b_u\backslash\mathcal{T}^b_{u,t})d\mathfrak{q}(\gamma).$$

But by construction of $\mathcal{T}^b_{u,t}$ we know that $\mathrm{m}_\gamma(\mathcal{T}^b_u\backslash\mathcal{T}^b_{u,t})$ is positive for every $\gamma \in Q$ if $t \in (0,1)$. Therefore, it follows $\mathfrak{q}(C_t) = 0$. □

Combining the above lemma with Corollary 3.7 gives

Corollary 3.9. *Let $\mathfrak{q}$ be associated to u. Then $\mathfrak{q}(\gamma \in Q : \gamma|_{(a_\gamma,b_\gamma)}$ is tangent to $Y) = 0$.*

Let us remark here that we *do not* claim that the set of geodesics in Q which are tangent to Y at one of the endpoints has measure zero. We suspect this is true but this is not needed for the applications.

As a first consequence of Proposition 3.2 and Corollary 3.9 we obtain the following corollary.

Corollary 3.10. *Let $x_1 \in X_1\backslash Y$. Then, for $\mathcal{H}^n_Z$-a.e. point $x_0 \in X_0$ the geodesic that connects x_0 and x_1 intersects with Y only finitely many times and in any intersection point $\Phi_Z|_Y$ is differentiable.*

4. Semiconcave functions on glued spaces

Lemma 4.1. *Let (X,d) be an n-dimensional Alexandrov space and let $\Phi : X \to \mathbb{R}$ be a double semi-concave function.*

Then Φ is semi-concave in the usual sense and for any $p \in \partial X$ it holds that $d\Phi(v) \leq 0$ for any normal vector $v \in \Sigma_p$.

Proof. A function $\Phi : X \to \mathbb{R}$ is semi-concave if $\Phi \circ P$ is semi-concave on the double space $\hat{X}$ that is an Alexandrov space without boundary. In particular, for any geodesic γ in $\hat{X}$ such that $\mathrm{Im}(\gamma) \subset X$ the composition $\Phi \circ \gamma$ is semi-concave.

Moreover, let $p \in X$ be a boundary point such that there exists $v \in \Sigma_p X$ normal to the boundary. Considering p in $\hat{X}$ we know that $\Sigma_p \hat{X} = \widehat{\Sigma_p X}$. Hence $v, -v \in \Sigma_p \hat{X}$ and $\angle(v, -v) = \pi$. Therefore, v and $-v$ generate a geodesic line in $T_p \hat{X} = C(\Sigma_p \hat{X})$. Since Φ is double semi-concave, its differential $d\Phi_p : T_p \hat{X} \to \mathbb{R}$ is concave, and by Lemma 2.19 it follows that $d\Phi(-v) \geq d\Phi(v)$. On the other hand we have $d\Phi(-v) = -d\Phi(v)$. This implies the claim. □

We continue to work with the setup from the previous section. Let $\Phi_i : X_i \to \mathbb{R}$, $i = 0, 1$, be semi-concave such that they agree on the boundaries ∂X_0 and ∂X_1 respectively identified via an isometry $\mathcal{I}$. Let Z be the glued space and let $\Phi_Z \colon Z \to \mathbb{R}$ be the naturally constructed glueing of Φ_0, Φ_1.

Lemma 4.2. *Let $\gamma : [0, L(\gamma)] \to Z$ be a constant speed geodesic with $\gamma(0) \in X_0 \backslash \partial X_0$ and $\gamma(L(\gamma)) \in X_1 \backslash \partial X_1$. Suppose γ intersects Y in a single point $p = \gamma(t_0)$. Suppose further that the following conditions hold:*

(i) p is a regular point in Z;

(ii) $d\Phi_0|_p(v_0) + d\Phi_1|_p(v_1) \leq 0 \quad \forall$ normal vectors $v_i \in \Sigma_p X_i, \ i = 0, 1$;

(iii) The restriction $\Phi|_Y$ is differentiable at p.

Then $\Phi_Z \circ \gamma : [0, L(\gamma)] \to \mathbb{R}$ is semi-concave. In particular

$$-d\Phi_0(\dot{\gamma}^-) = \frac{d^-}{dt} \Phi_0 \circ \gamma(t_0) \geq \frac{d^+}{dt} \Phi_1 \circ \gamma(t_0) = d\Phi_1(\dot{\gamma}^+). \tag{11}$$

where $t \in [0, L(\gamma)] \mapsto \gamma^-(t) = \gamma(L(\gamma) - t)$ and $\gamma^+ = \gamma$.

Proof. By Lemma 2.19 we only need to check (11). In the following we write $\gamma^{+/-}$ instead of $\dot{\gamma}^{+/-}$. By assumption we have that $\gamma([0, t_0]) \subset X_0$ and $\gamma([t_0, L(\gamma)]) \subset X_1$. By assumption $\gamma(t_0) =: p$ is a regular point in Z, that is $T_p Z = \mathbb{R}^n$ and $\Sigma_p Z = \mathbb{S}^{n-1}$. Moreover $\Sigma_p Z$ is the glued of $\Sigma_p X_0$ and $\Sigma_p X_1$ along their isometric boundary and $\partial \Sigma_p X_0 = \mathbb{S}^{n-1}_+$ and $\partial \Sigma_p X_1 = \mathbb{S}^{n-1}_-$ (see the remarks at the end of Subsection 2.3). In this case the north pole N and the south pole S are the unique normal vectors in $\Sigma_p X_0$ and $\Sigma_p X_1$, respectively.

If $\gamma^-(t_0) = N \in \Sigma_p X_0$, then by symmetry $\gamma^+(t_0) = S$ and by assumption it follows

$$-d\Phi_0(\gamma^-) \geq 0 \geq d\Phi_1(\gamma^+).$$

If $\gamma^-(t_0) \neq N$, then it also follows $\gamma^+(t_0) \neq S$. There exists a geodesic loop in $\Sigma_p Z$ that contains $\gamma^-(t_0)$ and $\gamma^+(t_0)$, and intersects with $\partial \Sigma_p X_0 = \Sigma_p Y = \mathbb{S}^{n-2}$ twice in w^- and w^+ such that $\angle(w^-, w^+) = \pi$.

Since $\angle(w^+, w^-) = \pi$, there exists a geodesic line in $T_p X$ of the form $s \in \mathbb{R} \mapsto (-sw^-) \star (sw^+)$ passing through 0 where $\star$ denotes the concatenation of curves. Since we assume $\Phi_Z|_Y$ is differentiable in p, $d\Phi_Z : T_p X_i \to \mathbb{R}$ is linear and therefore $d\Phi_Z(w^-) + d\Phi_Z(w^+) = 0$.

Let σ^0, $(\sigma^1)^{-1} : [0, \pi] \to \Sigma_p X_0, \Sigma_p X_1$ be the geodesics in $\Sigma_p X_0$ and $\Sigma_p X_1$ respectively connecting w^- and w^+ such that their concatenation is the geodesic loop S in Σ_p through w^+, w^-, γ^+ and γ^-.

Since $d\Phi_i : T_pX_i \to \mathbb{R}$ is concave, T_pX_i is the metric cone over Σ_pX_i and $d\Phi_i(rv) = rd\Phi_i(v)$, it follows that $d\Phi_i \circ \sigma_i = u^i : [0,\pi] \to \mathbb{R}$ satisfies

$$(u^i)'' + u^i \le 0,\ i = 0,1.$$

Moreover $v(s) = d\Phi_0 \circ \sigma_0(s) + d\Phi_1 \circ \sigma_1(s)$ satisfies

$$v'' + v \le 0, \quad v(0) = v(\pi) = d\Phi_0(w^-) + d\Phi_1(w^+) = 0.$$

Let r, s be the polar coordinates on $\mathbb{R}^2_+ = \{(x,y) : y \ge 0\}$ with $x = r\cos s, y = r\sin s$. Then the function $f(x,y) = rv(s)$ is concave. Since $f(-1,0) = f(1,0) = 0$ and $f(0,1) \le 0$ concavity of f implies that $f \le 0$. Therefore $v \le 0$ and hence $d\Phi_0(\gamma^-) + d\Phi_1(\gamma^+) \le 0$.

□

Theorem 4.3. *Let $\Phi_i : X_i \to \mathbb{R}$, $i = 0,1$, be semi-concave such that for any $p \in \partial X_i$ it holds that*

$$d\Phi_0|_p(v_0) + d\Phi_1|_p(v_1) \le 0 \quad \forall \text{ normal vectors } v_i \in \Sigma_pX_i,\ i = 0,1.$$

Then $\Phi_Z : Z \to \mathbb{R}$ is semiconcave.

Proof. It is obvious that we only need to check semi-concavity of Φ_Z near Y. By changing Z to a small convex neighborhood of a point $p \in Y$ we can assume that Φ_i are λ-concave on X_i for some real λ.

Let $\gamma\colon [0,L] \to Z$ be a unit speed geodesic. We wish to prove that $\Phi_Z(\gamma(t))$ is λ-concave. Fix an arbitrary $0 < \delta < L/10$ and let $x_0 = x_1 = \gamma(\delta), x_1 = \gamma(L-\delta)$. Let $y_i \to x_0, z_i \to x_1$ be such that $y_i, z_i \notin Y$ for any i. Let γ_i be a shortest unit speed geodesic from y_i to z_i. By Corollary 3.10 we can adjust z_i slightly so that that each γ_i intersects Y at most finitely many times, all intersection points are regular and $\Phi_Z|_Y$ is differentiable at those intersection points. Therefore by Lemma 4.2 we have that $\Phi_Z|_{\gamma_i}$ is λ-concave for every i. By passing to a subsequence we can assume that γ_i converge to a shortest geodesic from x_0 to x_1 and since Alexandrov spaces are nonbranching this geodesic must be equal to $\gamma|_{[\delta,L-\delta]}$. Therefore by continuity of Φ_Z we get that Φ_Z is λ-concave on $\gamma|_{[\delta,L-\delta]}$. Since this holds for arbitrary $0 < \delta < L/10$ we conclude that Φ_Z is λ-concave on all on γ. □

Corollary 4.4. *A function $\Phi : X \to \mathbb{R}$ is double semi-concave if and only if it is semi-concave in the usual sense and for any $p \in \partial X$ it holds that $d\Phi(v) \le 0$ for any normal vector $v \in \Sigma_p$.*

Let $\mathrm{m}_i = \Phi_i \mathcal{H}^n_{X_i}$ be measures on X_0 and X_1, respectively, for semi-concave function Φ_0 and Φ_1, and assume $\Phi_0|_{\partial X_0 \equiv \partial X_1} = \Phi_1|_{\partial X_0 \equiv \partial X_1}$.

Then, the metric measure glued space between the weighted Alexandrov spaces $(X_i, d_{X_i}, \mathrm{m}_i)$, $i = 0,1$, is given by

$$(X_0 \cup_{\mathcal{I}} X_1, \mathrm{m}_Z) \text{ where } \mathrm{m}_Z = (\iota_0)_\# \mathrm{m}_0 + (\iota_1)_\# \mathrm{m}_1 .$$

The maps $\iota_i : X_i \to Z$, $i = 0,1$, are the canonical inclusion maps. Note that $X_0 \cup_{\mathcal{I}} X_1$ is an n-dimensional Alexandrov space by Petrunin's glued space theorem. By Remark 2.12 it follows that

$$\left(\mathcal{H}^n_{X_0 \cup_{\mathcal{I}} X_1}\right)|_{X_i} = \mathcal{H}^n_{X_i},\ i = 0,1$$

we can write $\mathrm{m}_Z = \Phi_Z \mathcal{H}^n_{X_0 \cup_{\mathcal{I}} X_1}$.

5. Proof of Theorem 1.1

In this section we present the proof of the glued space theorem (Theorem 1.1).

Proof of Theorem 1.1

1. Let X_i, $i = 0, 1$, be Alexandrov spaces with curvature bounded from below by k_0 and k_1, respectively.

By Theorem 2.13 it follows that $X_0 \cup_\phi X_1 =: Z$ has curvature bounded from below by $\min\{k_0, k_1\} =: k$.

By Theorem 2.9 the metric measure space $(Z, d_Z, \mathcal{H}^n_Z)$ satisfies the condition $CD(k(n-1), n)$.

Hence, any 1-Lipschitz function $u : (Z, d_Z) \to \mathbb{R}$ induces a disintegration $\{\mathrm{m}_\gamma\}_{\gamma \in Q}$ that is strongly consistent with R^b_u, and for $\mathfrak{q}$-a.e. $\gamma \in Q$ the metric measure space $(\overline{X_\gamma}, \mathrm{m}_\gamma)$ satisfies the condition $CD(k(n-1), n)$ and hence $CD(k(n-1), N)$ by monotinicity in N . It follows that $\mathrm{m}_\gamma = h_\gamma \mathcal{H}^1|_{X_\gamma}$ and $h_\gamma : [a_\gamma, b_\gamma] \to \mathbb{R}$ satisfies

$$\frac{d^2}{dr^2} h_\gamma^{\frac{1}{N-1}} + k h_\gamma^{\frac{1}{N-1}} \le 0 \text{ on } (a_\gamma, b_\gamma) \text{ for } \mathfrak{q}\text{-a.e.}\gamma \in Q.$$

By Lemma 2.19 it follows

$$\frac{d^-}{dr} h_\gamma^{\frac{1}{N-1}} \ge \frac{d^+}{dr} h_\gamma^{\frac{1}{N-1}} \text{ everywhere on } (a_\gamma, b_\gamma).$$

2. Fix $0 < t < 1$. Define the set C_t as in Section 3.2.

Recall that regular points have full measure in Z. Hence, there exists $\hat{Q} \subset Q$ with full $\mathfrak{q}$-measure such that $\gamma(r)$ is a regular point for any $r \in [0, L(\gamma)]$ and for every $\gamma \in \hat{Q}$. Let $Q_t = \hat{Q} \backslash C_t$. By Lemma 3.3 and Lemma 3.6 we know that any $\gamma \in Q_t$ it holds that $\gamma|_{(ta_\gamma, tb_\gamma)}$ intersects Y in finitely many points. Further by Lemma 3.8 we know that Q_t has full measure in Q.

Let $p \in X_0 \backslash \partial X_0$ be arbitrary. By construction of the glued metric we can pick $\epsilon > 0$ that is sufficiently small such that $d_Z|_{B_\epsilon(p) \times B_\epsilon(p)} = d_{X_0}|_{B_\epsilon(p) \times B_\epsilon(p)}$. Moreover, since (X_0, d_{X_0}) is an Alexandrov space there exists an open domain $U_p \subset B_\epsilon$ that is geodesically convex. There is a countable set of points $\{p_i : i \in \mathbb{N}\}$ such that $\bigcup_{i \in \mathbb{N}} U_{p_i} = X_0 \backslash Y$. We pick $i \in \mathbb{N}$ and consider the corresponding U_{p_i}. In the following we drop the subscript p_i and work with $U = U_{p_i}$. Convexity of U implies that $(\overline{U}, d_{X_0}|_{\overline{U} \times \overline{U}}, \mathrm{m}_{X_0}|_{\overline{U}})$ satisfies the condition $CD(K, N)$, $u|_{\overline{U}}$ is 1-Lischitz and the set $\mathcal{T}_u \cap \overline{U} = \tilde{\mathcal{T}}_u$ is the transport set of u restricted to $\overline{U}$.

We obtain a decomposition of $\overline{U}$ via $X_\gamma \cap \overline{U} = \tilde{X}_\gamma$. The subset $\mathfrak{Q}(U) = \tilde{Q} \subset Q$ of geodesics in Q that intersect with U is measurable. We can pushforward the measure $\mathrm{m}|_{\overline{U}}$ w.r.t. the quotient map $\mathfrak{Q} : \overline{U} \to \tilde{Q}$ and we obtain a measure $\tilde{\mathfrak{q}}$ on $\tilde{Q}$. By the $1D$-localisation procedure applied to the metric measure space $\overline{U}$, there exists a disintegration $(\tilde{\mathrm{m}}_{\tilde{\gamma}})_{\gamma \in \tilde{Q}}$ where the geodesic $\tilde{\gamma}$ is defined as intersection of X_γ with $\overline{U}$. We also set $\mathrm{Im}(\tilde{\gamma}) =: X_{\tilde{\gamma}}$. Moreover, for $\tilde{\mathfrak{q}}$-a.e. $\tilde{\gamma}$ the metric measure space $(X_{\tilde{\gamma}}, \tilde{\mathrm{m}}_{\tilde{\gamma}})$ is $CD(K, N)$. That is, there exists a density $\tilde{h}_{\tilde{\gamma}}$ of $\tilde{\mathrm{m}}_{\tilde{\gamma}}$ w.r.t. $\mathcal{H}^1$ such that

$$\frac{d^2}{dr^2} \tilde{h}_{\tilde{\gamma}}^{\frac{1}{N-1}} + \frac{K}{N-1} \tilde{h}_{\tilde{\gamma}}^{\frac{1}{N-1}} \le 0 \text{ on } (a_{\tilde{\gamma}}, b_{\tilde{\gamma}}) \subset (a_\gamma, b_\gamma) \text{ for } \tilde{\mathfrak{q}}\text{-a.e. .} \tag{12}$$

More precisely, there exists a set $\mathcal{N} \subset \tilde{Q}$ with $\tilde{\mathfrak{q}}(N) = 0$ such that (12) holds for every $\tilde{\gamma} \in \tilde{Q} \backslash N$.

3. We show that $\tilde{\mathfrak{q}}$ is absolutely continuous w.r.t. $\mathfrak{q}$ on Q.

Recall $\tilde{\mathfrak{q}} = (\mathfrak{Q})_{\#} \operatorname{m}|_U$. Let $A \subset Q$ be a set such that $\mathfrak{q}(A) = 0$. Hence $0 = \operatorname{m}(\mathfrak{Q}^{-1}(A)) \geq \operatorname{m}(\mathfrak{Q}^{-1}(A) \cap U)$. Hence $\tilde{\mathfrak{q}}(A) = 0$.

Therefore, there exists a measurable function $G : Q \to [0, \infty)$ such that $\mathfrak{q} = G\tilde{\mathfrak{q}}$ and $\int_{Q \setminus \mathfrak{Q}^{-1}(U)} G d\tilde{\mathfrak{q}} = 0$. In particular, it follows that $\mathfrak{q}(\mathcal{N}) = 0$.

A unique and strongly consistent disintegration of $\operatorname{m}_Z|_{\mathcal{T}_u^b} = \Phi \mathcal{H}_Z^n|_{\mathcal{T}_u^b}$ is given by

$$\int_Q \Phi \operatorname{m}_\gamma d\mathfrak{q}$$

where $\Phi \operatorname{m}_\gamma = (\gamma)_{\#}[\Phi \circ \gamma h_\gamma \mathcal{H}^1]$. Then, it follows by uniqueness of the disintegration and since $\mathfrak{q} = G\mathfrak{q}$ that $G(\gamma)\tilde{h}_{\tilde{\gamma}} = (\Phi \circ \gamma)h_\gamma$ on $(a_{\tilde{\gamma}}, b_{\tilde{\gamma}})$ for $\tilde{\mathfrak{q}}$-a.e. $\tilde{\gamma}$.

4. We repeat the steps **2.** and **3.** for any U_{p_i}, $i \in \mathbb{N}$. We can find a set $\mathcal{N} \subset Q$ with $\mathfrak{q}(\mathcal{N}) = 0$ such that $\tilde{\mathfrak{q}}_{p_i}(\mathcal{N}) = 0$ for every $i \in \mathbb{N}$ and such that (12) holds for any $\gamma \in \mathfrak{Q}^{-1}(U_{p_i}) \backslash \mathcal{N}$ for any $i \in \mathbb{N}$.

We repeat all the previous steps again for X_1 instead of X_0 and find a corresponding set $\mathcal{N} \subset Q$ of $\mathfrak{q}$-measure 0.

We get that for every $\gamma \in Q \backslash \mathcal{N}$ the inequality (12) holds for h_γ for any interval $I \subset (a_\gamma, b_\gamma)$ as long $\gamma|_I$ is fully contained in U_{p_i} for some $i \in \mathbb{N}$.

From Lemma 2.19 and Lemma 4.2 it follows that inequality (12) holds for $\Phi \circ \gamma h_\gamma$ on (ta_γ, tb_γ) for any $\gamma \in Q_t \backslash N$. Since this holds for arbitrary $0 < t < 1$, we get that for q-almost all γ in Q it holds that $([a_\gamma, b_\gamma], m_\gamma)$ satisfies $CD(K, N)$. Since this holds for an arbitrary 1-Lipschitz function u we obtain that Z satisfies $CD^1_{lip}(K, N)$.

If m_Z is a finite measure Theorem 2.27 yields the condition $CD(K, N)$ for $(Z, d_Z, \operatorname{m}_Z)$.

If m_Z is a σ-finite measure we argue as follows. For $\overline{U}$ that is a geodesically convex and closed neighborhood with finite measure of some point $x \in Z$, it holds that the metric measure space $(\overline{U}, d_Z|_{\overline{U} \times \overline{U}}, \operatorname{m}_Z|_{\overline{U}})$ satisfies $CD^1_{lip}(K, N)$. Hence, by Theorem 2.27 it satisfies $CD(K, N)$ and also $CD^*(K, N)$. Finally by the globalisation theorem of CD^* [BS10] the space $(Z, d_Z, \operatorname{m}_Z)$ satisfies $CD^*(K, N)$. □

Example 5.1. Here we give another simple example that shows why Theorem 1.1 fails for the measure contraction property MCP.

We consider a metric space Z that is the cylinder $[0, \frac{3}{4}\epsilon] \times \mathbb{S}^{N-1}_\delta$ for $0 < \delta \ll \frac{\epsilon}{8}$ with one end closed by a disk. This space has nonnegative Alexandrov curvature and equipped with the N-dimensional Hausdorff measure is $CD(0, N)$ by Petrunin's theorem. It has diameter less than $\epsilon > 0$.

In [Stu06] (Remark 5.6) it was observed that there exists a constant $c_{N+1} \in (0, 1]$ such that $\forall \theta > 0$ with $N\theta^2 \leq c_{N+1}$ it holds

$$t^N \geq \tau^{(t)}_{N,N+1}(\theta)^{N+1} \ \forall t \in (0, 1). \tag{13}$$

Hence, provided $N\epsilon^2 \leq c_{N+1}$, Y will satisfy the $MCP(N, N + 1)$.

We show that if we pick ϵ sufficiently large, the double space does not satisfy this property. The function $\theta \mapsto \tau^{(t)}_{N,N+1}(\theta)^{N+1}$ is monotone increasing and $\tau^{(t)}_{N,N+1}(\theta)^{N+1} \to \infty$ for all $t \in (0, 1)$ if $\theta \uparrow \pi$. Therefore, the set

$$\Theta = \{\theta > 0 : t^N \geq \tau^{(t)}_{N,N+1}(\theta)^{N+1} \ \forall t \in (0, 1)\}$$

is nonempty and bounded by π and for $\theta \in \Theta$ and $\theta' \leq \theta$ it holds $\theta' \in \Theta$. We pick $\Theta \ni \epsilon \geq \frac{8}{9} \sup \Theta$ in the construction above. Then, by definition of Θ the space Y will satisfy $MCP(N, N + 1)$. The double space of Y is the cylinder $[0, \frac{3}{2}\epsilon] \times \mathbb{S}^{N-1}_\delta$

with both ends closed by a disk. Since $\frac{3}{2}\epsilon \geq \frac{4}{3}\sup\Theta$, it follows $t^N < \tau^{(t)}_{N,N+1}(\frac{5}{4}\epsilon)^{N+1}$ for some $t \in (0,1)$.

On the other hand, since $[0,\frac{3}{2}] \times \mathbb{S}^{N-1}_{\delta}$ is flat, one can find an optimal transport μ_t such that $\mu_1 = \delta_{x_1}$, $\mu_0 = \mathcal{H}^N(A)\mathcal{H}^N|_A$, $d(x_1, A) \geq \frac{5}{4}\epsilon$ and $\mu_t = t^N\mathcal{H}^N(A)\mathcal{H}^N|_A$. If the $MCP(N, N+1)$ holds, then $\mu_t \geq \tau^{(t)}_{N,N+1}(\frac{5}{4}\epsilon)^{N+1}\mathcal{H}^N(A)\mathcal{H}^N|_A$.

Together with the previous inequality we see that the $MCP(N, N+1)$ cannot be satisfied.

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