ON THE
TORSION IN THE CENTER
CONJECTURE

VITALI KAPOVITCH, ANTON PETRUNIN, AND WILDERICH TUSCHMANN

ABSTRACT. We present a condition for towers of fiber bundles which implies that the fundamental group of the total space has a nilpotent subgroup of finite index whose torsion is contained in its center. Moreover, the index of the subgroup can be bounded in terms of the fibers of the tower.

Our result is motivated by the conjecture that every almost nonnegatively curved closed m-dimensional manifold M admits a finite cover $\tilde{M}$ for which the number of leafs is bounded in terms of m such that the torsion of the fundamental group $\pi_1 \tilde{M}$ lies in its center.

INTRODUCTION

1.   We prove the following topological result, which is motivated by Conjecture 2.1 below.

Theorem 1.1. *Let* $F_1, F_2, \ldots, F_n$ *be an array of closed manifolds such that each* F_i *is either* $\mathbb{S}^1$ *or is simply connected. Assume* E *is the total space of a tower of fiber bundles over a point*

$$E = E_n \xrightarrow{F_n} E_{n-1} \xrightarrow{F_{n-1}} \cdots \xrightarrow{F_1} E_0 = \{pt\}$$

and each of the bundles $E_k \xrightarrow{F_k} E_{k-1}$ *is homotopically trivial over the* 1*-skeleton. Then the fundamental group* $\pi_1 E$ *contains a nilpotent subgroup* Γ *such that*

$$[\pi_1 E : \Gamma] \leqslant \operatorname{Const}(F_1, F_2, \ldots, F_n) \quad and \quad \operatorname{Tor}(\Gamma) \subset \mathrm{Z}(\Gamma),$$

where $\operatorname{Tor}(\Gamma)$ *and* $\mathrm{Z}(\Gamma)$ *denote the torsion and the center of* Γ*, respectively.*

Our proof is surprisingly involved. We would be very interested to see a proof based on a different idea; in particular, it might help to establish the conjecture in full generality.

MOTIVATION

2. **Conjectures.** Theorem 1 is motivated by the following conjecture from [KPT].

Main Conjecture 2.1. *For all dimensions* m *there is a constant* $C = C(m)$ *such that if* M^m *is an almost nonnegatively curved closed smooth* m*-manifold, then there is a nilpotent subgroup* $N \subset \pi_1 M$ *of index at most* C *whose torsion is contained in its center.*

The following earlier conjecture of Fukaya and Yamaguchi [FY] is closely related.

Conjecture 2.2 (Fukaya–Yamaguchi). *The fundamental group of a nonnegatively curved m-manifold M is $C(m)$-abelian: there is $C = C(m)$ such that if M^m is nonnegatively curved, then there is an abelian subgroup $A \subset \pi_1 M$ of index at most C.*

Let us show that our main conjecture (if true) implies Conjecture 2.2.

If the fundamental group of M is finite (in particular if M is positively curved), then the whole group is torsion. In this case Conjecture 2.1 and Theorem B in [KPT] imply Conjecture 2.2. The general case is more tricky. The following argument was suggested to us by Burkhard Wilking (compare to [KPT, Corollary 4.6.1]).

If $\sec(M^m) \geqslant 0$, then the universal cover $\tilde{M}$ of M is isometric to the product $\mathbb{R}^n \times K$, where K is a compact Riemannian manifold and the $\pi_1 M$ action on $\mathbb{R}^n \times K$ is diagonal [CG]. It follows from [W, Cor. 6.3] that one can deform the metric on M so that its universal cover is still isometric to $\mathbb{R}^n \times K$ and the induced action on K is finite. Let Γ_1 be the image of $\pi_1 M$ in $\mathrm{Iso}(\mathbb{R}^n)$ and Γ_2 be the image in $\mathrm{Iso}(K)$. By the above we can assume that Γ_2 is finite. Therefore Γ_1 is a crystallographic group. By the third Bieberbach theorem, it contains a subgroup of index $\leqslant C(n)$ which consists entirely of translations (and hence is abelian). Consider the action of Γ_2 on the frame bundle F of K. This action is free and by Cheeger's trick, F admits a sequence of almost nonnegatively curved metrics [FY]. Since Γ_2 is finite, by Conjecture 2.1 it contains an abelian subgroup of index $\leqslant C(m)$. Thus, both Γ_1 and Γ_2 are $C(m)$-abelian, and this yields Conjecture 2.2.

3. **About the reduction.** Let us explain how Theorem 1.1 is related to Conjecture 2.1.

An almost nonnegatively curved manifold can be defined as a closed smooth manifold M which admits a sequence of metrics g_n with a uniform lower curvature bound such that the sequence M_n converges to a point in Gromov–Hausdorff topology.

One approach to studying almost nonnegatively curved manifolds is by studying successive blow-ups of the collapsing sequence $M_n = (M, g_n)$, as was done in [KPT, Section 4.3]. Let us describe this construction.

The sequence $\{M_n\}$ converges to a point; this one-point space will be denoted by A_0.

Set $M_{n,1} := M_n$ and $\vartheta_{n,1} := \mathrm{diam}\, M_{n,1}$. Rescale now $M_{n,1}$ by $\frac{1}{\vartheta_{n,1}}$ so that

$$\mathrm{diam}(\tfrac{1}{\vartheta_{n,1}} \cdot M_{n,1}) = 1.$$

Passing to a subsequence if necessary, one has that the manifolds $\frac{1}{\vartheta_{n,1}} \cdot M_{n,1}$ converge to A_1, where A_1 is a compact nonnegatively curved Alexandrov space with diameter 1.

Now choose a regular point $\bar{p}_1 \in A_1$, and consider distance coordinates around $\bar{p}_1 \in U_1 \to \mathbb{R}^{k_1}$, where k_1 is the dimension of A_1. The distance functions can be lifted to $U_{n,1} \subset \frac{1}{\vartheta_{n,1}} \cdot M_{n,1}$. Denote by $M_{n,2}$ the level set of $U_{n,1} \to \mathbb{R}^{k_1}$ that corresponds to $\bar{p}_1$. Clearly, $M_{n,2}$ is a compact submanifold of codimension k_1. Set $\vartheta_{n,2} := \mathrm{diam}\, M_{n,2}$. Passing again to a subsequence, one has that the sequence $\frac{1}{\vartheta_{n,2}} \cdot M_{n,2}$ converges to some Alexandrov space A_2.

As before, A_2 is a compact nonnegatively curved Alexandrov space with diameter 1. Set $k_2 := k_1 + \dim A_2$. If one now chooses a marked point in $M_{n,2}$,

then, as $n \to \infty$, $\frac{1}{\vartheta_{n,2}} \cdot M_n$ converges in the pointed Gromov–Hausdorff topology to $A_2 \times \mathbb{R}^{k_1}$, which is of dimension k_2.

We repeat this procedure until, at some step, $k_\ell = m$, where $m = \dim M$.

As a result, we obtain a sequence $\{A_i\}$ of compact nonnegatively curved Alexandrov spaces with diameter 1 that satisfies

$$\dim A_i = k_i - k_{i-1} \quad \text{and therefore} \quad \sum_{i=1}^{\ell} \dim A_i = m.$$

We also obtain a sequence of rescaling factors $\vartheta_{n,i} = \operatorname{diam} M_{n,i}$, and a nested sequence of submanifolds

$$\{p_n\} = M_{n,\ell} \subset \cdots \subset M_{n,2} \subset M_{n,1} = M_n,$$

Now let us assume that *for all choices of $\bar{p}_i$* the Alexandrov spaces $A_1, \ldots, A_\ell$ are closed Riemannian manifolds. This condition will be called the *rescaled smoothness assumption*. Notice that this assumption definitely *does not need to hold* in general; it is only made to simplify the problem. Foremost, it makes it possible to apply the following fibration theorem of Yamaguchi [Y] to the successive blow-ups of the collapsing sequence under consideration.

Theorem 3.1 (Yamaguchi's Fibration Theorem). *Let M_n be a sequence of m-dimensional Riemannian manifolds with sectional curvature at least k and diameters at most D, which converges to a Riemannian manifold N in the Gromov–Hausdorff sense. Then for all large n there exists an almost Riemannian submersion $f_n \colon M_n \to N$. In particular, f_n is a fiber bundle.*

Let M_n as above satisfies the rescaled smoothness assumption. Then it should be true that after passing to a subsequence of M_n, each M_n is homeomorphic to the total space of a tower of fiber bundles

$$M_n = E_\ell \xrightarrow{A_\ell} E_{n-1} \xrightarrow{A_{\ell-1}} \cdots \xrightarrow{A_1} E_0 = \{pt\}.$$

Moreover, by using the Lipschitz properties of the gradient flow, it should be possible to pass to a finite cover of the total space with number of leafs controlled by the topology of A_i, so that the tower of bundles can be further refined to a tower of the form

$$E = E_n \xrightarrow{F_n} E_{n-1} \xrightarrow{F_{n-1}} \cdots \xrightarrow{F_1} E_0 = \{pt\}$$

such that

(i) Each F_i is either $\mathbb{S}^1$ or is simply connected.

(ii) Each of the bundles $E_k \xrightarrow{F_k} E_{k-1}$ is homotopically trivial over the 1-skeleton.

Note that given a tower of bundles satisfying these conditions it easily follows that $\pi_1(E)$ is nilpotent and is generated by $\leqslant n$ elements. We will repeatedly use this observation.

This finally leads us to the main topological result of this paper (Theorem 1.1 above), which says that if such a tower of bundles exists, then its total space satisfies the conclusion of Conjecture 2.1.

We have, however, not worked out the details of the reduction, as we do not see a geometric condition which would imply the rescaled smoothness assumption. In fact, it might well be possible to do the reduction even in the case if all A_i are

general Alexandrov spaces *without boundary*, but we have no idea what to do if at least one of them has nonempty boundary.

PROOF OF THE THEOREM

4. **Technical tools.** The proof of Theorem 1.1 uses the following result of Dror, Dwyer and Kan ([DDK]).

Theorem 4.1. *Let F be a finite simply connected CW complex and $\mathrm{Aut}_0(F)$ be the identity component of the space of all homotopy equivalences $F \to F$. Then $\mathrm{Aut}_0(F)$ is homotopy equivalent to a CW complex with a finite number of cells in each dimension.*

Let $\star = \mathrm{id}_F$ be the base point of $\mathrm{Aut}_0(F)$.

Furthermore, we consider $\mathbb{S}^1$ as the unit circle in the complex plane, so that $z \mapsto z^n$ defines a map $\mathbb{S}^1 \to \mathbb{S}^1$ of degree n.

Corollary 4.2. *Given a finite simply connected CW complex F, there is a natural number $n = n(F)$ such that the following holds.*

Assume that $\{\kappa_z : F \to F : z \in \mathbb{S}^1\}$ is a one parameter family of homotopy equivalences with $\kappa_1 = \mathrm{id}_F$ such that for some $N \in \mathbb{N}$ the family $\kappa_z^N = \kappa_{z^N}$ is homotopic to the constant family id_F. Then κ_z^n is homotopic to the constant family id_F.

Proof. We can view κ as a loop in $\mathrm{Aut}_0(F)$ based at $\star$. Since $\mathrm{Aut}_0(F)$ is an H-space, its fundamental group $\pi_1(E_0(F))$ is abelian.

By Theorem 4.1, $\pi_1(\mathrm{Aut}_0(F))$ is finitely generated, therefore $\mathrm{Tor}[\pi_1(\mathrm{Aut}_0(F))]$ is finite. Hence the corollary follows. $\qquad\square$

We denote by $\mathbb{D}$ the closed unit disc in the complex plane.

Technical Lemma 4.3. *Let F be a compact simply connected manifold and $h \colon \mathbb{D} \times \times F \to F$ be a $\mathbb{D}$-family of homotopy equivalences $F \to F$ such that $h(u, *) = \mathrm{id}_F$ for any $u \in \partial\mathbb{D}$. Consider the map $\beta_x \colon \mathbb{D} \to F$ defined by $\beta_x \colon u \mapsto h(u, x)$ for a fixed point $x \in F$. Then $[\beta_x] \in \mathrm{Tor}(\pi_2 F)$ and is independent of x.*

Moreover, given a map $f \colon \mathbb{D} \to F$ let $f_h \colon \mathbb{D} \to F$ be given by $f_h(u) = h(u, f(u))$. Note that $f_h|_{\partial\mathbb{D}} = f|_{\partial\mathbb{D}}$. In particular, we can view $[f_h] - [f]$ as an element of $\pi_2 F$.

Then $[f_h] - [f] = [\beta_x] \in \mathrm{Tor}(\pi_2 F)$.

Proof. Assume the contrary.

The map h induces a map $\mathbb{S}^2 \times F \to F$ which with a slight abuse of notation we'll still denote by h. Consider the induced map on cohomology

$$H^{*+2}(F) \xrightarrow{h^*} H^{*+2}(\mathbb{S}^2 \times F) \to H^*(F) \otimes H^2(\mathbb{S}^2) \cong H^*(F),$$

where the second map is the projection coming from the Künneth isomorphism. We'll denote the resulting map $H^{*+2}(F) \to H^*(F)$ by D_h. It's easy to see that it's an algebra derivation on $H^*(F)$. The same obviously holds true for $D_h^{\mathbb{Q}} \colon H^{*+2}(F, \mathbb{Q}) \to H^*(F, \mathbb{Q})$

We claim that $D_h^{\mathbb{Q}}$ vanishes on $H^2(F, \mathbb{Q})$. Suppose not and and that for some $\theta \in H^2(F, \mathbb{Q})$ we have $H^0(F, \mathbb{Q}) \ni D_h^{\mathbb{Q}}\theta \neq 0$. Note that

$$D_h^{\mathbb{Q}}\theta^k = k \cdot \theta^{k-1} \cdot D_h^{\mathbb{Q}}\theta.$$

Therefore
$$[\theta^{k-1}] \neq 0 \implies [\theta^k] \neq 0.$$
Thus, $[\theta^k] \neq 0$ for any positive integer k. In particular, F has infinite dimension, a contradiction. Thus, $D_h^{\mathbb{Q}}|_{H^2(F,\mathbb{Q})} \equiv 0$. Hence, $D_h|_{H^2(F)} \equiv 0$ and therefore $h^* = pr_2^*$ on $H^2(F)$ where $pr_2 \colon \mathbb{S}^2 \times F \to F$ is the second coordinate projection. This easily implies the statement of the Lemma. $\qquad\square$

Remark 4.4. Since $[\beta_x] \in \pi_2 F$ is independent of the choice of x, we can drop the subindex x and simply denote this element by $[\beta]$.

5. **Two claims.**

Claim 5.1. *Let $\mathbb{S}^1 \to E \to B$ be an oriented $\mathbb{S}^1$-bundle over a compact manifold B. Assume that the action of $\pi_1 B$ on $\pi_2 B$ is trivial. Then the action of $\pi_1 E$ on $\pi_2 E$ is trivial.*

Proof. Let $\gamma \in \pi_1(E, p)$ and $\alpha \in \pi_2(E, p)$. Let us denote by $\bar{\gamma} \in \pi_1(B, \bar{p})$ and $\bar{\alpha} \in \pi_2(B, \bar{p})$ their projection to B.

By our assumptions, $\bar{\alpha}^{\bar{\gamma}} = \bar{\alpha}$, or equivalently, there is a map $h \colon \mathbb{S}^2 \times \mathbb{S}^1 \to B$ such that
$$h(u, v) = \bar{p}, \quad h(u, *) \sim \bar{\gamma} \quad \text{and} \quad h(*, v) \sim \bar{\alpha}.$$
The induced bundle $\mathbb{S}^1 \to E' \to \mathbb{S}^2 \times \mathbb{S}^1$ is trivial. Therefore the $\alpha^{\gamma} = \alpha$ for any element in $\gamma \in \pi_1 E'$. $\qquad\square$

Claim 5.2. *Let $F \xrightarrow{i} E \to B$ be a fiber bundle with simply connected fiber F over a compact manifold B, which admits a trivialization over the 1-skeleton of B. Assume $\pi_1 B$ is almost nilpotent and the action of $\pi_1 B$ on $\pi_2 B$ is almost trivial. Then the action of $\pi_1 E$ on $\pi_2 E$ is almost trivial.*

Proof. By taking the pullback of $F \to E \to B$ to a finite cover of B we can assume that the action of $\pi_1 B$ on $\pi_2 B$ is trivial and $\pi_1 B$ is nilpotent.

Let $\gamma \in \pi_1(E, p)$ and $\alpha \in \pi_2(E, p)$. Let us denote by $\bar{\gamma} \in \pi_1(B, \bar{p})$ and $\bar{\alpha} \in \pi_2(B, \bar{p})$ their projections to B.

By assumptions, there is a map $h \colon \mathbb{S}^2 \times \mathbb{S}^1 \to B$ such that
$$h(u, v) = \bar{p}, \quad h(u, *) \sim \bar{\gamma} \quad \text{and} \quad h(*, v) \sim \bar{\alpha}.$$
The induced bundle $F \to E' \to \mathbb{S}^2 \times \mathbb{S}^1$ admits a trivialization over $u \times \mathbb{S}^1$. Let E'' be the induced bundle $F \to E'' \to \mathbb{S}^2 \times v \subset \mathbb{S}^2 \times \mathbb{S}^1$. The monodromy map $f \colon E'' \to E''$ around $\mathbb{S}^1$ induces the identity map on the base $\mathbb{S}^2$ and due the bundle being trivial over $u \times \mathbb{S}^1$, it can be chosen to be the identity on F_v.

Consider a map $m \colon \mathbb{D} \to \mathbb{S}^2$ such that
(1) m is injective in the interior of $\mathbb{D}$,
(2) $m(\partial \mathbb{D}) = v$.
Fix a trivialization $F \times \mathbb{D}$ of the induced bundle. Let $m' \colon F \times \mathbb{D} \to E''$ be the correspondent mapping. By construction, $m'(*, x)$ is a homeomorphism between $F \times x$ and the fiber $F_{m(x)}$. Then the map $f \colon E'' \to E''$ is completely described by the map
$$f' \colon F \times \mathbb{D} \to F \times \mathbb{D}$$
uniquely determined by the identity $m' \circ f' = f \circ m'$. Clearly $f'(*, x) = \mathrm{id}_{F \times x}$ for any $x \in \partial \mathbb{D}$.

By the Technical Lemma, there is an element $[\beta] \in \mathrm{Tor}(\pi_2 F)$ such that

$$[i(\beta)] = \alpha^\gamma - \alpha.$$

Since the bundle is trivial over $\mathbb{S}^1$,

$$[i(n \cdot \beta)] = \alpha^{\gamma^n} - \alpha$$

and therefore for $n(F) = |\mathrm{Tor}(\pi_2 F)|$ we have $\alpha^{\gamma^{n(F)}} - \alpha = 0$.

Since $\pi_1 B$ is nilpotent and finitely generated we have that the subgroup

$$\langle \gamma^{n(F)} | \gamma \in \pi_1 B \rangle < \pi_1 B$$

has finite index. $\square$

6. Reliable manifolds.

Definition 6.1. *A compact connected manifold M is called* reliable *if*

(1) *For any $\tau \in \mathrm{Tor}(\pi_1(M,p))$ and for any other element $\gamma \in \pi_1(M,p)$ there is a map of $h \colon \mathbb{S}^1 \times \mathbb{S}^1 \to M$ such that*

$$h(u,v) = p, \quad h(u,*) \sim \gamma \quad and \quad h(*,v) \sim \tau.$$

(2) *There is a finite cover $\iota \colon \tilde{T}^2 \to \mathbb{S}^1 \times \mathbb{S}^1$ and a map of solid torus $f \colon \mathbb{D} \times \mathbb{S}^1 \to$ $\to M$ and an isomorphism $i \colon \tilde{T}^2 \to \partial \mathbb{D} \times \mathbb{S}^1$ such that $f \circ i = h \circ \iota$.*

Evidently, if M is *reliable* then $\mathrm{Tor}(\pi_1 M) \subset \mathrm{Z}(\pi_1 M)$.

Claim 6.2. *Let $\mathbb{S}^1 \to E \to B$ be an oriented $\mathbb{S}^1$-bundle such that the base B is reliable and the action of $\pi_1 B$ on $\pi_2 B$ is almost trivial. Then E is reliable.*

Proof. Let $\gamma \in \pi_1(E,p)$ and $\tau \in \mathrm{Tor}(\pi_1(E,p))$. Let us denote by $\bar{\gamma} \in \pi_1(B, \bar{p})$ and $\bar{\tau} \in \pi_1(B, \bar{p})$ their projections to B.

From the assumptions we have that there is a map $\bar{h} \colon \mathbb{S}^1 \times \mathbb{S}^1 \to B$ such that

$$\bar{h}(u,v) = \bar{p}, \quad \bar{h}(u,*) \sim \bar{\gamma} \quad and \quad \bar{h}(*,v) \sim \bar{\tau},$$

which satisfy the definition of a reliable manifold.

Note that since B is reliable we have $[h] = 0 \in H_2(B, \mathbb{Q})$. Therefore the induced bundle $\mathbb{S}^1 \to E' \to \mathbb{S}^1 \times \mathbb{S}^1$ is trivial, in particular $E' = T^3$. This implies that γ and τ commute. Hence (1) is satisfied and we only need to check (2).

Since the bundle $\mathbb{S}^1 \to E' \to \mathbb{S}^1 \times \mathbb{S}^1$ is trivial there is a lifting $h \colon \mathbb{S}^1 \times \mathbb{S}^1 \to E$ of the map $\bar{h}$ such that

$$h(u,v) = p, \quad h(u,*) \sim \gamma \quad and \quad h(*,v) \sim \tau,$$

Consider the covering map $h \circ \iota \colon \tilde{T}^2 \to E$.

By the definition of a reliable manifold there is a covering $\iota \colon \tilde{T}^2 \to \mathbb{S}^1 \times \mathbb{S}^1$ and a map of solid torus $\bar{f} \colon \mathbb{D} \times \mathbb{S}^1 \to B$ and an isomorphism $i \colon \tilde{T}^2 \to \partial \mathbb{D} \times \mathbb{S}^1$ such that $\bar{f} \circ i = \bar{h} \circ \iota$. The pullback bundle $\mathbb{S}^1 \to \tilde{E}' \to \mathbb{D} \times \mathbb{S}^1$ is trivial. Let $s \colon \mathbb{D} \times \mathbb{S}^1 \to \tilde{E}'$ be a section.

From above, we have that $h \circ \iota$ is homotopic to a map $h' \colon \tilde{T}^2 \to E$ whose projection to B is the central $\mathbb{S}^1$ of $\mathbb{D} \times \mathbb{S}^1$. Therefore there are loops $\psi \subset B$ and φ in a fiber $\mathbb{S}^1$ such that $h \circ \iota$ is homotopic to a torus with meridians in $s(\psi)$ and φ.

Suppose $[\psi]$ has finite order in $\pi_1 B$. By changing the cover $\tilde{T} \to T$ we can assume that $[\psi] = 0$ in $\pi_1 B$. Then we can homotope h' to a single $\mathbb{S}^1$-fiber, and this easily gives the needed map of a solid torus.

Now suppose that $[\psi]$ has infinite order in $\pi_1 B$. Let $\tilde{\tau}$ be the homotopy class of a component in the preimage of τ under the covering $\tilde{T} \to T$. On the level of the fundamental groups we have that $\tilde{\tau} = [\varphi]^a \cdot [s(\psi)]^b$ for some integer a, b. Projecting to B we get that $\bar{\tau} = [\psi]^b$. Since τ is by assumption torsion and $[\psi]$ is not this means that $b = 0$. Thus $\tilde{\tau} = [\varphi]^a$ and hence $i(\varphi) \in \text{Tor}(\pi_1 E)$. Therefore passing to a finite cover of $\tilde{T}^2$ if necessary we get a map of the disc $\mathbb{D} \to E$ which contracts one of the generators $(k\varphi)$ of $\tilde{T}^2$. The projection of this disc to B gives an element $\alpha \in \pi_2 B$ and since $\pi_1 B$ acting on $\pi_2 B$ almost trivially we get that passing to finite cover of $\tilde{T}^2$ again (along ψ) we have a map of $\bar{g}: \mathbb{S}^2 \times \mathbb{S}^1 \to B$ with

$$\bar{g}(u, *) \sim \psi^l \quad \text{and} \quad \bar{g}(*, v) \sim \alpha.$$

We can lift this map to a map $g: \mathbb{D} \times \mathbb{S}^1 \to E$ in such a way that $g \circ i = h' \circ \iota$. Together with the homotopy above, the latter gives the needed map of $\mathbb{D} \times \mathbb{S}^1 \to$ $\to E$. $\qquad \square$

Claim 6.3. *There is a positive integer $n = n(F, \dim B)$ such that the following holds.*

Assume $F \to E \to B$ is a fiber bundle such that

(1) the fiber F is simply connected;
(2) the base B is reliable;
(3) the fundamental group of B is nilpotent and is generated by $\leqslant c(\dim B)$ elements;
(4) the bundle admits a trivialization over the 1-skeleton of B.

Then there is an at most n-fold cover $\tilde{E}$ of the total space E which is reliable.

Proof. The proof is divided into two steps.
Step 1. We will use the Corollary 4.2 to construct an at most $n = n(F, \dim B)$-fold cover $\tilde{B} \to B$ such that for any two elements $\bar{\gamma} \in \pi_1(\tilde{B}, \bar{p})$ and $\bar{\tau} \in \text{Tor} \, \pi_1(\tilde{B}, \bar{p})$ there is a map $\bar{h}: \mathbb{S}^1 \times \mathbb{S}^1 \to \tilde{B}$ such that

$$\bar{h}(u, v) = \bar{p}, \quad \bar{h}(u, *) \sim \bar{\gamma} \quad \text{and} \quad \bar{h}(*, v) \sim \bar{\tau},$$

which satisfy the condition in the definition of reliable manifold and such that the induced bundle $\bar{h}^*$ over $\mathbb{S}^1 \times \mathbb{S}^1$ with fiber F is homotopically trivial.

Let $\gamma \in \pi_1(E, p)$ and $\tau \in \text{Tor}(\pi_1(E, p))$. Let us denote by $\bar{\gamma} \in \pi_1(B, \bar{p})$ and $\bar{\tau} \in \pi_1(B, \bar{p})$ their projections to B.

From the assumptions we have that there is a map $\bar{h}: \mathbb{S}^1 \times \mathbb{S}^1 \to B$ such that

$$\bar{h}(u, v) = \bar{p}, \quad \bar{h}(u, *) \sim \bar{\gamma} \quad \text{and} \quad \bar{h}(*, v) \sim \bar{\tau},$$

which satisfy condition (1) in the definition of a reliable manifold.

The induced bundle f^* over $\mathbb{D} \times \mathbb{S}^1$ is trivial since the bundle over $\mathbb{S}^1$ is. Consider the induced bundle $\bar{h}^*$ over $\mathbb{S}^1 \times \mathbb{S}^1$. Pulling back by the finite cover $\iota: \tilde{T}^2 \to \mathbb{S}^1 \times \mathbb{S}^1$ the induced bundle $(\bar{h} \circ \iota)^*$ over $\tilde{T}^2$ becomes trivial, since it is also can be induced vial pull back by $i: \tilde{T}^2 \to \mathbb{D} \times \mathbb{S}^1$ and the bundle over $\mathbb{D} \times \mathbb{S}^1$ is trivial. The bundle over $\mathbb{S}^1 \times \mathbb{S}^1$ admits a trivialization on 1-skeleton for the standard product CW-structure. Therefore it can be induced from a bundle on $\mathbb{S}^2$ and a map of degree 1 which sends the 1-skeleton of $\mathbb{S}^1 \times \mathbb{S}^1$ to a single point of $\mathbb{S}^2$. Such a bundle over $\mathbb{S}^2$ can be described by its clutching map which is a one-parameter family of homeomorphisms $\kappa: \mathbb{S}^1 \times F \to F$.

Note that the homotopy type of κ does not depend on the choice of a trivialization over the 1-skeleton of $\mathbb{S}^1 \times \mathbb{S}^1$. Since the induced bundle over $\tilde{T}^2$ is trivial we

have that if $\iota\colon \tilde{T}^2 \to \mathbb{S}^1 \times \mathbb{S}^1$ is an N-fold cover then $\kappa^N\colon \mathbb{S}^1 \times F \to F$ is homotopic to the map $(z, x) \mapsto x$. By Corollary 4.2, $\kappa^{n(F)}$ is homotopic to the constant map $(z, x) \mapsto x$.

Since $\pi_1 B$ is nilpotent and generated by $\leqslant c(\dim B)$ elements, it follows that there is an at most $n' = n'(F, \dim B)$-fold cover $\widetilde{B}$ of B such that for any two elements $\bar{\gamma} \in \pi_1(\tilde{B}, \bar{p})$ and $\bar{\tau} \in \pi_1(\tilde{B}, \bar{p})$ there is a map $\bar{h}\colon \mathbb{S}^1 \times \mathbb{S}^1 \to B$ such that

$$\bar{h}(u, v) = \bar{p}, \quad \bar{h}(u, *) \sim \bar{\gamma} \quad \text{and} \quad \bar{h}(*, v) \sim \bar{\tau},$$

and such that the induced bundle $\bar{h}^*$ over $\mathbb{S}^1 \times \mathbb{S}^1$ and fiber F is homotopy trivial.

From now on we will assume that that $B = \tilde{B}$.

Step 2. Now, using Technical Lemma 4.3, we will prove that a section over $\bar{h}\colon \mathbb{S}^1 \times \times \mathbb{S}^1 \to B$, after finite cover (which is in fact a cover of $\tilde{T}^2$) bounds a solid torus; that is, E is reliable.

Let us summarize what we have:

(1) The induced bundle f^* over $\mathbb{D} \times \mathbb{S}^1$ is trivial since the bundle over $\mathbb{S}^1$ is.
(2) There is yet another trivialization over $\partial\mathbb{D} \times \mathbb{S}^1 = \tilde{T}^2$ which is induced from a trivialization on $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$.

We can assume that the bundle over $\mathbb{D} \times \mathbb{S}^1$ is smooth and that both of the above trivializations are smooth. Consider the vector field $\bar{V} = \frac{\partial}{\partial t}$ on $\mathbb{D} \times \mathbb{S}^1$; here t is the $\mathbb{S}^1$-coordinate and we assume $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$.

Let $p\colon E' \to \mathbb{D} \times \mathbb{S}^1$ be the pullback bundle over $\mathbb{D} \times \mathbb{S}^1$.

Let V be the horizontal lift of $\bar{V}$ over $\tilde{T}^2$ with respect to trivialization (2) above. Using partition of unity we can extend it to a vector field (which we'll still denote by V) on E' which projects to $\bar{V}$. Let $\varphi_t\colon E' \to E'$ be the time t integral flow of V. Note that by construction

$$(6.1) \qquad\qquad \varphi_1(q) = q \text{ for any } q \in p^{-1}(\partial\mathbb{D} \times \mathbb{S}^1).$$

Since E' is a trivial bundle we can identify its total space with $\mathbb{D} \times F \times \mathbb{S}^1$ via, say, trivialization (1) above. Consider the map $m\colon \mathbb{D} \times F \times \mathbb{R} \to E$ given by the formula

$$(u, x, t) \mapsto \varphi_t(u, x, [0]),$$

here $u \in \mathbb{D}$, $x \in F$ and $[0] \in \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$.

By (6.1) it follows that $m(u, x, n) = m(u, x, 0)$ for any $n \in \mathbb{Z}$, $u \in \partial\mathbb{D}$ and $x \in F$.

Note that there is a map $\hat{h}\colon \mathbb{D} \times F \to F$ for which the following identity holds

$$m(u, \hat{h}(u, x), 0) = m(u, x, 1).$$

Moreover $\hat{h}(u, x) = x$ for any $u \in \partial\mathbb{D}$ and $x \in F$. This means that we can apply Technical Lemma (4.3) to $\hat{h}$.

Consider a horizontal section of E' over $\partial\mathbb{D} \times \mathbb{S}^1$ with respect to (2). Fix $w \in \mathbb{S}^1$. This gives a with respect to (2) horizontal section over $\partial\mathbb{D} \times w$. Since F is simply connected this section can be extended to a section over $\mathbb{D} \times w$ which in trivialization (1) has the form $u \mapsto (u, w, f(u))$ for some map $f\colon \mathbb{D} \to F$. As in Technical Lemma (4.3) set $f_{\hat{h}}(u) = \hat{h}(u, f(u))$.

Then , by Technical Lemma (4.3)

$$(6.2) \qquad\qquad [f_{\hat{h}}] - [f] = [\beta] \in \mathrm{Tor}(\pi_2 F)$$

Consider the sequence of maps $f^n\colon \mathbb{D} \to F$ defined recursively, $f^0 = f$ and $f^{n+1} = f_{\hat{h}}^n$. By (6.2),

$$[f^n] - [f^0] = n\cdot[\beta] = 0 \tag{6.3}$$

for some positive integer n.

Note that $m(u, f^0(u), n) = m(u, f^n(u), 0)$. In particular, by (6.3), the two maps $\mathbb{D} \to E$ given by $u \mapsto m(u, f(u), 0)$ and $u \mapsto m(u, f(u), n)$ are homotopic rel. to $\partial\mathbb{D}$; denote by s_t, $t \in [0, 1]$ the homotopy. It follows that after taking an n-fold cover of $\widetilde{T}^2$ it has a section which can be contracted by a solid torus; the map of the solid torus is a concatenation of $(u, t) \mapsto m(u, f(u), t)$ for $t \in [0, n]$ and $(u, t) \to s_t(u)$ for $t \in [0, 1]$. $\qquad\square$

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Vitali Kapovitch, Department of Mathematics, University of Toronto, Toronto, Ontario, M5S 2E4, Canada.
E-mail address: vtk@math.toronto.edu

Anton Petrunin, Department of Mathematics, Pennsylvania State University, University Park, State College, PA 16802, USA.
E-mail address: petrunin@math.psu.edu

Wilderich Tuschmann, Arbeitsgruppe Differentialgeometrie, Institut für Algebra und Geometrie, Fakultät für Mathematik, Karlsruher Institut für Technologie, Englerstr. 2, D-76131 Karlsruhe, Deutschland.
E-mail address: tuschmann@kit.edu