Solution to problem 1d) on page 77.

Let M be the surface of revolution obtained by rotating the graph $z = x^2$ around z-axis. Then in polar coordinates the metric on M has the form $ds^2 = (1+4r^2)dr^2 + r^2 d\theta^2$. (In the notations of the book $r = v$ and $\theta = u$). We claim that every geodesic which is not a meridian intersects itself infinitely many times.

Step 1. Let us first prove that every such geodesic winds around z-axis infinitely many times as $t \to \infty$ and $t \to -\infty$. In other words, let $\gamma(t) = (r(t), \theta(t))$ be a unit speed geodesic with $\theta(t) \neq \mathit{const}$. Then we claim that $\theta(t) \to +\infty$ or $\theta(t) \to -\infty$ as $t \to \infty$ (and the same holds for $t \to -\infty$). By uniqueness of geodesics $\gamma(t)$ can not be tangent to a meridian for any t and hence $\theta(t)$ is monotone and $\theta'(t)$ has the same sign for all t. Therefore, it's enough to show that $\int_0^\infty \theta'(t)dt$ and $\int_{-\infty}^0 \theta'(t)dt$ and diverge.

Let us consider the case $t \geq 0$. The other half of the geodesic γ is treated similarly. We first observe that $r(t)$ satisfies the inequality

$$r(t) \leq \sqrt{r^2(0)+t} \text{ for all } t \geq 0 \tag{1}$$

Indeed, we have

$$\begin{aligned} 1 = |\gamma'(t)| = \sqrt{(1+4r^2)(r')^2 + r^2(\theta')^2} \geq \sqrt{1+4r^2}|r'| \geq \\ \geq |2rr'| = |(r^2)'| \end{aligned} \tag{2}$$

Integrating the inequality $|(r^2(t))'| \leq 1$ we get $|r^2(t) - r^2(0)| \leq \int_0^t |(r^2(s))'|ds \leq \int_0^t ds = t$ and hence $r^2(t) \leq r^2(0) + t$ as claimed.

By part c) of the problem we have that $\gamma(t)$ satisfies the Clairaut relation

$$r(t)\cos\beta(t) = C \tag{3}$$

for some constant $C \neq 0$. Here $\beta(t)$ is the angle between $\gamma'(t)$ and the parallel $r = \mathit{const}$ passing through $\gamma(t)$. In other words β is the angle between $\gamma'(t)$ and $\frac{\partial}{\partial\theta}|_{\gamma(t)}$. Since $|\gamma'(t)| = 1$ we have that $\cos\beta = \frac{\langle \gamma'(t), \frac{\partial}{\partial\theta}\rangle}{|\frac{\partial}{\partial\theta}|} = \frac{\langle r'(t)\frac{\partial}{\partial r} + \theta'(t)\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\rangle}{r(t)} = \theta'(t)r(t)$. Using Clairaut's equation this gives

$$\theta'(t) = \frac{C}{r^2(t)} \tag{4}$$

which by (1) implies
$|\theta'(t)| \geq \frac{|C|}{r^2(0)+t}$. Therefore

$$\int_0^\infty |\theta'(t)|dt \geq \int_0^\infty \frac{|C|}{r^2(0)+t} = +\infty$$

Step 2. Next, let us show that $r(t) \to \infty$ as $t \to \pm\infty$. Together with the fact that $\gamma(t)$ spirals around z-axis infinitely many times this easily implies that it has infinitely many self intersections.

Suppose $\beta(t_0) > 0$ for some t_0. Then Clairaut's relation easily implies that $\sin\beta(t) \geq \sin\beta(t_0)$ for all $t > t_0$ and hence $r'(t) > const > 0$ for $t > t_0$ which implies that $r(t) \to \infty$ as $t \to \infty$.

Now suppose $\beta(t) \leq 0$ for all t so that $r(t)$ is decreasing for all t. It's immediate from Clairaut's equation that we can not have $\lim_{t\to\infty} r(t) = 0$ and hence $\lim_{t\to\infty} r(t) = r_0 > 0$.

Geometrically this means that $\gamma(t)$ "spirals down" and approaches the parallel $r = r_0$ as $t \to \infty$ but never reaches it. We will show that this can not happen for a geodesic.

From (2) we have $1 = (1 + 4r^2)(r')^2 + r^2(\theta')^2 \overset{by\ (4)}{=} (1 + 4r^2)(r')^2 + \frac{C^2}{r^2}$. Therefore $\lim_{t\to\infty}(r')^2 = \frac{1-\frac{C^2}{r_0^2}}{1+4r_0^2}$. This of course means that $r_0^2 = C^2$ and $\lim_{t\to\infty} r'(t) = 0$ since otherwise $\lim_{t\to\infty} r'(t) = const < 0$ and hence $\lim_{t\to\infty} r(t) = -\infty$ which is a contradiction.

Thus $\lim_{t\to\infty} r'(t) = 0$. Finally, by the second geodesic equation from b) we have

$$r'' - \frac{(\theta')^2}{1 + 4r^2} + \frac{4r}{1 + 4r^2}(r')^2 = 0$$

which by (4) gives

$$r'' = \frac{C^2}{(1 + 4r^2)r^4} - \frac{4r}{1 + 4r^2}(r')^2$$

Since $r(t) \to r_0$ and $r'(t) \to 0$ as $t \to \infty$ this means that $r'' > \frac{C^2}{2(1+4r_0^2)r_0^4} > 0$ for all large t which implies that $r'(t) > 0$ for large enough t.

Lastly, applying the same argument to $\tilde{\gamma}(t) = \gamma(-t)$ we conclude that $\lim_{t\to-\infty} r(t) = +\infty$ as well.

$\square$