(1) Let $f\colon S_1 \to S_2$ be a smooth map between smooth surfaces in $\mathbb{R}^3$. It's called a *global isometry* if it's a local isometry and is a diffeomorphism.

Prove that if f is a global isometry then $d_{S_1}(p,q) = d_{S_2}(f(p), f(q))$ for any $p, q \in S_1$.

Is that true if f is only known to be a local isometry?

(2) Prove that for any $a, b, c, d \in \mathbb{R}^3$ the following equality holds

$$\langle a \times b, c \times d\rangle = \langle a, c\rangle \cdot \langle b, d\rangle - \langle a, d\rangle \cdot \langle b, c\rangle$$

(3) Let S be a smooth surface in $\mathbb{R}^3$. For any $p \in S$ and $r > 0$ define the metric ball of radius r around p in S to be the set

$$B_S(p,r) = \{x \in S | \text{such that } d_S(p,x) < r\}$$

Let $S = S^2 = \{x^2 + y^2 + z^2 = 1\}$.

Find the area of $B_S(p,r)$ for $p \in S$ and $r > 0$.

(4) Let U be an open subset in $S^2 = \{x^2 + y^2 + z^2 = 1\}$ and let V be an open subset in the xy-plane in $\mathbb{R}^3$.

Prove that U and V are not locally isometric.