Solution to problem 2.3.6 from homework 3

The only new part of this problem is to show that for a unit speed curve $\gamma(s)$ in $\mathbb{R}^2$ the centre of the circle through the points $\gamma(s_0), \gamma(s_0-\varepsilon), \gamma(s_0-\varepsilon)$ converges to $O = \gamma(s_0) + \frac{1}{k_{s_0}} n_{s_0}$ and its radius converges to $R = \frac{1}{k_{s_0}}$ as $\varepsilon \to 0$.

Without loss of generality we can assume that $s_0 = 0$. Let $\gamma_1(s)$ be the unit circle of radius R centred at O parameterized in such a way that $\gamma(0) = \gamma_1(0)$ and $\gamma'(0) = \gamma_1'(0)$.

Then, as was shown on the previous homework , $\gamma(s)$ and $\gamma_1(s)$ have the same Taylor polynomials of order 2 at 0 so that $\gamma(s) = \gamma_1(s) + o_{s\to 0}(s^2)$.

(Recall that the notation $f(s) = o_{s\to 0}(g(s))$ means that there exists $\alpha(s) {\to} 0$ as $s \to 0$ with $f(s) = \alpha(s)g(s)$.)

Let $O(s)$ be the centre of the circle through $\gamma(s), \gamma(-s), \gamma(0)$ and $O_1(s)$ be the centre of the circle through $\gamma_1(s), \gamma_1(-s), \gamma_1(0)$. Let $R(s)$ and $R_1(s)$ be the radii of these circles. Obviously, $O_1(s) = O$ and $R_1(s) = R$ for all s because γ_1 is a circle.

We will show that $O(s) - O_1(s) = o_{s\to 0}(1)$ which obviously implies that $\lim_{s\to 0} O(s) = \lim_{s\to 0} O(s) = O$.

Recall that the centre of the circle through points $A, B, C \in \mathbb{R}^2$ lies on the intersection of the perpendiculars to $[A, B]$ and $[A, C]$ through their midpoints. Also by the sine law the radius of this circle is equal to $\frac{|BC|}{2\sin\alpha}$ where α is the angle of the triangle ΔABC at the vertex A.

Now , let $A = \gamma(0), B = \gamma(s)$ and $C = \gamma(-s)$. The midpoint of $[A, B]$ is $m(s) = (\gamma(0) + \gamma(s))/2$. Obviously, $m(s) = m_1(s) + o_{s\to 0}(s^2)$ where $m_1(s)$ is the midpoint of $[\gamma_1(0), \gamma_1(s)]$. As mentioned above, the line through $m(s)$ and $O(s)$ is perpendicular to $[\gamma(0), \gamma(s)]$ and so its direction is given by rotating $v(s) = \frac{\gamma(s)-\gamma(0)}{|\gamma(s)-\gamma(0)|}$ by $\pi/2$ (define $v_1(s)$ similarly). Since $\gamma(s) = \gamma_1(s) + o_{s\to 0}(s^2)$ it's easy to see that $v(s) = v_1(s) + o_{s\to 0}(s)$. Thus, to prove that $O(s) - O_1(s) = o_{s\to 0}(1)$ it's enough to show that $|m(s) - O(s)| - |m_1(s) - O_1(s)| \to 0$ as $s \to 0$. Since $|m(s) - \gamma(0)| \to 0$ as $s \to 0$ this is equivalent to showing that $R(s) \to R$.

An easy computation shows that for the circle γ_1 we have that $|\gamma_1(s) - \gamma_1(-s)| = 2R\sin(s/R)$ and the angle $\angle\gamma_1(s)\gamma_1(0)\gamma_1(-s) = \pi - s/R$.

Since $\gamma(s) = \gamma_1(s) + o_{s\to 0}(s^2)$ this easily implies that $|\gamma(s) - \gamma(-s)| = 2R\sin(s/R) + o(s^2)$ and $\angle\gamma(s)\gamma(0)\gamma(-s) = \pi - s/R + o(s)$. therefore, by the sine law,

$$R(s) = \frac{|\gamma(s) - \gamma(-s)|}{2\sin\angle\gamma(s)\gamma(0)\gamma(-s)} = \frac{2R\sin(s/R) + o(s^2)}{2\sin(\pi - s/R + o(s))}$$

and hence

$$\lim_{s\to 0} R(s) = \lim_{s\to 0} \frac{2R\sin(s/R) + o(s^2)}{2\sin(\pi - s/R + o(s))} = \lim_{s\to 0} \frac{2R\sin(s/R)}{2\sin(\pi - s/R)} = R$$