(1) (10 pts)

Let $\gamma(t) = (t - \sin t, 1 - \cos t)$ be the cycloid obtained by tracing a point on a circle of radius 1 rolling along the x-axis.

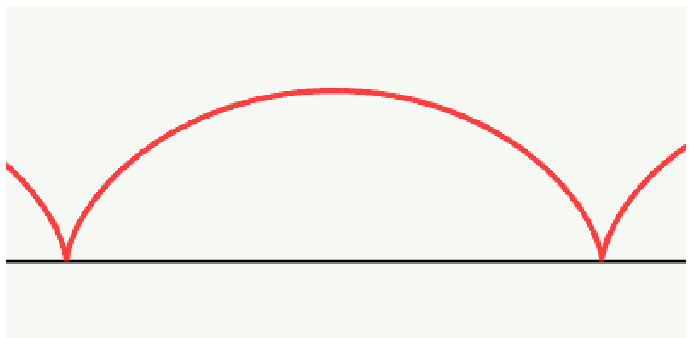

FIGURE 1. Cycloid

(a) Is γ regular?

Solution

We compute $\gamma'(t) = (1-\cos t, \sin t)$. We see that $\gamma'(2\pi k) = (1-\cos 2\pi k, \sin 2\pi k) = (0,0)$ which means that γ is not regular.

(b) Find the length of one arch of γ.

Solution

The cycloid touches the x-axes when $1 - \cos t = 0$ i.e. for $t = 2\pi k$ where $k \in Z$. Therefore we need to compute the length of γ on the interval $[0, 2\pi]$. We have $L(\gamma|_{[0,2\pi]}) = \int_0^{2\pi} |\gamma'(t)|dt = \int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t}dt = \int_0^{2\pi} \sqrt{1 + \cos^2 t - 2\cos t + \sin^2 t}dt = \int_0^{2\pi} \sqrt{2(1-\cos t)}dt = \int_0^{2\pi} \sqrt{4\sin^2(t/2)}dt =$

$= \int_0^{2\pi} |2\sin(t/2)|dt = \int_0^{2\pi} 2\sin(t/2)dt = -4\cos(t/2)|_0^{2\pi} = 8.$

Answer: $L = 8$.

(2) (10 pts) Let $\gamma(t)$ be a regular cure in $\mathbb{R}^3$ such that its tangent line at $\gamma(t)$ (i.e. the line passing through $\gamma(t)$ and parallel to $\gamma'(t)$) passes through 0 for all t.

Prove that there is a line l in $\mathbb{R}^3$ such that $\gamma(t) \in l$ for all t.

Hint: Differentiate $\gamma(t) \times \gamma'(t)$.

Solution

By the assumption $\gamma(t)$ and $\gamma'(t)$ are proportional which implies that $\gamma(t) \times \gamma'(t) = 0$ for all t. Differentiating this equality we get

$0 = \gamma'(t) \times \gamma'(t) + \gamma(t) \times \gamma''(t) = \gamma(t) \times \gamma''(t)$ since $\gamma'(t) \times \gamma'(t) = 0$.

Thus $\gamma(t) \times \gamma''(t) = 0$ for all t.

Fix some t_0. If $\gamma(t_0) \neq 0$ then $\gamma(t_0) = \lambda\gamma'(t_0)$ for some $\lambda \neq 0$ and hence $0 = \gamma(t) \times \gamma''(t) = \lambda\gamma'(t_0) \times \gamma''(t_0)$. Therefore $\gamma'(t_0) \times \gamma''(t_0) = 0$. Hence $k(t_0) = \frac{|\gamma'(t_0) \times \gamma''(t_0)|}{|\gamma'(t_0|^3} = 0$.

Thus the curvature of γ is zero for all points t such that $\gamma(t) \neq 0$. Suppose $\gamma(t_0) = 0$. Since γ is regular $\gamma'(t_0) \neq 0$ and hence $\gamma(t) \neq 0$ for all $t \neq t_0$ near t_0. By above, for all such t we have that $k(t) = 0$. By continuity of $k(t)$ this implies that $k(t_0) = 0$ as well.

Thus, $k(t) = 0$ for all t which implies that γ is contained in a line by a theorem from class.

(3) (10 pts) Let $\gamma(t)$ be a regular curve in $\mathbb{R}^2$.

Let $A(t)$ be the 2×2 matrix with columns $\gamma'(t), \gamma''(t)$.

Prove that the signed curvature k_t of γ is positive if and only if $\det A(t) > 0$.

Hint: Reduce to the case when γ is unit speed.

Solution

Let s be the arc length parameter of γ then we have the general formulas $\frac{d\gamma}{ds} = \frac{\gamma'}{|\gamma'|}, \frac{d^2\gamma}{ds^2} = \frac{|\gamma'|^2\gamma'' - \langle\gamma',\gamma''\rangle\gamma'}{|\gamma'|^4} = \frac{\gamma''}{|\gamma'|^2} - \frac{\langle\gamma',\gamma''\rangle\gamma'}{|\gamma'|^4}$.

Therefore, $\det(\frac{d\gamma}{ds}, \frac{d^2\gamma}{ds^2}) = \det(\frac{\gamma'}{|\gamma'|}, \frac{\gamma''}{|\gamma'|^2} - \frac{\langle\gamma',\gamma''\rangle\gamma'}{|\gamma'|^4}) = \det(\frac{\gamma'}{|\gamma'|}, \frac{\gamma''}{|\gamma'|^2}) = \frac{1}{|\gamma'|^3}\det(\gamma', \gamma'')$. Since $|\gamma'| > 0$ we have that $\det(\frac{d\gamma}{ds}, \frac{d^2\gamma}{ds^2})$ and $\det(\gamma', \gamma'')$ have the same sign. Thus it's enough to show that for a unit speed curve $\gamma(s)$, the curvature has the same sign as $\det(\frac{d\gamma}{ds}, \frac{d^2\gamma}{ds^2})$.

From now on we will assume that γ is unit speed. To finish the proof it's enough to establish the following

Claim. Let $\gamma(s)$ be a unit speed curve in $\mathbb{R}^2$. Then its signed curvature k_s is equal to $\det(\gamma', \gamma'')$

Indeed, recall that the signed curvature k_s is defined as follows. Let $\vec{\tau} = \gamma'$ be the unit tangent. Then the unit normal $\vec{n}$ is defined by rotating $\vec{\tau}$ by $\pi/2$ counterclockwise. Then k_s is determined by the formula $\gamma'' = k_s\vec{n}$. Observe that counterclockwise rotation $R_{\pi/2}$ in $\mathbb{R}^2$ sends $v = (x, y)$ to $(-y, x)$. Therefore $\det(v, R_{\pi/2}(v)) = \det \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = x^2 + y^2$. In particular, if $|v| = 1$ then $\det(v, R_{\pi/2}(v)) = 1$.

Applying this to $v = \vec{\tau}$ we get $\det(\gamma', \gamma'') = \det(\vec{\tau}, k_s\vec{n}) = k_s \det(\vec{\tau}, \vec{n}) = k_s \det(\vec{\tau}, R_{\pi/2}(\vec{\tau})) = k_s \cdot 1 = k_s$. This proves the Claim. $\square$.

(4) (10 pts) Let $\gamma(t) = (e^t, t, e^{-t})$.

Find the curvature and the torsion of γ at $t = 0$.

Solution

We compute $\gamma'(t) = (e^t, 1, -e^{-t}), \gamma''(t) = (e^t, 0, e^{-t}), \gamma'''(t) = (e^t, 0, -e^{-t})$. Plugging in $t = 0$ this gives $\gamma'(0) = (1, 1, -1), \gamma''(0) = (1, 0, 1), \gamma'''(0) = (1, 0, -1)$.

We have the general formulas

$$k = \frac{|\gamma' \times \gamma''|}{|\gamma'|^3} \qquad \tau = \frac{\langle \gamma''', \gamma' \times \gamma'' \rangle}{|\gamma' \times \gamma''|^2}$$

Plugging the values of $\gamma'(0), \gamma'''(0), \gamma'''(0)$ in these formulas we get $\gamma'(0) \times \gamma''(0) = (1, -2, -1)$, $|\gamma'(0) \times \gamma''(0)| = \sqrt{6}$, $|\gamma'(0)| = \sqrt{3}$.

hence

$$k(0) = \frac{\sqrt{6}}{(\sqrt{3})^3} = \frac{\sqrt{2}}{3} \qquad \tau(0) = \frac{2}{6} = \frac{1}{3}$$

(5) (8 pts) Give the following definitions:

(a) Signed curvature of a regular curve in $\mathbb{R}^2$.
(b) An open subset of a surface in $\mathbb{R}^3$.
(c) Tangent space at a point on a smooth surface in $\mathbb{R}^3$.

Solution

(a) Let $\gamma(s)$ be arc length parameterization of γ. Let $\vec{\tau} = \gamma'$ be the unit tangent. Then the unit normal $\vec{n}$ is defined by rotating $\vec{\tau}$ by $\pi/2$ counterclockwise. Then the signed curvature k_s is determined by the formula $\gamma'' = k_s\vec{n}$.

(b) An open subset of a surface S in $\mathbb{R}^3$ is an intersection of an open subset of $\mathbb{R}^3$ with S.

(c) Let S be surface in $\mathbb{R}^3$. Let $p \in S$ be a point. let $f\colon U \to V$ be a smooth surface patch where U is an open subset of $\mathbb{R}^2$, V is an open subset of S and

$p = f(q)$ for some $q \in U$. Since S is a subset of $\mathbb{R}^3$ we can view f as a map $f\colon U \to \mathbb{R}^3$. Then the tangent space to S at p is defined by the formula

$$T_pS := df_q(\mathbb{R}^2)$$

(6) (12 pts) Let $S = \{(x, y, z) \in \mathbb{R}^3 |$ such that $x^2 + 2y^2 - 3z^2 = 0$ and $z \geq 0\}$.
(a) Prove that S is a not smooth surface in $\mathbb{R}^3$.
(b) Prove that $S_1 = S \backslash \{(0, 0, 0)\}$ is a smooth surface in $\mathbb{R}^3$.
(c) Let $p = (1, 1, 1)$. Find T_pS_1.

Solution

(7) Recall a theorem from class that a subset $S \subset \mathbb{R}^3$ is a smooth surface if an only if it is locally given by either a graph of a smooth function $z = z(x, y)$ or $y = y(x, z)$ or $x = x(y, z)$.

Let $p = (0, 0, 0) \in S$. We claim that near p neither of these 3 possibilities hold.

Note that if $(x_0, y_0, z_0) \in S$ then $(-x_0, y_0, z_0) \in S$ also, therefore S is not a graph of $x = x(y, z)$ near 0 since in any small ball around zero we can find $(\pm x_0, y_0, z_0) \in S$ with $x_0 \neq 0$. Likewise, if $(x_0, y_0, z_0) \in S$ then $(x_0, -y_0, z_0) \in S$ too and hence S is not a graph of $y = y(x, z)$ near 0. The restriction $z \geq 0$ means that S IS a graph of $z = z(x, y)$ near 0 (and, in fact, globally). Namely, it's a graph of $z = +\sqrt{\frac{x^2+2y^2}{3}}$. However, this function is not smooth at zero. Therefore, S is not a smooth surface.

(8) let $U = \mathbb{R}^3 \backslash \{(0, 0, 0)\}$. Then U is an open subset of $\mathbb{R}^3$. Let $f\colon U \to \mathbb{R}$ be given by $f(x, y, z) = x^2 + 2y^2 - 3z^2$. Then $S_1 = \{f = 0\}$. We compute $\nabla f = (2x, 4y, -6z) \neq 0$ on U. hence 0 is a regular value of f and $S_1 = \{f = 0\}$ is a smooth surface.

(9) By part (b) we have that $S_1 = \{f = 0\}$. By the general formula, $T_pS_1 = \nabla f_p^{\perp} = (2, 4, -6)^{\perp} = \{2x + 4y - 6z = 0\}$.