(1) Consider the IVP

$$\begin{cases} y' = y^2 + 1 \\ y(0) = 0 \end{cases}$$

Find the biggest interval $(-a, a)$ for which the solution is guaranteed to exist by the existence theorem.

(2) (a) Show that if $f(t) \geq 0$ satisfies

$$|f'(t)| \leq a \cdot f(t)$$

for some constant $a > 0$ then $f(t) \leq f(0)e^{a|t|}$

(b) Show that if $y(t)$ is a vector-valued function satisfying

$$|y'(t)| \leq a \cdot |y(t)|$$

then $|y(t)| \leq |y(0)|e^{a|t|}$

Hint: Put $f(t) = |y(t)|^2$ and use a).

(3) Show that the solution of the IVP

$$\begin{cases} y' = \sin(y) + 1 \\ y(t_0) = y_0 \end{cases}$$

exists for all real t.

Hint: Show that the existence interval guaranteed by the existence theorem is the same for all initial conditions t_0, y_0.