1. Let A be the following matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ -2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & 1 \end{pmatrix}$$

Find a basis of the solution space for the system $y' = Ay$ where y is a column vector.

2. Consider a higher order linear ODE with constant coefficients

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0 y = 0 \tag{1}$$

Let

$$f(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_0$$

be its characteristic polynomial.

(a) Convert the equation to a 1st order linear system with constant coefficients $x' = Ax$.

(b) Show that the characteristic polynomial of A is given by

$$P_A(\lambda) = (-1)^n f(\lambda)$$

Hint: Expand along the first row and use induction.

(c) Suppose λ_1 is a root of $P_A(\lambda)$ of multiplicity n.

(i) Show that the Jordan form J of A consists of a single Jordan block of size $n \times n$.
Hint: Put $v_n = (0, \ldots, 0, 1)^t$ and verify that $(A - \lambda I)v_n \neq 0, \ldots, (A - \lambda I)^{n-1}v_n \neq 0$. To see this show that the $(n-i)$th coordinate of $(A - \lambda I)^i v_n$ is $\neq 0$ for all $i = 0, \ldots, n-1$.

(ii) Find a basis of the solution space of (1).
Hint: Use that the columns of Te^{tJ} give a basis of the solution space of $x' = Ax$ where the columns of T are given by the Jordan basis found in (i).

(d) Show that if the characteristic polynomial $g(\lambda)$ of an equation

$$y^{(m)} + b_{m-1}y^{(m-1)} + \ldots + b_0 y = 0 \tag{2}$$

divides $f(\lambda)$ then any solution of (2) also solves (1).
Hint: First show that this is true if $f(\lambda) = \lambda g(\lambda)$.

(e) Suppose $f(\lambda)$ can be factored as

$$f(\lambda) = (\lambda - \lambda_1)^{l_1} \cdot \ldots \cdot (\lambda - \lambda_k)^{l_k}$$

where λ_is are distinct.
Find a basis of the solution space of (1).